APPLICATION OF THE FINITE ELEMENT METHOD TO OPEN CHANNEL FLOW

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PAPER NO. 30

APRIL 1973
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SUMMARY

In this paper the finite element method and Galerkin's principle are applied to open channel flow. Solutions obtained from the nonstationary and nonlinear equations for onedimensional flow are presented for the case of lateral inflow superposed on an initial flow.

1. INTRODUCTION

In the field of hydraulics open channel flow is a subject of detailed experimental and theoretical research. The theoretical work concerns the investigation of special types of waves by analytical methods as well as the application of numerical methods for solving the complete timedependent equations. Such a numerical method is the finite element technique, described for instance in [1] and [2]*. This method has some advantages as compared with the finite difference technique ([3]) and is applied frequently to stationary and nonstationary boundary value problems in mathematical physics in recent years.

This paper deals with the application of the finite element method (f.e.m.) in combination with Galerkin's principle ([4], [5]) to the non-stationary and nonlinear equations for the onedimensional flow in open channels as given in the handbooks ([6], [7], [8], [9]). We consider a slightly inclining channel with lateral inflow perpendicular to the channel and superposed on an initial flow. The channel is of finite length and has a uniform, rectangular cross section. However these restrictions are not essential, the method may also be applied to channels with a nonuniform and nonrectangular cross section. We assume that at the upstream end of the channel the discharge is prescribed and that at the downstream end Froude's number equals 1 (free overfall weir). The resistance is assumed to be governed by Manning's law with a constant friction coefficient.

The differential equations for open channel flow are expressed in two dependent variables. These ones may be selected from the discharge, the mean velocity and the waterdepth in every cross section. The choise of mean velocity and waterdepth is preferred since it yields the most simple forms of the differential equations. However computations showed that the solutions obtained for these variables with the f.e.m. were slightly unstable. For that reason the equations are expressed in the discharge and the waterdepth. Now a stable solution is obtained. Only the analysis for the latter case is described in this paper.

* Numbers between square brackets refer to the references.

2. FORMULATION OF THE PROBLEM

The equations governing the problem may be written as ([10]),

$$\frac{\partial q}{\partial x} + B\frac{\partial y}{\partial t} = Q \qquad \text{...... (2.1.a)}$$

$$S_f - S_o + (1 - F^2)\frac{\partial y}{\partial x} + \frac{2q}{gB^2y^2}\frac{\partial q}{\partial x} + \frac{1}{gBy}\frac{\partial q}{\partial t} = 0 \qquad \text{...... (2.1.b)}$$

Here x represents the coordinate along the channel, t the time, q the discharge, y the waterdepth, B the width of the channel, Q the lateral inflow which is supposed to be directed perpendicular to the channel, S_f the friction slope, S_o the bottom slope, F is Froude's number and g the acceleration of gravity. B is a constant; q, y and Q are functions of x and t. The upstream end of the channel is denoted by x=0 and the downstream end by x=L.

For Froude's number we have the expression

$$F^2 = \frac{q^2}{gB^2y^3} \qquad \text{...... (2.2)}$$

Manning's formule gives for the friction slope S_f, approximating the hydraulic radius by y,

$$S_f = \frac{q^2}{B^2k_m^2y^{10/3}} \qquad \text{...... (2.3)}$$

Here k_m is a constant.

The boundary conditions read

$$q = q_o \text{ (constant)}, \; x=0 \qquad \text{...... (2.4.a)}$$

$$F^2 = 1 \rightarrow q^2 = gB^2y^3, \; x=L \qquad \text{...... (2.4.b)}$$

At t=0 we have a stationary initial flow with $q = q_o$. The initial waterdepth is derived from the stationary form of (2.1),

$$S_f - S_o + (1 - F^2)\frac{dy}{dx} = 0 \qquad \text{...... (2.5)}$$

with

$$y^3 = \frac{q_o^2}{gB^2}, \; x=L \qquad \text{...... (2.6)}$$

The equations and conditions (2.1)-(2.4) are written in nondimensional form by introducing the quantities

$$\bar{x} = x/L, \ \bar{q} = q/q_o, \ \bar{y} = y/h, \ \bar{t} = t/T \qquad \ldots\ldots (2.7)$$

where $h^3 = q_o^2/B^2g$. T represents a unit of time, which will be specified later on. Substituting (2.7) into (2.1)-(2.6), multiplying (2.1.b) by $\bar{y}^3$, and dropping the bars, we obtain

$$\frac{\partial q}{\partial x} + \alpha \frac{\partial y}{\partial t} - P = 0 \qquad \ldots\ldots (2.8.a)$$

$$\beta \frac{q^2}{y^{1/3}} - \gamma y^3 + (y^3 - q^2) \frac{\partial y}{\partial x} + 2\, qy \frac{\partial q}{\partial x} + \alpha y^2 \frac{\partial q}{\partial t} = 0 \qquad (2.8.b)$$

Here

$$P = \frac{LQ}{q_o} \qquad \ldots\ldots (2.9)$$

and α, β and γ are constants,

$$\alpha = \frac{BhL}{q_o T}, \quad \beta = \frac{Lg}{h^{4/3} k_m^2}, \quad \gamma = \frac{LS_o}{h} \qquad \ldots\ldots (2.10)$$

The boundary conditions read in dimensionless form

$$q = 1, \quad x = 0 \qquad \ldots\ldots (2.11.a)$$

$$y^3 = q^2, \quad x = 1 \qquad \ldots\ldots (2.11.b)$$

The initial waterdepth is the solution of

$$\frac{\beta}{y^{1/3}} - \gamma y^3 + (y^3 - 1) \frac{dy}{dx} = 0 \qquad \ldots\ldots (2.12)$$

satisfying the condition

$$y = 1, \quad x = 1 \qquad \ldots\ldots (2.13)$$

3. THE STATIONARY SOLUTION

According to the f.e.m. we divide the range $[0, 1]$ into N subranges $[x_{n-1}, x_n]$, n = 1 .... N with $x_o = 0$ and $x_N = 1$. We define a set of base-functions $f_m(x)$, m = 0 .... N, on $[0, 1]$ in the following manner (fig. 1)

$$f_o(x) = \begin{cases} \dfrac{x_1 - x}{g_1}, & 0 \leq x \leq x_1 \\ 0, & x > x_1 \end{cases} \qquad \text{...... (3.1.a)}$$

$$f_m(x) = \begin{cases} 0, & x < x_{m-1},\ x > x_{m+1} \\ \dfrac{x - x_{m-1}}{g_m}, & x_{m-1} \leq x \leq x_m \\ \dfrac{x_{m+1} - x_m}{g_{m+1}}, & x_m \leq x \leq x_{m+1} \end{cases} \qquad m = 1 \text{ .... } N-1 \qquad (3.1.b)$$

$$f_N(x) = \begin{cases} 0, & x < x_{N-1} \\ \dfrac{x - x_{N-1}}{g_N}, & x_{N-1} \leq x \leq 1 \end{cases} \qquad \text{...... (3.1.c)}$$

Here the length of a sub-interval is denoted by g_m,

$$g_m = x_m - x_{m-1} \qquad \text{...... (3.2)}$$

Galerkin's principle states that an approximate solution of (2.12) is obtained from the system of equations

$$\int_0^1 \{ \frac{\beta}{y^{1/3}} - \gamma y^3 + (y^3 - 1) \frac{dy}{dx} \} f_m(x)dx = 0, \quad m = 0 \ldots N-1 \qquad (3.3)$$

with y approximated by

$$y = \sum_{k=0}^{N} y_k f_k(x) \qquad \text{...... (3.4)}$$

The quantities y_k are unknown constants, representing $y(x_k)$, except y_N, which is determined by the boundary condition (2.13),

$$y_N = 1 \qquad \text{...... (3.5)}$$

Since $f_k(x)$ vanishes for $x < x_{k-1}$ and $x > x_{k+1}$, the boundaries in (3.3) may be replaced by x_{m-1} and x_{m+1}. For m = 0 the boundaries are 0 and x_1. The right hand side of (3.4) represents a linear function on each interval $[x_{k-1}, x_k]$. In such a range (3.4) yields

$$y = y_{k-1} f_{k-1}(x) + y_k f_k(x) \qquad \text{...... (3.6)}$$

A constant waterdepth is represented exactly by (3.4), since

$$f_{k-1}(x) + f_k(x) = 1 \qquad \text{...... (3.7)}$$

For any set of base-functions $f_k(x)$ equation (3.7) has to be satisfied in order that the approximation (3.4) converges to the exact solution as $N \to \infty$.

Substituting (3.4) into (3.3) a system of N equations for the N unknowns y_m, m = 0 .... N-1, is obtained from which the y_m's may be computed.

For reason of convenience the factor $y^{-1/3}$ in (3.3) is expanded in a Taylor series,

$$y^{-1/3} = \{y_m + (y_m - y_{m-1})(f_m - 1)\} \sim$$

$$y_m^{-1/3} \{1 - \frac{y_m - y_{m-1}}{3y_m} (f_m - 1)\}, \quad x_{m-1} \leq x \leq x_m \qquad \text{...... (3.8)}$$

For sufficient small subranges $[x_{m-1}, x_m]$ the error introduced by this expansion is negligible. In the same manner we derive

$$y^{-1/3} \sim y_m^{-1/3} \{1 - \frac{y_m - y_{m+1}}{3y_m} (f_m - 1)\}, \quad x_m \leq x \leq x_{m+1} \qquad \text{... (3.9)}$$

By means of simple calculus the integrations in (3.3) are performed for every m, m = 0 .... N-1, yielding the following non-linear equations for y_m, m = 0 .... N-1,

$$10\ \beta g_1 y_o^{-4/3} (10\ y_o - y_1) + 9(y_1 - y_o - \gamma g_1)(y_1^3 + 2\ y_1^2 y_o +$$

$$3\ y_1 y_o^2 + 4\ y_o^3) - 90(y_1 - y_o) = 0, \qquad \text{...... (3.10.a)}$$

$$10\ \beta\ y_m^{-4/3} \{g_m(10\ y_m - y_{m-1}) + g_{m+1}(10\ y_m - y_{m+1})\}$$

$$+ 9(y_m - y_{m-1} + \gamma g_m)(y_{m-1}^3 + 2\ y_{m-1}^2\ y_m + 3\ y_{m-1}\ y_m^2 + 4\ y_m^3)$$

$$+ 9(y_{m+1} - y_m - \gamma g_{m+1})(y_{m+1}^3 + 2\ y_{m+1}^2\ y_m + 3\ y_{m+1}\ y_m^2 + 4\ y_m^3)$$

$$- 90(y_{m+1} - y_{m-1}) = 0 \quad , m = 1 \ .... \ N-1 \qquad \text{...... (3.10.b)}$$

These equations are solved by means of Newton-Raphson's technique. Denoting the left hand sides of (3.10) by $h_m(y_o \ .... \ y_{N-1})$, m = 0 .... N-1, the iteration scheme reads

$$\underline{y}^{(k+1)} = \underline{y}^{(k)} - \left[\frac{\partial \underline{h}}{\partial \underline{y}} (\underline{y}^{(k)})\right]^{-1} \cdot \underline{h}(\underline{y}^{(k)}) \qquad \text{...... (3.11)}$$

Here $\underline{y}^{(k)}$ represents the vector $[y_o \ .... \ y_N]^T$ after k iterations and $\underline{h}$ the vector $[h_o, \ .... \ h_N]^T$. Further we define

$$\frac{\partial \underline{h}}{\partial \underline{y}} = \begin{bmatrix} \frac{\partial h_o}{\partial y_o} & \frac{\partial h_o}{\partial y_1} & \cdots\cdots & \frac{\partial h_o}{\partial y_{N-1}} \\ \vdots & & & \vdots \\ \frac{\partial h_{N-1}}{\partial y_o} & \cdots\cdots & & \frac{\partial h_{N-1}}{\partial y_{N-1}} \end{bmatrix}$$

The starting vector $\underline{y}^{(o)}$ is given as $\{1, 1, ... 1\}^T$.

The waterdepth is determined for

$$\beta = 171.843, \quad \gamma = 16.4993 \qquad ...... (3.12)$$

These values are obtained from (2.10) for the following parameters in SI units,

$$L = 12.00, B = 1.02, S_o = .005, k_m = 35, q_o = 7.10^{-4} \qquad ...... (3.13)$$

The computations are performed for a lot of possible subdivisions of the interval $[0, 1]$. From (2.14) we conclude that the exact solution has an infinite derivative for x = 1. Hence the intervals $[x_{m-1}, x_m]$ must be very small near this point in order to approximate y sufficiently accurate by means of (3.4). A satisfactory division is given by (figure 2)

$$x_m = \sin \frac{m\pi}{2N}, \quad m = 0 .... N \qquad ...... (3.14)$$

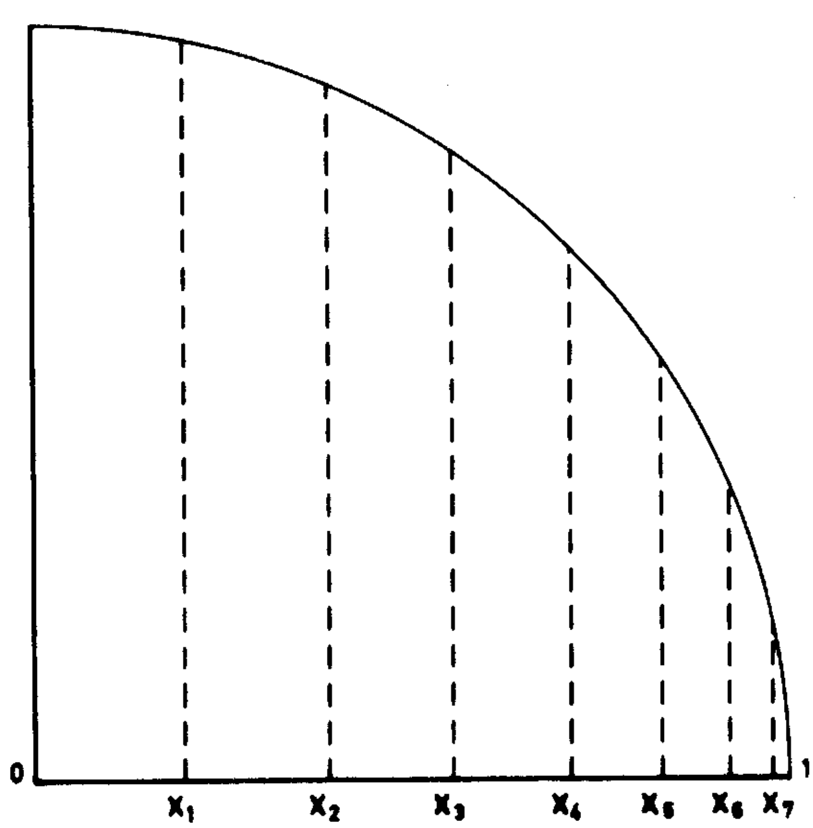

Figure 2

It appeared that for $N > 10$ the waterdepth is computed in the points x_m with five significant digits: redoubling of N yields the same values in the corresponding points. The number of iterations equals 7 for this accuracy.

In figure 4 a plot of the stationary waterdepth is given (curve denoted by $n = 0$). For a large part of the channel the waterdepth has the constant value 2.0198, near $x = 1$ it tends rapidly to 1.

The number of intervals may be decreased if $f_N(x)$ and $f_{N-1}(x)$ are replaced by functions fitting more accurately the behaviour of y for $[x_{N-1}, 1]$. A good choise appeared to be

$$f_{N-1}(x) = \left(\frac{1 - x}{g_N}\right)^{\frac{1}{2}} \qquad \ldots\ldots (3.15.a)$$

$$f_N(x) = 1 - f_{N-1}(x) \qquad \ldots\ldots (3.15.b)$$

We remark that solving equation (2.12) or (2.14) by a classical method is complicated due to the infinite derivative at $x = 1$ and the constant behaviour in a large part of the channel.

4. THE NONSTATIONARY SOLUTION

The nonstationary equations (2.8) are treated in the same manner as the stationary equation in the preceding section. The functions q(x, t) and y(x, t) are approximated by

$$q = \sum_{k=0}^{N} q_k(t) f_k(x) \qquad \ldots\ldots (4.1.a)$$

$$y = \sum_{k=0}^{N} y_k(t) f_k(x) \qquad \ldots\ldots (4.1.b)$$

Now q_k and y_k are assumed to be functions of time.

We approximate equations (2.8) by the system of equations,

$$\int_0^1 \left(\frac{\partial q}{\partial x} + \alpha \frac{\partial y}{\partial t} + P\right) f_m \, dx = 0, \quad m = 1 \ldots\ldots N \qquad \ldots\ldots (4.2.a)$$

$$\int_0^1 \left(\beta \frac{q^2}{y^{1/3}} - \gamma y^3 + (y^3 - q^2)\frac{\partial y}{\partial x} + 2qy \frac{\partial q}{\partial x} + \alpha y^2 \frac{\partial q}{\partial t}\right) f_m \, dx = 0$$

$$m = 0 \ldots\ldots N-1 \qquad (4.2.b)$$

In virtue of the boundary conditions the set of integers does not include m = 0 in (4.2.a) and m = N in (4.2.b).

Substituting (4.1) into (4.2) a set of first order ordinary differential equations depending on time, is obtained. These equations are solved simultaniously by means of an implicit, finite difference scheme, yielding

$$\Big[3(q_{m+1} - q_{m-1}) + \alpha\{g_m(y_{m-1} + 2y_m) + g_{m+1}(y_{m+1} + 2y_m)\}\Big]\Big|_{t=n} =$$

$$\alpha\{g_m(y_{m-1} + 2y_m) + g_{m+1}(y_{m+1} + 2y_m)\}\big|_{t=n-1} + 6 \int_{n-1}^{n} \int_{x_{m-1}}^{x_m} P f_m \, dx dt$$

$$m = 1 \ldots\ldots N-1 \qquad (4.3.a)$$

$$\{3(q_N - q_{N-1}) + \alpha g_N(y_{N-1} + 2y_N)\}\big|_{t=n} =$$

$$\alpha g_N(y_{N-1} + 2y_N)\big|_{t=n-1} + 6 \int_{n-1}^{n} \int_{x_{N-1}}^{1} P f_N \, dx dt \qquad \ldots\ldots (4.3.b)$$

$$\Big[\beta g_1 y_o^{-4/3}\{-y_1(3q_1^2 + 4q_1q_o + 3q_o^2) + y_o(18q_1^2 + 34q_1q_o + 48q_o^2)\}$$

$$+9(y_1 - y_o - \gamma g_1)\{y_1^3 + 2y_1^2 y_o + 3y_1 y_o^2 + 4y_o^3\} + 45y_o(q_1^2 +$$

$$2q_1q_o - q_o^2) + 15y_1(q_1^2 - 2q_1q_o - 5q_o^2)\Big]\Big|_{t=n} + 3\alpha g_1\Big[(3y_1^2 + 4y_1y_o +$$

$$3y_o^2)\big|_{t=n}\{q_1(n) - q_1(n-1)\} + (2y_1^2 + 6y_1y_o + 12y_o^2)\big|_{t=n}$$

$$\{q_o(n) - q_o(n-1)\}\Big] = 0 \qquad \text{...... (4.3.c)}$$

$$\Big[\beta y_m^{-4/3}\{-g_m y_{m-1}(3q_{m-1}^2 + 4q_m q_{m-1} + 3q_m^2) + g_m y_m(18q_{m-1}^2 + 34q_{m-1}q_m$$

$$+ 48q_m^2) - g_{m+1}y_{m+1}(3q_{m+1}^2 + 4q_m q_{m+1} + 3q_m^2) + g_{m+1}y_m(18q_{m+1}^2 +$$

$$34q_{m+1}q_m + 48q_m^2)\} + 9(y_m - y_{m-1} - \gamma g_m)(y_{m-1}^3 + 2y_{m-1}^2\, y_m +$$

$$3y_{m-1}y_m^2 + 4y_m^3) + 9(y_{m+1} - y_m - \gamma g_{m+1})(y_{m+1}^3 + 2y_{m+1}^2\, y_m +$$

$$3y_{m+1}y_m^2 + 4y_m^3) + 15y_{m-1}(-q_{m-1}^2 + 2q_{m-1}q_m + 5q_m^2) + 45y_m$$

$$\{q_{m+1}^2 - q_{m-1}^2 + 2q_m(q_{m+1} - q_{m-1})\} + 15y_{m+1}(q_{m+1}^2 - 2q_{m+1}q_m - 5q_m^2)\Big]$$

$$\Big|_{t=n} + 3\alpha g_m\Big[(3y_{m-1}^2 + 4y_{m-1}y_m + 3y_m^2)\big|_{t=n}\{q_{m-1}(n) - q_{m-1}(n-1)\} +$$

$$(2y_{m-1}^2 + 6y_{m-1}y_m + 12y_m^2)\big|_{t=n}\{q_m(n) - q_m(n-1)\}\Big] +$$

$$3\alpha g_{m+1}\Big[(3y_{m+1}^2 + 4y_{m+1}y_m + 3y_m^2)\big|_{t=n}\{q_{m+1}(n) - q_{m+1}(n-1)\} +$$

$$(2y_{m+1}^2 + 6y_{m+1}y_m + 12y_m^2)\big|_{t=n}\{q_m(n) - q_m(n-1)\}\Big] = 0, \quad m = 1 \ldots N-1$$

...... (4.3.d)

In deriving these equations we used (3.8).
From the boundary conditions (2.11) we find,

$$q_o(n) = 1 \qquad \text{...... (4.3.e)}$$

$$y_N^3 - q_N^2 = 0, \quad t = n \qquad \text{...... (4.3.f)}$$

Now a set of 2N + 2 algebraic equations for the 2N + 2 unknowns $q_o(t_n)$ .... $q_N(t_n)$, $y_o(t_n)$ .... $y_N(t_n)$ is obtained, which may be solved again by Newton-Raphson's technique for n = 1, 2, 3, ... For t = 0 we have the stationary solution discussed in the preceding section; the solution for t = n-1 is the first approximation in Newton-Raphson's scheme for the solution for t = n.

Uptil now only the numerical accuracy and convergence of the nonlinear system is investigated. For that reason computations are performed for a constant lateral inflow during 1 or 2 minutes and a lot of values of T and N, using (3.14). The parameters β and γ are given by (3.12). It appeared that for N = 11 and T = 6 sec., yielding

$$\alpha = 10.598 \qquad \text{...... (4.4)}$$

four significant digits could be obtained for P = 20/7, corresponding with a total inflow of .002 m^3/sec. Remaining computations have been performed for these values of N and T.

The most interesting quantity is the discharge at the downstream end. In figure 3 a plot of q_N is given for P = 20/7, 0 < t < 20, P = 0, t > 20 (curve I) and P = 20/7, 0 < t < 10, P = 0, t > 10 (curve II). Also y is plotted as a function of x in the first case for n = 0, 5, 20 and 25. The waterdepth for n = 20 represents approximately the stationary case with a lateral inflow (figure 4).

Also solutions for P = 10/7 are derived. The results show that for the mentioned values of the parameters

$$q_N(P = 20/7) = 1 + 2.q_N(P = 10/7)$$

within an accuracy 0.065 (maximal deviation < 4%).

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