WEIGHTED WEIERSTRASS’ THEOREM WITH FIRST DERIVATIVES

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ABSTRACT

We characterize the set of functions which can be approximated by continuous functions with the norm
\[ \| f \|_{L^\infty(w)} \]
for every weight \( w \). This fact allows to determine the closure of the space of polynomials in
\( L^\infty(w) \) for every weight \( w \) with compact support. We characterize as well the set of functions which can be
approximated by smooth functions with the norm
\[ \| f \|_{W^{1,\infty}(w_0,w_1)} := \| f \|_{L^\infty(w_0)} + \| f' \|_{L^\infty(w_1)}, \]
for a wide range of (even non-bounded) weights \( w_j \)’s. We allow a great deal of independence among the
weights \( w_j \)’s.

Key words and phrases: Weierstrass’ theorem; weight; Sobolev spaces; weighted Sobolev spaces.

1. INTRODUCTION

If \( I \) is any compact interval, Weierstrass’ Theorem says that \( C(I) \) is the largest set of functions which
can be approximated by polynomials in the norm \( L^\infty(I) \), if we identify, as usual, functions which are equal
almost everywhere. There are many generalizations of this theorem (see e.g. the monographs [L], [P], and
the references therein).

In [R1] and [PQRT1] we study the same problem with the norm \( L^\infty(I, w) \) defined by
\[ \| f \|_{L^\infty(I, w)} := \text{ess sup}_{x \in I} |f(x)|w(x), \]
where \( w \) is a weight, i.e. a non-negative measurable function and we use the convention \( 0 \cdot \infty = 0 \). Notice
that (1.1) is not the usual definition of the \( L^\infty \) norm in the context of measure theory, although it is the
correct one when working with weights (see e.g. [BO] and [DMS]). In [PQRT1] we improve the theorems
in [R1], obtaining sharp results for a large class of weights. Here we also study this problem both with the
norm (1.1) for every weight \( w \), and with the Sobolev norm \( W^{1,\infty}(I, w_0, w_1) \) defined by
\[ \| f \|_{W^{1,\infty}(I,w_0,w_1)} := \| f \|_{L^\infty(I,w_0)} + \| f' \|_{L^\infty(I,w_1)}, \]
since in many situations it is natural to consider the simultaneous approximation of a function and its first
derivative.

Considering weighted norms \( L^\infty(w) \) has been proved to be interesting mainly because of two reasons:
on the one hand, it allows to enlarge the set of approximable functions (since the functions in \( L^\infty(w) \) can
have singularities where the weight tends to zero); and, on the other one, it is possible to find functions
which approximate \( f \) whose qualitative behaviour is similar to the one of \( f \) at those points where the weight
tends to infinity.

Weighted Sobolev spaces are an interesting topic in many fields of Mathematics, as Approximation The-
ory, Partial Differential Equations (with or without Numerical Methods), and Quasiconformal and Quasireg-
ular maps (see e.g. [HKM], [IKNS1], [IKNS2], [K], [Ku], [KO] and [KS]). In particular, in [IKNS1] and
[IKNS2], the authors showed that the expansions with Sobolev orthogonal polynomials can avoid the Gibbs
phenomenon which appears with classical orthogonal series in \( L^2 \). In [ELW1], [EL] and [ELW2] the au-
thors study some examples of Sobolev spaces for \( p = 2 \) with respect to general measures instead of weights,

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in relation with ordinary differential equations and Sobolev orthogonal polynomials. The papers [RARP1], [RARP2], [R1], [R2] and [R3] are the beginning of a theory of Sobolev spaces with respect to general measures for $1 \leq p \leq \infty$. This theory plays an important role in the location of the zeroes of the Sobolev orthogonal polynomials (see [LP], [LPP], [RARP2] and [R2]). The location of these zeroes allows to prove results on the asymptotic behaviour of Sobolev orthogonal polynomials (see [LP]). The papers [APRR], [BFM], [CM], [FMP], [LPP] and [RY] deal with Sobolev spaces on curves and more general subsets of the complex plane.

In this paper we characterize the set of functions which can be approximated by continuous functions in $L^\infty(I, w)$, for any weight $w$ (see Theorem 2.1); as a consequence of this result, we obtain the set of functions which can be approximated by polynomials in $L^\infty(I, w)$, for any weight $w$ with compact support. Theorem 2.1 is an improvement over the previous result obtained in [PQRT1, Theorem 2.1]; while the conclusion of the theorems are the same, we have completely removed the technical hypothesis on the weight required in [PQRT1]. We also characterize the set of functions which can be approximated by $C^1$ functions in $W^{1,\infty}(I, w_0, w_1)$, for a wide range of (possibly unbounded) weights $w_0, w_1$, which have a great deal of independence among them. It is a remarkable fact that this last characterization depends on the value $L(a):=\limsup_{x\to a}|x-a|w_0(x)$ at every singular point $a$ of $w_1$ (see definitions 2.4 and 2.6 below). Depending on the value $L(a) = 0$, $0 < L(a) < \infty$ or $L(a) = \infty$, theorems 4.2, 4.3 and 4.4 describe, respectively, the set of functions which can be approximated by $C^1$ functions in $W^{1,\infty}(I, w_0, w_1)$, when there is just one singular point of $w_1$. Furthermore, some of the conditions appearing in the characterizations are not obvious at all. Besides, we would like to remark that our methods of proof are constructive.

The main result in Sobolev approximation is Theorem 4.5, which gives the characterization with infinitely many singular points of $w_1$ (even for non-bounded intervals), combining the results of theorems 4.2, 4.3 and 4.4.

We use these results in order to study the approximation by $C^\infty$ functions as well (see Theorem 5.2).

Some other results about weighted approximation with $k$ derivatives can be found in [PQRT2] and [PQRT3].

The outline of the paper is as follows: In Section 2 we find the closure of continuous functions in $L^\infty(I, w)$. Section 3 is dedicated to definitions and preliminary results. Section 4 presents the theorems on approximation by $C^1$ functions in $W^{1,\infty}(I, w_0, w_1)$. We prove the results on approximation by $C^\infty$ functions in Section 5.

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2. APPROXIMATION IN $L^\infty(I, w)$

Let us start with some definitions.

**Definition 2.1.** A weight $w$ is a measurable function $w : \mathbb{R} \to [0, \infty]$. If $w$ is only defined in $A \subset \mathbb{R}$, we set $w := 0$ in $\mathbb{R} \setminus A$.

**Definition 2.2.** Given a measurable set $A \subset \mathbb{R}$ and a weight $w$, we define the space $L^\infty(A, w)$ as the space of equivalence classes of measurable functions $f : A \to \mathbb{R}$ with respect to the norm

$$
\|f\|_{L^\infty(A, w)} := \text{ess sup}_{x \in A} |f(x)|w(x).
$$

We always consider the space $L^1(A)$, with respect to the restriction of the Lebesgue measure on $A$.

The theorems in this paper can be applied to functions $f$ with complex values, splitting $f$ into its real and imaginary parts. From now on, if we do not specify the set $A$, we are assuming that $A = \mathbb{R}$; analogously, if we do not specify the weight $w$, we are assuming that $w \equiv 1$.

**Definition 2.3.** Given a measurable set $A$, we define the essential closure of $A$, as the set

$$
\text{ess cl} A := \{ x \in \mathbb{R} : |A \cap (x-\delta, x+\delta)| > 0, \ \forall \delta > 0 \},
$$

where $|E|$ denotes the Lebesgue measure of $E$. 

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Definition 2.4. If $A$ is a measurable set, $f$ is a function defined on $A$ with real values and $a \in \text{ess cl } A$, we say that $\text{ess lim}_{x \rightarrow a} f(x) = l \in \mathbb{R}$ if for every $\varepsilon > 0$ there exists $\delta > 0$ such that $|f(x) - l| < \varepsilon$ for almost every $x \in A \cap (a-\delta,a+\delta)$. In a similar way we can define $\text{ess lim}_{x \rightarrow a} f(x) = \infty$ and $\text{ess lim}_{x \rightarrow a} f(x) = -\infty$. We define the essential superior limit and the essential inferior limit on $A$ as follows:

$\text{ess lim sup } f(x) := \inf_{\delta > 0} \sup_{x \in A \cap (a-\delta,a+\delta)} f(x),$ 

$\text{ess lim inf } f(x) := \sup_{\delta > 0} \inf_{x \in A \cap (a-\delta,a+\delta)} f(x).$

Remarks.

1. The essential superior (or inferior) limit of a function $f$ does not change if we modify $f$ on a set of zero Lebesgue measure.

2. When we say that there exists an essential limit (or essential superior limit or essential inferior limit), we are assuming that it is finite.

3. It is well known that

$\text{ess lim sup } f(x) \geq \text{ess lim inf } f(x),$ 

$\text{ess lim } f(x) = l$ if and only if $\text{ess lim sup } f(x) = \text{ess lim inf } f(x) = l.$

4. We impose the condition $a \in \text{ess cl } A$ in order to have the unicity of the essential limit. If $a \notin \text{ess cl } A$, then every real number is an essential limit for any function $f$.

Definition 2.5. Given a weight $w$, the support of $w$, denoted by $\text{supp } w$, is the complement of the largest open set $G \subset \mathbb{R}$ with $w = 0$ a.e. on $G$.

Definition 2.6. Given a weight $w$ we say that $a \in \text{supp } w$ is a singularity of $w$ (or singular for $w$) if

$\text{ess lim inf } w(x) = 0.$

We say that a singularity $a$ of $w$ is of type 1 if $\text{ess lim}_{x \rightarrow a} w(x) = 0$.

We say that a singularity $a$ of $w$ is of type 2 if $0 < \text{ess lim sup}_{x \rightarrow a} w(x) < \infty$.

We denote by $S(w)$ and $S_i(w)$ ($i = 1, 2$), respectively, the set of singularities of $w$ and the set of singularities of $w$ of type $i$.

We say that $a \in S^+(w)$ (respectively $a \in S^-(w)$) if $\text{ess lim inf}_{x \in \text{supp } w, x \rightarrow a^+} w(x) = 0$ (respectively $\text{ess lim inf}_{x \in \text{supp } w, x \rightarrow a^-} w(x) = 0$).

We say that $a \in S_i^+(w)$ (respectively $a \in S_i^-(w)$) if $a$ verifies the property in the definition of $S_i(w)$ when we take the limit as $x \rightarrow a^+$ (respectively $x \rightarrow a^-$).

Definition 2.7. Given a weight $w$, we define the right regular and left regular points of $w$, respectively, as

$R^+(w) := \{ a \in \text{supp } w : \text{ess lim inf } w(x) > 0 \},$ 

$R^-(w) := \{ a \in \text{supp } w : \text{ess lim inf } w(x) > 0 \}.$

The following result characterizes the set of functions which can be approximated by continuous functions in $L^\infty(w)$, for any weight $w$.

Theorem 2.1. Let $w$ be any weight and

$H_0 := \{ f \in L^\infty(w) : f$ is continuous to the right at every point of $R^+(w),$ 

$f$ is continuous to the left at every point of $R^-(w),$ 

for each $a \in S^+(w),$ $\text{ess lim}_{x \rightarrow a^+} |f(x) - f(a)| w(x) = 0,$ 

for each $a \in S^-(w),$ $\text{ess lim}_{x \rightarrow a^-} |f(x) - f(a)| w(x) = 0 \}.$

Then:

(a) The closure of $C(\mathbb{R}) \cap L^\infty(w)$ in $L^\infty(w)$ is $H_0$.

(b) If $w \in L^1_{\text{loc}}(\mathbb{R})$, then the closure of $C^\infty(\mathbb{R}) \cap L^\infty(w)$ in $L^\infty(w)$ is also $H_0$.

(c) If supp $w$ is compact and $w \in L^\infty(\mathbb{R})$, then the closure of the space of polynomials is $H_0$ as well.

(d) If $f \in H_0 \cap L^1(\text{supp } w)$, $S_1^+(w) \cup S_1^-(w) \cup S_2^+(w) \cup S_2^-(w)$ is countable and $|S(w)| = 0$, then $f$ can be approximated by functions in $C(\mathbb{R})$ with the norm $\| f \|_{L^\infty(w)} + \| f \|_{L^1(\text{supp } w)}$. 

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Remark. Recall that we identify functions which are equal almost everywhere.

As a consequence of this result and Theorem A below, we characterize the set of functions which can be approximated by polynomials in $L^{\infty}(w)$, for any weight $w$ with compact support.

**Definition 2.8.** Given a weight $w$ with compact support, a polynomial $p \in L^{\infty}(w)$ is said to be the *minimal polynomial* for $w$ if it is 0 or it is monic, and every polynomial in $L^{\infty}(w)$ is a multiple of $p$. We denote by $p_{w}$ the minimal polynomial for $w$.

It is clear that there always exists the minimal polynomial for $w$ (although it can be 0): it is sufficient to consider the monic polynomial in $L^{\infty}(w)$ of minimal degree.

**Theorem A.** [PQRT1, Theorem 2.2] Let us consider a weight $w$ with compact support. If $p_{w} \equiv 0$, then the closure of the space of polynomials in $L^{\infty}(w)$ is $\{0\}$. If $p_{w}$ is not identically 0, the closure of the space of polynomials in $L^{\infty}(w)$ is the set of functions $f$ such that $f/p_{w}$ is in the closure of the space of polynomials in $L^{\infty}(|p_{w}|w)$.

Remark. The weight $|p_{w}|w$ is bounded (since $p_{w} \in L^{\infty}(w)$) and has compact support. Then we know which is the closure of the space of polynomials in $L^{\infty}(|p_{w}|w)$ by Theorem 2.1.

In the proof of Theorem 2.1 we need the following lemma.

**Lemma 2.1.** Let us consider a weight $w$ with $a \in S_{w}^{+}(w) \cup S_{w}^{-}(w)$. Let us fix $\eta > 0$ and a function $f$ with $f \in L^{\infty}(w)$ such that $\limsup_{x \to a^{+},w} f(x) - f(a)(x) = 0$. Then, there exists $b_{3} \in (a, a + 1)$ such that for any $a < b_{1} < b_{2} < b_{3}$ there exist $b_{0} \in (b_{1}, b_{2})$ and a function $g \in L^{\infty}(w) \cap C([a, b_{0}])$, with $g = f$ in $R \setminus (a, b_{0})$, $\|f - g\|_{L^{\infty}(w)} < \eta$ (and $\|f - g\|_{L^{1}(\supw)} < \eta$ if $f \in L^{1}(\supw)$).

Remark. A similar result is true if $a \in S_{w}^{+}(w) \cup S_{w}^{-}(w)$.

**Proof.** Let us fix $\varepsilon > 0$. Since $a \in S_{w}^{+}(w) \cup S_{w}^{-}(w)$, $\limsup_{x \to a^{+},w} w(x) = m \in [0, \infty)$. It follows that there exists $\delta_{1} > 0$ such that $w(x) \leq m + 1$, a.e. $x \in (a, a + \delta_{1})$.

If $f \in L^{1}(\supw)$, there exists $\delta_{2} > 0$, such that $\|f - f(a)\|_{L^{1}([a, a + \delta_{2}] \cap \supw)} < \varepsilon$. If $f \notin L^{1}(\supw)$, we take $\delta_{2} := 1$.

By hypothesis, there exists $0 < \delta < \min\{\delta_{1}, \delta_{2}, 1\}$ such that $|f(x) - f(a)(x)w(x)| < \varepsilon$, a.e. $x \in (a, a + \delta)$.

Let us define $b_{3} := a + \delta$ and let us consider $a < b_{1} < b_{2} < b_{3}$. Let us consider $c := \inf_{x \in (b_{1}, b_{2})} |f(x) - f(a)|$. Then, there exists $b_{0} \in (b_{1}, b_{2})$ such that $|f(b_{0}) - f(a)| < c + \varepsilon$. Let us choose $s > 0$ small enough such that $(b_{0} - s, b_{0}) \subseteq (b_{1}, b_{2})$. Then, we define the function $g$ as

$$g(x) := \begin{cases} f(a), & \text{if } x \in (a, b_{0} - s), \\ f(b_{0}) + (f(b_{0}) - f(a))(x - b_{0})/s, & \text{if } x \in (b_{0} - s, b_{0}), \\ f(x), & \text{if } x \notin (a, b_{0}). \end{cases}$$

Let us remark that $g$ is continuous in $[a, b_{0}]$ and $g = f$ in $R \setminus (a, b_{0})$.

It is obvious that $|f(a) - g(x)| \leq |f(a) - g(b_{0})| = |f(a) - f(b_{0})|$ for every $x \in [a, b_{0}]$.

$$\|f - g\|_{L^{\infty}(w)} = \|f - g\|_{L^{\infty}([a, b_{0}], w)} \leq \|f - f(a)\|_{L^{\infty}([a, b_{0}], w)} + \|f(a) - g\|_{L^{\infty}([a, b_{0}], w)} \leq \|f - f(a)\|_{L^{\infty}([a, b_{0}], w)} + \|f - f(b_{0})\|_{L^{\infty}([a, b_{0}], w)} \leq 2\|f - f(a)\|_{L^{\infty}([a, b_{0}], w)} + \|\varepsilon\|_{L^{\infty}([a, b_{0}], w)} \leq 2\varepsilon + (m + 1)\varepsilon = (3 + m)\varepsilon.$$
Proof of Theorem 2.1. This result is an improvement over a previous result in [PQRT1, Theorem 2.1]; this result is better because we have removed the technical hypothesis on \( w \) which was necessary in [PQRT1], and that essentially meant that the regular points were dense in \( R \).

Items (b), (c) and (d) are direct consequences of (a) (see the proof in [PQRT1, Proposition 2.1 and Theorem 2.1]). The proof of the inclusion of the closure of \( C(R) \cap L^\infty(w) \) in \( H_0 \) is not difficult (see the proof in [PQRT1, Proposition 2.1 and Theorem 2.1]). So far, the proof coincides with the one in [PQRT1], since no additional hypothesis on the weight were needed there in this part of the proof.

In order to prove the other inclusion, let us fix \( f \in H_0 \). The proof has several ingredients: Lemma 2.1 allows to modify \( f \) in a neighborhood of each singular point in \( S^+_1(w) \cup S^+_2(w) \cup S^-_1(w) \cup S^-_2(w) \); then we need to paste these modifications in an appropriate way.

Fix \( \eta > 0 \). Let us assume that \( a \in (S^+_1(w) \cup S^+_2(w)) \cap (S^-_1(w) \cup S^-_2(w)) \). Then Lemma 2.1 gives intervals \( [b^+_0, a], [a, b^-_0] \) and functions \( g^- \in L^\infty(w) \cap C([b^-_0, a]) \), \( g^+ \in L^\infty(w) \cap C([a, b^+_0]) \), with \( g^- = f \) in \( R \setminus (b^-_0, a) \), \( \|f - g^-\|_{L^\infty(w)} < \eta \), \( g^+ = f \) in \( R \setminus (a, b^+_0) \), \( \|f - g^+\|_{L^\infty(w)} < \eta \). Without loss of generality we can assume that \( r^- := a - b^-_0 \leq b^+_0 - a \). If \( b^+_0 - a \leq 21r^-/20 \), we define \( r^+ := b^+_0 - a \) and \( g_0 := g^+ \). If \( b^+_0 - a > 21r^-/20 \), Lemma 2.1 allows to find \( r^+ \in [r^-, 21r^-/20) \) and a function \( g_0 \in L^\infty(w) \cap C([a, a + r^+]) \), with \( g_0 = f \) in \( R \setminus (a, a + r^+) \), \( \|f - g_0\|_{L^\infty(w)} < \eta \). Hence, the function \( g \) defined by

\[
g(x) := \begin{cases} 
g^-(x), & \text{if } x \in [a - r^-, a], 
g_0(x), & \text{if } x \in [a, a + r^+], 
f(x), & \text{in other case}, \end{cases}
\]

verifies \( g \in L^\infty(w) \cap C([a - r^-, a + r^+]) \), \( g = f \) in \( R \setminus (a - r^-, a + r^+) \) and \( \|f - g\|_{L^\infty(w)} < \eta \).

If \( a \in (S^+_1(w) \cup S^+_2(w)) \cap R^+(w) \) (or if \( a \in (S^-_1(w) \cup S^-_2(w)) \cap R^-(w) \)), we can also obtain such an interval and such an approximating function. Using this result, we can follow the arguments of the proofs of [PQRT1, Proposition 2.1 and Theorem 2.1] in order to obtain a way to “paste” the approximations to \( f \) in each singular point (in these arguments it is crucial to have \( 20/21 \leq r^+/r^- \leq 21/20 \)). This finishes the proof of the theorem.

We have finished the proof of Theorem 2.1 following the same argument as in [PQRT1] thanks to Lemma 2.1. This is due to the fact that the hypothesis on the density of regular points that was crucial in [PQRT1] was only necessary to get approximations of \( f \) in a neighborhood of points belonging to \( S^+_1(w) \cup S^+_2(w) \) (see [PQRT1, Lemma 2.2 and Lemma 2.3]).

Notice that whereas in [PQRT1] the point \( b_0 \) used as a key tool in the construction of the approximation has to be regular (and, hence, regular points must be dense), Lemma 2.1 does not require that hypothesis any more.

3. SOBOLEV SPACES AND PREVIOUS RESULTS

We state here an useful technical result which was proved in [PQRT1].

**Lemma A.** [PQRT1, Lemma 2.1] Let us consider a weight \( w \) and \( a \in \text{supp } w \). If \( \text{ess lim sup}_{x \to a} w(x) = l \in (0, \infty] \), then for every function \( f \) in the closure of \( C(R) \cap L^\infty(w) \) with the norm \( L^\infty(w) \), we have that

\[
\text{ess lim}_{x \to a, w(x) \geq \eta} f(x) = f(a), \quad \text{for every } 0 < \eta < l.
\]

**Remark.** A similar result is true if we change both limits when \( x \to a \) by \( x \to a^+ \) (or \( x \to a^- \)).

In order to control a function from its derivative, we need the following version (see a proof in [RARP1, Lemma 3.2]) of Muckenhoupt inequality (see [Mu], [M, p.44]).

**Lemma B.** Let us consider \( w_0, w_1 \) weights in \([\alpha, \beta]\) and \( a \in [\alpha, \beta] \). Then there exists a positive constant \( c \) such that

\[
\left\| \int_a^x g(t) \, dt \right\|_{L^\infty([\alpha, \beta], w_0)} \leq c \|g\|_{L^\infty([\alpha, \beta], w_1)}.
\]
for any measurable function \( g \) in \([\alpha, \beta]\), if and only if
\[
\text{ess sup}_{a<x<\beta} w_0(x) \left| \int_a^x 1/w_1 \right| < \infty.
\]

We deal now with the definition of Sobolev spaces \( W^{1,\infty}(w_0, w_1) \).

We follow the approach in [KO]. First of all, notice that the distributional derivative of a function \( f \) in an interval \( I \) is a function belonging to \( L^1_{\text{loc}}(I) \). If \( f' \in L^\infty(I, w_1) \), in order to get the inclusion
\[
L^\infty(I, w_1) \subseteq L^1_{\text{loc}}(I),
\]
a sufficient condition, is that the weight \( w_1 \) satisfies \( 1/w_1 \in L^1_{\text{loc}}(I) \) (see e.g. the proof of Proposition 4.3 below). Consequently, \( f \in AC_{\text{loc}}(I) \), i.e. \( f \) is an absolutely continuous function on every compact interval contained in \( I \), if \( 1/w_1 \in L^1_{\text{loc}}(I) \).

Given two weights \( w_0, w_1 \), let us denote by \( \Omega \) the largest set (which is a union of intervals) such that \( 1/w_1 \in L^1_{\text{loc}}(\Omega) \). We always require that \( \text{supp } w_1 = \Omega \). We define the Sobolev space \( W^{1,\infty}(w_0, w_1) \), as the set of all (equivalence classes of) functions \( f \in L^\infty(w_0) \cap AC_{\text{loc}}(\Omega) \) such that their weak derivative \( f' \) in \( \Omega \) belongs to \( L^\infty(w_1) \).

With this definition, the weighted Sobolev space \( W^{1,\infty}(w_0, w_1) \) is a Banach space (see [KO, Section 3]). In general, this is not true without our hypotheses (see some examples in [KO]).

4. APPROXIMATION BY \( C^1 \) FUNCTIONS IN \( W^{1,\infty}(I, w_0, w_1) \)

The main result of this section is Theorem 4.5, which characterizes the functions which can be approximated by \( C^1 \) functions in \( W^{1,\infty}(w_0, w_1) \), under very weak hypotheses on \( w_0, w_1 \). We obtain it by means of some auxiliary lemmas and theorems.

**Lemma 4.1.** Let us consider \( \lambda \in \mathbb{R} \) and a function \( u \) defined in \([a - \delta_0, a]\), such that \( u \in C([a - \delta_0, a]) \) and \( u(a) \) is finite. For each \( 0 < \delta < \delta_0 \) there exists \( v \in C([a - \delta_0, a]) \) with \( v(x) = u(x) \) if \( x \notin (a - \delta, a) \), \( |v(x) - u(a)| \leq 2|u(x) - u(a)| \) for every \( x \in [a - \delta_0, a] \), and there exists \( \eta > 0 \) with \( v(x) = u(a) \) if \( x \in [a - \eta, a] \).

Furthermore, if we define \( U(x) := \int_{a-\delta_0}^x u, \ V(x) := \int_{a-\delta_0}^x v \), we also have:

1. \( V(a) = U(a-) \text{ and } |V(x) - U(a-)| \leq |U(x) - U(a-)| + 2|u(a)||x - a| \) for every \( x \in [a - \delta_0, a] \), if there exists \( U(a-) := \lim_{x \to a^-} U(x) \),
2. \( V(a) = \lambda \text{ and } |V(x) - \lambda| \leq |U(x) - \lambda| + 2|u(a)||x - a| \) for every \( x \in [a - \delta_0, a] \), if \( \lim_{x \to a^-} U(x) \) does not exist.

**Remarks.**

1. Notice that the value \( u(a) \) does not need to have any relation with the values of \( u \) in \([a - \delta_0, a]\).
2. A similar result is true for \( u \in C((a, a + \delta_0]) \).

**Proof.** Our goal is to construct a function \( V \) which approximates \( U \), which is equal to \( U \) far away from \( a \) and whose graph is a straight line \( r \) near \( a \). In order to do this, we will make two changes of \( u \) : the first one, \( v_1 \), will have a primitive intersecting \( r \), and the second one, \( v_2 \), will make smooth the connection with \( r \).

It is clear that we can assume that \( a = 0 \). We only consider the case \( u(0) > 0 \); the case \( u(0) < 0 \) is similar and the case \( u(0) = 0 \) is easier.

1. \( \text{(i) Let us assume that there exists } U(0-) := \lim_{x \to 0^-} U(x). \)

(1) Consider first the case \( U(x) = r(x) := U(0-) + u(0)x \), for every point in some interval \((-\delta', 0)\), with \( \delta' < \delta_0 \). If \( u(x) = u(0) \) for every \( x \) in a left neighborhood of \( 0 \), it is sufficient to take \( v := u \). If this is not so, it is possible to choose \( 0 < \delta_2 < \delta_1 < \min \{ \delta, \delta' \} \) with \( u(x) \neq u(0) \) for every \( x \in [-\delta_1, -\delta_2] \). Without loss of generality we can assume that \( u(x) < u(0) \) for every \( x \in [-\delta_1, -\delta_2] \) (since the case \( u(x) > u(0) \) is similar).

So there exists a positive constant \( \nu \) such that \( u(0) - u(x) \geq \nu \) for every \( x \in [-\delta_1, -\delta_2] \). Let us choose a function \( \phi \in C(\mathbb{R}) \) with \( \text{supp } \phi = [-\delta_1, -\delta_2] \) and \( 0 < \phi \leq \nu \) in \((-\delta_1, -\delta_2)\). If we define \( v_1 := u - \phi \), then \( v_1(x) = u(x) - \phi(x) < u(x) < u(0) \) for every \( x \in (-\delta_1, -\delta_2) \), and
\[
|v_1(x) - u(0)| = u(0) - v_1(x) = u(0) - u(x) + \phi(x) \leq u(0) - u(x) + \nu \\
\leq 2(u(0) - u(x)) = 2|u(x) - u(0)|,
\]
for every $x \in (-\delta_1, -\delta_2)$. Therefore, $v_1(x) = u(x)$ if $x \notin (-\delta_1, -\delta_2)$, $v_1(x) < u(x)$ if $x \in (-\delta_1, -\delta_2)$, $|v_1(x) - u(0)| \leq 2|u(x) - u(0)|$ for every $x$. If we define $V_1(x) := \int_{-\delta_0}^x v_1$, then $V_1(x) \leq U(x)$ for every $x$. It is clear that $\lim_{x \to 0^-} V_1(x) = U(0^-)$, and consequently there exists a minimum $-\delta_3 \in (-\delta_1, 0)$ with $V_1(-\delta_3) = r(-\delta_3)$; this implies that $V_1^t(-\delta_3) = v_1(-\delta_3) \leq u(0) = r'(-\delta_3)$, since $V_1(-\delta_1) = U(1(-\delta_1) > r(-\delta_1)$.

If this is not so, it is possible to choose $0 < \delta_2 < \delta_1 < \min\{\delta, \delta'\}$ and a function $v_1 \in C([-\delta_0, 0])$ with $v_1(x) = u(x)$ if $x \notin (-\delta_1, -\delta_2)$, $v_1(x) < u(x)$ if $x \in (-\delta_1, -\delta_2)$, $|v_1(x) - u(0)| \leq 2|u(x) - u(0)|$ for every $x$; then $V_1(x) \leq U(x)$ for every $x$, if $V_1(x) := \int_{-\delta_0}^x v_1$. It is clear that $\lim_{x \to 0^-} V_1(x) < U(0^-)$, and consequently there exists a minimum $-\delta_3 \in (-\delta_1, 0)$ with $V_1(-\delta_3) = r(-\delta_3)$; this implies that $V_1^t(-\delta_3) = v_1(-\delta_3) \leq u(0) = r'(-\delta_3)$. Therefore there exists $\tau_0 > 0$ such that

$$S(\tau) = \int_{-\delta_3 - \epsilon_1}^{-\delta_3 + \epsilon_2}s_{\tau_0} = \int_{-\delta_3}^{\delta_3} (u(0) - v_1).$$

Therefore there exists $\tau_0 > 0$ such that $S(\tau_0) = \int_{-\delta_3 - \epsilon_1}^{\delta_3} (u(0) - v_1)$. If we define $s := s_{\tau_0}$, then $0 \leq s \leq u(0) - v_1$, $s(-\delta_3 - \epsilon_1) = 0$, $s(-\delta_3 + \epsilon_2) = u(0) - v_1(-\delta_3 + \epsilon_2) > 0$, and $\int_{-\delta_3 - \epsilon_1}^{\delta_3} s = \int_{-\delta_3}^{\delta_3} (u(0) - v_1 - s).

If we define $v_2 := v_1 + s$, then $v_2 \in C([-\delta_3 - \epsilon_1, -\delta_3 + \epsilon_2])$ with $v_2 \leq 0 \leq u(0), v_2(-\delta_3 - \epsilon_1) = v_1(-\delta_3 - \epsilon_1), v_2(-\delta_3 + \epsilon_2) = u(0)$, and $V_2(-\delta_3 - \epsilon_1) - V_1(-\delta_3 - \epsilon_1) = \int_{-\delta_3 - \epsilon_1}^{\delta_3} (u(0) - v_2) \leq u(0)(\delta_2)/2$. We define $v(x) := v_2(x)$ if $x < -\delta_3 - \epsilon_1$, $v(x) := v_2(x)$ if $x \in [-\delta_3 - \epsilon_1, -\delta_3 + \epsilon_2]$, and $v(x) := u(x)$ if $x > -\delta_3 + \epsilon_2$. It is clear that $v \in C([-\delta_0, 0])$ and $|v(x) - u(0)| \leq |v_1(x) - u(0)| \leq 2|u(x) - u(0)|$ for every $x$.

If $V(x) := \int_{-\delta_0}^x v$, notice that $V(x) = V_1(x) = U(x)$ if $x \leq -\delta_1$, and $V(x) = V_1(x)$ if $x \in [-\delta_1, -\delta_3 - \epsilon_3]$. It is obvious that $r(x) \leq V_1(x) \leq U(x)$ if $x \in [-\delta_1, -\delta_3]$, consequently

$$u(0)x \leq V_1(x) - U(0^-) \leq U(x) - U(0^-),$$

$$|V_1(x) - U(0^-)| \leq \max\{|U(x) - U(0^-)|, |u(0)x|\} \leq |U(x) - U(0^-) + |u(0)x|,$$

if $x \in [-\delta_1, -\delta_3]$; now it is direct that this inequality also holds for $x \in [-\delta_0, -\delta_3]$. Therefore $|V(x) - U(0^-)| = |V_1(x) - U(0^-)| \leq |U(x) - U(0^-) + |u(0)x|$, if $x \in [-\delta_0, -\delta_3 - \epsilon_1]$.

Let us consider $x \in [-\delta_3 - \epsilon_1, -\delta_3]$; on the one hand, if $x$ satisfies $V(x) \leq U(0^-)$, we have that $|V(x) - U(0^-)| \leq |V_1(x) - U(0^-)| \leq |U(x) - U(0^-) + |u(0)x|$, since $V_1(x) \leq V(x)$; on the other hand, if $x$ satisfies $V(x) > U(0^-)$, then

$$-u(0)x \geq u(0)\delta_3/2 \geq \int_{-\delta_3 - \epsilon_1}^{\delta_3} (v_2 - v_1) \geq \int_{-\delta_3 - \epsilon_1}^{\delta_3} (v_2 - v_1) = V(x) - V_1(x),$$

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and so
\[
V(x) - U(0-) \leq V_1(x) - U(0-) - u(x) \leq U(x) - U(0-) - u(0)x \leq |U(x) - U(0-)| + |u(0)x|;
\]
it follows, in any case, that \(|V(x) - U(0-)| \leq |U(x) - U(0-)| + |u(0)x|\) if \(x \in [-\delta_3 - \varepsilon_1, -\delta_3]\).

If \(x \in [-\delta_3, -\delta_3 + \varepsilon_2]\), then \(V(x) \geq V_1(x)\): it is clear that
\[
-u(0)x \geq u(0)(\delta_3 - \varepsilon_2) \geq u(0)\delta_3/2 \geq \int_{-\delta_3}^{-\delta_3+\varepsilon_2} (v_2 - v_1) = V(-\delta_3) - V_1(-\delta_3) = V(-\delta_3) - r(-\delta_3) \geq V(x) - r(x),
\]
if \(x \in [-\delta_3, -\delta_3 + \varepsilon_2]\) (since \((V(x) - r(x))' = v_2(x) - u(0) \leq 0\)), and hence \(V(x) - U(0-) \leq 0\); we also have
\[
-u(0)x \geq \int_{-\delta_3}^{-\delta_3+\varepsilon_2} (u(0) - v_2) \geq \int_{-\delta_3}^{x} (u(0) - v_2) = r(x) - r(-\delta_3) - V(x) + V(-\delta_3)
\]
\[
\geq r(x) - r(-\delta_3) - V(x) + V_1(-\delta_3) = r(x) - V(x),
\]
if \(x \in [-\delta_3, -\delta_3 + \varepsilon_2]\), and hence \(V(x) - U(0-) \geq r(x) - U(0-) + u(0)x = 2u(0)x\) in this interval; it follows that \(|V(x) - U(0-)| \leq 2|u(0)x|\) if \(x \in [-\delta_3, -\delta_3 + \varepsilon_2]\).

If \(x \in [-\delta_3 + \varepsilon_2, 0]\), then \(V(x) = r(x)\), since \(V'(x) = v(x) = u(0) = r'(x)\) in this interval, and
\[
r(-\delta_3 + \varepsilon_2) - V(-\delta_3 + \varepsilon_2) = \int_{-\delta_3}^{-\delta_3+\varepsilon_2} (u(0) - v_2) + r(-\delta_3) - V(-\delta_3)
\]
\[
= \int_{-\delta_3}^{-\delta_3+\varepsilon_2} (u(0) - v_2) - (V(-\delta_3) - V_1(-\delta_3))
\]
\[
\geq 0.
\]
Hence \(V(x) - U(0-) = u(0)x\) and \(|V(x) - U(0-)| = |u(0)x|\) if \(x \in [-\delta_3 + \varepsilon_2, 0]\).

(1.2) If \(v_1(-\delta_3) = u(0)\), we define \(v(x) := v_1(x)\) if \(x \leq -\delta_3\) and \(v(x) := u(0)\) if \(x > -\delta_3\). We can argue as in the case \(v_1(-\delta_3) < u(0)\).

(2) If \(U(x) = r(x) := U(0-) + u(0)x\), for every point in a left neighborhood of 0, we can use a similar construction of \(v\) (taking \(v_1 = u\) and \(-\delta_3 = x_n\) for some \(n\) large enough).

(ii) Let us assume now that \(\lim_{x \to 0} - U(x)\) does not exist; then \(u \notin L^1([-\delta_0, 0])\).

(1) Consider first the case \(U(x) > r(x) := \lambda + u(0)x\), for every point in a left neighborhood of 0. The function \(u_0 := u(0) - |u - u(0)|\) verifies \(|u_0 - u(0)| = |u - u(0)|\) and \(\lim_{x \to 0} - \int_{-\delta_0} u_0 = -\infty\). It is clear that \(u_0(x) = u(x)\) for any \(x\) with \(u(x) \leq 0\).

If there exists some \(x_0 \in (-\delta, 0)\) with \(u(x_0) \leq 0\), then let us define
\[
v_1(x) := \begin{cases} u(x), & \text{if } x \in (-\delta, x_0], \\ u_0(x), & \text{if } x \in (x_0, 0]. \end{cases}
\]

If \(u(x) > u(0)\), for every \(x \in (-\delta, 0)\), then \(u_0(x) = 2u(0) - u(x)\) for every \(x \in (-\delta, 0)\). For any \(0 < \delta_2 < \delta_1 < \delta\), we define
\[
v_1(x) := \begin{cases} x + x_0, & \text{if } x \in (-\delta, x_0], \\ \left(\frac{x + \delta_1}{\delta_1 - \delta_2}\right) u_0(x), & \text{if } x \in (-\delta_1, -\delta_2), \\ u_0(x), & \text{if } x \in [-\delta_2, 0]. \end{cases}
\]

If we take \(\delta_1 := \delta_2 := -x_0\) in the first case, by the definition of \(v_1\), we obtain (in both cases) that \(v_1 \in C([-\delta_0, 0])\), \(v_1(x) = u(x)\) if \(x \leq -\delta_1\), \(v_1(x) = u_0(x)\) if \(x \geq -\delta_2\), and \(|v_1(x) - u(0)| \leq 2|u(x) - u(0)|\) for every \(x\). If \(V_1(x) := \int_{-\delta_0}^{x} v_1\), it is clear that \(\lim_{x \to 0} - V_1(x) = -\infty\), and consequently there exists a minimum \(-\delta_3 \in (-\delta_1, 0)\) with \(V_1(-\delta_3) = r(-\delta_3)\).

Now it is sufficient to choose the functions \(v_2\) and \(v\) as in the case (i), and do the same computations.
(2) If \( U(x) < r(x) := \lambda + u(0)x \), for every point in a left neighborhood of 0, we can repeat the argument with \( u_1 := u(0) + |u - u(0)| \) instead of \( u_0 \).

(3) If \( U(x_n) = r(x_n) \), for a sequence \( x_n \to 0 \), it is also possible to use a similar construction of \( v \) (taking \( v_1 = u \) and \(-\delta_1 = x_n \) for some \( n \) large enough). 

**Definition 4.1.** Let us consider a weight \( w_1 \) such that \( S(w_1) \cap [a - \delta, a + \delta] = \{a\} \) for some \( \delta > 0 \). We say that \( w_1 \) is left-dominated at \( a \) if there exists a constant \( c \) such that any function \( F \in C([a - \delta, a]) \) with \( 0 \leq F \leq 1/w_1 \) a.e. verifies \( \int_{a-\delta}^{a} F \leq c \). We say that \( w_1 \) is right-dominated at \( a \) if there exists a constant \( c \) such that any function \( F \in C([a, a + \delta]) \) with \( 0 \leq F \leq 1/w_1 \) a.e. verifies \( \int_{a}^{a+\delta} F \leq c \). We denote by \( D^-w_1 \) (respectively, \( D^+w_1 \)) the set of left-dominated (respectively, right-dominated) points of \( w_1 \).

**Remarks.**

1. Every weight \( w_1 \) with \( 1/w_1 \in L^1([a, a + \delta]) \) is right-dominated at \( a \).

2. There exists weights \( w_1 \) right-dominated at \( a \), with \( 1/w_1 \notin L^1([a, a + \delta]) \): Let us consider a Borel set \( E \subset [0, 1] \) with \( |E| \leq |I| \) for every interval \( I \subset [0, 1] \) (see e.g. [Ru, Chapter 2]). Since \( \int_{E} dx/x + \int_{[0,1]\setminus E} dx/x = \int_{[0,1]} dx/x = \infty \), without loss of generality we can assume that \( \int_{E} dx/x = \infty \) (in other case we can take \([0,1]\setminus \) \( E \)) instead of \( E \). Then, \( w_1(x) := x \chi_{E}(x) + x\chi_{[0,1]\setminus E}(x) \) is right-dominated at 0 and \( 1/w_1 \notin L^1([0, 1]) \).

**Lemma 4.2.** Let us consider a weight \( w_1 \) in \([a - \delta, a]\) with \( S(w_1) = \{a\} \). Then a \( a \notin D^-w_1 \) if and only if there exists a function \( F \in C([a - \delta, a]) \) with \( 0 \leq F \leq 1/w_1 \) a.e. and \( \int_{a-\delta}^{a} F = \infty \).

**Proof.** Let us assume that there exists a function \( F \in C([a - \delta, a]) \) with \( 0 \leq F \leq 1/w_1 \) a.e. and \( \int_{a-\delta}^{a} F = \infty \). For each \( n \) we can consider a function \( F_n \in C([a - \delta, a]) \) with \( 0 \leq F_n \leq F \leq 1/w_1 \) a.e. and \( F_n = F \) in \([a - \delta, a - 1/n] \). Let us assume \( \lim_{n \to \infty} \int_{a-\delta}^{a} F_n \to \infty \) and \( a \notin D^-w_1 \). Then, for each \( n \) there exists a function \( F_n \in C([a - \delta, a]) \) with \( 0 \leq F_n \leq 1/w_1 \) a.e. and \( \int_{a-\delta}^{a} F_n > 1 \). Let us choose \( a_n \in (a - 1/n, a) \) with \( \int_{a_n}^{a} F_n > n \). Since \( S(w_1) = \{a\} \), then \( 1/w_1 \notin L^1_{\loc}([a - \delta, a]) \), and consequently \( \int_{a-\delta}^{a} F_n \leq \int_{a-\delta}^{a} 1/w_1 \in C([a - \delta, a]) \). Therefore, there exists a subsequence \( \{a_n\}_k \) with \( \int_{a_n}^{a} F_n > 1 \), and hence we can construct a function \( F \in C([a - \delta, a]) \) with \( 0 \leq F \leq 1/w_1 \) a.e. and \( \int_{a-\delta}^{a} F = \infty \).

**Lemma 4.3.** Let us consider two weights \( w_0, w_1 \) in \([a - \delta, a]\) with \( \ess\lim_{x \to a-} w_0(x) = 0 \) and \( a \notin D^-w_1 \). Then for each \( f \in W^{1,\infty}(w_0, w_1) \cap C^1([a - \delta, a]) \) with \( \|f - g\|_{W^{1,\infty}(w_0, w_1)} < \varepsilon \), \( g(x) = f(x) \) if \( x \notin [a - \delta, a] \), \( g' = f' \) in some neighborhood of \( a \), and \( g(a) = s \).

**Remark.** A similar result is true for \( f \in W^{1,\infty}(w_0, w_1) \cap C^1([a, a + \delta]) \).

**Proof.** By Lemma 4.2, there exists a function \( F \in C([a - \delta, a]) \) with \( 0 \leq F \leq 1/w_1 \) a.e. and \( \int_{a-\delta}^{a} F = \infty \). Without loss of generality, we can assume that \( a = 0 \) and \( s > f(0) \): the case \( s < f(0) \) is similar, and the case \( s = f(0) \) is trivial (it is sufficient to take \( g = f \)). Since \( \ess\lim_{x \to 0-} w_0(x) = 0 \), then there exists \( 0 < \delta_1 < \delta \) with \( (s - f(0))w_0(x) < \varepsilon/3 \) for almost every \( x \in (-\delta_1, 0) \).

Since \( F \in C([-\delta_1, 0]) \), \( F \geq 0 \) and \( \int_{-\delta_1}^{\delta_1} F = \infty \), it is clear that we can find a function \( J \in C_c([-\delta_1, 0]) \) (i.e. \( J \in C_c([-\delta_1, 0]) \) and \( \supp J \subset [-\delta_1, 0] \)) with \( 0 \leq J \leq \varepsilon F/2 \) and \( \int_{-\delta_1}^{\delta_1} J = s - f(0) \). Let us define \( h(x) := \int_{-\delta_1}^{\delta_1} J + g := f + h \). Then we have \( h(x) \leq s - f(0) \). It is clear that \( g(x) = f(x) \) if \( x \notin (-\delta, 0) \), \( g' = f' \) in some neighborhood of 0, and \( g(0) = s \). We only need to check that \( \|h\|_{W^{1,\infty}(w_0, w_1)} < \varepsilon \), and this fact is a consequence of

\[
\|h\|_{L^\infty(w_0)} = \ess\sup_{x \in [-\delta_0, 0]} h(x)w_0(x) \leq \ess\sup_{x \in [-\delta_0, 0]} (s - f(0))w_0(x) \leq \varepsilon/3 < \varepsilon/2,
\]

\[
\|h\|_{L^\infty(w_1)} = \ess\sup_{x \in [-\delta_0, 0]} J(x)w_1(x) \leq \varepsilon/2 \ess\sup_{x \in [-\delta_0, 0]} F(x)w_1(x) \leq \varepsilon/2.
\]

**Lemma 4.4.** Let us consider two weights \( w_0, w_1 \) in \([a - \delta_0, a]\) with \( S(w_1) = \{a\} \) and \( a \in D^-w_1 \). Let us assume that there exists \( f \in W^{1,\infty}(w_0, w_1) \) and \( \{g_n\}_n \in W^{1,\infty}(w_0, w_1) \cap C^1([a - \delta_0, a]) \) converging to \( f \) in \( W^{1,\infty}(w_0, w_1) \). Then \( \{g'_n\}_n \) converges to \( f' \) in \( L^1([a - \delta_0, a]) \) and \( f \) is continuous to the left in \( a \).
Remark. A similar result is true if we change \([a - \delta_0, a]\) by \([a, a + \delta_0]\) everywhere.

Proof. Since \(S(w_1) = \{a\}\), then \(1/w_1 \in L^1(\mathbb{R})\). For any \(0 < \delta < \delta_0\), we obtain

\[
\|f' - g_n'\|_{L^1([a - \delta, a - \delta])} = \int_{a - \delta}^{a - \delta} |f' - g_n'| \frac{w_1}{w_2} \leq \|f' - g_n'\|_{L^\infty(w_1)} \int_{a - \delta}^{a - \delta} \frac{1}{w_1}.
\]

Then, \(\{g_n'\}\) converges to \(f'\) in \(L^1([a - \delta, a - \delta])\), for any \(0 < \delta < \delta_0\). Furthermore, \(\{g_n'\}\) is a Cauchy sequence in \(L^1([a - \delta_0, a - \delta])\): Since \(a \in D^-(w_1)\), there exists a constant \(c\) such that any function \(F \in C([a - \delta_0, a])\) satisfies \(0 \leq F \leq 1/w_1\) a.e. verifies \(\int_{a - \delta_0}^{a - \delta} F \leq c\). We have \(|g_n' - g_m'|/\|g_n' - g_m'\|_{L^\infty(w_1)} \leq 1/w_1\) a.e., and hence \(\int_{a - \delta_0}^{a - \delta} |g_n' - g_m'| \leq c\|g_n' - g_m'\|_{L^\infty(w_1)}\). Therefore \(\{g_n'\}\) converges to \(f'\) in \(L^1([a - \delta_0, a])\).

Let us consider \(\overline{g_n}(x) := g_n(x) - g_n(a - \delta_0) + f(a - \delta_0) \in C^1([a - \delta_0, a])\). Then \(|f(x) - \overline{g_n}(x)| = |\int_{a - \delta_0}^{a - \delta} (f' - g_n')| \leq \|f' - g_n'\|_{L^1([a - \delta, a])}\) for every \(x \in [a - \delta_0, a]\). Consequently \(\{\overline{g_n}\}\) converges uniformly to \(f\) in \([a - \delta_0, a]\) and \(f\) is continuous to the left in \(a\).

The following definition makes sense because of Lemma A.

**Definition 4.2.** Let us consider a weight \(w_1\). For each \(f\) with \(f' \in C^1(\mathbb{R}) \cap L^\infty(w_1)\), let us define \(u_f(a) := 0\) if \(a \in S_1(w_1)\), and \(u_f(a) := \text{ess lim}_{x \to a} w_1(x) \geq \eta|f'(x)|\) for any \(\eta > 0\) small enough if \(a \notin S_1(w_1)\).

Let us remark that \(u_f(a)\) is finite by Lemma A. We can state now our first theorem in this section.

**Theorem 4.1.** Let us consider two weights \(w_0, w_1\), in \([\alpha, \beta]\) such that \(S(w_1) = \{a\}\), and \(d > 0\). Then every function in

\[
H_1 := \{f \in W^{1,\infty}(w_0, w_1) : f \in C^1(\mathbb{R}) \cap L^\infty(w_0) \cap L^\infty(w_1), f' \in C^1(\mathbb{R}) \cap L^\infty(w_1), f \text{ is continuous to the right if } a \in D^+(w_1), f \text{ is continuous to the left if } a \in D^-(w_1), \text{ ess lim}_{x \to a} f(x) - f(a)|w_0(x) = 0, \text{ ess lim}_{x \to a} u_f(a)(x - a)|w_0(x) = 0\},
\]

can be approximated by functions \(\{g_n\}\) in \(C^1(\mathbb{R}) \cap W^{1,\infty}(w_0, w_1)\) with the norm of \(W^{1,\infty}(w_0, w_1)\) and \(g_n(x) = f(x)\) if \(x \notin (a - d, a + d)\). Furthermore, if \(f\) also satisfies \(\text{ess lim}_{x \to a} |f'(x) - u_f(a)|w_1(x) = 0\), each function \(g_n\) is a polynomial of degree at most 1 in a neighborhood of \(a\).

**Remarks.**

1. Notice that the hypothesis \(\text{ess lim}_{x \to a} u_f(a)(x - a)|w_0(x) = 0\) for every function \(f\) with \(f' \in C^1(\mathbb{R}) \cap L^\infty(w_1)\), is a consequence of any of the following conditions:
   - \(\text{ess lim}_{x \to a} (x - a)|w_0(x) = 0\),
   - \(a \notin S_1(w_1)\), i.e. \(\text{ess lim}_{x \to a} w_1(x) = 0\) or \(\text{ess lim sup}_{x \to a} w_1(x) = \infty\) (in both cases, \(u_f(a) = 0\)).

2. Either of the following conditions guarantees \(\text{ess lim}_{x \to a} |f(x) - f(a)|w_0(x) = 0\) for every function \(f \in C^1(\mathbb{R}) \cap L^\infty(w_1)\):
   - \(a \in S^+(w_0) \cap S^-(w_0)\), i.e. \(\text{ess lim}_{x \to a, +} w_0(x) = \text{ess lim}_{x \to a, -} w_0(x) = 0\),
   - \(a \in S^+(w_0)\) and \(w_0 \in L^\infty([a - \epsilon, a])\), for some \(\epsilon > 0\),
   - \(a \in S^-(w_0)\) and \(w_0 \in L^\infty([a, a + \epsilon])\), for some \(\epsilon > 0\),
   - \(w_0 \in L^\infty([a - \epsilon, a + \epsilon])\), for some \(\epsilon > 0\).

3. Either of the following conditions guarantees \(\text{ess lim}_{x \to a} |f'(x) - u_f(a)|w_1(x) = 0\) for every function \(f\) with \(f' \in C^1(\mathbb{R}) \cap L^\infty(w_1)\):
   - \(a \in S^+(w_1) \cap S^-(w_1)\), i.e. \(\text{ess lim}_{x \to a, +} w_1(x) = \text{ess lim}_{x \to a, -} w_1(x) = 0\),
   - \(a \in S^+(w_1)\) and \(w_1 \in L^\infty([a - \epsilon, a])\), for some \(\epsilon > 0\),
   - \(a \in S^-(w_1)\) and \(w_1 \in L^\infty([a, a + \epsilon])\), for some \(\epsilon > 0\),
   - \(a = \alpha\) or \(a = \beta\) (since \(a \in S(w_1)\)).

4. Notice that we do not have any hypothesis about the singularities of \(w_0\).
The heart of the proof is to use Lemma 4.1 in the approximation in \([a, a]\) and the “right version” of Lemma 4.1 in the approximation in \([a, b]\). If these two approximations do not glue in a continuous way, we must use Lemma 4.3 in order to obtain a continuous function. Without loss of generality, we can assume that \(a = (a, b)\), since the cases \(a = a\) and \(a = b\) are easier (in these cases we do not use Lemma 4.3).

If \(a \in S^{-}(w_{1}) \cap R^{+}(w_{1})\), then every \(f \in H_{1}\) belongs to \(C^{1}([a, b])\), and we only need to apply Lemma 4.1; if \(a \in S^{+}(w_{1}) \cap R^{-}(w_{1})\), then every \(f \in H_{1}\) belongs to \(C^{0}([a, b])\), and we only need to apply the “right version” of Lemma 4.1; then, without loss of generality, we can assume that \(a \in S^{+}(w_{1}) \cap S^{-}(w_{1})\), since the other cases are easier. In this case \(a \in S^{+}(w_{1}) \cap S^{-}(w_{1})\), every \(f \in H_{1}\) satisfies \(\lim_{x \to a} |f'(x) - u(f)|w_{1}(x) = 0\) (see Theorem 2.1 and Lemma A; in the case \(a \in S(w_{1})\) we have in fact \(\lim_{x \to a} |f'(x) - \lambda|w_{1}(x) = 0\) for any \(\lambda \in \mathbb{R}\), since \(\lim_{x \to a} w_{1}(x) = 0\).

Let us consider any \(f \in H_{1}\) and \(\varepsilon > 0\). Let us define \(u := f'\) in \([a, b] \setminus \{a\}\) and \(u(a) := u_{f}(a)\). Since \(f \in H_{1}\), it is possible to choose \(0 < \delta < d\) with

\[
3\|f' - u(a)\|_{L^{\infty}([a-\delta,a+\delta],w_{1})} < \frac{\varepsilon}{6}, \quad 4\|f - f(a)\|_{L^{\infty}([a-\delta,a+\delta],w_{0})} < \frac{\varepsilon}{6}, \quad 4\|u(a)\|_{L^{\infty}([a-\delta,a+\delta],w_{0})} < \frac{\varepsilon}{6}.
\]

We also require from \(\delta\) that

\[
|f(x) - f(a)| \leq |f(x) - u(a)| \text{ for } x \in [a - \delta, a] \text{ if there exists } f(a) \neq f(a), \text{ and}
\]

\[
|f(x) - f(a)| \leq |f(x) - u(a)| \text{ for } x \in (a, a + \delta] \text{ if there exists } f(a) \neq f(a).
\]

Let us define \(U(x) := f(x) - f(a) = \int_{a}^{x} f'\) if \(x \in [a, a]\), and \(U(x) := f(x) - f(b) = \int_{a}^{x} f'\) if \(x \in (a, b]\). Consider the function \(v \in C([a, a])\) in Lemma 4.1 satisfying \(v(x) = u(x)\) if \(x \notin (a - \delta, a)\), \(|v(x) - u(a)| \leq 2|u(x) - u(a)|\) for every \(x \in [a, a]\),

\[
V(a) = \begin{cases} f(a) - f(a), & \text{if there exists } f(a), \\ f(a) - f(a), & \text{in other case,} \end{cases}
\]

and \(|V(x) - V(a)| \leq |U(x) - V(a)| + 2|u(a)|x - a|\) for every \(x \in [a, a]\), if \(V(x) := \int_{a}^{x} v\). Consider also the function \(\tilde{v} \in C([a, b])\) in the “right version” of Lemma 4.1 satisfying \(\tilde{v}(x) = u(x)\) if \(x \notin (a, a + \delta)\), \(|\tilde{v}(x) - u(a)| \leq 2|u(x) - u(a)|\) for every \(x \in [a, b]\),

\[
\tilde{V}(a) = \begin{cases} f(a +) - f(\beta), & \text{if there exists } f(a +), \\ f(a) - f(\beta), & \text{in other case,} \end{cases}
\]

and \(|\tilde{V}(x) - \tilde{V}(a)| \leq |U(x) - \tilde{V}(a)| + 2|u(a)|x - a|\) for every \(x \in (a, b]\), if \(\tilde{V}(x) := \int_{a}^{x} \tilde{v}\).

Let us consider the function \(g_{0}\) given by \(g_{0}(x) := V(x) + f(a)\) if \(x \in [a, a]\), and \(g_{0}(x) := \tilde{V}(x) + f(\beta)\) if \(x \in (a, b]\). Notice that \(g_{0} \in C^{1}([a, b] \setminus \{a\})\) and \(g_{0}'(a) = g_{0}'(a +) = u(a)\). In fact, \(g_{0}\) is a polynomial of degree at most 1 in a left (respectively right) neighborhood of \(a\), since \(g_{0}(x) = u(a)\) there (by Lemma 4.1).

This function also satisfies \(g_{0}(x) = f(x)\) if \(x \notin (a - \delta, a + \delta)\), and \(|g_{0}(x) - u(a)| \leq 2|f'(x) - u(a)|\) for
every \( x \in [\alpha, \beta] \setminus \{a\} \). It follows that \( g_0 \) verifies

\[
\|f - g_0\|_{W^{1,\infty}(w_0, w_1)} = \|f - g_0\|_{L^\infty(w_0)} + \|f' - g_0'\|_{L^\infty(w_1)}
\]

\[
= \max \{\|U - \tilde{V}\|_{L^\infty([\alpha - \delta, \alpha], w_0)}, \|U - \tilde{V}\|_{L^\infty([\alpha, a + \delta], w_0)}\} + \|f' - g_0'\|_{L^\infty([\alpha - \delta, a + \delta], w_1)}
\]

\[
\leq \|U - \tilde{V}(a)\|_{L^\infty([\alpha - \delta, a], w_0)} + \|V - \tilde{V}(a)\|_{L^\infty([\alpha - \delta, a], w_0)}
\]

\[
+ \|f' - u(a)\|_{L^\infty([\alpha - \delta, a], w_1)} + \|g_0' - u(a)\|_{L^\infty([\alpha - \delta, a + \delta], w_1)}
\]

\[
\leq 2\|U - \tilde{V}(a)\|_{L^\infty([\alpha - \delta, a], w_0)} + 2\|u(a)\|_{\|x - a\|_{L^\infty([\alpha - \delta, a], w_0)}} + 3\|f' - u(a)\|_{L^\infty([\alpha - \delta, a + \delta], w_1)}
\]

\[
\leq 4\|f - f(a)\|_{L^\infty([\alpha - \delta, a], w_0)} + 2\|u(a)\|_{\|x - a\|_{L^\infty([\alpha - \delta, a], w_0)}} + 3\|f' - u(a)\|_{L^\infty([\alpha - \delta, a + \delta], w_1)}
\]

\[
\leq \|f - f(a)\|_{L^\infty([\alpha - \delta, a], w_0)} + 2\|f\|_{\|x - a\|_{L^\infty([\alpha - \delta, a], w_0)}} + 3\|f' - u(a)\|_{L^\infty([\alpha - \delta, a + \delta], w_1)}
\]

\[
\leq \frac{\varepsilon}{6} + \frac{\varepsilon}{6} + \frac{\varepsilon}{6} = \varepsilon,
\]

where we have used (4.1) in the third inequality. In order to finish the proof we only need to construct a function \( g \in C^1((\alpha, \beta]) \) with \( \|g - g_0\|_{W^{1,\infty}([\alpha, a + \delta], w_0, w_1)} < \varepsilon/2 \), \( g(x) = g_0(x) = f(x) \) if \( x \notin (a - d, a + d) \) and \( g' = g_0' = u(a) \) in a neighborhood of \( a \).

Let us recall that \( g_0(a -) = f(a -) \) if there exists \( f(a -) \) and \( g_0(a -) = f(a) \) in other case, \( g_0(a +) = f(a +) \) if there exists \( f(a +) \) and \( g_0(a +) = f(a) \) in other case. We also have \( g_0'(a -) = g_0'(a+) = u(a) \). Hence, \( g_0 \in C^1((\alpha, \beta]) \) if and only if \( g_0(a -) = g_0(a+) \); in this case, it is sufficient to take \( g := g_0 \).

We analyse now the different cases:

1. If \( a \in D^-(w_1) \cap D^+(w_1) \), then \( f \in C([\alpha, \beta]) \). Therefore we can take \( g := g_0 \).

2. Let us assume now that \( a \notin D^-(w_1) \cap D^+(w_1) \).

2.1 If there exist neither \( f(a -) \) nor \( f(a+) \), then we also have \( g_0 \in C([\alpha, \beta]) \).

2.2 Let us assume that there exists \( f(a -) \) and \( f(a+) \) does not exist (the case in which there exists \( f(a+) \) and \( f(a -) \) does not exist is similar). If \( f(a -) = f(a) \), it follows that \( g_0 \in C([\alpha, \beta]) \). If \( f(a -) \neq f(a) \), it follows that \( \text{ess lim}_{x \to a-} w_0(x) = 0 \) and \( a \notin D^-(w_1) \): if \( \text{ess lim}_{x \to a-} w_0(x) > 0 \), then Lemma A and its remark imply that \( f(a) = \text{ess lim}_{x \to a-} w_0(x) = f(x) = f(a-) \), for any \( \eta > 0 \) small enough, which is a contradiction; if \( a \notin D^-(w_1) \), then \( f \) is continuous to the left at \( a \), which is a contradiction. Consequently we can apply Lemma 4.3 to \( g_0 \mid_{[a, a]} \) in order to obtain a function \( g \in C^1([\alpha, a]) \) with \( \|g - g_0\|_{W^{1,\infty}([\alpha, a], w_0, w_1)} < \varepsilon/2 \), \( g'(a-) = g_0'(a-) = g_0'(a+) \), \( g(a) = g_0(a+) \) and \( g(x) = g_0(x) = f(x) \) if \( x \notin (a - d, a] \); if we define \( g := g_0 \) in \( (a, \beta] \), this \( g \) is the required function.

Notice that lemmas 4.1 and 4.3 guarantee that \( g \) is a polynomial of degree at most 1 in a neighborhood of \( a \), since \( g' \) is constant in a neighborhood of \( a \).

2.3 Finally, let us assume that there exist \( f(a -) \) and \( f(a+) \). If \( f(a-) = f(a+) \), it follows that \( g_0 \in C([\alpha, \beta]) \). If \( f(a-) \neq f(a+) \), we consider two cases:

If \( \text{ess lim}_{x \to a-} w_0(x) = 0 \), without loss of generality, we can assume that \( a \notin D^-(w_1) \) (the case \( a \notin D^+(w_1) \) is similar). Consequently we can apply Lemma 4.3 as in the case (2.2).

If \( \text{ess lim}_{x \to a-} w_0(x) > 0 \), without loss of generality, we can assume that \( a \notin D^-(w_1) \) (the case \( a \notin D^+(w_1) \) is similar). Then, Lemma A and its remark imply that \( f(a) = \text{ess lim}_{x \to a-} w_0(x) = f(x) = f(a-) \), it follows that \( \text{ess lim}_{x \to a-} w_0(x) = 0 \), since if this is not so, \( f(a) = \text{ess lim}_{x \to a-} w_0(x) = f(x) = f(a-) \) and hence \( f(a+) = f(a-) \), which is a contradiction. We also have \( a \notin D^-(w_1) \), since if this is not so, \( f \) is continuous to the left at \( a \), which is a contradiction. Consequently we can apply Lemma 4.3 as in the case (2.2).

This finishes the proof of the theorem.

**Lemma 4.5.** Let us consider a weight \( w_0 \) with \( \text{ess lim}_{x \to a} w_0(x) = \infty \) and \( \text{ess lim}_{x \to a} |x - a| w_0(x) = 0 \). If \( f \in L^\infty(w_0) \) and \( \|f\|_{L^\infty([\alpha - \delta, a + \delta], w_0)} \geq c > 0 \) for every \( \delta > 0 \), then \( \text{dist}_{L^\infty(w_0)}(f, C^1(R) \cap L^\infty(w_0)) \geq c \).
Theorem 4.2. Let us consider two weights \( w_0, w_1 \), in \([\alpha, \beta]\) such that \( S(w_1) = \{a\} \) and \( \text{ess lim}_{x \to a} |x - a|w_0(x) = 0 \). Then the closure of \( C^1(\mathbb{R}) \cap W^{1,\infty}(w_0, w_1) \) in \( W^{1,\infty}(w_0, w_1) \) is equal to

\[
H_2 := \{ f \in W^{1,\infty}(w_0, w_1) : f \in C(\mathbb{R}) \cap L^\infty(w_0) \}, \quad f' \in C(\mathbb{R}) \cap L^\infty(w_1) \}
\]

Furthermore, if \( w_0, w_1 \in L^\infty([\alpha, \beta]) \), then the closure of the space of polynomials in \( W^{1,\infty}(w_0, w_1) \) is also \( H_2 \). In fact, for each \( f \in H_2 \) and \( d > 0 \) there exist \( \{g_n\}_n \) in \( C^1(\mathbb{R}) \) with \( \text{lim n} \to \infty \| f - g_n \|_{W^{1,\infty}(w_0, w_1)} = 0 \) and \( g_n(x) = f(x) \) if \( x \notin (a - d, a + d) \).

Remarks.

1. It is a remarkable fact that the approximation method is constructive.

2. Notice that we require \( \text{ess lim}_{x \to a} |f(x) - f(a)|w_0(x) = 0 \) in \( H_2 \), even if \( a \notin S(w_0) \).

Proof. If \( f \) is in the closure of \( C^1(\mathbb{R}) \cap W^{1,\infty}(w_0, w_1) \) in \( W^{1,\infty}(w_0, w_1) \), it follows that \( f \in W^{1,\infty}(w_0, w_1) \), \( f \in C(\mathbb{R}) \cap L^\infty(w_0) \), and \( f' \in C(\mathbb{R}) \cap L^\infty(w_1) \). Lemma 4.4 implies that \( f \) is continuous to the right if \( a \in D^+(w_1) \) and \( f \) is continuous to the left if \( a \in D^-(w_1) \). If \( \text{ess lim sup}_{x \to a} w_0(x) < \infty \), we can deduce that \( \text{ess lim}_{x \to a} |f(x) - f(a)|w_0(x) = 0 \). We see that \( \text{ess lim}_{x \to a} |f(x) - f(a)|w_0(x) = 0 \) (the left limit is similar); it is a consequence of Theorem 2.1 if \( a \in S^+(w_0) \) and if this is not so, \( f \) is continuous to the right at \( a \), as a consequence of \( f \in C(\mathbb{R}) \cap L^\infty(w_0) \) and Theorem 2.1. If \( \text{ess lim sup}_{x \to a} w_0(x) = \infty \), we have \( f(a) = 0 \), and Lemma 4.5 implies that there no longer exists \( c > 0 \) with \( \| f \|_{L^\infty([a - \delta, a + \delta], w_0)} \geq c \) for every \( \delta > 0 \); therefore we obtain \( \text{ess lim}_{x \to a} |f(x) - f(a)|w_0(x) = 0 \) also in this case. Then \( f \in H_2 \).

It is clear that \( H_2 \) is contained in the closure of \( C^1(\mathbb{R}) \cap W^{1,\infty}(w_0, w_1) \) in \( W^{1,\infty}(w_0, w_1) \), since \( f \in H_1 \); \( u_f(a) \) is finite and we have the hypothesis \( \text{ess lim}_{x \to a} |a - a|w_0(x) = 0 \), and consequently \( \text{ess lim}_{x \to a} u_f(x)|x - a|w_0(x) = 0 \). Then it is possible to apply Theorem 4.1, which allows to choose \( \{g_n\}_n \) in \( C^1(\mathbb{R}) \) with \( \text{lim n} \to \infty \| f - g_n \|_{W^{1,\infty}(w_0, w_1)} = 0 \) and \( g_n(x) = f(x) \) if \( x \notin (a - d, a + d) \).

If \( w_0, w_1 \in L^\infty([\alpha, \beta]) \), the closure of the polynomials is \( H_2 \) as well, as a consequence of Bernstein’s proof of Weierstrass’ Theorem (see e.g. [D, p.113]), which gives a sequence of polynomials converging uniformly up to the k-th derivative for any function in \( C^k([\alpha, \beta]) \).

Proposition 4.1. Let us consider two weights \( w_0, w_1 \), in \([\alpha, \beta]\), with \( \text{ess lim sup}_{x \to a} |x - a|w_0(x) > 0 \) and \( a \in S(w_1) \).

(a) If \( f \) belongs to the closure of \( C^1(\mathbb{R}) \cap L^\infty(w_0) \) in \( L^\infty(w_0) \), then for each \( \eta > 0 \) small enough there exists \( \varepsilon := \text{ess lim}_{x \to a, |x - a|w_0(x) \geq \eta} f(x)/(x - a) \). We also have \( \text{lim n} \to \infty g_n(a) = l \), for any sequence \( \{g_n\}_n \subseteq C^1(\mathbb{R}) \cap L^\infty(w_0) \) converging to \( f \) in \( L^\infty(w_0) \).

(b) If \( f \) belongs to the closure of \( C^1(\mathbb{R}) \cap W^{1,\infty}(w_0, w_1) \) in \( W^{1,\infty}(w_0, w_1) \) and \( a \notin S_1(w_1) \), then \( u_f(a) = l \).

Furthermore, if there exists \( f'(a) \), then \( u_f(a) = f'(a) \).

(c) If \( f' \) belongs to the closure of \( C^1(\mathbb{R}) \cap L^\infty(w_1) \) in \( L^\infty(w_1) \) and \( a \notin S_1(w_1) \), then \( u_f(a) = \text{lim n} \to \infty h_n(a) \), if \( \{h_n\}_n \subseteq C(\mathbb{R}) \cap L^\infty(w_1) \) converges to \( f' \) in \( L^\infty(w_1) \).
Proof. Let us fix $0 < \eta < \text{ess lim sup}_{x \to a} |x - a|w_0(x)$. Seeking a contradiction, suppose that
\[
\text{ess lim inf}_{x \to a, |x - a|w_0(x) \geq \eta} \frac{f(x)}{x - a} = c_1 < c_2 = \frac{1}{\text{ess lim sup}_{x \to a, |x - a|w_0(x) \geq \eta} f(x)}.
\]
If $g$ is any function in $C^1(\mathbb{R}) \cap L^\infty(w_0)$, it follows that $g(a) = 0$ (by \text{ess lim sup}_{x \to a} w_0(x) = \infty) and
\[
\|f - g\|_{L^\infty(w_0)} \geq \eta \left( \frac{\|f(x) - g(x)\|}{x - a} \right)_{L^\infty(\{|x - a|w_0(x)\geq \eta\})} \geq \eta \text{ max } \{ |c_1 - g'(a)|, |c_2 - g'(a)| \} \geq \eta \frac{c_2 - c_1}{2}.
\]
This is a contradiction with $f$ belonging to the closure of $C^1(\mathbb{R}) \cap L^\infty(w_0)$ in $L^\infty(w_0)$.

Let us choose $g_n \in C^1(\mathbb{R}) \cap L^\infty(w_0)$ with \( \|f - g_n\|_{L^\infty(w_0)} \leq 1/n \). Hence
\[
\frac{\eta}{n} \left| \frac{f(x) - g_n(x)}{x - a} \right| \leq \frac{\|f(x) - g_n(x)\|}{x - a} \leq \|f - g_n\|_{L^\infty(w_0)} \leq \frac{1}{n},
\]
for almost every $x$ with $|x - a|w_0(x) \geq \eta$. Therefore, it follows that $\eta l - g_n(a) \leq 1/n$, for every $n$, since $g_n(a) = 0$ (by \text{ess lim sup}_{x \to a} w_0(x) = \infty). Hence $l$ is finite and $\lim_{n \to \infty} g_n'(a) = l$.

Let us assume now that $f$ belongs to the closure of $C^1(\mathbb{R}) \cap W^{1,\infty}(w_0, w_1)$ in $W^{1,\infty}(w_0, w_1)$ and $a \notin S_1(w_1)$. Notice that Lemma A gives that there exists $u_f(a) := \text{ess lim sup}_{x \to a, w_1(x) \geq \eta} f'(x)$, for each $\eta > 0$ small enough, since $a \notin S_1(w_1)$. We have that there exists $n \in C^1(\mathbb{R}) \cap W^{1,\infty}(w_0, w_1)$ with $\|f - g_n\|_{W^{1,\infty}(w_0, w_1)} \leq 1/n$. Hence
\[
\frac{\eta}{n} |f'(x) - g_n'(x)| \leq |f'(x) - g_n'(x)|_{W^{1,\infty}(w_1)} \leq \frac{1}{n},
\]
for almost every $x$ with $w_1(x) \geq \eta$. Consequently, it follows that $\eta u_f(a) - g_n'(a) \leq 1/n$, for every $n$, and we deduce that $l = \lim_{n \to \infty} g_n'(a) = u_f(a)$. (The same argument allows to deduce that $\lim_{n \to \infty} h_n(a) = u_f(a)$, for any sequence $(h_n) \subset C(\mathbb{R}) \cap L^\infty(w_1)$ converging to $f'$ in $L^\infty(w_1)$. This proves (c).

Let us assume now that there exists $f'(a)$. Then it follows that $f'(a) = l$ and consequently $f'(a) = l = u_f(a)$. #

Proposition 4.2. Let us consider two weights $w_0, w_1$, in $[a, \beta]$, with $\text{ess lim sup}_{x \to a} |x - a|w_0(x) = \infty$ and $a \in S(w_1)$. If $f$ belongs to the closure of $C^1(\mathbb{R}) \cap W^{1,\infty}(w_0, w_1)$ in $W^{1,\infty}(w_0, w_1)$, then $u_f(a) = 0$.

Proof. We only need to consider the case $a \in S(w_1) \setminus S_1(w_1)$, since $u_f(a) = 0$ if $a \in S_1(w_1)$ (recall Definition 4.2).

If we take $g_n \in C^1(\mathbb{R}) \cap W^{1,\infty}(w_0, w_1)$ with $\|f - g_n\|_{W^{1,\infty}(w_0, w_1)} \leq 1/n$, then parts (a) and (b) of Proposition 4.1 imply that $\lim_{n \to \infty} g_n'(a) = u_f(a)$.

Since $\text{ess lim sup}_{x \to a} |x - a|w_0(x) = \infty$, for each $m$\n\[
m \left| \frac{g_n(x)}{x - a} \right| \leq \frac{|g_n(x)|w_0(x)}{|x - a|} \leq \|g_n\|_{L^\infty(w_0)} \leq \frac{\|f\|_{L^\infty(w_0)} + 1}{n},
\]
for almost every $x$ with $|x - a|w_0(x) \geq m$. Then $m|g_n'(a)| \leq \|f\|_{L^\infty(w_0)} + 1/n$ for every $m$, since $g_n(a) = 0$. Consequently, it follows that $g_n'(a) = 0$ and $u_f(a) = 0$. #

Definition 4.3. Let us consider a weight $w_0$ in $[a, \beta]$, with $\text{ess lim sup}_{x \to a} |x - a|w_0(x) > 0$ and $a \in S(w_1)$, and a function $f$ in the closure of $C^1(\mathbb{R}) \cap L^\infty(w_0)$ in $L^\infty(w_0)$. We define the derivative of $f$ in $a$ through $\{ |x - a|w_0(x) \geq \eta \}$ as $\text{ess lim sup}_{x \to a, |x - a|w_0(x) \geq \eta} f'(x)/(x - a)$, for any $0 < \eta < \text{ess lim sup}_{x \to a} |x - a|w_0(x)$.

Theorem 4.3. Let us consider two weights $w_0, w_1$, in $[a, \beta]$ such that $S(w_1) = \{ a \}$ and $0 < \text{ess lim sup}_{x \to a} |x - a|w_0(x) < \infty$. Then the closure of $C^1(\mathbb{R}) \cap W^{1,\infty}(w_0, w_1)$ in $W^{1,\infty}(w_0, w_1)$ is equal to
\[
H_3 := \{ f \in W^{1,\infty}(w_0, w_1) : f \in C(\mathbb{R}) \cap L^\infty(w_0), j' \in C(\mathbb{R}) \cap L^\infty(w_1), f \text{ is continuous to the right if } a \in D^+(w_1), f \text{ is continuous to the left if } a \in D^-(w_1), \exists \lim (f(a) - f(x)) \text{ if } x \notin S_1(w_1), \text{ and } \text{ess lim sup}_{x \to a} j (x - a) = 0, a \notin S_1(w_1), \text{ then } u_f(a) = \{ f(a) \} \}
\]

In fact, for each $f \in H_3$ and $d > 0$ there exist $g_n$ in $C^1(\mathbb{R})$ with $\lim_{n \to \infty} \|f - g_n\|_{W^{1,\infty}(w_0, w_1)} = 0$ and $g_n(x) = f(x)$ if $x \notin \{ a - d, a + d \}$. 

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Remark. Condition “if \( a \notin S_1(w_1) \), then \( u_f(a) = l(f,a) \)” shows the interaction that must exist between \( f \), \( w_0 \) and \( w_1 \) in order to approximate \( f \) by smooth functions (compare with Theorem 4.2). The example after the proof of Theorem 4.3 shows that this condition is independent of the other hypotheses in the definition of \( H_3 \).

Proof. If \( f \) is in the closure of \( C^1(\mathcal{R}) \cap W^{1,\infty}(w_0, w_1) \) in \( W^{1,\infty}(w_0, w_1) \), we will see that it belongs to \( H_3 \). It is clear that \( f \in \overline{C(\mathcal{R}) \cap L^\infty(w_0)} \), and \( f' \in \overline{C(\mathcal{R}) \cap L^\infty(w_1)} \). Lemma 4.4 allows to deduce that \( f \) is continuous to the right if \( a \in D^+(w_1) \) and \( f \) is continuous to the left if \( a \in D^-(w_1) \). Proposition 4.1 implies that if \( a \notin S_1(w_1) \), then \( u_f(a) = l(f,a) \). Let us choose a sequence \( \{g_n\} \subset C^1(\mathcal{R}) \cap W^{1,\infty}(w_0, w_1) \) converging to \( f \) in \( W^{1,\infty}(w_0, w_1) \). By Proposition 4.1 it follows that \( l(f,a) = \text{ess lim}_{x \to a} |x - a|w_0(x) \geq \eta \ f(x)/(x - a) = \lim_{n \to \infty} g_n(a) \), for \( \eta > 0 \) small enough.

Let us fix \( \varepsilon > 0 \). It is clear that

\[
\text{ess lim}_{x \to a} |x - a|w_0(x) < \varepsilon; \text{ then there exists } \delta_1 > 0 \text{ with }
\|f(x) - l(f,a)(x - a)\|_{L^\infty([a - \delta_1, a + \delta_1] \cap \{|x - a|w_0(x) \geq \eta\}, w_0)} < \varepsilon.
\]

Now, it is sufficient to prove that \( \|f(x) - l(f,a)(x - a)\|_{L^\infty([a - \delta_1, a + \delta_1] \cap \{|x - a|w_0(x) < \eta\}, w_0)} < \varepsilon \), for some \( 0 < \delta \leq \delta_1 \). Proposition 4.1 allows to choose \( n \) with \( \|f - g_n\|_{L^\infty(w_0)} < \varepsilon/2 \) and \( g_n(a) - l(f,a) \eta < \varepsilon/2 \); hence, there exists \( 0 < \delta \leq \delta_1 \) with \( |g_n(x)/(x-a) - l(f,a)\eta| < \varepsilon/2 \) for every \( 0 < |x - a| < \delta \). Consequently

\[
\|g_n(x) - l(f,a)(x - a)\|_{L^\infty([a - \delta, a + \delta] \cap \{|x - a|w_0(x) < \eta\}, w_0)} < \varepsilon/2,
\]

We also have \( \|f - g_n\|_{L^\infty(w_0)} < \varepsilon/2 \); therefore \( \|f(x) - l(f,a)(x - a)\|_{L^\infty([a - \delta, a + \delta] \cap \{|x - a|w_0(x) < \eta\}, w_0)} < \varepsilon \), and \( \|f(x) - l(f,a)(x - a)\|_{L^\infty([a - \delta, a + \delta] \cap \{|x - a|w_0(x) < \eta\}, w_0)} < \varepsilon \), then \( f \in H_3 \).

Let us fix now \( f \in H_3 \). The hypothesis \( \text{ess lim sup}_{x \to a} |x - a|w_0(x) < \infty \) implies that there exists \( 0 < \delta_0 < a/2 \) such that \( x - a \in L^\infty([a - 2\delta_0, a + 2\delta_0], w_0) \); if \( \text{ess lim sup}_{x \to a} \text{ w}_1(x) < \infty \), we also require \( w_1 \in L^\infty([a - 2\delta_0, a + 2\delta_0]) \). Let us fix \( \phi \in C_c^\infty([a - 2\delta_0, a + 2\delta_0]) \) with \( 0 \leq \phi \leq 1 \) and \( \phi = 1 \) in \( [a - \delta_0, a + \delta_0] \). We see now that \( l(f,a)(x - a)\phi(x) \in C_c^\infty([a - 2\delta_0, a + 2\delta_0]) \cap W^{1,\infty}(w_0, w_1) \): it is clear that it belongs to \( L^\infty(w_1) \); its derivative is in \( L^\infty(w_1) \) if \( \text{ess lim sup}_{x \to a} \text{w}_1(x) < \infty \); if this is not so, \( a \notin S_1(w_1) \), and it follows that \( l(f,a) = 0 \); if \( \{h_n\} \subset C(\mathcal{R}) \cap L^\infty(w_1) \) converges to \( f' \) in \( L^\infty(w_1) \), part (c) of Proposition 4.1 implies that \( u_f(a) = \lim_{n \to \infty} h_n(a) \); the fact \( \text{ess lim sup}_{x \to a} h_n(x) = \infty \) implies \( h_n(a) = 0 \), and we have \( u_f(a) = l(f,a) \), since \( f \in H_3 \).

We consider the function \( g(x) := f(x) - l(f,a)(x - a)\phi(x) \). Since \( l(f,a)(x-a)\phi(x) \) is a smooth function in \( W^{1,\infty}(w_0, w_1) \), it is sufficient to show that \( g \) can be approximated by \( C^1 \) functions in \( W^{1,\infty}(w_0, w_1) \). We have that \( f(a) = g(a) = 0 \) since \( \text{ess lim sup}_{x \to a} \text{w}_0(x) = \infty \); then \( \text{ess lim sup}_{x \to a} (g(x) - g(a))w_0(x) = 0 \), since \( f \in H_3 \). Notice that \( u_f(a) = 0 \) if \( a \in S_1(w_1) \); if \( a \notin S_1(w_1) \), it follows that \( u_f(a) = \text{ess lim}_{x \to a} \text{ w}_1(x) \geq \eta \ f'(x) - l(f,a) = u_f(a) - l(f,a) = 0 \). Then Theorem 4.1 implies that \( g \) can be approximated by functions \( \{g_n\} \) in \( C^1 \cap W^{1,\infty}(w_0, w_1) \), with \( g_n(x) = g(x) = f(x) \) if \( x \notin (a - d, a + d) \).

Example. There exist weights \( w_0, w_1 \), and a function \( f \) such that \( a \not\in S_1(w_1) \), \( u_f(a) \neq l(f,a) \), and verifying the other hypotheses in the definition of \( H_3 \).

Let us consider the function \( f(x) = x^2 \sin(1/x) \) and the weights in \([0,1]\),

\[
w_0(x) = \frac{1}{x^3}, \quad w_1(x) = \begin{cases} 1, & \text{if } x \in \left(\frac{1}{2\pi n + 1/n + 1}, \frac{1}{2\pi n - 1/n}\right), \\ 1, & \text{if } x \in \left(\frac{1}{2\pi n - 1/n}, \frac{1}{2\pi n - 1/n + 1}\right), \\ \end{cases}
\]

It is clear that \( a = 0, a \notin S_1(w_1) \), \( f \in C([0,1]) \), \( f' \in C((0,1]) \), \( l(f,0) = f'(0) = 0 \) and \( \text{ess lim}_{x \to 0} f(x)w_0(x) = 0 \). A direct computation shows that \( u_f(0) = -1 \) and \( \text{ess lim}_{x \to 0} [f'(x) + 1]w_1(x) = 0 \) (then \( f' \) belongs to the closure of \( C(\mathcal{R}) \cap L^\infty(w_1) \) in \( L^\infty(w_1) \)).
We can deduce the following result from Theorem 4.3. We say that two functions \( u, v \) are comparable in the set \( A \) if there are positive constants \( c_1, c_2 \) such that \( c_1 v(x) \leq u(x) \leq c_2 v(x) \) for almost every \( x \in A \).

**Corollary 4.1.** Let us consider two weights \( w_0, w_1 \), in \([\alpha, \beta]\) such that \( S(w_1) = \{a\} \) and \( w_0 \) is comparable to \( 1/|x-a| \) in a neighborhood of \( a \). Then the closure of \( C^1(\mathbb{R}) \cap W^{1,\infty}(w_0, w_1) \) in \( W^{1,\infty}(w_0, w_1) \) is equal to

\[
\left\{ f \in W^{1,\infty}(w_0, w_1) : f \in C^1(\mathbb{R}) \cap L^\infty(w_0), f' \in C^1(\mathbb{R}) \cap L^\infty(w_1) \right\},
\]

where

\[
f \text{ is continuous to the right if } a \in D^+(w_1),
\]

\[
f \text{ is continuous to the left if } a \in D^-(w_1),
\]

\[
\exists f'(a) \text{ and if } a \notin S_1(w_1), \text{ then } u_f(a) = f'(a).
\]

**Proof.** It is clear that \( l(f, a) = f'(a) \), since \( w_0 \) is comparable to \( 1/|x-a| \), and it follows that \( \text{ess lim}_{x \to a} |f(x) - f'(a)(x-a)|w_0(x) = 0 \), since \( f \) is differentiable in \( a \).

We introduce now the following condition which will be essential in the characterization of the functions \( f \) which can be approximated by smooth functions in \( W^{1,\infty}(w_0, w_1) \) in the last case:

Let us consider two weights \( w_0, w_1 \), in \([\alpha, \beta]\) such that \( S(w_1) = \{a\} \) and \( \text{ess lim}_{x \to a} |x-a|w_0(x) = \infty \), and \( f \in W^{1,\infty}(w_0, w_1) \).

For some \( d_0 > 0 \) and each \( n \in \mathbb{N} \),

\[
\text{there exists } \phi_n \in C^1([a-d_0, a+d_0]) \cap W^{1,\infty}([a-d_0, a+d_0], w_0, w_1) \text{ such that } \text{ess lim sup}_{x \to a} |f(x) - \phi_n(x)|w_0(x) < 1/n.
\]

**Lemma 4.6.** Let us consider two weights \( w_0, w_1 \), in \([\alpha, \beta]\) such that \( S(w_1) = \{a\} \) and \( \text{ess lim}_{x \to a} |x-a|w_0(x) = \infty \). If \( f \) verifies condition (4.2), then for each \( 0 < d \leq d_0 \) we can choose the functions \( \phi_n \) with the additional property \( \phi_n = 0 \) in a neighborhood of \( a-d \). The argument in a neighborhood of \( a+d \) is similar.

Let us assume first that \( \text{ess lim}_{x \to a} w_1(x) = \infty \) for every \( t \in [a-d, a] \). Then \( \phi_n' = 0 \) in \([a-d, a] \), and \( \phi_n(a) = 0 \) since \( \text{ess lim}_{x \to a} w_1(x) = \infty \). Hence, \( \phi_n = 0 \) in \([a-d, a] \).

In other case, there exists \( t \in [a-d, a] \) with \( \text{ess lim}_{x \to a} w_1(x) < \infty \). Then, there exists a closed interval \( A = [a_1, a_2] \subset (a-d, a) \) with \( w_1 \in L^\infty(A) \). Let us fix \( \varphi \in C^1(\mathbb{R}) \) with \( \varphi = 0 \) in \((-\infty, a_1] \) and \( \varphi = 1 \) in \([a_2, \infty) \). It is clear that \( \varphi \phi_n \in W^{1,\infty}([a-d, a+d], w_0, w_1) \) since \( w_1 \in L^\infty(A) \). Hence, we can substitute \( \phi_n \) by \( \varphi \phi_n \).

**Theorem 4.4.** Let us consider two weights \( w_0, w_1 \), in \([\alpha, \beta]\) such that \( S(w_1) = \{a\} \) and \( \text{ess lim}_{x \to a} |x-a|w_0(x) = \infty \). Then the closure of \( C^1(\mathbb{R}) \cap W^{1,\infty}(w_0, w_1) \) in \( W^{1,\infty}(w_0, w_1) \) is equal to

\[
H_4 := \left\{ f \in W^{1,\infty}(w_0, w_1) : f \in C^1(\mathbb{R}) \cap L^\infty(w_0) \cap L^\infty(w_1),
\right\}
\]

\[
f \text{ is continuous to the right if } a \in D^+(w_1),
\]

\[
f \text{ is continuous to the left if } a \in D^-(w_1),
\]

\[
f \text{satisfies (4.2) and } u_f(a) = 0
\]

In fact, for each \( f \in H_4 \) and \( d > 0 \) there exist \( \{g_n\}_n \in C^1(\mathbb{R}) \) with \( \lim_{n \to \infty} \|f - g_n\|_{W^{1,\infty}(w_0, w_1)} = 0 \) and \( g_n(x) = f(x) \) if \( x \notin (a-d, a+d) \).

**Remarks.**

1. Although (4.2) is not a condition so clean than those in \( H_2 \) or \( H_3 \), it simplifies notably the approximation problem, since it is a local condition and there is no reference to \( f' \) (we do not need to approximate simultaneously \( f \) and \( f' \)). Condition (5.1) below implies (4.2), and Proposition 5.2 characterizes (5.1) in many situations.

2. Condition (4.2) shows the interaction that must exist between \( f \), \( w_0 \) and \( w_1 \) in order to approximate \( f \) by smooth functions (notice that \( \phi_n \in C^1(\mathbb{R}) \cap W^{1,\infty}(w_0, w_1) \)).
If \( f(a) \in W^{1,\infty}(w_0, w_1) \) for any \( f \in C^1(\mathbb{R}) \cap L^\infty(w_0) \) with \( f' \in C^1(\mathbb{R}) \cap L^\infty(w_1) \) (in particular, if \( w_0 \in L^\infty([\alpha, \beta]) \)), then condition (4.2) can be removed (since \( \text{ess lim}_{x \to a} |f(x) - f(a)| w_0(x) = 0 \)).

If we are in some of the following situations (see Remark 2 to Theorem 4.1):

(a) \( a \in S^+(w_0) \cap S^-(w_0) \), i.e., \( \text{ess lim inf}_{x \to a} w_0(x) = \text{ess lim inf}_{x \to a} w_0(x) = 0 \),

(b) \( a \in S^+(w_0) \) and \( w_0 \in L^\infty([a - \varepsilon, a]) \), for some \( \varepsilon > 0 \),

(c) \( a \in S^-(w_0) \) and \( w_0 \in L^\infty([a, a + \varepsilon]) \), for some \( \varepsilon > 0 \),

(d) \( w_0 \in L^\infty([a - \varepsilon, a + \varepsilon]) \), for some \( \varepsilon > 0 \).

Therefore, in this situation, the statement of Theorem 4.4 is nicer.

**Proof.** It is clear that if \( f \) belongs to the closure of \( C^1(\mathbb{R}) \cap W^{1,\infty}(w_0, w_1) \) in \( W^{1,\infty}(w_0, w_1) \), then \( f \in C^1(\mathbb{R}) \cap L^\infty(w_0) \) and \( f' \in C^1(\mathbb{R}) \cap L^\infty(w_1) \). Lemma 4.4 implies that \( f \) is continuous to the right if \( a \in D^+(w_1) \) and \( f \) is continuous to the left if \( a \in D^-(w_1) \). Proposition 4.2 implies that \( uf(a) = 0 \). We prove now that \( f \) satisfies (4.2): Seeking a contradiction, suppose that \( f \) does not satisfy (4.2); then there exist positive constants \( c, d \) such that \( \text{ess lim sup}_{x \to a} |f(x) - \phi(x)| w_0(x) \geq c \) for every \( \phi \in C^1([a - d, a + d]) \cap W^{1,\infty}([a - d, a + d]) \). This means that \( \|f - \phi\|_{L^\infty([a - \delta, a + \delta], w_0)} \geq c \) for every \( \delta > 0 \). Hence, \( \|f - \phi\|_{L^\infty(w_0)} \geq \|f - \phi\|_{L^\infty([a - \delta, a + \delta], w_0)} \geq c \) for every \( \phi \in C^1(\mathbb{R}) \cap W^{1,\infty}(w_0, w_1) \), which provides the expected contradiction. Then \( f \in H_q \).

Let us see now that \( H_q \) is contained in the closure of \( C^1(\mathbb{R}) \cap W^{1,\infty}(w_0, w_1) \) in \( W^{1,\infty}(w_0, w_1) \). By Lemma 4.6, given \( f_0 \in H_4, d > 0 \) and \( \varepsilon > 0 \) we can choose \( \phi \in C^1([a - d, a + d]) \cap W^{1,\infty}(w_0, w_1) \) such that the function defined by \( f := f_0 - \phi \) verifies \( \text{ess sup}_{x \to a} |f(x)| w_0(x) < \varepsilon / 24 \); besides, \( f(x) = f_0(x) \) if \( x \notin (a - d, a + d) \). Then there exists \( \delta > 0 \) with \( 4\|f - f_0\|_{L^\infty([a - \delta, a + \delta], w_0)} < \varepsilon / 6 \) (recall that \( f(a) = 0 \) since \( \text{ess lim sup}_{x \to a} w_0(x) = \infty \)).

Since \( uf(a) = 0 \), then applying the argument in the proof of Theorem 4.1 it is possible to find \( g \in C^1(\mathbb{R}) \cap W^{1,\infty}(w_0, w_1) \) with \( \|f - g\|_{W^{1,\infty}(w_0, w_1)} < \varepsilon \) and \( g(x) = f(x) \) if \( x \notin (a - d, a + d) \). Hence, if \( g_0 := g + \phi \), it follows that \( \|f_0 - g_0\|_{W^{1,\infty}(w_0, w_1)} < \varepsilon \) and \( g_0(x) = f_0(x) \) if \( x \notin (a - d, a + d) \).

The following result allows to reduce the global approximation problem in \( W^{1,\infty}(I, w_0, w_1) \) by smooth functions to a local approximation problem, under some technical conditions.

**Theorem B.** [R1, Theorem 5.2] Let us consider strictly increasing sequences of real numbers \( \{\alpha_n\}_{n \in J}, \{\beta_n\}_{n \in J} \) (\( J \) is either a finite set, \( \mathbb{Z}, \mathbb{Z}^+ \) or \( \mathbb{Z}^- \)) with \( \alpha_{n+1} < \beta_n < \alpha_{n+2} \) for every \( n \). Let \( w_0, w_1 \) be weights in the interval \( I := \cup_n [\alpha_n, \beta_n] \). Assume that for each \( n \) there exists an interval \( I_n \subset [\alpha_n, \beta_n] \) with \( w_1 \in L^\infty(I_n) \) and \( \int_{I_n} w_0 > 0 \). Then \( f \) can be approximated by functions of \( C^1(I) \) in \( W^{1,\infty}(I, w_0, w_1) \) if and only if it can be approximated by functions of \( C^1(I, \alpha_n, \beta_n) \) in \( W^{1,\infty}(I, \alpha_n, \beta_n, w_0, w_1) \) for each \( n \). The same result is true if we replace \( C^1 \) by \( C^\infty \) in both cases.

**Remarks.**

1. The proof of this theorem in [R1] is constructive and the main idea is natural: it suffices to consider functions \( g_n \) which approximate \( f \) in \( [\alpha_n, \beta_n] \) and to obtain a function \( g \) which approximate \( f \) in \( I \) by “pasting” \( \{g_n\}_n \) with an appropriate partition of unity. Since the pasting process occurs in \( \cup_n I_n \), we have \( g = g_n \) in \( [\beta_n - 1, \beta_n + 1] \), and if there exists a last index \( n \) in \( J \), then \( g = g_n \) in \( [\beta_n, \alpha_{n+1}] \) in particular, \( g(\alpha_n) = g_{n+1}(\alpha_n) \) and \( g(\beta_n) = g_n(\beta_n) \).

2. Condition \( \alpha_{n+1} < \beta_n \) means that \( (\alpha_n, \beta_n) \) and \( (\alpha_{n+1}, \beta_{n+1}) \) overlap; \( (\alpha_n, \beta_n) \cap (\alpha_{n+2}, \beta_{n+2}) = \emptyset \) since \( \beta_n < \alpha_{n+2} \).

In fact, Theorem 5.2 in [R1] is a more general result, but the statement we present here is good enough for our purposes.

**Definition 4.4.** The weights \( w_0, w_1 \) are jointly admissible on the interval \( I \), if there exist strictly increasing sequences of real numbers \( \{\alpha_n\}_{n \in J}, \{\beta_n\}_{n \in J} \) (\( J \) is either a finite set, \( \mathbb{Z}, \mathbb{Z}^+ \) or \( \mathbb{Z}^- \)) with \( \alpha_{n+1} < \beta_n < \alpha_{n+2} \) for every \( n \) and \( I := \cup_n [\alpha_n, \beta_n] \), and verifying the following conditions:

- There exists a partition \( J_1, J_2, J_3 \) of \( J \), such that
  - (a1) if \( n \in J_1 \), then \( w_0 \in L^\infty([\alpha_n, \beta_n]) \) and \( 1/w_1 \in L^1([\alpha_n, \beta_n]) \),
  - (a2) if \( n \in J_2 \), then \( S(w_1) \cap [\alpha_n, \beta_n] = \{\alpha_n\} \),
  - (a3) if \( n \in J_3 \), then \( S(w_1) \cap [\alpha_n, \beta_n] = \emptyset \).
Remark. Without loss of generality we can assume that \( a_n \in (\beta_{n-1}, \alpha_{n+1}) \) if \( n \in J_2 \): if \( a_n \in (\alpha_n, \beta_n) \) and \( a_n \leq \beta_{n-1} \), it suffices to take \( \beta_{n-1} \) smaller; if \( a_n \in (\alpha_n, \beta_n) \) and \( \alpha_{n+1} \leq a_n \), it suffices to take \( \alpha_{n+1} \) bigger; if \( a_n = \alpha_n \), it suffices to take \( \alpha_n \) bigger (and then \( n \in J_2 \)); if \( a_n = \beta_n \), it suffices to take \( \beta_n \) smaller (and then we also have \( n \in J_3 \)). We always assume this property.

Now, we can state the main result of this section. Notice that we do not have any hypothesis about the singularities of \( \omega_0 \), that the weights \( \omega_0, \omega_1 \) have a great deal of independence among them, and that the interval \( I \) is not required to be bounded.

**Theorem 4.5.** Let us consider two weights \( \omega_0, \omega_1 \) which are jointly admissible on the interval \( I \). Then the closure of \( C^1(I) \cap W^{1,\infty}(\omega_0, w_1) \) in \( W^{1,\infty}(\omega_0, w_1) \) is equal to

\[
H := \{ f \in W^{1,\infty}(\omega_0, w_1) : f \in C(I) \cap L^\infty(\omega_0), f' \in C(I) \cap L^\infty(\omega_1), \text{ for each } \{ a_n \} = S(\omega_1) \cap [\alpha_n, \beta_n], \text{ with } n \in J_2, \text{ we have } \\
\text{if } f \text{ is continuous to the right if } a_n \in D^+(w_1), \\
\text{if } f \text{ is continuous to the left if } a_n \in D^-(w_1), \\
\text{if } \text{ess lim } |x - a_n|\omega_0(x) = 0, \text{ ess lim } |f(x) - f(a_n)|\omega_0(x) = 0, \\
\text{if } 0 < \text{ess lim sup } |x - a_n|\omega_0(x) < \infty, \\
\exists l(f, a_n) \text{ and } \text{ess lim } |f(x) - l(f, a_n)(x - a_n)|\omega_0(x) = 0, \\
\text{if } a_n \in S(I(\omega_1)), \text{ then } u_f(a_n) = l(f, a_n), \\
\text{if } \text{ess lim sup } |x - a_n|\omega_0(x) = \infty, \text{ } f \text{ satisfies (4.2) and } u_f(a_n) = 0 \}. \]

**Remarks.**

1. Notice that this theorem has a wide range of application. Let us consider the particular case of Jacobi weights: \( \omega_0(x) = (1 + x)^{\alpha_1}(1 - x)^{\beta_1}, \omega_1(x) = (1 + x)^{\alpha_2}(1 - x)^{\beta_2} \), in \([-1, 1]\). Theorem 4.5 describes the closure of \( C^1([-1, 1]) \cap W^{1,\infty}([-1, 1], \omega_0, \omega_1) \) in \( W^{1,\infty}([-1, 1], \omega_0, \omega_1) \) for every possible value of the exponents; if \( t_1 \leq 0 \) (respectively \( t_2 \leq 0 \)), then \(-1\) (respectively \(1\)) is a regular point of \( \omega_1 \).

It is obvious that Theorem 4.5 also describes the closure of \( C^2 \) functions with weights with many singular points, as \( \omega_0(x) = |x - a_1|^{\alpha_1}|x - a_2|^{\alpha_2} \cdots |x - a_n|^{\alpha_n}, \omega_1(x) = |x - b_1|^{\beta_1}|x - b_2|^{\beta_2} \cdots |x - b_n|^{\beta_n} \). The same is true if we change each power \( |x - a|^\beta \) by any function with a singularity in \( \alpha \), and even if we consider weights defined in some interval \( I \) such that \( S(\omega_1) \) has no accumulation point in the interior of \( I \).

2. Let us observe that in Theorem 4.5 we do not have as hypotheses the technical conditions which appear in the statement of Theorem B.

In order to prove Theorem 4.5, we need two preliminary results.

**Proposition 4.3.** Let us consider two weights \( \omega_0, \omega_1 \), in \( A = [\alpha, \beta] (-\infty \leq \alpha < \beta \leq \infty) \), with \( \omega_0 \in L^\infty(A) \) and \( 1/\omega_1 \in L^1(A) \). Then

\[
\overline{C^1(A) \cap W^{1,\infty}(\omega_0, \omega_1)}^{W^{1,\infty}(\omega_0, \omega_1)} = \{ f \in W^{1,\infty}(A, \omega_0, \omega_1) : f' \in C(A) \cap L^\infty(\omega_1, \omega_1^w) \}. \]

Furthermore, if \( f \in C^1(A) \cap W^{1,\infty}(\omega_0, \omega_1) \), we can obtain a sequence of functions \( \{ F_n \} \subset C^1(A) \cap W^{1,\infty}(\omega_0, \omega_1) \) converging to \( f \) in \( W^{1,\infty}(A, \omega_0, \omega_1) \) with \( F_n(\alpha) = f(\alpha) \) and \( F_n(\beta) = f(\beta) \). The same result is true if we replace \( C^1(A) \) and \( C(A) \) by \( C^\infty(A) \) everywhere.

**Proof.** We prove the non-trivial inclusion. If \( f' \in C(A) \cap L^\infty(\omega_1, \omega_1^w) \), let us consider a sequence \( \{ g_n \} \subset C(A) \cap L^\infty(\omega_1, \omega_1) \) which converges to \( f' \) in \( L^\infty(A, \omega_1) \). Notice that \( f' \in L^\infty(A, \omega_1) \) and \( 1/\omega_1 \in L^1(A) \) imply that \( f' \in L^1(A) \) and hence \( f \) is an absolutely continuous function on \( A \). Then the functions \( G_n(x) := f(\alpha) + \int_{\alpha}^{x} g_n \) belongs to \( C^1(A) \), satisfy \( G_n(\alpha) = f(\alpha) \) and

\[
|f(x) - G_n(x)| = \left| \int_{\alpha}^{x} (f' - g_n) \right| \leq \int_{A} |f' - g_n| \frac{w_1}{w_1} \leq \| f' - g_n \|_{L^\infty(A, \omega_1)} \int_{A} \frac{1}{w_1}.
\]
Let us consider strictly increasing sequences of real numbers \( w \) with every \( w \in C^1(A) \cap W^{1,\infty}(A, w_0, w_1) \) verifies \( g' = 0 \) in \( A \), and therefore is constant. Hence the closure of \( C^1(A) \cap W^{1,\infty}(A, w_0, w_1) \) is the space of constants, and then the last conclusion of the proposition is direct.

If we do not have ess\( \limsup_{t \to \infty} w_1(x) = \infty \) for every \( t \in A \), then there exists an interval \( B \subset A \) with \( w \in L^\infty(B) \). Let us consider a function \( h \in C(A) \) with supp \( h \subset B \) and \( h = 1 \). In this case we can define the functions \( F_n(x) := G_n(x) + (f(\beta) - G_n(\beta)) \int_0^x h \in C^1(A) \cap W^{1,\infty}(A, w_0, w_1) \), which verify \( F_n(\alpha) = f(\alpha) \) and \( F_n(\beta) = f(\beta) \). Since \( \lim_{n \to \infty} (f(\beta) - G_n(\beta)) = 0 \), we also have that \( \{F_n\} \) converges to \( f \) in \( W^{1,\infty}(A, w_0, w_1) \).

If we replace \( C^1(A) \) and \( C(A) \) by \( C^\infty(A) \) everywhere in this proof, we obtain that

\[
C^\infty(A) \cap W^{1,\infty}(A, w_0, w_1) = \left\{ f \in W^{1,\infty}(A, w_0, w_1) : f' \in C^\infty(A) \cap L^\infty(A, w_1) \right\}.
\]

**Proposition 4.4.** Let us consider strictly increasing sequences of real numbers \( \{\alpha_n\}_{n \in J}, \{\beta_n\}_{n \in J} \) (\( J \) is either a finite set, \( \mathbb{Z}, \mathbb{Z}^+ \) or \( \mathbb{Z}^- \)) with \( \alpha_{n+1} < \beta_n < \alpha_{n+2} \) for every \( n \). Let \( w_0, w_1 \) be weights in the interval \( I := \cup_n [\alpha_n, \beta_n] \). Let us fix \( f \in W^{1,\infty}(I, w_0, w_1) \). Assume that for each \( n \) \( \liminf_{t \rightarrow \infty} x_t(\alpha_n, \beta_n, w_0, w_1) = 0 \). Let \( \{g_{n,k}\}_{k \in I} \) be \( C^1([\alpha_n, \beta_n]) \cap W^{1,\infty}([\alpha_n, \beta_n], w_0, w_1) \) with \( \lim_{k \rightarrow \infty} \|f - g_{n,k}\|_{W^{1,\infty}([\alpha_n, \beta_n], w_0, w_1)} = 0 \). Consequently, \( g_{n,k}^{(1)}(x) \rightarrow f(x) \) for every \( x \in [\alpha_{n+1}, \beta_n] \), and \( g_{n,k+1}^{(1)}(x) \rightarrow f(x) \) for every \( x \in [\alpha_{n+1}, \beta_{n+1}] \). Then \( f \) belongs to the closure of \( C^1(I) \cap W^{1,\infty}(0, w_1) \) in \( W^{1,\infty}(w_0, w_1) \). The same result is true if we replace \( C^1 \) by \( C^\infty \) in both cases.

**Proof.** For each \( n \), let us consider \( g_{n,k}^{(1)} \) in \( C^1([\alpha_n, \beta_n]) \) with \( \|f - g_{n,k}^{(1)}\|_{W^{1,\infty}([\alpha_n, \beta_n], w_0, w_1)} < 1/k \). Since \( \liminf_{t \rightarrow \infty} x_t(\alpha_n, \beta_n, w_0, w_1) = 0 \), we have that \( \{g_{n,k}\}_k \) is a sequence in \( [\alpha_{n+1}, \beta_n] \), and consequently, \( g_{n,k}^{(1)}(x) \rightarrow f(x) \) for every \( x \in [\alpha_{n+1}, \beta_n] \), and \( g_{n,k+1}^{(1)}(x) \rightarrow f(x) \) for every \( x \in [\alpha_{n+1}, \beta_{n+1}] \). Then \( f \) belongs to the closure of \( C^1(I) \cap W^{1,\infty}(w_0, w_1) \) in \( W^{1,\infty}(w_0, w_1) \). It is clear now that the same result is true if we replace \( C^1 \) by \( C^\infty \) in both cases.

**Proof of Theorem 4.5.** Theorems 4.2, 4.3 and 4.4, and Proposition 4.3 allow to deduce that any function in the closure of \( C^1(I) \) in \( W^{1,\infty}(w_0, w_1) \) belongs to \( H \). Let us observe that the closure of \( C^1([\alpha_n, \beta_n]) \cap W^{1,\infty}([\alpha_n, \beta_n], w_0, w_1) \) in \( W^{1,\infty}([\alpha_n, \beta_n], w_0, w_1) \) is \( C^1([\alpha_n, \beta_n]) \cap W^{1,\infty}([\alpha_n, \beta_n], w_0, w_1) \) if \( n \in J_3 \), since the closure of \( C([\alpha_n, \beta_n]) \cap L^\infty([\alpha_n, \beta_n], w_0) \) in \( L^\infty([\alpha_n, \beta_n], w_0) \) is \( C([\alpha_n, \beta_n]) \cap L^\infty([\alpha_n, \beta_n], w_0) \), by Theorem 2.1.

We prove now the other inclusion. Let us consider the sequences \( \{\alpha_n\}_{n \in J} \) and \( \{\beta_n\}_{n \in J} \) in the definition of jointly admissible weights. Recall that \( \alpha_n \in (\beta_{n-1}, \alpha_{n+1}) \) if \( n \in J_2 \). This fact allows to take the approximations in theorems 4.2, 4.3 and 4.4 with the same values of the approximated function in \( \alpha_n \) and \( \beta_n \).

We show that each function \( f \in H \) can be approximated by functions of \( C^1(I) \) in \( W^{1,\infty}(I, w_0, w_1) \) if it can be approximated by functions of \( C^1([\alpha_n, \beta_n]) \) in \( W^{1,\infty}([\alpha_n, \beta_n], w_0, w_1) \) for each \( n \); then we can apply theorems 4.2, 4.3 and 4.4, and Proposition 4.3, which show that any function in \( H \) belongs to the closure of \( C^1([\alpha_n, \beta_n]) \) in \( W^{1,\infty}([\alpha_n, \beta_n], w_0, w_1) \) for every \( n \). We use an argument with two steps, using Theorem B and Proposition 4.4.

Let us assume firstly that for each \( n \) there exists an interval \( I_n \subset [\alpha_{n+1}, \beta_n] \) with \( w \in L^\infty(I_n) \). Let us remark that \( n \notin I_n \) if \( n \in J_2 \), since \( \alpha_n < \alpha_{n+1} \). Then every function \( f \) in \( H \) belongs to \( C^1(I_n) \): if \( n \in J_2 \cup J_3 \), then \( S(w_0) \cap I_n = \emptyset \) and \( f \in C^1(I_n) \); if \( n \in J_1 \), then \( f \in L^1(I_n) \) and \( f \in AC(I_n) \). For each \( f \in H \), let us define \( c_n := \|f\|_{L^\infty(I_n)} \) if \( \|f\|_{L^\infty(I_n)} > 0 \) and \( c_n := 0 \) otherwise. Then \( f \in L^\infty(w_0) \), where \( w_0 := w_0 + \sum c_n \chi_{I_n} \), since \( \|f\|_{L^\infty(w_0)} \leq \|f\|_{L^\infty(w_0)} + 1 \). We also have \( \int_{I_n} w_0' > 0 \) for each \( n \in J \). Hence, theorems B, 4.2, 4.3 and 4.4, and Proposition 4.3 finish the proof of Theorem 4.5 in this case, since the closures of \( C^1([\alpha_n, \beta_n]) \) in \( W^{1,\infty}([\alpha_n, \beta_n], w_0, w_1) \) and in \( W^{1,\infty}([\alpha_n, \beta_n], w_0, w_1) \) are the same (recall that any \( f \) in the closure of \( C^1([\alpha_n, \beta_n]) \) in \( W^{1,\infty}([\alpha_n, \beta_n], w_0, w_1) \) belongs to \( C^1(I_n) \)).

In the general case, there are some \( n \)'s with \( \limsup_{t \rightarrow \infty} x_t(\alpha_n, \beta_n, w_0, w_1) = \infty \) for every \( t \in [\alpha_{n+1}, \beta_n] \). The simplified version of Theorem 4.5 which we have proved allows to joint some intervals in a single interval (recall the first remark to Theorem B); therefore, we can assume that \( \limsup x_t(\alpha_n, \beta_n, w_0, w_1) = \infty \) for every \( t \in [\alpha_{n+1}, \beta_n] \) and every \( n \). Then, Proposition 4.4, theorems 4.2, 4.3 and 4.4, and Proposition 4.3 finish the proof.
5. APPROXIMATION BY $C^\infty$ FUNCTIONS IN $W^{1,\infty}(I, w_0, w_1)$

We are also interested in approximation by more regular functions. With some additional hypothesis we can use Theorem 4.1 in order to approximate by $C^\infty$ functions.

**Theorem 5.1.** Let us consider two weights $w_0, w_1$, in $[\alpha, \beta]$ such that $S(w_1) = \{a\}$ and $w_0, w_1 \in L^\infty_{loc}(\alpha, \beta \setminus \{a\})$. Then every function in

$$H_5 := \{f \in W^{1,\infty}(w_0, w_1) : f \in C(R) \cap L^\infty(w_0)^n, f' \in C(R) \cap L^\infty(w_1)^n, f$$

is continuous to the right if $a \in D^+(w_1)$,

\begin{align*}
&f \text{ is continuous to the left if } a \in D^-(w_1), \\
&\text{ ess lim } f(x) - f(a) |w_0(x)| = 0, \quad \text{ ess lim } u_f(a)(x-a) |w_0(x)| = 0, \\
&\text{ and } \quad \text{ ess lim } f'(x) - u_f(a) |w_1(x)| = 0, \\
\end{align*}

\begin{align*}
\text{can be approximated by functions } \{g_n\}_n \text{ in } C^\infty(R) \cap W^{1,\infty}(w_0, w_1) \text{ with the norm of } W^{1,\infty}(w_0, w_1), \text{ with } g_n(\alpha) = f(\alpha) \text{ if } a \neq \alpha \text{ and with } g_n(\beta) = f(\beta) \text{ if } a \neq \beta.
\end{align*}

**Remark.** In the remark after Theorem 4.1 appear simple conditions which guarantee the properties which define $H_5$.

**Proof.** Let us consider $f \in H_5$ and $\varepsilon > 0$. Theorem 4.1 implies that there exists $g_0 \in C^1(R)$ with

$$\|f - g_0\|_{W^{1,\infty}(w_0, w_1)} < \varepsilon/2,$$

such that $g_0$ is a polynomial of degree at most 1 in $[a - 2\delta, a + 2\delta]$ for some $\delta > 0$.

Let us choose an even function $\phi \in C^\infty([-1, 1])$ with $\phi \geq 0$ and $\int \phi = 1$. For each $t > 0$, we define $\phi_t(x) := t^{-1}\phi(x/t)$ and $\phi_t := \phi_t * \phi_t$; these functions satisfy $\phi_t \in C^\infty([-t, t])$, $\phi_t \geq 0$ and $\int \phi_t = 1$.

It is well known that $g_t \in C^\infty(R)$, and that $g_t$ (respectively $g_t'$) converges uniformly in $[\alpha, \beta]$ to $g_0$ (respectively $g_0'$) when $t \to 0$.

Notice that if $h$ is a polynomial of degree at most 1, then $h * \phi_t = h$, since $1 * \phi_t = \int \phi_t = 1$ and $x * \phi_t = x$: it is sufficient to notice that $(x * \phi_t)(0) = \int_0^x \phi_t(y) dy = 0$ and $(x * \phi_t)' = 1 * \phi_t = 1$.

Consequently, $g_t = g_0$ in $[a - \delta, a + \delta]$, for $0 < t < \delta$, since under this hypothesis, the integral defining $g_t$ only takes into account the values of $g_0$ in which it is a polynomial of degree at most 1.

Since $w_0, w_1 \in L^\infty_{loc}(\alpha, \beta \setminus \{a\})$, there exists a constant $M$ with $w_0, w_1 \leq M$ in $\alpha, \beta \setminus (a - \delta, a + \delta)$. Therefore

$$\|g_t - g_0\|_{W^{1,\infty}(w_0, w_1)} = \|g_t - g_0\|_{W^{1,\infty}([\alpha, \beta \setminus (a - \delta, a + \delta), w_0, w_1)} \leq M\|g_t - g_0\|_{W^{1,\infty}([\alpha, \beta \setminus (a - \delta, a + \delta)}) < \frac{\varepsilon}{2},$$

if $t$ is small enough, since $g_t$ and $g_t'$ converge uniformly in $[\alpha, \beta]$ to $g_0$ and $g_0'$ respectively.

Then $\|f - g_t\|_{W^{1,\infty}(w_0, w_1)} < \varepsilon$ if $t$ is small enough.

Let us assume that $a \neq \alpha$. Fix $\varphi \in C^\infty(R)$ with $\varphi = 1$ in $(-\infty, a)$ and $\varphi = 0$ in $[a - \delta, \infty)$. Since $g_t$ converges uniformly to $g_0$ in $[\alpha, \beta]$, $g_0(\alpha) = f(\alpha)$ and $w_0, w_1 \leq M$ in $[a, a - \delta]$, we can choose $t$ with the additional condition $|f(\alpha) - g_t(\alpha)| \|\varphi\|_{W^{1,\infty}(w_0, w_1)} < \varepsilon$. Then $\int \frac{g_t(\alpha)}{w_0} + (f(\alpha) - g_t(\alpha)) \varphi$ verifies

\begin{align*}
\int \frac{g_t(\alpha)}{w_0} + (f(\alpha) - g_t(\alpha)) \varphi &< \varepsilon, \\
\int g_t &< \varepsilon|a - \alpha| \\
|f(\alpha) - g_t(\alpha)| \|\varphi\|_{W^{1,\infty}(w_0, w_1)} &< 2\varepsilon.
\end{align*}

If $a \neq \beta$, we use a similar argument in a neighborhood of $\beta$.

**Definition 5.1.** We say that a weight $w_1$ in $[\alpha, \beta]$ is balanced at $a \in [\alpha, \beta]$, if it verifies some of the following conditions:

1. $a \in S^+(w_1) \cap S^-(w_1)$, i.e., $\text{ ess lim inf}_{x \to a-} w_1(x) = \text{ ess lim inf}_{x \to a-} w_1(x) = 0$,
2. $a \in S^+(w_1)$ and $w_1 \in L^\infty([a - \varepsilon, a])$, for some $\varepsilon > 0$,
3. $a \in S^-(w_1)$ and $w_1 \in L^\infty([a + \varepsilon, a])$, for some $\varepsilon > 0$,
4. $a = \alpha$ or $a = \beta$.

Theorem 5.1 and Remark 3 to Theorem 4.1, give the following result.
Corollary 5.1. Let us consider two weights \( w_0, w_1 \), in \([\alpha, \beta] \) such that \( S(w_1) = \{a\} \), \( w_1 \) is balanced at \( a \), and \( w_0, w_1 \in L^\infty_{\text{loc}}(a, \beta) \setminus \{a\} \). Then every function in \( H_1 \) can be approximated by functions \( \{g_n\}_n \) in \( C^\infty(R) \cap W^{1,\infty}(w_0, w_1) \) with the norm of \( W^{1,\infty}(w_0, w_1) \), with \( g_n(a) = f(a) \) if \( a \neq \alpha \) and with \( g_n(\beta) = f(\beta) \) if \( a = \beta \).

We introduce now the following condition which plays the same role that (4.2) in the approximation by functions in \( C^\infty \):

Let us consider two weights \( w_0, w_1 \), in \([\alpha, \beta] \) such that \( S(w_1) = \{a\} \) and \( \text{ess lim sup}_{x \to a} |x-a|w_0(x) = \infty \), and \( f \in W^{1,\infty}(w_0, w_1) \).

For some \( d_0 > 0 \) and each \( n \in \mathbb{N} \),

\[
\phi_n \in C^\infty([a-d_0, a+d_0]) \cap W^{1,\infty}([a-d_0, a+d_0], w_0, w_1)
\]

there exists \( \phi_n \in C^\infty([a-d_0, a+d_0]) \cap W^{1,\infty}([a-d_0, a+d_0], w_0, w_1) \)

such that \( \text{ess lim sup}_{x \to a} |f(x) - \phi_n(x)|w_0(x) < 1/n \).

Remarks.

1. We will see in propositions 5.1 and 5.2 that condition (5.1) can be substituted in many cases by simpler conditions which only involve \( f \).

2. The same argument as that in the proof of Lemma 4.6 allows to deduce that if \( f \) verifies condition (5.1), then for each \( 0 < d \leq d_0 \) we can choose the functions \( \phi_n \) with the additional property \( \phi_n \in C^\infty_\mathbb{C}((a-d, a+d)) \).

Let us assume that \( w_0, w_1 \in L^\infty_{\text{loc}}([\alpha, \beta] \setminus \{a\}) \), \( S(w_1) = \{a\} \), and \( w_1 \) is balanced at \( a \). The argument in the proof of Theorem 4.2 (using Corollary 5.1) gives that if \( \text{ess lim sup}_{x \to a} |x-a|w_0(x) = 0 \), then the closure of \( C^\infty(R) \cap W^{1,\infty}(w_0, w_1) \) in \( W^{1,\infty}(w_0, w_1) \) is \( H_2 \). In a similar way, if \( 0 < \text{ess lim sup}_{x \to a} |x-a|w_0(x) < \infty \), then the closure of \( C^\infty(R) \cap W^{1,\infty}(w_0, w_1) \) in \( W^{1,\infty}(w_0, w_1) \) is \( H_3 \). We also have that, if \( \text{ess lim sup}_{x \to a} |x-a|w_0(x) = \infty \), then the closure of \( C^\infty(R) \cap W^{1,\infty}(w_0, w_1) \) in \( W^{1,\infty}(w_0, w_1) \) is \( H_4 \), if we change (4.2) by (5.1). We also obtain that if \( f \in H_j \) (2 \( \leq j \leq 4 \)), then it can be approximated by functions \( \{g_n\}_n \) in \( C^\infty(R) \cap W^{1,\infty}(w_0, w_1) \) with the norm of \( W^{1,\infty}(w_0, w_1) \), with \( g_n(a) = f(a) \) if \( a \neq \alpha \) and with \( g_n(\beta) = f(\beta) \) if \( a \neq \beta \).

Definition 5.2. The weights \( w_0, w_1 \) are strongly jointly admissible on the interval \( I \), if they verify the conditions in the definition of jointly admissible (Definition 4.4), with \( J_3 = \emptyset \) and replacing (a2) by

(a2') if \( n \in J_2 \), then \( S(w_1) \cap [\alpha_n, \beta_n] = \{a_n\} \), \( w_0, w_1 \in L^\infty_{\text{loc}}([\alpha_n, \beta_n] \setminus \{a_n\}) \), and \( w_1 \) is balanced at \( a_n \).

Remark. We choose \( J_3 = \emptyset \), since in this context we must require \( w_0, w_1 \in L^\infty([\alpha_n, \beta_n]) \) additionally in (a3), and these facts imply the hypothesis in (a1). Hence, \( J_1 \) plays here the role of \( J_1 \cup J_3 \) in Definition 4.4.

The following is the main result of this section.

Theorem 5.2. Let us consider two weights \( w_0, w_1 \) which are strongly jointly admissible on the interval \( I \). Then the closure of \( C^\infty(I) \cap W^{1,\infty}(w_0, w_1) \) in \( W^{1,\infty}(w_0, w_1) \) is equal to

\[
H_6 := \{ f \in W^{1,\infty}(w_0, w_1) : f \in C(I) \cap L^\infty(w_0)^L^\infty(w_0), f' \in C(I) \cap L^\infty(w_1)^L^\infty(w_1), \]

\[
f' \in C^\infty([\alpha_n, \beta_n]) \cap L^\infty([\alpha_n, \beta_n], w_1)^L^\infty([\alpha_n, \beta_n], w_1), \text{ for any } n \in J_1, \]

\[
f \text{ is continuous to the right if } a_n \in D^+(w_1), \]

\[
f \text{ is continuous to the left if } a_n \in D^-(w_1), \]

\[
\text{if } \text{ess lim } |x-a_n|w_0(x) = 0, \text{ ess lim } |x-a_n|w_0(x) = 0, \]

\[
\text{if } 0 < \text{ess lim sup } |x-a_n|w_0(x) < \infty, \]

\[
\exists l(f, a_n) \text{ and } \text{ess lim } f(x) - f(a_n)w_0(x) = 0, \]

\[
\text{if } a_n \notin S_1(w_1), \text{ then } \text{ess lim } |x-a_n|w_0(x) = \infty, \text{ if } a_n \notin S_1(w_1) \}

Remark. In Theorem 2.1 and in [PQRT1] we characterize \( C^\infty \cap L^\infty(w)^L^\infty(w) \) for a general kind of weights.
Proof. We only need to follow the argument in the proof of Theorem 4.5, replacing the functions in $C$ or $C^1$, by functions in $C^\infty$. This is the reason why we need to require that $f'$ belongs to the closure of $C^\infty([a_n, b_n]) \cap L^2([a_n, b_n], w_1)$ in $L^2([a_n, b_n], w_1)$ for any $n \in J$.

In many situations we can simplify condition (5.1).

**Proposition 5.1.** Let us consider two weights $w_0, w_1$, in $[\alpha, \beta]$ such that $S(w_1) = \{a\}$, and ess lim sup$_{x \to a} |x-a|w_0(x) = \infty$. Let us assume that for some function $s$ verifying $0 < m \leq |s(x)| \leq M < \infty$ a.e., there exists ess lim$_{x \to a} \phi(x)s(x)w_0(x)$ for every $\phi \in C^\infty(R) \cap W^{1, \infty}(w_0, w_1)$. Let us denote by $D(w_0, a)$ the set of values of these limits when we consider every $\phi \in C^\infty(R) \cap W^{1, \infty}(w_0, w_1)$ $(D(w_0, a)$ is either $\{0\}$ or $R)$. Then (5.1) is equivalent to the following: for any $f \in W^{1, \infty}(w_0, w_1)$ the limit ess lim$_{x \to a} f(x)s(x)w_0(x)$ there exists and belongs to $D(w_0, a)$.

**Remarks.**

1. By Remark 2 behind (5.1), the functions in $C^\infty(R) \cap W^{1, \infty}(w_0, w_1)$ can be substituted by $C^\infty([a - d, a + d]) \cap W^{1, \infty}([a - d, a + d], w_0, w_1)$ everywhere in Proposition 5.1, for some (or for every) $d > 0$.

2. The conclusion of Proposition 5.1 also holds if we substitute (5.1) by (4.2) and $C^\infty$ by $C^1$ everywhere.

3. A natural choice for is $s(x) := 1$ or $s(x) := \text{sgn}(x - a)$ (see the proof of Proposition 5.2).

**Proof.** Let us fix $f \in W^{1, \infty}(w_0, w_1)$. If the limit $d := \text{ess lim sup}_{x \to a} f(x)s(x)w_0(x)$ exists and belongs to $D(w_0, a)$, we have (5.1) with $\phi_n := \phi$, where $\phi$ is a function in $C^\infty(R) \cap W^{1, \infty}(w_0, w_1)$ with ess lim$_{x \to a} \phi(x)s(x)w_0(x) = d$, since then ess lim$_{x \to a} f(x) - \phi(x)w_0(x)$ is $\leq m - \text{ess lim sup}_{x \to a} f(x)s(x)w_0(x) - \phi(x)s(x)w_0(x) = 0$. If $d \notin D(w_0, a)$, then $D(w_0, a)$ is $\{0\}$, and consequently $d \neq 0$; hence, ess lim sup$_{x \to a} f(x) - \phi(x)w_0(x) \geq |d|/M > 0$, for every $\phi \in C^\infty(R) \cap W^{1, \infty}(w_0, w_1)$. Then the limit ess lim$_{x \to a} f(x)s(x)w_0(x)$ does not exist, a similar argument implies that there exists a constant $c = c(f, M) > 0$ such that ess lim sup$_{x \to a} f(x) - \phi(x)w_0(x) \geq c > 0$, for every $\phi \in C^\infty(R) \cap W^{1, \infty}(w_0, w_1)$.

**Definition 5.3.** We say that a weight $w_0$ has potential growth at $a$, if ess lim sup$_{x \to a} |x-a|^m w_0(x) < \infty$, for some natural number $m$. If $w_0$ has potential growth at $a$, we say that the degree of $w_0$ at $a$ is $m$, if $m$ is the minimum natural number with ess lim sup$_{x \to a} |x-a|^m w_0(x) < \infty$.

**Proposition 5.2.** Let us consider two weights $w_0, w_1$, in $[\alpha, \beta]$ such that $S(w_1) = \{a\}$, ess lim sup$_{x \to a} |x-a|^m w_0(x) = \infty$ and $w_0$ has potential growth at $a$. Let us assume that $m$ is the degree of $w_0$ at $a$.

1. If ess lim$_{x \to a} |x-a|^m w_0(x) = 0$, then (5.1) is equivalent to ess lim$_{x \to a} f(x)w_0(x) = 0$.

2. If ess lim sup$_{x \to a} |x-a|^m w_0(x) > 0$ and ess lim sup$_{x \to a} |x-a|^m-1 w_0(x) < \infty$, then we can substitute (5.1) by the following condition: there exists $l_n(f, a) := \text{ess lim sup}_{x \to a} |x-a|^m w_0(x) \geq \eta f(x)/(x-a)^m$ for $\eta$ small enough, and ess lim$_{x \to a} f(x) - l_n(f, a)(x-a)^m w_0(x) = 0$.

3. If $w_0(x)$ is comparable with $|x-a|^{-m}$ in a neighborhood of $a$, for some positive integer $m$, then (5.1) is equivalent to the existence of ess lim$_{x \to a} f(x)/(x-a)^m$.

**Proof.** (1) Let us fix $f \in W^{1, \infty}(w_0, w_1)$ with ess lim$_{x \to a} |f(x)|w_0(x) = 0$; then (5.1) holds with $\phi_n := 0$.

In order to see the other implication, let us fix $f \in W^{1, \infty}(w_0, w_1)$ (5.1). Let us consider $\phi \in C^\infty(R) \cap W^{1, \infty}(w_0, w_1)$. Condition ess lim sup$_{x \to a} |x-a|^m w_0(x) = \infty$ implies $\phi(a) = \phi'(a) = \cdots = \phi^{(m-1)}(a) = 0$; then $\phi(x) \approx \phi^{(m)}(a)/m!(x-a)^m$, and condition ess lim$_{x \to a} |x-a|^m w_0(x) = 0$ gives ess lim$_{x \to a} \phi(x)w_0(x) = 0$ for every $\phi \in C^\infty(R) \cap W^{1, \infty}(w_0, w_1)$. Hence ess lim sup$_{x \to a} f(x)w_0(x) = \text{ess lim sup}_{x \to a} f(x) - \phi_n(x)w_0(x) < 1/n$ for every $n$.

(2) Let us fix $f \in W^{1, \infty}(w_0, w_1)$ satisfying (5.1). An argument similar to the one in the proof of part (a) of Proposition 4.4 implies that there exists $l_n(f, a)$ for $0 < \eta < \text{ess lim sup}_{x \to a} |x-a|^m w_0(x)$, and that $\phi_n^{(m)}(a)/m! \to l_n(f, a)$ as $n \to \infty$. In order to finish the proof of this implication, it is sufficient to follow the proof of the implication in the proof of the first part of Theorem 4.4, taking the function $l_n(f, a)(x-a)^m$ instead of $l(f, a)(x-a)$.

We deal with the other implication. Let us consider $f \in W^{1, \infty}(w_0, w_1)$ such that there exists $l_n(f, a)$ for $\eta$ small enough, and ess lim$_{x \to a} |f(x) - l_n(f, a)(x-a)^m|w_0(x) = 0$. In order to verify (5.1), it is sufficient to take as $\phi_n = \phi$ the function $l_n(f, a)(x-a)^m$ multiplied by an appropriate smooth function with compact support which is equal to 1 in a neighborhood of $a$ ($\phi$ belongs to $W^{1, \infty}(w_0, w_1)$ by hypothesis).

(3) It is sufficient to apply Proposition 5.1 with $s(x) := (x-a)^{-m}/w_0(x)$. 

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References.


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