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SOLUTION OF INTEGRALS NECESSARY TO DETERMINE RAINFALL INTERSTATION CORRELATION FUNCTIONS

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1. INTRODUCTION

The analytic solution to the rainfall interstation correlation function depends on the solution of some integrals (STOL, 1977a).

Although the solution is a straightforward application of integral calculus the structure of the integrals is rather complicated (STOL, 1977b) and need some comments to simplify the elaborations.

In this report the solution of the required integrals will be given for reference. The integrals are solved step by step. The final result is the rainfall interstation correlation function for time series. They are given too and briefly commented with respect to their specific properties.

Main formulas are given in boxes to clearly distinguish between definitions or final results and intermediate expressions.

2. GENERAL DEFINITIONS

The meteo-hydrological background of the elaborations to be dealt with, will not be paid attention to. They can be found in publications mentioned in the list of references.

Instead, the mathematical treatment is subject of our considerations. The development of the required model will be given in mathematical terms only. However, symbols are chosen such that they correspond with those used in the applications, so results need not be transformed or encoded.

For the same reason functions and variables are given names according to their hydrological meaning.

Let a model for a storm be given by the stormfunction

$$\boxed{h = f(x)}$$

consisting of two parts

$$h = {}^{1}f(x) \quad \text{if} \quad 0 \leq x < B'$$

$$h = {}^{2}f(x) \quad \text{if} \quad B' \leq x \leq B$$

(See Fig. 1)

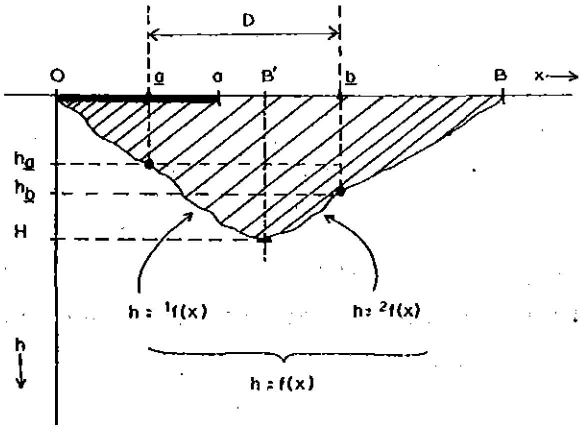

Fig. 1. Schematic illustration of the relationship between symbols defined in this report

Values of h obtained by specific values of x are defined by

$$\boxed{h_a = f(a)}$$

Particular values are

$${}^{1}f(0) = 0, \quad {}^{1}f(B') = {}^{2}f(B') = H, \quad {}^{2}f(B) = 0$$

We assume that x = a is obtained by a random process so x = $\underline{a}$*), where $\underline{a}$ is supposed to be uniformly distributed on the interval $[0, B]$

*) stochastic variables will be denoted by underlining the symbol

The function ${}^{1}f(x)$ is assumed to be monotonic increasing while ${}^{2}f(x)$ is assumed to be monotonic decreasing. Now we can define the probability P by

$$P(\underline{a} \leqq a) = P(h_{\underline{a}} < h_a), \quad 0 \leqq a < B'$$

Using intervals, to define probabilities, we have

$$P(h_{\underline{a}} < h_a) = \frac{a}{B}, \quad 0 \leqq a < B'$$

Dropping indices, and expressing intervals on x in terms of h we can write

$$P(\underline{h} < h) = \frac{{}^{1}f^{-1}(h)}{B}, \quad 0 \leqq a < B'$$

The density of this distribution reads

$$\boxed{\frac{dP(\underline{h} < h)}{dh} = \frac{1}{B}\frac{d}{dh}\,{}^{1}f^{-1}(h), \quad 0 \leqq a < B'}$$

which can be used to derive expectations.

Function values are connected by the 'interstation' distance D, such that

$$b = a + D, \quad 0 \leqq D \leqq B - a$$

If coordinates are obtained randomly, we have

$$\underline{b} = \underline{a} + D$$

consequently, conventionally, in general

$$h_{\underline{b}} = f(\underline{b}) = f(\underline{a} + D)$$

but, depending on the magnitudes of the intervals, probable outcomes are

$$h_{\underline{b}} = {}^1f\{{}^1f^{-1}(h_{\underline{a}}) + D\} \tag{1}$$

$$h_{\underline{b}} = {}^2f\{{}^1f^{-1}(h_{\underline{a}}) + D\} \tag{2}$$

$$h_{\underline{b}} = {}^2f\{{}^2f^{-1}(h_{\underline{a}}) + D\} \tag{3}$$

which depend on the location of $\underline{a}$ and $\underline{b}$ with respect to each of the defined functions.

3. SPECIAL PROPERTIES OF STORM FUNCTIONS

In this Section some special properties are introduced.

In the first place we take

$$B' = \tfrac{1}{2}B$$

and assume the storm function to be symmetric about $x = \frac{1}{2}B$, so

$$ {}^1f(x) = {}^2f(B-x) \tag{4}$$

Consider a pair of points moving along the l e f t b r a n c h of the storm function according to

$$\{{}^1f(x),\quad {}^1f(x+D)\} \quad \text{from } x = 0 \quad \text{to} \quad x = \tfrac{1}{2}B-D$$

where ${}^1f(x)$, because of the trivial argument, is called the leading point.

After having passed the center the move becomes

$$\{{}^2f(x),\quad {}^2f(x+D)\} \quad \text{from } x = \tfrac{1}{2}B \quad \text{to} \quad x = B-D$$

change the direction of the move, giving

$$\{{}^2f(x),\quad {}^2f(x+D)\} \quad \text{from} \quad x = B-D \quad \text{to} \quad x = \tfrac{1}{2}B$$

change leading points, first by arguments

$$\{{}^2f(x-D),\ {}^2f(x)\} \quad \text{from } x = B \ \text{ to } \ x = \tfrac{1}{2}B+D$$

then by their occurrence in the set

$$\{{}^2f(x),\ {}^2f(x-D)\} \quad \text{from } \ x = B \ \text{ to } \ x = \tfrac{1}{2}B+D$$

Now we use x = B as a new origin and define y = B−x, so

$$\{{}^2f(B-y),\ {}^2f(B-y-D)\} \quad \text{from } \ y = 0 \ \text{ to } y = \tfrac{1}{2}B-D$$

The last step is that we make use of the symmetry by (4) and so the move along the r i g h t b r a n c h can be written

$$\{{}^1f(y),\ {}^1f(y+D)\} \quad \text{from } y = 0 \ \text{ to } \ y = \tfrac{1}{2}B-D$$

But here we arrive at the same structure as the one given for the move along the left branch. Since x and y are dummies in the sense that their values are defined by the 'from-to' statements we see that the moves - under the condition of symmetry - are identical.

All combinations of points on one branch and at distance D apart ($0 < D < \frac{1}{2}B$) produce a combination on the other branch with the same function values.

This also means that

$$P(h_{\underline{a}} < h_a | a \in [0, B]) =$$

$$P(h_{B-\underline{a}} < h_{B-a} | a \in [B, \tfrac{1}{2}B])$$

so,

$$P(h_{\underline{a}} < h_a) = 2P(h_{\underline{a}} < h_a | a \in [0, B]) = 2P({}^1f(\underline{a}) < {}^1f(a))$$

Thus, for symmetric univariate moves, we can use the density

$$\boxed{\frac{2}{B}\frac{d}{dh}\;{}^{1}f^{-1}(h)}$$

to calculate all statistical parameters.

For shortness we introduce in this Section symbols to denote integrals by the following convention. An integrand will be denoted by i, the integral by I. Subscripts refer to the occurrence of a first: 1, and a second: 2, value of h, while, if no values of h are present, the subscript equals 0. Superscripts are used to indicate functions of the left branch: 1, and the right branch: 2. Consequently, to start with

$$\boxed{i_o = \frac{d}{dh}\;{}^{1}f^{-1}(h)} \qquad (5)$$

we can check the formula for the density by the relationships

$$\boxed{I_o = \int_0^H i_o \, dh} \qquad (6)$$

$$P(\underline{h} \leq H) = \frac{2}{B} I_o = 1 \qquad (7)$$

The univariate density itself is given by

$$\frac{dP(h)}{dh} = \frac{2}{B} i_o$$

but this formula will not be used explicitely.

The mathematical expectation will be defined by respectively

$$\boxed{I_1 = \int_0^H h \, i_o \, dh \quad \text{and} \quad \mu = \frac{2}{B} I_1} \qquad (8)$$

The variance then is given by

$$I_{11} = \int_0^H h^2 i_o \, dh \quad \text{and} \quad \sigma^2 = \frac{2}{B} I_{11} - \mu^2 \qquad (9)$$

or

$$\sigma^2 = \frac{2}{B} I_{11} - \left(\frac{2}{B} I_1\right)^2$$

Note that a more general formula for the integral reads

$$I_o = \int_{{}^1f(0)}^{{}^1f(\frac{1}{2}E)} i_o \, dh \qquad (6a)$$

which defines the boundary values in terms of ${}^1f(x)$.

In the same way the covariance between h_a and h_b can be obtained. We should consider a bivariate density. However, since values of h_b are determined by h_a according to $h_b = h_a + D$, densities for h_a are required only. Because of the symmetry they can be defined on ${}^1f(x)$. The second branch need not be used for this purpose.

According to equations (1), (2) and (3) we have to consider several combinations with which h_a and h_b can occur. This is subject-matter of next Section.

4. RELEVANT h_a AND h_b COMBINATIONS

The relevant situations that need concern are sketched in Fig. 2.

Increasing values of D give rise to distinguish between three cases, viz.

Case I :	$0 \leq D < \frac{1}{2}B < B$	(large)
Case II :	$0 < \frac{1}{2}B \leq D < B$	(medium)
Case III:	$0 < \frac{1}{2}B < B \leq D$	(small)

(The qualification refers to storm sizes)

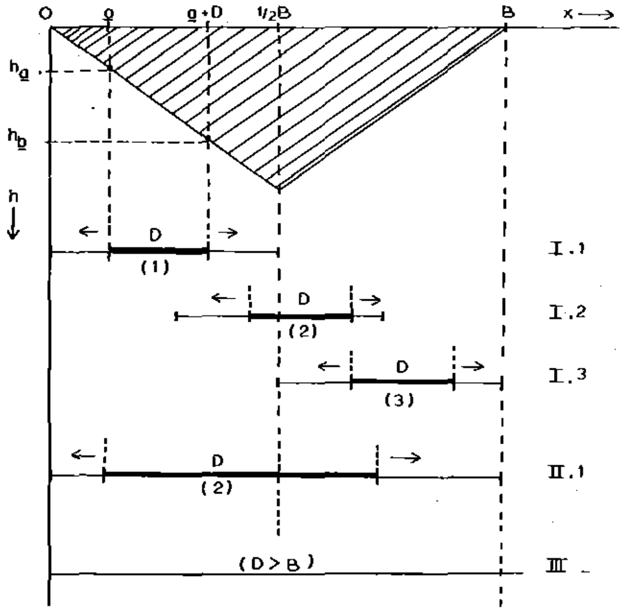

Fig. 2. Schematic illustration of the three cases (Roman figures), the three situations (Arabic figures) and the three equations (figures between parentheses) to be distinguished for combinations of $h_{\underline{a}}$ and $h_{\underline{b}}$ for increasing values of D

Since the expectation from which the covariance is obtained vanishes when one of the variables takes on the value zero, only those combinations of $h_{\underline{a}}$ and $h_{\underline{b}}$ need considered in which both variables are greater than zero. This means that in Case I there are three relevant situations to be treated with the formulas (1), (2) and (3), (see Fig. 2); Case II with only one relevant situation to be treated with formula (2); and Case III with no relevant situation.

Because of the assumed symmetry Case I.1 can be treated in the same way as Case I.3, which means that the corresponding integral in Case I.1 can be taken twice.

A further special property is that Case I.2 and Case II.1 can be treated with the same integral, although different boundary expressions have to be used.

From these considerations we see that the formulas for the covariance are:

Case I:

$$I_{12}^{11} = \int_{^1f(0)}^{^1f(\frac{1}{2}B-D)} h.\,^1f\{^1f^{-1}(h) + D\}.\; i_o\; dh$$

$$I_{12}^{21} = \int_{^1f(\frac{1}{2}B-D)}^{^1f(\frac{1}{2}B)} h.\,^2f\{^1f^{-1}(h) + D\}.\; i_o\; dh$$

$$I_{12} = 2\, I_{12}^{11} + I_{12}^{21} \qquad (10)$$

$$Cov(h_{\underline{a}}\; h_{\underline{b}}) = \frac{1}{B}\, I_{12} - \mu^2 \qquad (11)$$

Case II:

$$I_{12}^{21'} = \int_{^1f(0)}^{^1f(B-D)} h.\,^2f\{^1f^{-1}(h) + D\}.\; i_o\; dh$$

$$I_{12} = I_{12}^{21'} \qquad (12)$$

$$Cov(h_{\underline{a}}\; h_{\underline{b}}) = \frac{1}{B}\, I_{12} - \mu^2 \qquad (13)$$

Case III:

$$I_{12} = 0 \qquad (14)$$

$$Cov(h_{\underline{a}}\; h_{\underline{b}}) = \frac{1}{B}\, I_{12} - \mu^2 = -\mu^2 \qquad (15)$$

From these results, to be worked with specific storm functions, the correlation coefficient is obtained by dividing the covariance by the variance since $Var(h_a) = Var(h_b)$, which easily can be verified with the aid of the foregoing Sections.

It should be noted that the above mentioned correlation refers to the storm function. The ultimate goal, the interstation correlation for time series, can be obtained by further transformations (STOL, 1977a). They are briefly discussed in Annex 1.

5. PARTICULAR STORM FUNCTIONS

A. The rectangular type

A.1. Definition

The rectangular type is defined by

$$h = {}^{1}f(x) = H, \quad 0 \leqq x < \tfrac{1}{2}B$$

$$h = {}^{2}f(x) = H, \quad \tfrac{1}{2}B \leqq x \leqq B$$

The probability that $\underline{h} \leqq H$ equals $\frac{B}{B} = 1$, because h is constant.

For combination of values we have to determine probabilities instead of densities since h is constant. On the basis of intervals we have (see Fig. 2)

Case	Probability
I.1	$(\frac{1}{2}B-D)$ / B
I.2	(D) / B
I.3	$(\frac{1}{2}B-D)$ / B
II.1	(B-D) / B
III	0

Since h = H for $0 \leqq x \leqq B$ we can combine the cases by adding, giving

Case	Probability	
I	(B-D) / B	(also for D = 0)
II	(B-D) / B	
III	0	

A.2. Expectation

The mathematical expectation is found to be

$$\mu = E(\underline{h}_a) = \frac{B}{B} \cdot H = H$$

A.3. Variance

The variance is found by

$$\sigma^2 = E(\underline{h}_a^2) - \mu^2 = \frac{B}{B} \cdot H^2 - H^2 = 0$$

A.4. Covariance

A.4.1. Case I

The covariance between $\underline{h}_a = H$ and $\underline{h}_b = H$ is obtained by

$$Cov(\underline{h}_a \ \underline{h}_b) = E(\underline{h}_a \ \underline{h}_b) - \mu^2$$

$$= \frac{B-D}{B} H^2 - H^2$$

so

$$Cov(\underline{h}_a \ \underline{h}_b) = \frac{-D}{B} H^2$$

A.4.2. Case II

$$Cov(\underline{h}_a \ \underline{h}_b) = \frac{B-D}{B} H^2 - H^2 = \frac{-D}{B} H^2$$

A.4.3. Case III

$$Cov(\underline{h}_a \ \underline{h}_b) = - H^2$$

B. The triangular type

B.1. Definition

The triangular type is already sketched in Fig. 2. The definition is

$$h = {}^{1}f(x) = \frac{2H}{B}x, \qquad 0 \leq x < \tfrac{1}{2}B$$

$$h = {}^{2}f(x) = 2H - \frac{2H}{B}x, \qquad \tfrac{1}{2}B \leq x \leq B$$

The inverse of the first function is

$$x = {}^{1}f^{-1}(h) = \frac{Bh}{2H}$$

and so, by (5) and (6):

$$i_o = \frac{B}{2H} \quad \text{and} \quad I_o = \frac{B}{2H} \cdot h\Big|_0^H = \frac{B}{2}$$

The check according to (7) reads

$$\frac{2}{B} I_o = 1$$

B.2. Expectation

First the integral (8) has to be solved which gives

$$I_1 = \int_0^H h \cdot \frac{B}{2H}\, dh = \frac{B}{2H} \cdot \tfrac{1}{2}h^2\Big|_0^H = \frac{BH}{4}$$

and so

$$\mu = \frac{2}{B} \cdot \frac{BH}{4} = \frac{H}{2}$$

B.3. Variance

Inserting i_o in (9) produces

$$I_{11} = \int_0^H h^2\, i_o\, dh = \frac{B}{2H} \cdot \frac{1}{3} h^3 \Big|_0^H = \frac{BH^2}{6}$$

and

$$\sigma^2 = \frac{2}{B} \cdot \frac{BH^2}{6} - \frac{H^2}{4} = \frac{H^2}{12}$$

B.4. Covariance

Elements to be used in the determination of the covariance according to formulas given in Section 4 are, putting $2HD/B = K$,

$$^1f(0) = 0; \quad ^1f(\tfrac{1}{2}B-D) = H - K$$

$$^1f\{^1f^{-1}(h) + D\} = {}^1f\{\frac{Bh}{2H} + D\} = h + K$$

$$\frac{d}{dh}\, {}^1f^{-1}(h) = i_o = \frac{B}{2H}$$

$$^1f(\tfrac{1}{2}B-D) = H-K; \quad ^1f(\tfrac{1}{2}B) = H$$

$$^2f\{^1f^{-1}(h) + D\} = {}^2f\{\frac{Bh}{2H} + D\} = 2H - h - K$$

$$^1f(0) = 0; \quad ^1f(B-D) = 2H - K$$

with which the integrals can be obtained.

B.4.1. Case I

$$I_{12}^{11} = \int_0^{H-K} h.(h+K) . \frac{B}{2H} dh$$

$$= \frac{D}{K} (\frac{1}{3} h^3 + \tfrac{1}{2} h^2 K) \Big|_0^{H-K}$$

$$= \frac{D}{K} \{\frac{1}{3}(H-K)^3 + \tfrac{1}{2}K(H-K)^2\}$$

$$= \frac{D}{6K}(2H^3 - 6H^2K + 6HK^2 - 2K^3 + 3H^2K - 6HK^2 + 3K^3)$$

$$\boxed{I^{11}_{12} = \frac{D}{6K}(2H^3 - 3H^2K + K^3)}$$

and for the second part

$$I^{21}_{12} = \int_{H-K}^{H} h.(2H - h - K).\frac{B}{2H}\,dh$$

$$= \frac{D}{K}\left\{\tfrac{1}{2}h^2(2H - K) - \frac{1}{3}h^3\right\}\Big|_{H-K}^{H}$$

$$= \frac{D}{6K}\{3H^2\ (2H-K) - 2H^3 - 3(H-K)^2(2H-K) + 2(H-K)^3\}$$

$$= \frac{D}{6K}\{6H^3 - 3H^2K - 2H^3 - (H-K)^2(4H-K)\}$$

$$\boxed{I^{21}_{12} = \frac{D}{6K}(6H^2K - 6HK^2 + K^3)}$$

which finally leads to (see (10)):

$$I_{12} = \frac{D}{6K}(4H^3 - 6H^2K + 2K^3 + 6H^2K - 6HK^2 + K^3)$$

$$= \frac{D}{6K}(4H^3 - 6HK^2 + 3K^3)$$

$$= \frac{B}{12H}\left(4H^3 - \frac{24H^3D^2}{B^2} + \frac{24H^3D^3}{B^3}\right)$$

$$\boxed{I_{12} = \frac{H^2}{3B^2}(B^3 - 6BD^2 + 6D^3)}$$

giving for the covariance, according to (11):

$$\mathrm{Cov}(h_{\underline{a}}\ h_{\underline{b}}) = \frac{1}{B}I_{12} - \mu^2$$

$$\boxed{\mathrm{Cov}(h_{\underline{a}}\ h_{\underline{b}}) = \frac{H^2}{3B^3}(B^3 - 6BD^2 + 6D^3) - \frac{H^2}{4}}$$

B.4.2. Case II

We can use the second integral of Case I but have to change the boundary values, so we have to work out

$$I_{12}^{21'} = \frac{D}{K} \{ \tfrac{1}{2} h^2 (2H-K) - \frac{1}{3} h^3 \} \Big|_0^{2H-K}$$

$$= \frac{D}{K} \{ \tfrac{1}{2}(2H-K)^3 - \frac{1}{3}(2H-K)^3 \}$$

$$= \frac{D}{6K}(2H - \frac{2HD}{B})^3 = \frac{B}{12H} \cdot \frac{8H^3}{B^3} (B - D)^3 = I_{12}$$

which, according to (12) yields

$$\boxed{I_{12} = \frac{2H^2}{3B^2} (B - D)^3}$$

and according to (13) finally gives

$$\boxed{\mathrm{Cov}(\underline{h}_a\ \underline{h}_b) = \frac{2H^2}{3B^3} (B - D)^3 - \frac{H^2}{4}}$$

B.4.3. Case III

No integrals need be solved since in this case

$$\boxed{I_{12} = 0}$$

so we have at once with (14) and (15):

$$\boxed{\mathrm{Cov}(\underline{h}_a\ \underline{h}_b) = -\frac{H^2}{4}}$$

C. The exponential type

C.1. Definition

The exponential type is defined by the following equations in which b is a storm parameter,

$$h = {}^1f(x) = He^{2b(x-\frac{1}{2}B)}, \quad 0 \leqq x < \tfrac{1}{2}B$$

$$h = {}^2f(x) = He^{2b(\frac{1}{2}B-x)}, \quad \tfrac{1}{2}B \leqq x \leqq B$$

Here, ${}^1f(0) = {}^2f(B) = He^{-bB} \neq 0$, but b can be taken large enough to make h_o and h_B small.

The inverse of the first function is

$$x = {}^1f^{-1}(h) = \frac{1}{2b} \ln \frac{h}{H} + \tfrac{1}{2}B$$

and so, by (5) and (6):

$$i_o = \frac{1}{2bh} \quad \text{and} \quad I_o = \frac{1}{2b} \ln h \Big|_{He^{-bB}}^{H}$$

which yields

$$I_o = \frac{1}{2b} \ln H - \frac{1}{2b} \ln H + \frac{1}{2b} \cdot bB = \tfrac{1}{2}B$$

The check according to (7) reads

$$\frac{2}{B} I_o = 1$$

C.2. Expectation

First the integral (8) has to be solved which gives

$$I_1 = \int_{He^{-bB}}^{H} h.\frac{1}{2bh}\,dh = \frac{h}{2b}\Big|_{He^{-bB}}^{H}$$

$$\boxed{I_1 = \frac{H}{2b}(1-e^{-bB})}$$

and so

$$\boxed{\mu = \frac{2}{B}\,I_1 = \frac{H}{bB}(1-e^{-bB})}$$

It will appear to be convenient to define

$$\boxed{\begin{array}{l} u = 1 - e^{-bB} \\ v = 1 + e^{-bB} \end{array}}$$

so

$$\boxed{\mu = \frac{H}{bB}u}$$

C.3. Variance

Inserting i_o in (9) produces

$$I_{11} = \int_{He^{-bB}}^{H} h^2 . \frac{1}{2bh}dh = \frac{h^2}{4b}\Big|_{He^{-bB}}^{H}$$

$$\boxed{I_{11} = \frac{H^2}{4b}(1 - e^{-2bB}) = \frac{H^2}{4b}\,uv}$$

and so, since $\sigma^2 = \frac{2}{B}\,I_{11} - \mu^2$

$$\sigma^2 = \frac{H^2}{2bB}\,uv - \frac{H^2}{b^2B^2}u^2 \quad \text{or} \quad \boxed{\sigma^2 = \tfrac{1}{2}\left(\frac{H}{bB}\right)^2 u(bBv - 2u)}$$

C.4. Covariance

Elements to be used in the determination of the covariance according to formulas given in Section 4 are

$$^{1}f(0) = He^{-bB}; \quad ^{1}f(\tfrac{1}{2}B-D) = He^{-2bD}$$

$$^{1}f\{^{1}f^{-1}(h) + D\} = {}^{1}f\{\frac{1}{2b}\ln\frac{h}{H} + \tfrac{1}{2}B + D\}$$

$$= H \exp 2b(\frac{1}{2b}\ln h - \frac{1}{2b}\ln H + D)$$

$$= H \exp(\ln h - \ln H + 2bD)$$

$$= h\, e^{2bD}$$

$$^{1}f(\tfrac{1}{2}B-D) = He^{-2bD}; \quad ^{1}f(\tfrac{1}{2}B) = H$$

$$^{2}f\,\{^{1}f^{-1}(h) + D\} = {}^{2}f\,\{\frac{1}{2b}\ln\frac{h}{H} + \tfrac{1}{2}B + D\}$$

$$= H \exp 2b(-\frac{1}{2b}\ln\frac{h}{H} - D)$$

$$= \frac{H^2}{h}\, e^{-2bD}$$

$$^{1}f(0) = He^{-bB}; \quad ^{1}f(B-D) = H \exp 2b(\tfrac{1}{2}B-D)$$

with which the integrals can be obtained.

C.4.1. Case I

$$I_{12}^{11} = \int_{He^{-bB}}^{He^{-2bD}} h.he^{2bD}.\frac{1}{2bh}\,dh$$

$$= \frac{e^{2bD}}{2b} \tfrac{1}{2} \{H^2 e^{-4bD} - H^2 e^{-2bB}\}$$

$$\boxed{I_{12}^{11} = H^2 \frac{e^{2bD}}{4b} (e^{-4bD} - e^{-2bB})}$$

and for the second part

$$I_{12}^{21} = \int_{He^{-2bD}}^{H} h . \frac{H^2}{h} e^{-2bD} . \frac{1}{2bh} \, dh$$

$$= \frac{H^2 e^{-2bD}}{2b} (\ln H - \ln H - \ln e^{-2bD})$$

$$= \frac{H^2 e^{-2bD}}{2b} . 2bD = H^2 D \; e^{-2bD}$$

which finally leads to (see (10)):

$$\boxed{I_{12} = \frac{H^2 e^{2bD}}{2b} (e^{-4bD} - e^{-2bB} + 2bDe^{-4bD})}$$

giving for the covariance, according to (11):

$$\boxed{Cov(\underline{h}_a \; \underline{h}_b) = \frac{1}{B} I_{12} - \mu^2}$$

C.4.2. Case II

We can use the second integral of Case I but have to change the boundary values, so we have to work out

$$I_{12}^{21'} = \frac{H^2 e^{-2bD}}{2b} (\ln h) \Big|_{He^{-bB}}^{He^{bB-2bD}}$$

$$= \frac{H^2 e^{-2bD}}{2b} (bB - 2bD + bB)$$

so using the equallity given in (12):

$$\boxed{I_{12} = H^2 e^{-2bD} (B-D)}$$

giving for the covariance, according to (11):

$$\boxed{Cov(h_{\underline{a}}\ h_{\underline{b}}) = H^2 \{\frac{e^{-2bD}}{B} (B-D) - \frac{u^2}{b^2 B^2}\}}$$

C. 4.3. Case III

No integrals need be solved since in this case

$$\boxed{I_{12} = 0}$$

so we have at once with (14) and (15):

$$\boxed{Cov(h_{\underline{a}}\ h_{\underline{b}}) = - \frac{H^2 u^2}{b^2 B^2}}$$

REFERENCES

STOL, PH.TH., 1977a. Principles underlying an anlytical model for the process of measuring rainfall amounts and the determination of interstation correlations. ICW Nota 992

______ 1977b. Rainfall interstation correlation functions: an analytic approach (in press)

ANNEX 1

MODIFIED SOLUTIONS FOR TIME SERIES

The solutions obtained in Section 5 refer-in the hydrological sense- to statistical parameters in storms. It is usefull to have solutions for time series. The relationship between both is given by Stol (1977a). With the aid of the integrals the modified solutions can be obtained easily.

Consider the correlation coefficient

$$\rho = \frac{A\ \mathrm{Cov}(\underline{h}_a \underline{h}_b) + (A-B)\ \mu^2}{A\ \mathrm{Var}(\underline{h}_a) + (A-B)\ \mu^2} \qquad \text{(STOL, 1977a, Annex 1)}$$

where A = L + B, where L is a constant. Actually L plays the role of area length which, in the mathematical treatment, is immaterial.

We may write for symmetric storm functions, according the basic integrals (8), (9) and (11):

$$\rho = \frac{A(\frac{1}{B} I_{12} - \frac{4}{B^2} I_1^2) + (A-B)\frac{4I_1^2}{B^2}}{A(\frac{2}{B} I_{11} - \frac{4}{B^2} I_1^2) + (A-B)\frac{4I_1^2}{B^2}}$$

which equals

$$\boxed{\rho = \frac{AI_{12} - 4I_1^2}{2AI_{11} - 4I_1^2}}$$

a. Rectangular storms

For rectangular storms the first equation is more convenient. So with results obtained in Section A.2 through A.4 we have:

Case I and Case II:

$$\rho_I = \rho_{II} = \frac{A(\frac{-D}{B}H^2) + (A-B)\ H^2}{(A-B)\ H^2}$$

$$= \frac{-AD + B(A-B)}{B(A-B)} = \frac{-AD + BL}{BL}$$

or

$$\boxed{\rho_I = \rho_{II} = 1 - \frac{L + B}{LB} D}$$

Case III:

$$\rho_{III} = \frac{A(-H^2) + (A-B)\ H^2}{(A-B)\ H^2} = \frac{-B}{L}$$

$$\boxed{\rho_{III} = 1 - \frac{L + B}{L}}$$

b. Triangular storms

For triangular storms we apply the last equation for ρ. So with results obtained in Section B we have

Case I:

$$\rho_I = \frac{A\frac{H^2}{3B^2}(B^3 - 6BD^2 + 6D^3) - 4\frac{B^2H^2}{16}}{2A\frac{BH^2}{6} - 4\ \frac{B^2H^2}{16}}$$

$$= \frac{4A(B^3-6BD^2+6D^3) - 3B^4}{4AB^3 - 3B^4}$$

$$= 1 - \frac{24AD^2\ (B-D)}{B^3(4A - 3B)}$$

and finally

$$\boxed{\rho_I = 1 - 24(L+B)D^2 \frac{B - D}{B^3(4L + B)}}$$

Case II:

$$\rho_{II} = \frac{A\frac{2H^2}{3B^2}(B - D)^3 - 4\frac{B^2H^2}{16}}{2A\frac{BH^2}{6} - 4\frac{B^2H^2}{16}}$$

$$= \frac{8A\ (B - D)^3 - 3B^4}{4AB^3 - 3B^4}$$

and finally

$$\boxed{\rho_{II} = 1 - 4(L + B)\frac{B^3 - 2(B - D)^3}{B^3(4L + B)}}$$

Case III:

$$\rho_{III} = \frac{-4\frac{B^2H^2}{16}}{2A\frac{BH^2}{6} - 4\frac{B^2H^2}{16}}$$

$$= \frac{-\ 3B}{4A - 3B}$$

$$\boxed{\rho_{III} = 1 - \frac{4(L + B)}{4L + B}}$$

c. E x p o n e n t i a l   s t o r m s

For the exponential storms we apply the equation for ρ with the results obtained in Section C. We have

Case I:

We have to divide

$$\frac{AH^2\, e^{2bD}}{2b}\left(e^{-4bD} - e^{-2bB} + 2bDe^{-4bD}\right) - 4\frac{H^2}{4b^2}u^2$$

by

$$\frac{2AH^2}{4b}\, uv - \frac{4H^2}{4b^2}u^2$$

which gives

$$\rho_I = \frac{Abe^{2bD}(e^{-4bD} - e^{-2bB} + 2bDe^{-4bD}) - 2\,u^2}{Abuv - 2\,u^2}$$

$$= 1 - Ab\,\frac{1 - e^{-2bB} - e^{-2bD} + e^{-2b(B-D)} - 2bDe^{-2bD}}{u(Abv - 2u)}$$

which, possibly not can be written in a better way than by

$$\boxed{\rho_I = 1 - Ab\,\frac{uv - \{e^{-2bD}\ (1 - e^{-2b(B-2D)}\}\ -2bDe^{-2bD}}{u(Abv - 2u)}}$$

(where A = L + B) to match the following expressions.

Case II:

$$\rho_{II} = \frac{AH^2 e^{-2bD}(B-D) - 4\,\dfrac{H^2}{4b^2}\,u^2}{2A\,\dfrac{H^2}{4b}\,uv - 4\,\dfrac{H^2}{4b^2}\,u^2}$$

$$= \frac{2Ab^2 e^{-2bD}(B-D) - 2u^2}{Abuv - 2u^2}$$

finally

$$\boxed{\rho_{II} = 1 - Ab\,\frac{uv - 2b(B-D)e^{-2bD}}{u(Abv - 2u)}}$$

Case III:

$$\rho_{III} = \frac{-4 \dfrac{H^2}{4b^2} u^2}{2A\dfrac{H^2}{4b} uv - 4 \dfrac{H^2}{4b^2} u^2}$$

$$= \frac{-2u^2}{Abuv - 2u^2}$$

so

$$\boxed{\rho_{III} = 1 - Ab \frac{uv}{u(Abv - 2u)}}$$

ANNEX 2.

SPECIFIC PROPERTIES OF THE CORRELATION FUNCTIONS

The correlation function for time series as derived in Annex 1, have some special properties with respect to the transition from ρ_I to ρ_{II} to ρ_{III}. A few comments will be given here.

a. The rectangular storm function

It was found that $\rho_I(D) = \rho_{II}(D)$
Further we have

$$\rho_I(0) = 1$$

$$\rho_I(\tfrac{1}{2}B) = \rho_{II}(\tfrac{1}{2}B), \quad \text{(same function)}$$

$$\rho_{II}(B) = 1 - \frac{L+B}{L} = -\frac{B}{L}$$

$$\rho_{III}(B) = -\frac{B}{L}, \quad \text{(constant)}$$

So the complete function is continuous at the transition points from one case to another.

The derivatives are:

$$\frac{d\rho_I(D)}{dD} = \frac{d\rho_{II}(D)}{dD} = -\frac{L+B}{LB}$$

$$\frac{d\rho_{III}(D)}{dD} = 0$$

The complete function is not 'smooth' in the sense that the first derivatives are continuous. The complete function consists of two straight lines that intersect at D = B.

b. The triangular storm function

There are three different correlation functions, $\rho_I(D)$, $\rho_{II}(D)$ and $\rho_{III}(D)$, pertaining the cases I, II and III respectively.

We have

$$\rho_I(0) = 1$$

$$\rho_I(\tfrac{1}{2}B) = 1 - 24(L + B)\frac{(\tfrac{1}{2}B)^3}{B^3(4L + B)}$$

$$= 1 - 3\,\frac{L + B}{4L + B}$$

and

$$\rho_{II}(\tfrac{1}{2}B) = 1 - 4\,(L + B)\frac{B^3 - 2(\tfrac{1}{2}B)^3}{B^3(4L + B)}$$

$$= 1 - 3\,(L + B)\,\frac{1}{4L + B}$$

Finally,

$$\rho_{II}(B) = 1 - \frac{4(L + B)}{4L + B}$$

and

$$\rho_{III}(B) = 1 - \frac{4(L + B)}{4L + B}, \qquad \text{(constant)}$$

The complete function is continuous at the transition points.

The derivatives are:

$$\rho_I'(D) = \frac{-24(L + B)}{B^3(4L + B)}(2BD - 3D^2)$$

$$\rho_{II}'(D) = \frac{8(L + B)}{B^3(4L + B)}\,.3.(B - D)^2(-1)$$

$$= \frac{-24(L + B)}{B^3(4L + B)}(B - D)^2$$

$$\rho_{III}'(D) = 0$$

The complete function is 'smooth' since

$$\rho_I'(\tfrac{1}{2}B) = \frac{-24(L + B)}{B^3(4L + B)}(B^2 - \tfrac{3}{4}B^2)$$

$$\rho_{II}'(\tfrac{1}{2}B) = \frac{-24(L + B)}{B^3(4L + B)}(\tfrac{1}{2}B)^2$$

$$\rho_{II}'(B) = 0$$

$$\rho_{III}'(B) = 0$$

giving

$$\rho_I'(\tfrac{1}{2}B) = \rho_{II}'(\tfrac{1}{2}B) \quad \text{and} \quad \rho_{II}'(B) = \rho_{III}'(B)$$

Further we observe that

$$\rho_I'(0) = 0$$

c. The exponential storm function

There are, again, three different correlation functions. We have, with

$$u = 1 - e^{-bB}$$
$$v = 1 + e^{-bB}$$
$$uv = 1 - e^{-2bB}$$

the results

$$\rho_I(0) = 1 - Ab\frac{uv - uv - 0}{u(Abv - 2u)} = 1$$

$$\rho_I(\tfrac{1}{2}B) = 1 - Ab\,\frac{uv - \{e^{-bB}(1-1)\} - bBe^{-bB}}{u(Abv - 2u)}$$

$$\rho_{II}(\tfrac{1}{2}B) = 1 - Ab\,\frac{uv - bBe^{-bB}}{u(Abv - 2u)}$$

$$\rho_{III}(B) = 1 - Ab\frac{uv}{u(Abv - 2u)}, \quad \text{(constant)}, = \rho_{II}(B)$$

The complete function is continuous at the transition points $D = \frac{1}{2}B$ and $D = B$.

The first derivatives are:

$$\rho'_I(D) = \frac{+Ab}{u(Abv - 2u)} \cdot g'_1(D)$$

where

$$g_1(D) = e^{-2bD} - e^{-2b(B-D)} + 2bD\ e^{-2bD}$$

and

$$g'_1(D) = -2b\ e^{-2bD} - 2be^{-2b(B-D)} + 2be^{-2bD} - 4b^2De^{-2bD}$$

so

$$\rho'_I(D) = \frac{-4Ab^3}{u(Abv - 2u)} (D + \frac{e^{-2b(B-2D)}}{2b})\ e^{-2bD}$$

Next we have

$$\rho'_{II}(D) = \frac{+2Ab^2}{u(Abv - 2u)} \cdot g'_2(D)$$

where

$$g_2(D) = Be^{-2bD} - De^{-2bD}$$

and

$$g'_2(D) = -2bBe^{-2bD} - e^{-2bD} + 2bDe^{-2bD}$$

$$= -2be^{-2bD}(B - D) - e^{-2bD}$$

giving

$$\rho'_{II}(D) = \frac{-4Ab^3}{u(Abv - 2u)}(B - D + \frac{1}{2b})\ e^{-2bD}$$

and finally

$$\rho'_{III}(D) = 0$$

The complete function is 'smooth' only at the first transition $D = \frac{1}{2}B$, namely it is readily seen that

$$\rho'_{I}(\tfrac{1}{2}B) = \rho'_{II}(\tfrac{1}{2}B)$$

but also that

$$\rho'_{II}(B) = \frac{-2Ab^2}{u(Abv - 2u)} e^{-2bB}$$

which only equals $\rho'_{III}(D) = 0$ for $b = 0$

which has no physical meaning.

Finally we observe that

$$\rho'_{I}(0) = \frac{-2Ab^2}{u(Abv - 2u)} e^{-2bB}$$

which surprisingly equals $\rho'_{II}(B)$.

ANNEX 3.

SUMMARY OF GENERAL FORMULAS

It is assumed that the storm function has two branches and that it is symmetric about the center.

Definitions,

b = storm parameter (if present)

H = storm maximum, in center

B = storm diameter

L = length of gaged area

D = inter-station distance

h = rainfall amount in storm

x = storm coordinate

$h_a = f(a)$ for $x = a$

Case I: $0 \leqq D < \frac{1}{2}B < B$ (large)

Case II: $0 < \frac{1}{2}B \leqq D < B$ (medium)

Case III: $0 < \frac{1}{2}B < B \leqq D$ (small)

Left branch: $h = {}^1f(x)$, $0 \leqq x \leqq \frac{1}{2}B$

Right branch: $h = {}^2f(x)$, $\frac{1}{2}B \leqq x \leqq B$

Properties,

$$^1f(0) = {}^2f(B)$$

$$^1f(\tfrac{1}{2}B) = {}^2f(\tfrac{1}{2}B) = H$$

$$^1f(x) = {}^2f(B - x)$$

Integrals,

$$i_o = \frac{d}{dh}\,{}^1f^{-1}(h)$$

$$I_o = \int_{{}^1f(0)}^{{}^1f(\frac{1}{2}B)} i_o \, dh$$

$$I_1 = \int_{{}^1f(0)}^{{}^1f(\frac{1}{2}B)} h \, i_o \, dh$$

$$I_{11} = \int_{{}^1f(0)}^{{}^1f(\frac{1}{2}B)} h^2 \, i_o \, dh$$

$$I_{12}^{11} = \int_{{}^1f(0)}^{{}^1f(\frac{1}{2}B-D)} h.\,{}^1f\{{}^1f^{-1}(h) + D\}.\, i_o \, dh$$

$$I_{12}^{21} = \int_{{}^1f(\frac{1}{2}B-D)}^{{}^1f(\frac{1}{2}B)} h.\,{}^2f\{{}^1f^{-1}(h) + D\}\,.\, i_o \, dh$$

$$I_{12}^{21'} = \int_{{}^1f(0)}^{{}^1f(B-D)} h.\,{}^2f\{{}^1f^{-1}(h) + D\}.\, i_o \, dh$$

$$I_{12} = 2\, I_{12}^{11} + I_{12}^{21} \qquad \text{(Case I)}$$

$$I_{12} = I_{12}^{21'} \qquad \text{(Case II)}$$

$$I_{12} = 0 \qquad \text{(Case III)}$$

Total probability: $\frac{2}{B}\, I_o = 1$

Expectation: $\mu = \frac{2}{B} I_1$

Variance: $\sigma^2 = \frac{2}{B} I_{11} - \mu^2$

Covariance: $Cov(h_{\underline{a}}\ h_{\underline{b}}) = \frac{1}{B} I_{12} - \mu^2$

Correlation: $\rho_{h_{\underline{a}}\ h_{\underline{b}}} = \frac{\text{Covariance}}{\text{Variance}}$

Correlation for time series:

$A = L + B$

$$\boxed{\rho(D) = \frac{AI_{12} - 4\ I_1^2}{2AI_{11} - 4\ I_1^2}}$$

Alternative form:

$$\boxed{\rho(D) = 1 - A\ \frac{2I_{11} - I_{12}}{2AI_{11} - 4\ I_1^2}}$$

First derivative:

$$\frac{d\rho(D)}{dD} = \frac{A}{2AI_{11} - 4\ I_1^2}\ \frac{dI_{12}(D)}{dD}$$

If $D = 0$ then $I_{12}^{21} = 0$

and $I_{12}^{11} = I_{11}$

giving $I_{12} = 2\ I_{11}$ and $\rho(0) = 1$