DISJUNCTIVE LINEAR OPERATORS AND PARTIAL MULTIPLICATIONS IN RIESZ SPACES

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PROEFSCHRIFT
TER VERKRIJGING VAN DE GRAAD VAN
DOCTOR IN DE LANDBOUWWETENSCHAPPEN,
OP GEZAG VAN DE RECTOR MAGNIFICUS
DR. H.C. VAN DER PLAS,
HOOGLERAAR IN DE ORGANISCHE SCHEIKUNDE,
IN HET OPENBAAR TE VERDEDIGEN
OP WOENSDAG 17 DECEMBER 1980
DES NAMIDDAGS TE VIER UUR IN DE AULA
VAN DE LANDBOUWHOGESCHOOL TE WAGENINGEN.

Krips Repro - Meppel

Want het dwaze Gods
is wijzer dan de mensen

(1 Korinthe 1 : 25a)

NNO8201,830

STELLINGEN

I

Iedere semi-norm ρ op een eindig dimensionale Archimedische Riesz ruimte E zodanig dat $\rho(x) = \rho(|x|)$ voor alle $x \in E$ is een Riesz semi-norm.
De verwijzing die Aliprantis en Burkinshaw voor deze stelling geven is niet terecht omdat deze stelling reeds in 1961 is bewezen door Bauer, Stoer en Witzgall.

Aliprantis, C.D. en Burkinshaw, O.
Locally solid Riesz spaces, Academic Press (1978).

Bauer, F.L., Stoer, J. en Witzgall, C.
Absolute and monotonic norms,
Numer. Math., Vol. 3, pp. 257-264 (1961).

II

Als p een reguliere vectoriële pseudonorm: $\mathbb{C}^n \to \mathbb{R}^k_+$ is, $N(p) = \{x \in \mathbb{C}^n;\ p(x) = 0\}$ en $R(p) = \{p(x);\ x \in \mathbb{C}^n\}^{\perp\perp}$, dan is $\dim R(p) \leqslant n - \dim N(p)$.

III

Als op een Archimedische Riesz algebra E een L-norm $\|\cdot\|$ gedefinieerd is dan bestaat er op E een semi-inproduct.
Als E een Archimedische Φ-algebra is en $(E, \|\cdot\|)$ een AL-ruimte, dan is er een norm $\|\cdot\|'$ op E zodanig dat $(E, \|\cdot\|')$ een Hilbert ruimte is.

IV

De lineaire ruimte B(X,Y) met operatornorm van alle norm begrensde lineaire operatoren van X naar Y, waarbij $X \neq \{0\}$ en Y genormeerde lineaire ruimten over $\mathbb{R}$ zijn, is juist dan norm compleet als Y norm compleet is.
Het analogon van deze stelling in Riesz ruimten, met orde begrensd in plaats van norm begrensd en Dedekind compleet in plaats van norm compleet, is geldig onder de beperking dat de orde duale $X^\sim$ van X voldoet aan $X^\sim \neq \{0\}$.

Pfaffenberger, W.E.
A converse to a completeness theorem,
Amer. Math. Monthly, Vol. 87, no. 3, p. 216 (1980).

Aliprantis, C.D.
On order properties of order bounded transformations,
Canad. J. Math., Vol. 27, no. 3, pp. 666-678.

V

Als X een lineaire ruimte over $\mathbb{R}$ is, Y een Dedekind complete Riesz rui
en p: $X \to Y^+$ een sublineaire operator van de vorm $|Lx|$ (L een lineaire operator van X naar Y), dan is voor iedere $x \in X$ de verzameling $K_x = \{y \in X;\ p(x + y) = p(x) + p(y)\}$ een pre-kegel in X.

VI

Zij voor alle $n \in \mathbb{N}$ de $n \times n$ matrix A_n gegeven door $A_n(i,j) = e^{-|i-j|}$ $(i, j = 1, \ldots , n)$.
Voor de spectraalstraal $\rho(A_n)$ van A_n geldt $\lim_{n\to\infty} \rho(A_n) = \frac{e + 1}{e - 1}$.

VII

Veksler en Geiler hebben bewezen dat iedere voorwaardelijk lateraal complete Archimedische Riesz ruimte de projectie eigenschap bezit.
Bernau zegt van deze stelling een generalisatie te geven in tralie-geordende groepen. In tegenstelling tot zijn bewering is zijn resultaat voor Riesz ruimten zwakker dan dat van Veksler en Geiler.

Veksler, A.I. en Geiler, V.A.
Order and disjoint completeness of linear partially ordered spaces (Russisch),
Sibirsk Mat. Ž., Tom. 13, pp. 43-51,
English transl.: Siberian Math. J., Vol. 13, pp. 30-35.

Bernau, S.J.
Lateral and Dedekind completion of archimedean lattice groups,
J. London Math. Soc. (2), Vol. 12, pp. 320-322.

VIII

De statistische selectie- en rangschikkingstechnieken, waarvoor recent in de statistische literatuur veel belangstelling aan de dag is gelegd, verdienen in het landbouwkundig onderzoek grote aandacht.

IX

Het verdient aanbeveling de uitdrukking "een x-aantal" niet te bezigen zonder specificatie van x.

X

De overheid dient ten aanzien van alternatieve groeperingen geen alternatieve gedragslijn te volgen.

Stellingen bij het proefschrift "Disjunctive linear operators and parti
multiplications in Riesz spaces".

B. van Putten
Wageningen, 17 december 19

VOORWOORD

De aanleiding tot het schrijven van dit proefschrift was een projectvoorstel, gericht aan de Vaste Commissie voor de Wetenschapsbeoefening van de Landbouwhogeschool, dat als titel had "vectorial norms on linear spaces". Dit was tevens de titel van mijn afstudeerverslag (mei 1976). Dit projectvoorstel werd goedgekeurd en in september 1977 werd begonnen met de uitwerking van dit project.
Daartoe werden normen bestudeerd, die waarden aannemen in Dedekind complete Riesz ruimten. Al spoedig kreeg ik daarbij te maken met de operatoren, die in dit proefschrift een belangrijke plaats innemen en die zeer de moeite waard leken om te worden bestudeerd. Voor verdere studie van het onderwerp was kennis van deze operatoren noodzakelijk en zo langzamerhand werden de ruimten die het bereik zijn van deze normen het domein van studie. Intussen bleek dat de operatoren die mijn belangstelling hadden ook bestudeerd werden door enkele anderen, meestal vanuit verschillende achtergronden. Het resultaat is nu een proefschrift op het gebied van de zuivere wiskunde; de titel ervan heeft met de titel van het projectvoorstel toch nog twee woorden gemeen.

Bij dezen dank ik mijn ouders voor hun zorg door alle jaren heen en ook mijn overige familie dank ik voor hun belangstelling.
Professor Dr. A. van der Sluis dank ik vriendelijk voor de fijne afstudeerperiode en voor het vele dat ik in die tijd van hem heb geleerd.
Mijn promotor, Professor Dr. B. van Rootselaar, dank ik hartelijk voor zijn hulp en zijn aanmoediging om de sprong in het oneindig dimensionale te maken.
Ook de overige leden van de Vakgroep Wiskunde dank ik voor hun belangstelling, in het bijzonder Drs. B.R. Damsté voor het corrigeren van de tekst in de zeer korte tijd die daarvoor nog overig was.
I would like to thank Professor Dr. W.A.J. Luxemburg (Pasadena, California) and Professor Dr. H.H. Schaefer (Tübingen, B.R.D.) for their kind invitation to visit the conference about "Riesz spaces and order bounded operators", which was held in Oberwolfach, from June 24 till June 30, 1979.
Jodien Houwers en Ans van der Lande-Heij dank ik voor het typen van dit proefschrift.
Mijn vrouw Riet dank ik voor haar steun en voor het accepteren van mijn dag- en avondindeling.

AMS(MOS) subject classifications (1980)
06F20, 47B55.

CONTENTS page

Chapter I

RIESZ SPACES

In this chapter we give an exposition of the elements of the theory of Riesz space; for a short historical introduction we refer to the books of Aliprantis and Burkinshaw [1978] and Luxemburg and Zaanen [1971].

1. Order relation

An order relation on a non-empty set S is a subset $\leqslant$ of the Cartesian product S x S of S with the following properties:
$\leqslant$ is transitive, i.e. if $(x,y) \in \leqslant$ and $(y,z) \in \leqslant$ for $x,y,z \in S$, then also $(x,z) \in \leqslant$, $\leqslant$ is reflexive, i.e. $(x,x) \in \leqslant$ for all $x \in S$, $\leqslant$ is anti-symmetric, i.e. $(x,y) \in \leqslant$ and $(y,x) \in \leqslant$ for $x,y \in S$ implies $x = y$.

It is common to write $x \leqslant y$ instead of $(x,y) \in \leqslant$.

A partially ordered set is now defined as a pair $(S,\leqslant)$ where S is a non-empty set and $\leqslant$ is an order relation on S.
Two elements x and y are called comparable if at least one of the statements $x \leqslant y$, $y \leqslant x$ holds, otherwise they are called incomparable.
A non-empty subset X of a partially ordered set $(S,\leqslant)$ is called a chain in $(S,\leqslant)$ if all pairs of elements of X are comparable.
$(S,\leqslant)$ is said to be totally ordered if S itself is a chain in $(S,\leqslant)$.
If x is an element of a partially ordered set $(S,\leqslant)$ such that $x \leqslant y$ for $y \in S$ implies $x = y$, then x is called a maximal element of $(S,\leqslant)$.
x is a minimal element of $(S,\leqslant)$ if $y \leqslant x$ for $y \in S$ implies $y = x$.

For elements x,y,z of a partially ordered set $(S,\leqslant)$ the following notations are used:
$x < y$ for $x \leqslant y$ & $x \neq y$;
$y \geqslant x$ for $x \leqslant y$;
$y > x$ for $x < y$;
$x \leqslant y \leqslant z$ for $x \leqslant y$ & $y \leqslant z$;
$x,y \leqslant z$ for $x \leqslant z$ & $y \leqslant z$;
$[x,y]$ for the order interval $\{z \in S;\ x \leqslant z \leqslant y\}$.

If X and Y are non-empty subsets of a partially ordered set $(S,\leqslant)$, then X is majorized by Y, in formula $X \leqslant Y$, if $x \leqslant y$ holds for all $x \in X$ and $y \in Y$. In that case Y is called minorized by X. If Y is a singleton $\{y\}$, then we write $X \leqslant y$ instead of $X \leqslant \{y\}$. y is said to be a majorant of X then.
Dually, if X is a singleton $\{x\}$, then we write $x \leqslant Y$ instead of $\{x\} \leqslant Y$ and x is said to be a minorant of Y.
A non-empty subset X of $(S,\leqslant)$ is called majorized in $(S,\leqslant)$, in formula $X \leqslant$, if there exists a $z \in S$ such that $X \leqslant z$, minorized in $(S,\leqslant)$, in formula $\leqslant X$, if there exists a $y \in S$ such that $y \leqslant X$ and bounded, in formula $\leqslant X \leqslant$, if X is majorized and minorized at the same time.
As a consequence, we have that a non-empty subset X is bounded if and only if there exist $y,z \in S$ such that $X \subset [y,z]$.

Now we can formulate Zorn's lemma, which we give in the following form.

1.1. Zorn's lemma. *If every chain in a partially ordered set* $(S,\leqslant)$ *is majorized in* $(S,\leqslant)$ *then* $(S,\leqslant)$ *contains at least one maximal element.*

An element x of a partially ordered set $(S,\leqslant)$ is called a supremum of a non-empty subset X of S if x is a majorant of X and at the same time a minorant of the set of all majorants of X, in formula $X \leqslant x$ and if $X \leqslant y$ for some $y \in S$ then $x \leqslant y$. It follows from the anti-symmetry of the order relation that a supremum is unique.
The supremum x of $X \neq \phi$ is denoted by $\sup X$.
Dually, $z \in S$ is called an infimum of X if z is a minorant of X and at the same time a majorant of the set of all minorants of X, in formula $z \leqslant X$ and if $y \leqslant X$ for some $y \in S$ then $y \leqslant z$. Also an infimum is unique and is denoted by $\inf X$.

1.2. Definition. *A lattice is a partially ordered set* $(S,\leqslant)$ *with the property that for all* $x,y \in S$ *holds that* $\sup\{x,y\}$ *and* $\inf\{x,y\}$ *exist in* S.

In the sequel we write $x_1 \vee x_2 \vee \ldots \vee x_n$ for $\sup\{x_1,x_2,\ldots,x_n\}$ and $x_1 \wedge x_2 \wedge \ldots \wedge x_n$ for $\inf\{x_1,x_2,\ldots,x_n\}$.

A lattice $(S,\leqslant)$ is called distributive if for all $x,y,z\in S$ holds that $(x\vee y)\wedge z = (x\wedge z)\vee(y\wedge z)$.
A lattice $(S,\leqslant)$ is distributive if and only if for all $x,y,z\in S$ holds that $(x\wedge y)\vee z = (x\vee z)\wedge(y\vee z)$ (cf. e.g. Birkhoff [1967, I. 6 thm 9].

1.3. Definition. *A lattice* $(S,\leqslant)$ *is called*

(a) (order) complete if $\sup X$ *and* $\inf X$ *exist in* S *for every non-empty subset* X *of* S.

(b) Dedekind complete if $\sup X$ *exists for every majorized subset* X *of* S *and* $\inf Y$ *exists for every minorized subset* Y *of* S.

(c) Dedekind σ*-complete if* $\sup X$ *exists for every majorized countable subset* X *of* S *and* $\inf Y$ *exists for every minorized countable subset* Y *of* S.

1.4. Definition. *A distributive lattice* $(S,\leqslant)$ *is called a Boolean algebra if* $1 := \sup S$ *and* $0 := \inf S$ *exist and if for every* $x\in S$ *there exists a (necessarily unique)* $x'\in S$ *such that* $x\wedge x' = 0$ *and* $x\vee x' = 1$. *In that case* x' *is called the complement of* x.

2. Partially ordered linear spaces and Riesz spaces

2.1. Definition. *A tripel* $(E,+,\leqslant)$ *is called a (partially) ordered linear space if*

(a) $(E,+)$ *is a linear space over* $\mathbb{R}$

(b) $(E,\leqslant)$ *is a partially ordered set*

(c1) $x\leqslant y$ *implies* $x+z\leqslant y+z$ *for all* $x,y,z\in E$

(c2) $0\leqslant x$ *and* $\lambda\geqslant 0$ *implies* $0\leqslant\lambda x$ *for all* $\lambda\in\mathbb{R}$ *and* $x\in E$.

In the sequel we abbreviate $(E,+,\leqslant)$ to E for fixed $+$ and $\leqslant$.
An element x of a partially ordered linear space E is called infinitely small with respect to $0\leqslant y\in E$ if $nx\leqslant y$ and $-nx\leqslant y$ hold for all $n\in\mathbb{N}$. The set of all $x\in E$ such that x is infinitely small with respect to a given $0\leqslant y\in E$ is denoted by $IS(y)$. We abbreviate $\cup\{IS(y);\ 0\leqslant y\in E\}$ to $IS(E)$. E is called Archimedean if $IS(E)=\{0\}$.

2.2. Definition. *A partially ordered linear space* $(E,+,\leqslant)$ *is called a Riesz space if* $(E,\leqslant)$ *is a lattice. A Riesz space* $(E,+,\leqslant)$ *is called Dedekind complete if* $(E,\leqslant)$ *is Dedekind complete, and Dedekind* σ*-complete if* $(E,\leqslant)$ *is Dedekind* σ*-complete.*

2.3. Proposition. *A Riesz space* E *is Archimedean if and only if* $nx \leqslant y$ *for* $x,y \in E^+$ *and all* $n \in \mathbb{N}$ *implies* $x = 0$.

Proof: $\rightarrow$: follows directly from the definition

$\leftarrow$: if $nx \leqslant y$ and $-nx \leqslant y$ hold for certain $x \in E$, $0 \leqslant y \in E$ and all $n \in \mathbb{N}$, then $n.\sup(x,-x) \leqslant y$ holds for all $n \in \mathbb{N}$. Now it follows from $\sup(x,-x) \geqslant x$, $\sup(x,-x) \geqslant -x$ that $2\sup(x,-x) \geqslant 0$, so $\sup(x,-x) \geqslant 0$. Hence $\sup(x,-x) = 0$ or $x = 0$.

Next we give some examples of partially ordered linear spaces. All examples which are given here are well known in the literature and there exists a reasonable uniformity in the literature about a standard name of most of them.

2.4. Examples.

(a) $\mathbb{R}^2$ is the real plane partially ordered componentwise, i.e. $(x_1,x_2) \leqslant (y_1,y_2)$ if $x_1 \leqslant y_1$ and $x_2 \leqslant y_2$ at the same time. Provided with the usual algebraic operations $\mathbb{R}^2$ is a Riesz space, where $(z_1,z_2) = (x_1,x_2)\vee(y_1,y_2)$ if z_i is the maximum of x_i and y_i $(i = 1,2)$. Note that $\mathbb{R}^2$ is even Dedekind complete.

(b) $(\mathbb{R}^2, \text{lex})$ is the real plane ordered lexicographicly, i.e. $(x_1,x_2) \leqslant (y_1,y_2)$ if $[x_1 < y_1]$ or $[x_1 = y_1 \;\&\; x_2 \leqslant y_2]$. With the usual algebraic operations $(\mathbb{R}^2, \text{lex})$ is a Riesz space which is totally ordered. However $(\mathbb{R}^2, \text{lex})$ is not Archimedean, because $n(0,1) \leqslant (1,0)$ for all $n \in \mathbb{N}$.

(c) $P(\mathbb{R})$ is the linear space of all real polynomials on the real axis with pointwise linear operations and pointwise partial ordering, i.e. $x \leqslant y$ in $P(\mathbb{R})$ if $x(t) \leqslant y(t)$ for all $t \in \mathbb{R}$. $P(\mathbb{R})$ is a partially ordered linear space, which is moreover Archimedean. However, $P(\mathbb{R})$ is not a Riesz space because $x\vee y$ does not exist for x and y incomparable, which obviously exist in $P(\mathbb{R})$, e.g. x and y with $x(t) = 1$ and $y(t) = t$ for all $t \in \mathbb{R}$ are incomparable.

(d) $C(X)$ is the linear space of all continuous real valued functions on a topological space X with pointwise linear operations and pointwise partial ordering.
$C(X)$ is an Archimedean Riesz space, but in general not Dedekind complete. If $C(X)$ is Dedekind complete then X is called extremally disconnected.

(e) (cf. e.g. Luxemburg and Zaanen [1971, ex.11.2(ix)]).
If H is a Hilbert space over the complex numbers, with inner product $(.,.)$, then by $\mathcal{H}$ we denote the real linear space of all bounded Hermitean operators on H, provided with the partial ordering given by $S \leqslant T$ for $S,T \in \mathcal{H}$ if $(Sx,x) \leqslant (Tx,x)$ for all $x \in H$.
$\mathcal{H}$ is an Archimedean partially ordered linear space. $\mathcal{H}$ is a Riesz space only if the dimension of H is 0 or 1.

(f) (Compare e.g. Aliprantis and Burkinshaw [1978, ex. 2.13 (2)]).
If (X,Γ,μ) is a measure space, i.e. a non-empty set X and a σ-field Γ of subsets of X on which is defined a non-negative countably additive measure μ, then let $M(X,\mu)$ be the linear space of all real measurable functions on X. If we provide $M(X,\mu)$ with the partial ordering given by $x \leqslant y$ if $x(t) \leqslant y(t)$ for all $t \in X$, then $M(X,\mu,\leqslant)$ is a Dedekind σ-complete Riesz space.

(g) (cf. e.g. Luxemburg and Zaanen [1971, ex. 11.2(v)]). If in $M(X,\mu)$ from example (f) x is called equivalent with y ($x \sim y$) if $x = y$ μ-almost everywhere, then $\sim$ is an equivalence relation on $M(X,\mu)$. The linear space $\mathcal{M}(X,\mu)$ of all equivalence classes $[x]$ in $M(X,\mu)$ (natural algebraic operations) can be provided with a partial ordering given by $[x] \leqslant [y]$ if $x \leqslant y$ μ-almost everywhere. $\mathcal{M}(X,\mu,\leqslant)$ is a Dedekind complete Riesz space.

(h) s is the linear space of all sequences of real numbers. With pointwise partial ordering s is a Dedekind complete Riesz space.

(i) b is the linear space of all bounded sequences of real numbers. With pointwise partial ordering b is a Dedekind complete Riesz space.

(j) c is the linear space of all sequences of real numbers which converge. With pointwise partial ordering c is an Archimedean Riesz space, which is not Dedekind complete, because if $A = \{(1,0,0,0,\ldots),(1,0,1,0,0,\ldots),(1,0,1,0,1,0,0,\ldots), \ldots\}$, then $A \leq (1,1,1,\ldots)$. however $\sup A$ does not exist.

(k) c_0 is the linear space of all sequences of real numbers which converge to 0. With pointwise partial ordering c_0 is a Dedekind complete Riesz space.

(l) c_{00} is the linear space of all sequences of real numbers which are eventually 0. With pointwise partial ordering c_{00} is a Dedekind complete Riesz space.

(m) FR is the linear space of all sequences of real numbers which have a finite range. With pointwise partial ordering FR is an Archimedean Riesz space which is not Dedekind complete, because if $A = \{(1,0,0,0,\ldots),(1,\frac{1}{2},0,0,0,\ldots),(1,\frac{1}{2},\frac{1}{3},0,0,..),\ldots\}$ then $A \leq (1,1,1,\ldots)$, however $\sup A$ does not exist.

3. Elementary properties of Riesz spaces

In this section some abbreviations are given, most of which are commonly used, further some elementary properties are derived.

For an element x of a Riesz space E the positive part x^+ of x is defined by $x^+ = x \vee 0$, the negative part x^- of x by $x^- = (-x) \vee 0$ and the absolute value $|x|$ of x by $|x| = x \vee (-x)$.
x is said to be orthogonal to y, or disjoint to y, in formula $x \perp y$, if $|x| \wedge |y| = 0$. The orthogonal complement $X^\perp$ of a subset X of E is defined by $X^\perp = \{y \in E; y \perp x \text{ for all } x \in X\}$; $\{x\}^\perp$ is abbreviated to $x^\perp$. A subset X of E and a subset Y of E are said to be orthogonal, or disjoint, in formula $X \perp Y$ if $x \perp y$ for all $x \in X$ and $y \in Y$. $\{x\} \perp Y$ is abbreviated to $x \perp Y$.

3.1. Definition. *A subset* P *of a Riesz space* E *is called a polar of* E *if* $P = P^{\perp\perp}$.

It is known that for every subset X of a Riesz space E the equality $X^{\perp} = X^{\perp\perp\perp}$ holds (cf. e.g. Luxemburg and Zaanen [1971, thm 19.2(ii)]). This implies that every subset of the form $X^{\perp}$ (for some $X \subset E$) is a polar of E; it is evident that conversely every polar is of this form. Every polar of E is a linear subspace of E (cf. e.g. Luxemburg and Zaanen [1971, thm 14.2]).

For subsets X and Y of a Riesz space E we use $X^+ = \{x^+; x \in X\}$, $X^- = \{x^-; x \in X\}$ $|X| = \{|x|; x \in X\}$, $X + Y = \{x + y; x \in X, y \in Y\}$, $X - Y = \{x - y; x \in X, y \in Y\}$, $X \vee Y = \{x \vee y; x \in X, y \in Y\}$ and $X \wedge Y = \{x \wedge y; x \in X, y \in Y\}$. We abbreviate $\{x\} + Y$ to $x + Y$, similar abbreviations are made in the other cases. For $\lambda \in \mathbb{R}$ we denote $\{\lambda x; x \in X\}$ by λX. In every Riesz space E the equality $E = E^+ - E^+$ holds (cf. e.g. Schaefer [1974, p. 58]). E^+ is called the positive cone of E; the elements of E^+ are called the positive elements of E.

3.2. Proposition. *For* x *and* y *positive elements of a Riesz space* E *the inclusion* $x^{\perp\perp} + y^{\perp\perp} \subset (x + y)^{\perp\perp}$ *holds.*

Proof: If $x + y \perp s$ for some $s \in E$ then from $0 \leq x$, $0 \leq y$ it follows that $x \perp s$ and $y \perp s$. But then also $u \perp s$ and $v \perp s$ for all $u \in x^{\perp\perp}$ and $v \in y^{\perp\perp}$, hence $u + v \perp s$, which implies $u + v \in (x + y)^{\perp\perp}$.

3.3. Theorem. (cf. e.g. Aliprantis and Burkinshaw [1978, thm. 1.1], Luxemburg and Zaanen [1971, cor. 12.3] and Schaefer [1974, II, prop. 1.4, cor. 1, cor. 2]).

For x,y,z *elements of a Riesz space* E *we have*

(a) $x = x^+ - x^-$, $|x| = x^+ + x^-$, $x^+ \wedge x^- = 0$

(b) $x \vee y = -((-x)\wedge(-y))$, $x \wedge y = -((-x)\vee(-y))$

(c) $x + (y \vee z) = (x + y)\vee(x + z)$, $x + (y \wedge z) = (x + y)\wedge(x + z)$

(d) $\lambda(x \vee y) = (\lambda x)\vee(\lambda y)$ *for all* $\lambda \in \mathbb{R}^+$; $|\lambda x| = |\lambda||x|$ *for all* $\lambda \in \mathbb{R}$

(e) $(x \vee y)\vee z = x \vee(y \vee z)$, $(x \wedge y)\wedge z = x \wedge(y \wedge z)$

(f) $x + y = x \vee y + x \wedge y$, $|x - y| = x \vee y - x \wedge y$

(g) $(x - y)^+ = x$ *and* $(x - y)^- = y$ *if* $x \wedge y = 0$

(h) $x \perp y$ *if and only if* $|x + y| = |x - y|$

(i) (Birkhoff's identity) $|x \vee z - y \vee z| + |x \wedge z - y \wedge z| = |x - y|$

(j) $(x + y)\wedge z \leq (x \wedge z) + (y \wedge z)$ *if* $x,y,z \in E^+$

(k) $x \leqslant y$ *is equivalent to* $x^+ \leqslant y^+$ *&* $y^- \leqslant x^-$

(l) $||x| - |y|| \leqslant |x + y| \leqslant |x| + |y|$, $(x + y)^+ \leqslant x^+ + y^+$, $(x + y)^- \leqslant x^- + y^-$

(m) *if* $x \perp y$ *then* $|x + y| = |x| + |y|$, $(x + y)^+ = x^+ + y^+$, $(x + y)^- = x^- + y^-$

(n) $0 \leqslant x \wedge y \leqslant x \vee y \leqslant x + y$ *if* $x, y \in E^+$

The following theorem is frequently used in Riesz space theory.

3.4. Theorem. (compare e.g. Luxemburg and Zaanen [1971, cor. 15.6])

(a) (Dominated decomposition property) If $x_1, \ldots, x_n$ *and* y *are positive elements of a Riesz space* E *such that* $y \leqslant x_1 + \ldots + x_n$ *holds, then there exist* $y_1, \ldots, y_n$ *in* E^+ *such that* $y = y_1 + \ldots + y_n$ *and* $y_i \leqslant x_i$ *for all* $i = 1, \ldots, n$.

(b) (Riesz interpolation property) If $x, y, z, x_1, \ldots, x_n$ *are positive elements of a Riesz space* E *such that* $x = y + z = x_1 + \ldots + x_n$ *then there exist* $y_1, \ldots, y_n, z_1, \ldots, z_n$ *in* E^+ *such that* $y = y_1 + \ldots + y_n$ *and* $z = z_1 + \ldots + z_n$ *and* $x_i = y_i + z_i$ *for all* $i = 1, \ldots, n$.

3.5. Theorem. (cf. e.g. Schaefer [1974, II thm 1.5])

If X *is a non-empty subset of a Riesz space* E *such that* $\sup X$ *exists in* E, *then for every* $x \in E$ *also* $\sup(x \wedge X)$ *exists in* E *and the equality* $\sup(x \wedge X) = x \wedge \sup X$ *holds.*

Now we collect some notions of extreme importance in Riesz space theory.

3.6. Definition. *If* E *is a Riesz space, then*

(a) a linear subspace R *of* E *is called a Riesz subspace of* E *if for* $x, y \in R$ *holds that* $x \wedge y \in R$.

(b) a linear subspace J *of* E *is called an (order) ideal of* E *if* $|x| \leqslant |y|$ *for* $x \in E$ *and* $y \in J$ *implies* $x \in J$.

(c) an ideal B *of* E *is called a band of* E *if the following holds: if* X *is a subset of* B *such that* $\sup X$ *exists, then* $\sup X \in B$.

(d) an ideal J *of* E *is called a* σ*-band if for every countable subset* X *of* J *for which* $\sup X$ *exists holds that* $\sup X \in J$.

(e) an ideal J *of* E *is called a principal ideal if there exists a* $z \in E$ *such that* $J = \{x \in E;\ |x| \leqslant \lambda z \text{ for some } \lambda \in \mathbb{R}\}$.

(f) a band B *of* E *is called a principal band if there exists a* $z \in E$ *such that* $B = z^{\perp\perp}$.

(g) *a band* B *of* E *is called a projection band if* $B + B^{\perp} = E$.

(h) *a band* B *of* E *which is a principal band and a projection band is called a principal projection band.*

For an element z of a Riesz space E the ideal $\{x \in E; |x| \leq \lambda z$ for some $\lambda \in \mathbb{R}\}$ is called the principal ideal generated by z, and is denoted by I_z. The band $z^{\perp\perp}$ is called the principal band generated by z, and is denoted by B_z.

If z is an element of a Riesz space E, then the set IS(z) of all infinitely smalls with respect to z (cf. sect. 2) is an ideal of E, because if $x,y \in IS(z)$ and $\lambda,\mu \in \mathbb{R}$ then for all $n \in \mathbb{N}$ we have $n|x| \leq z$ and $n|y| \leq z$ hence, for all $n \in \mathbb{N}, n|\lambda||x| \leq z$ and $n|\mu||y| \leq z$, so $n|\lambda||x| + n|\mu||y| \leq 2z$, which implies that $n|\lambda x + \mu y| \leq n|\lambda||x| + n|\mu||y| \leq z$ holds for all $n \in \mathbb{N}$, hence $\lambda x + \mu y \in IS(z)$. If $x \in IS(z)$ and $|y| \leq |x|$, then also $n|y| \leq z$ for all $n \in \mathbb{N}$, hence $y \in IS(z)$. Also IS(E) is an ideal in E, which can be proved likewise.

Note that for a Riesz subspace R of a Riesz space E also $x \vee y \in R$ whenever x and y are in R. Hence, any Riesz subspace R of a Riesz space E, with the linear space structure and the order structure inherited from E, is a Riesz space by itself. Note further that every ideal is a Riesz subspace.

It follows from Luxemburg and Zaanen [1971, thm 19.2(i)] that every polar of a Riesz space E is a band; the converse implication holds under the additional assumption that E is Archimedean (cf. Luxemburg and Zaanen [1971, thm 22.3]).

As an immediate consequence of the definitions we have (cf. Luxemburg and Zaanen [1971, thm 17.4]).

3.7. Theorem. *Any arbitrary non-empty set-theoretic intersection of Riesz subspaces (or ideals, or bands, or polars) is a Riesz subspace (or an ideal, or a band, or a polar).*

By $\mathbf{L}(E)$, respectively $\mathbf{R}(E)$, $\mathbf{I}(E)$, $\mathbf{B}(E)$, $\mathbf{P}(E)$ we denote the set of all linear subspaces, respectively Riesz subspaces, ideals, bands, polars of E, each partially ordered by inclusion.

As a consequence of thm 3.7 all these sets form lattices under their ordering. For, the infimum fo two elements is the intersection of these elements, the supremum of two elements is the intersection of all linear subspaces, respectively Riesz subspaces, ideals, bands, polars of E which contain these two elements.

It is also an immediate consequence of thm 3.7 that all these lattices are complete, moreover $\mathbf{I}(E)$, $\mathbf{B}(E)$ and $\mathbf{P}(E)$ are distributive (cf. Schaefer [1974, II prop. 2.3] for $\mathbf{I}(E)$, Luxemburg and Zaanen [1971, thm 22.6] for $\mathbf{B}(E)$, and e.g. Bernau [1965 a, thm 1] for $\mathbf{P}(E)$); $\mathbf{L}(E)$ and $\mathbf{R}(E)$ are not distributive in general.

We give a simple example for $\mathbf{R}(E)$: Take $E = \mathbb{R}^2$, $R = \{(x_1,x_2) \in E;\ x_1 = x_2\}$, $S = \{(x_1,0) \in E;\ x_1 \in \mathbb{R}\}$, $T = \{(0,x_2) \in E;\ x_2 \in \mathbb{R}\}$ then $(R\nabla S)\Delta(R\nabla T) = E$, but $R\nabla(S\Delta T) = R$, if supremum and infimum in $\mathbf{R}(E)$ are denoted by ∇ and Δ respectively. $\mathbf{P}(E)$ is a complete Boolean algebra (cf. Sik [1956] or Bernau [1956a]); $\mathbf{I}(E)$ and $\mathbf{B}(E)$ in general are not.

3.8. Definition. *An ideal* J *of an Archimedean Riesz space* E *is called order dense if for all* $f \in E$ *with* $0 < f$ *there exists a* $g \in J$ *with* $0 < g \leqslant f$.

In some representation theories of Riesz spaces the following two notions play an important role. (cf. [Schaefer, III def. 2.1, II def. 3.2]).

3.9. Definition. *An ideal* J *of a Riesz space* E *is called a prime ideal if* $x \in E, y \in E$ *and* $x\Delta y \in J$ *imply* $x \in J$ *or* $y \in J$.

3.10. Definition. *An ideal* M *of a Riesz space* E *is called a maximal ideal if* $M \neq E$ *and there is no ideal in* E *properly between* M *and* E *(i.e. any ideal* J *such that* $M \subset J \subset E$ *holds, satisfies either* $J = M$ *or* $J = E$*).*

3.11. Definition. (cf. A.L. Peressini [1967, chap II, prop. 5.13b]) *A Riesz space* E *is called countably bounded if* E^+ *contains a countable subset* C *with the property that for each* $x \in E^+$ *there exist* $e \in C$ *and* $\lambda \in \mathbb{R}$ *such that* $x \leqslant \lambda e$. E *is called bounded if there exists an* $e \in E$ *such that for each* $x \in E^+$ *there exists a* $\lambda \in \mathbb{R}$ *such that* $x \leqslant \lambda e$. *In the last case* e *is called a strong order unit of* E.

It is evident that every bounded Riesz space is countably bounded; the converse does not hold, for the Riesz space c_{00} of all sequences of real numbers with only a finite number of components not equal to 0, is countably bounded (let C be the subset of c_{00} whose elements are (1,0,0,0,..), (2,2,0,0,..), (3,3,3,0,...), ...), but c_{00} is not bounded.

In a bounded Riesz space E the order unit e has the property that $e^{\perp} = \{0\}$, for if $x \perp e$, then, if $|x| \leq \lambda e$, we have $|x| \wedge \lambda e = 0$, which implies $x = 0$.

3.12. Definition. *A Riesz space* E *is called weakly bounded if there exists an* $e \in E^{+}$ *with the property that* $e^{\perp} = \{0\}$. *In that case* e *is called a weak order unit of* E.

It is evident that every bounded Riesz space is weakly bounded and every strong order unit is a weak order unit. The converse does not hold because the Riesz space E of all sequences of real numbers has a weak order unit $e = (1,1,1,\ldots)$, but no strong order unit.
The foregoing two examples can serve to demonstrate that a Riesz space E can be countably bounded without being weakly bounded, and weakly bounded without being countably bounded.

The following theorem gives an important characterization of bounded Archimedean Riesz spaces.

3.13. Theorem (cf. e.g. Luxemburg and Zaanen [1971, thm 27.6]).
The intersection of all maximal ideals of a bounded Archimedean Riesz space consists of the zero element only.

3.14. Definition. *A Riesz space* E *is said to have the projection property (abbreviated to* PP*) if every band of* E *is a projection band.* E *is said to have the principal property (abbreviated to* PPP*) if every principal band of* E *is a (principal) projection band.*

Now we can state an important theorem for Riesz spaces.

3.15. Theorem (cf. Luxemburg and Zaanen [1971, thm 25.1])
With obvious notational abbreviations the following implications hold in any Riesz space E.

$$\begin{array}{ccccc} & \nearrow & \text{Ded. } \sigma\text{-complete} & \searrow & \\ \text{Ded. compl.} & & & & \text{PPP} \Rightarrow \text{Arch.} \\ & \searrow & \text{PP} & \nearrow & \end{array}$$

No implication in the converse direction holds; further E *can have* PP *without being Dedekind* σ*-complete and conversely; Dedekind* σ*-completeness and* PP *together imply Dedekind completeness.*

Finally, we discuss briefly the notion of lateral completeness.

3.16. Definition. *A Riesz space* E *is called (conditionally) lateral complete if for every (bounded) set* D *in* E *of pairwise disjoint elements* sup D *exists in* E.

We remark that the notion of lateral completeness was already defined in Nakano [1950] for Dedekind complete Riesz spaces.
A fundamental breakthrough was achieved by Veksler and Geiler [1972], who proved that every Archimedean conditionally lateral complete Riesz space has PP. Futher contributions are by Aliprantis and Burkinshaw [1977], Bernau [1966], [1975], [1976], Bleier [1976], Conrad [1969], Fremlin [1972], Jakubik [1975], [1978] and Wickstead [1979].

For a Riesz space E lateral completeness and being Archimedean are independent properties, for ($\mathbb{R}^2$, lex) is lateral complete but not Archimedean, $C[0,1]$ is Archimedean but not lateral complete.
For an Archimedean Riesz space E lateral completeness and Dedekind completeness are independent properties, because the Riesz space of all bounded sequences of real numbers is Dedekind complete, but not lateral complete. The Riesz space E of all real functions on $\mathbb{R}$ which are right locally constant in every $t \in \mathbb{R}$, (i.e. $x \in E$ if for all $t \in \mathbb{R}$ there exists an $\varepsilon > 0$ such that x is constant in $[t, t+\varepsilon)$) is an example of a lateral complete Riesz space, which is not Dedekind complete. Aliprantis and Burkinshaw [1978, ex. 23.30] and Wickstead [1979] give examples of such

a Riesz space; especially the example in Aliprantis and Burkinshaw [1978] is rather complicated.

3.17. Proposition. *Every Dedekind complete Riesz space* E *is conditionally lateral complete.*
Proof: Evident

3.18. Proposition. *Every lateral complete Riesz space* E *contains weak order units.*
Proof: Let S be the set of all subsets X of E of pairwise disjoint elements. $S \neq \phi$, for $\{0\} \in S$. We suppose S to be ordered by inclusion. Application of Zorn's lemma gives that there exists a maximal element M in S. Now e: = sup M is a weak order unit of E, because $e \perp x$ for some $x \neq 0$ would imply that S could be enlarged with $|x|$, contradiction.

3.19. Definition. *A Riesz space which is Dedekind complete and at the same time lateral complete is called universally complete or inextensible.*

Universally complete Riesz spaces are very important in Riesz space theory; every Archimedean Riesz space admits a unique universal completion (cf. e.g. Conrad [1971]).

4. Linear operators

In this section, E and F are arbirary Riesz spaces. The zero operator from E to F will be denoted by 0. The identity operator on E will be denoted by I_E, or simply by I. For a linear operator T from E to F the nullspace N(T) is defined by $N(T) = \{x \in E; Tx = 0\}$. A linear operator T from E to F is called positive, in formula $T \geq 0$, if $T(E^+) \subset F^+$.
By $\mathcal{L}(E,F)$ we denote the linear space of all linear operators from E to F, provided with the partial ordering $S \leq T$ if and only if $T - S \geq 0$, the socalled operator ordering.
In the case E = F the space $\mathcal{L}(E,F)$ can be given moreover an algebra structure by composition. In that case $\mathcal{L}(E,F)$ is a partially ordered algebra, i.e. an algebra which is at the same time a partially ordered linear space, such that the product of two positive elements is posi-

tive again. A linear operator $T \in \mathcal{L}(E,F)$ is called a Jordan operator if T is the difference of two positive linear operators. The class $\mathcal{L}^J(E,F)$ of Jordan operators is a linear subspace of $\mathcal{L}(E,F)$. $\mathcal{L}^J(E,F)$ is a partially ordered linear space under the operator ordering and in the case $E = F$ a partially ordered algebra. A linear operator T from E to F is called order bounded if the image of every order bounded subset of E under T is an order bounded subset of F. Also the class $\mathcal{L}^b(E,F)$ of all order bounded linear operators is a linear subspace of $\mathcal{L}(E,F)$ and a partially ordered linear space under the operator ordering and in the case $E = F$ a partially ordered algebra under composition.

4.1. Lemma. *A mapping* t *from* E^+ *to* F *which satisfies*

(a) $t(x + y) = t(x) + t(y)$ *for all* $x, y \in E^+$

(b) $t(\lambda x) = \lambda t(x)$ *for all* $x \in E^+$ *and* $\lambda \geq 0$ *admits a unique extension to a linear operator* T *from* E *to* F. *If moreover the range of* t *is contained in* F^+, *then* T *is positive.*

Proof: Let $T\colon E \to F$ be defined by $Tx = t(x^+) - t(x^-)$, then T is a linear operator, which is an extension of t. If S is another linear operator which is an extension of t, then $Sx = Sx^+ - Sx^- = t(x^+) - t(x^-) = Tx$, hence $S = T$. If $t(E^+) \subset F^+$ then $T(E^+) \subset F^+$, hence T is positive.

4.2. Theorem. (cf. e.g. Schaefer [1974, IV prop. 1.2]). *For a linear operator* T *from a Riesz space* E *to a Riesz space* F *for the assertions*

(a) T^+ *exists in* $\mathcal{L}(E,F)$

(b) T *is a Jordan operator*

(c) T *is order bounded*

holds that (a) $\Rightarrow$ (b), (b) $\Rightarrow$ (c). *In the case* F *is Dedekind complete all assertions are equivalent.*

4.3. Theorem. (Riesz-Kantorovič, cf. e.g. Schaefer [1974, IV prop. 1.3]). *If* E *and* F *are Riesz spaces and* F *is Dedekind complete, then* $\mathcal{L}^b(E,F)$ *is a Dedekind complete Riesz space, in which* $\sup \mathcal{T}$ *for* $\mathcal{T} \subset \mathcal{L}^b(E,F)$ *such that* $\mathcal{T} \leq$ *is given by* $(\sup \mathcal{T})(x) = \sup \{T_1x_1 + \ldots + T_nx_n;\ \{T_1, \ldots, T_n\}$ finite subset of $\mathcal{T}$, $x_1, \ldots, x_n \geq 0$ and $x = x_1 + \ldots + x_n\}$ $(x \geq 0)$

Krengel [1963] gives an example of a Jordan operator T from $C[-1, 1]$ to $C[-1,1]$ for which T^+ does not exist. No answer seems to have been given in the literature to the question whether PP for F is already sufficient to garantee that $\mathfrak{L}^J(E,F)$ is a Riesz space.
The following example shows that PP is not sufficient.

<u>4.4. Example.</u> If E is the Riesz subspace of s generated by $e = (1,1,1,\ldots)$ and c_{00}, then E, with the Riesz space structure induced by s, is an Archimedean Riesz space, namely the Riesz space of all eventually constant real sequences. Note that e is a strong order unit of E.
If for all $m \in \mathbb{N}$ the element e_m of E is defined by $e_m(n) = \delta_{m,n}$ for all $n \in \mathbb{N}$ (where δ is the Kronecker function), then the elements $e, e_1, e_2, \ldots$ form a basis B of E. Let a linear operator T from E to FR be given by $Te = 0$, $Te_1 = e_1$ and $Te_n = \frac{1}{n} e_n - \frac{1}{n-1} e_{n-1}$ for all $n \geq 2$.
For an element $y \in FR$ we write y_n or $(y)_n$ in stead of $y(n)$ $(n \in \mathbb{N})$.
Let $0 \leq x \in E$ be arbitrary, say

$$x = \lambda e + \sum_{i=1}^{\infty} \lambda_i e_i, \text{ then } \lambda \geq 0 \text{ and } \lambda_i \geq -\lambda \text{ for all } i \in \mathbb{N}; \text{ almost}$$

all λ_i are equal to 0.

T is a Jordan operator, because $T = I - (I - T)$, where I is the canonical embedding operator from E into FR, and $I - T$ is positive because

$$(I - T)x = \lambda e + \sum_{i=2}^{\infty} \lambda_i e_i - \sum_{i=2}^{\infty} \lambda_i Te_i = \lambda e + \sum_{i=2}^{\infty} (\lambda_i e_i + \frac{\lambda_i}{i-1} e_{i-1} - \frac{\lambda_i}{i} e_i),$$

so for $n \in \mathbb{N}$ we have $((I - T)x)_n = \lambda + \lambda_n(\frac{n-1}{n}) + \lambda_{n+1}(\frac{1}{n}) \geq \lambda - \lambda(\frac{n-1}{n}) - \lambda(\frac{1}{n}) = 0.$

Suppose T^+ exists in $\mathfrak{L}^J(E,FR)$, then for all $n \in \mathbb{N}$ we have $T^+ e_n \geq (Te_n)^+ = \frac{1}{n} e_n$, hence $(T^+ e_n)_m \geq \frac{1}{n} \delta_{n,m}$ for all $m \in \mathbb{N}$.

If for certain pair $(N,M) \in \mathbb{N} \times \mathbb{N}$ holds that $N \neq M$ and $(T^+ e_N)_M > 0$, then let $S \in \mathfrak{L}^J(E,FR)$ be defined by $(Se)_m = (T^+ e)_m$ for all $m \neq M$, $(Se)_M = (T^+ e)_M - (T^+ e_N)_M$, $(Se_n)_m = (T^+ e_n)_m$ for all $(n,m) \in \mathbb{N} \times \mathbb{N}$ such that $(n,m) \neq (N,M)$ and $(Se_N)_M = 0$.
$S \geq 0$ because for all $m \neq M$ we have $(Sx)_m = (T^+ x)_m \geq 0$ and

$$(Sx)_M = \lambda(Se)_M + \sum_{i=1}^{\infty} \lambda_i(Se_i)_M = \lambda(Se)_M + \sum_{\substack{i=1\\ i\neq N}}^{\infty} \lambda_i(Se_i)_M + \lambda_N(Se_N)_M =$$

$$= \lambda(T^+e)_M - \lambda(T^+e_N)_M + \sum_{\substack{i=1\\ i\neq N}}^{\infty} \lambda_i(T^+e_i)_M = (T^+(\lambda e - \sum_{\substack{i=1\\ i\neq N}}^{\infty} \lambda_i T^+ e_i - \lambda(T^+e_N))_M \geq 0$$

$S \geq T$ because for all $m \neq M$ we have $(Sx)_m = (T^+x)_m \geq (Tx)_m$ and

$$(Sx)_M = \lambda(Se)_M + \sum_{i=1}^{\infty} \lambda_i(Se_i)_M = \lambda(Se)_M + \lambda_M(Se_M)_M + \sum_{\substack{i=1\\ i\neq M}}^{\infty} \lambda_i(Se_i)_M$$

$$= \lambda(Se_M)_M + \lambda(S(e - e_M))_M + \lambda_M(Se_M)_M + \sum_{\substack{i=1\\ i\neq M}}^{\infty} \lambda_i(Se_i)_M \geq \lambda(Se_M)_M + \lambda_M(Se_M)_M \geq$$

$$\frac{\lambda + \lambda_M}{M} \geq \frac{-\lambda_{M+1} + \lambda_M}{M} = (Tx)_M.$$

$S \leq T^+$ because for $m \neq M$ we have $(Sx)_m = (T^+x)_m$ and

$$(Sx)_M = \lambda(Se)_M + \sum_{i=1}^{\infty} \lambda_i(Se_i)_M = \lambda(T^+e)_M - \lambda(T^+e_N)_M + \sum_{\substack{i=1\\ i\neq N}}^{\infty} \lambda_i(Se_i)_M + \lambda_N(Se_N)_M$$

$$= \lambda(T^+e)_M - \lambda(T^+e_N)_M + \sum_{\substack{i=1\\ i\neq N}}^{\infty} \lambda_i(T^+e_i)_M$$

$$\leq \lambda(T^+e)_M + \lambda_N(T^+e_N)_M + \sum_{\substack{i=1\\ i\neq N}}^{\infty} \lambda_i(T^+e_i)_M = T^+x.$$

$S < T^+$, because $0 = (Se_N)_M < (T^+e_N)_M$.
It follows that for all $n,m \in \mathbb{N}$ that $(T^+e_n)_m = 0$ if $n \neq m$.
If for certain $K \in \mathbb{N}$ we have that $(T^+e_K)_K > \frac{1}{K}$, then let $R \in \mathcal{L}^J(E,FR)$ be such that $(Re)_n = (T^+e)_n$ for $n \neq K$, $(Re)_K = \frac{1}{K}$, $Re_n = T^+e_n$ for all $n \neq K$ and $Re_K = \frac{1}{K} e_K$.
$R \geq 0$, because $(Rx)_n = (T^+x)_n$ for all $n \neq K$ and

$$(Rx)_K = \lambda(Re)_K + \sum_{i=1}^{\infty} \lambda_i(Re_i)_K = \lambda(Re)_K + \sum_{\substack{i=1\\ i\neq K}}^{\infty} \lambda_i(Re_i)_K + \lambda_K(Re_K)_K$$

$$= \lambda(Re)_K + \sum_{\substack{i=1\\ i\neq K}}^{\infty} \lambda_i(T^+e_n)_K + \lambda_K(Re_K)_K = \frac{\lambda}{K} + \frac{\lambda_K}{K} \geq \frac{\lambda}{K} - \frac{\lambda}{K} = 0.$$

$R \geq T$ because $(Rx)_n = (T^+x)_n \geq (Tx)_n$ for $n \neq K$ and $(Rx)_K = \lambda(Re)_K + \sum_{i=1}^{\infty} \lambda_i(Re_i)_K =$

$$\lambda(Re_K)_K + \lambda(R(e - e_K))_K + \sum_{\substack{i=1\\ i\neq K}}^{\infty} \lambda_i(Re_i)_K + \lambda_K(Re_K)_K \geq \lambda(Re_K)_K + \lambda_K(Re_K)_K \geq$$

$$\frac{\lambda}{K} + \frac{\lambda_K}{K} \geq \frac{\lambda_K}{K} - \frac{\lambda_{K+1}}{K} = (Tx)_K .$$

$R \leq T^+$ because $(Rx)_n = (T^+x)_n$ for all $n \neq K$ and $(Rx)_K = \lambda(Re)_K + \sum_{i=1}^{\infty} \lambda_i(Re_i)_K =$

$$\frac{\lambda}{K} + \frac{\lambda_K}{K} \leq (\lambda + \lambda_K)(T^+e_K)_K \leq$$

$$\lambda(T^+e_K)_K + \lambda(T^+(e - e_K))_K + \lambda_K(T^+e_K)_K + \sum_{\substack{i=1\\ i\neq K}}^{\infty} \lambda_i(T^+e_i)_K =$$

$$\lambda(T^+e_K)_K + \lambda(T^+(e - e_K))_K + \sum_{i=1}^{\infty} \lambda_i(T^+e_i)_K = \lambda(T^+e)_K + \sum_{i=1}^{\infty} \lambda_i(T^+e_i)_K = (T^+x)_K$$

$R < T^+$ because $\frac{1}{K} = (Re_K)_K < (T^+e_K)_K$.

It follows that $(T^+e_n)_n = \frac{1}{n}$ for all $n \in \mathbb{N}$.

If for certain $P \in \mathbb{N}$ we have $(T^+e)_P > (T^+e_P)_P$, then let $W \in \mathcal{L}^J(E,FR)$ be such that $(We)_n = (T^+e)_n$ for all $n \neq P$, $(We)_P = \frac{1}{P}$ and $We_n = T^+e_n$ for all $n \in \mathbb{N}$.

$W \geq 0$ because for $n \neq P$ we have $(Wx)_n = \lambda(We)_n + \sum_{i=1}^{\infty} \lambda_i(We_i)_n =$

$(T^+x)_n \geq 0$ and $(Wx)_P = \lambda(We)_P + \sum_{i=1}^{\infty} \lambda_i(We_i)_P = \frac{\lambda}{P} + \sum_{i=1}^{\infty} \lambda_i(T^+e_i)_P =$

$$\frac{\lambda}{P} + \lambda_P(T^+e_P)_P = \frac{\lambda}{P} + \frac{\lambda_P}{P} \geq 0.$$

$W \geqslant T$ because $(Wx)_n = (T^+x)_n \geqslant (Tx)_n$ for all $n \neq P$ and

$$(Wx)_P = \lambda(We)_P + \sum_{i=1}^{\infty} \lambda_i (We_i)_P = \frac{\lambda}{P} + \frac{\lambda_P}{P} \geqslant \frac{-\lambda_{P+1} + \lambda_P}{P} = (Tx)_P \ .$$

$W \leqslant T^+$ because $(Wx)_n = (T^+x)_n$ for $n \neq P$ and $(Wx)_P = \lambda(We)_P + \sum_{i=1}^{\infty} \lambda_i (We_i)_P$

$$= \frac{\lambda}{P} + \sum_{i=1}^{\infty} \lambda_i (T^+e_i)_P \leqslant \lambda(T^+e)_P + \sum_{i=1}^{\infty} \lambda_i (T^+e_i)_P = (T^+x)_P.$$

$W < T^+$ because $\frac{1}{P} = (We)_P < (T^+e)_P$.

It follows that $(T^+e)_n = \frac{1}{n}$ for all $n \in \mathbb{N}$, but this is in contradiction with the finite range of T^+e, hence T^+ does not exist.

Note that FR is not Dedekind complete, but FR has PP (Aliprantis and Burkinshaw [1978, Ex. 2.13 (3)]).

Chapter II
SOME TYPES OF CONVERGENCE

In this chapter three types of convergence are given; some attention is paid to the relations between them and finally continuity of linear operators with respect to these types of convergence is defined.

5. Sequences in Riesz spaces

In this section E is an arbitrary Riesz space.
A sequence in E is a mapping f from $\mathbb{N}$ to E. A sequence f in E is called increasing, in formula $\uparrow f$, if $f(n) \leqslant f(n+1)$ holds for all $n \in \mathbb{N}$, and decreasing, in formula $\downarrow f$, if $f(n) \geqslant f(n+1)$ holds for all $n \in \mathbb{N}$.
We write $f\uparrow x$ if $\uparrow f$ and $\sup f(\mathbb{N}) = x$, $f\downarrow y$ if $\downarrow f$ and $\inf f(\mathbb{N}) = y$.
The class of all sequences in E is denoted by seq(E). On seq(E) we define a linear structure by $(f+g)(n) = f(n) + g(n)$, $(\lambda f)(n) = \lambda f(n)$ for all $n \in \mathbb{N}$, if $\lambda \in \mathbb{R}$ and $f,g \in$ seq(E).
A partial ordering on seq(E) is defined by $f \leqslant g$ if and only if $f(n) \leqslant g(n)$ for all $n \in \mathbb{N}$. (seq(E), $\leqslant$) is a Riesz space in which $(f\vee g)(n) = f(n)\vee g(n)$ for all $n \in \mathbb{N}$.
In the sequel (seq(E), $\leqslant$) is abbreviated to seq(E).

For $x \in E$ and $f \in$ seq(E) we define $x + f \in$ seq(E) by $(x+f)(n) = x + f(n)$ for all $n \in \mathbb{N}$.
The sequence in E with range $\{0\}$ is denoted by 0.
For $f \in$ seq(E) and $n \in \mathbb{N}$ we write occasionally f_n or $(f)_n$ instead of $f(n)$.

If f is a sequence in E and $\sigma: \mathbb{N} \to \mathbb{N}$ is a strictly increasing function, then the sequence $f\circ\sigma$ is called a subsequence of f.

By $\Pi(E)$ we denote the power set of E, i.e. the set $\{X; X \subset E\}$.

6. Order convergence

The first type of convergence we discuss is order convergence.

In this section E is an arbitrary Riesz space.

We define a mapping ${}^{o}L$, called order limit, from seq(E) to $\Pi(E)$ which assigns to $f \in$ seq(E) the element of $\Pi(E)$ consisting of all $x \in E$ such that there exists a $g \in$ seq(E) with the property that $|f - x| \leq g$ and $g \downarrow 0$.

In the case ${}^{o}Lf \neq \phi$, it is well known (cf. e.g. Luxemburg and Zaanen [1971, thm 16.1 (i)]) that there exists exactly one $x \in E$ such that ${}^{o}Lf = \{x\}$. In that case f is called order convergent, or more precisely, order convergent to x, and we write ${}^{o}Lf = x$.

6.1. Theorem (compare Luxemburg and Zaanen [1971, thm 16.1]).

For $f,g \in$ seq(E) *and* $x,y \in E$, $\lambda,\mu \in \mathbb{R}$ *it holds that*

(a) if $f\uparrow x$ *or* $f\downarrow x$ *then* ${}^{o}Lf = x$

(b) if $\uparrow f$ *or* $\downarrow f$ *and* ${}^{o}Lf = x$ *then* $f\uparrow x$ *or* $f\downarrow x$ *respectively*

(c) if ${}^{o}Lf = x$ *and* ${}^{o}Lg = y$, *then* ${}^{o}L(\lambda f + \mu g) = \lambda x + \mu y$

(d) if ${}^{o}Lf = x$ *and* ${}^{o}Lg = y$ *then* ${}^{o}L(f \vee g) = x \vee y$ *and* ${}^{o}L(f \wedge g) = x \wedge y$

(e) if f' *is a subsequence of* f *and* ${}^{o}Lf = x$ *then* ${}^{o}Lf' = x$

(f) if $0 \leq f \leq g$ *and* ${}^{o}Lg = 0$ *then* ${}^{o}Lf = 0$

From this theorem it follows that the class of all order convergent sequences is a Riesz subspace of seq(E), and the class of all sequences order convergent to 0 is an ideal of seq(E).

6.2. Theorem (compare Luxemburg and Zaanen [1971, exc. 16.10])

For a Riesz space E *the following assertions are equivalent*

(a) E *is Archimedean*

(b) if for $x \in E$, $\lambda \in \mathbb{R}$, $f \in$ seq(E) *and* $a \in$ seq($\mathbb{R}$) *holds that* ${}^{o}Lf = x$ *and* ${}^{o}La = \lambda$, *then* ${}^{o}L(af) = \lambda x$.

Proof: *(a)* $\rightarrow$ *(b)*: There exists a sequence g in E such that $|f - x| \leq g$ and $g \downarrow 0$. If $a_0 = \sup |a(\mathbb{N})|$, then we have for all $n \in \mathbb{N}$ that $0 \leq |a(n)f(n) - \lambda x| \leq |a(n)f(n) - a(n)x| + |a(n)x - \lambda x|$.

Further it holds that $|af - ax| \leq a_0 g$ and $a_0 g \downarrow 0$, hence by thm 6.1 *(f)* it follows that ${}^{o}L(|af - ax|) = 0$.

We also have that ${}^{o}L(|ax - \lambda x|) = 0$.

By thm 6.1 *(c)* it follows now that ${}^{o}L(|af - ax| + |ax - \lambda x|) = 0$. One more application of thm 6.1 *(f)* gives ${}^{o}L(|af - \lambda x|) = 0$, hence ${}^{o}L(af) = \lambda x$.

(b) $\rightarrow$ *(a)*: Suppose $nx \leq y$ for certain $x,y \in E^{+}$ and all $n \in \mathbb{N}$. If $f,g \in$ seq(E)

and $a \in \operatorname{seq}(\mathbb{R})$ are such that $f(n) = x$, $g(n) = y$ and $a(n) = \frac{1}{n}$ for all $n \in \mathbb{N}$, then ${}^{o}La = 0$, hence ${}^{o}L(ag) = 0y = 0$.
From $0 \leqslant f \leqslant ag$ it follows now by thm 6.1*(f)* that ${}^{o}Lf = 0$, hence $x = 0$.

7. Regulator convergence

In this section E is an arbitrary Riesz space. We define a mapping ${}^{r}L$, called regulator limit, from the Cartesian product $\operatorname{seq}(E) \times E^{+}$ to $\Pi(E)$ which assigns to $(f,u) \in \operatorname{seq}(E) \times E^{+}$ the element of $\Pi(E)$ consisting of all $x \in E$ such that for all $\varepsilon > 0$ there exists an $N_{\varepsilon} \in \mathbb{N}$ such that $|f(n) - x| \leqslant \varepsilon u$ holds for all $n \geqslant N_{\varepsilon}$.
${}^{r}L(f,u)$ is called the regulator limit of f with respect to regulator u.
For $\cup\{{}^{r}L(f,u);\ u \in E^{+}\}$ we write ${}^{r}Lf$.
If for some pair $(f,u) \in \operatorname{seq}(E) \times E^{+}$ there is an $x \in E$ such that $\{x\} = {}^{r}L(f,u)$, then we write $x = {}^{r}L(f,u)$.
$f \in \operatorname{seq}(E)$ is called regulator convergent if ${}^{r}Lf \neq \phi$.
It follows directly from the definitions that for all $u \in E^{+}$ holds that ${}^{r}L(0,u) = IS(u)$. If u is a strong order unit of E, then ${}^{r}L(0,u) = IS(E)$, because if $n|x| \leqslant y$ for some $x \in E$, $y \in E^{+}$ and all $n \in \mathbb{N}$, and $m \in \mathbb{N}$ is such that $y \leqslant mu$, then $n|x| \leqslant mu$ for all $n \in \mathbb{N}$, hence $n|x| \leqslant u$ for all $n \in \mathbb{N}$, so $x \in IS(u)$. Hence, $IS(u) = IS(E)$.

7.1. Theorem (compare Luxemburg and Zaanen [1971, thm 16.2 (ii)])
If x *and* y *are elements of a Riesz space* E *and if* $f,g \in \operatorname{seq}(E)$ *such that* $x \in {}^{r}Lf$ *and* $y \in {}^{r}Lg$, *then*
(a) $\lambda x + \mu y \in {}^{r}L(\lambda f + \mu g)$ *for all* $\lambda,\mu \in \mathbb{R}$.
(b) $x \vee y \in {}^{r}L(f \vee g)$ *and* $x \wedge y \in {}^{r}L(f \wedge g)$
(c) if f' *is a subsequence of* f *and* ${}^{r}Lf = x$ *then* ${}^{r}Lf' = x$
(d) if $0 \leqslant f \leqslant g$ *and* $0 \in {}^{r}Lg$ *then* $0 \in {}^{r}Lf$.
From this theorem it follows that the class of all regulator convergent sequences is a Riesz subspace of $\operatorname{seq}(E)$ and the class of all sequences which are regulator convergent to 0, is an ideal of $\operatorname{seq}(E)$.

7.2. Proposition. *If* $x \in E$, $u \in E^{+}$ *and* $f \in \operatorname{seq}(E)$ *such that* $x \in {}^{r}L(f,u)$ *then* ${}^{r}L(f,u) = x + IS(u)$.
Proof: if $z \in IS(u)$, then by definition $n|z| \leqslant u$ for all $n \in \mathbb{N}$.

$x \in {}^rL(f,u)$, hence for all $\varepsilon > 0$ there exists an $N_\varepsilon \in \mathbb{N}$ such that $|f(n) - x| \leq \varepsilon u$ holds for all $n \geq N_\varepsilon$. If $\varepsilon > 0$ then for all $n \in \mathbb{N}$ such that $n \geq \max(N_{\frac{1}{2}\varepsilon}, \text{entier}(\frac{1}{2\varepsilon}))$ we have that
$|f(n) - (x + z)| \leq |f(n) - x| + |z| \leq \frac{1}{2}\varepsilon u + \frac{1}{2}\varepsilon u = \varepsilon u$, hence $x + z \in {}^rL(f,u)$.
Conversely, if $y \in {}^rL(f,u)$, then for all $\varepsilon > 0$ there exists an $M_\varepsilon \in \mathbb{N}$ such that $|f(n) - y| \leq \varepsilon u$ holds for all $n \geq M_\varepsilon$.
Let $\varepsilon > 0$. For all $n \geq \max(N_{\frac{1}{2}\varepsilon}, M_{\frac{1}{2}\varepsilon})$ we have that
$|x - y| \leq |f(n) - x| + |f(n) - y| \leq \frac{1}{2}\varepsilon u + \frac{1}{2}\varepsilon u = \varepsilon u$, hence $x - y \in {}^rL(0,u) = IS(u)$.

7.3. Example. If $E = (\mathbb{R}^2, \text{lex})$, $u = (0,1)$ and $f \in \text{seq}(E)$ is such that $f(n) = (0,\frac{1}{n})$ for all $n \in \mathbb{N}$, then ${}^rL(f,u) = \{0\}$ because $0 \in {}^rL(f,u)$ and $IS(u) = \{0\}$. However $IS(E) = \{(0,\lambda);\ \lambda \in \mathbb{R}\}$.
Note that ${}^rL(f,(1,1)) = IS(E)$ because $(1,1)$ is a strong order unit of E.

7.4. Proposition. *If* $x \in E$ *and* $f \in \text{seq}(E)$ *such that* $x \in {}^rLf$, *then* ${}^rLf = x + IS(E)$.
Proof: if $z \in IS(E)$, then there exists a $y \in E$ such that $n|z| \leq y$ holds for all $n \in \mathbb{N}$. $x \in {}^rLf$ implies that there exists a $u \in E^+$ such that for all $\varepsilon > 0$ there exists an $N_\varepsilon \in \mathbb{N}$ such that $|f(n) - x| \leq \varepsilon u$ holds for all $n \geq N_\varepsilon$. If $\varepsilon > 0$ then for all $n \geq N_\varepsilon$ we have that
$|f(n) - (x + z)| \leq |f(n) - x| + |z| \leq \varepsilon u + \varepsilon y = \varepsilon(u + y)$, hence $x + z \in {}^rLf$.
Conversely, if $y \in {}^rLf$, then by thm 7.1.*(a)* we have $x - y \in {}^rL0$, hence $x - y \in IS(E)$.

7.5. Theorem. *A Riesz space* E *is Archimedean if and only if for* $x \in E$ *and* $f \in \text{seq}(E)$ *it follows from* $x \in {}^rLf$ *that* $x = {}^rLf$.
Proof: $\rightarrow$ if $x,y \in {}^rLf$, then $x - y \in {}^rL0 = IS(E) = \{0\}$, hence $x = y$.
$\leftarrow$ $0 \in {}^rL0$, hence $\{0\} = {}^rL0 = 0 + IS(E) = IS(E)$ by prop. 7.4.

7.6. Theorem (cf. e.g. Luxemburg and Zaanen [1971, thm 16.2 (i)]).
In an Archimedean Riesz space E *it follows from* $x = {}^rLf$ *for* $x \in E$ *and* $f \in \text{seq}(E)$ *that* $x = {}^oLf$.

If E is an Archimedean Riesz space and $f \in \text{seq}(E)$, $u, v \in E^+$, then ${}^rL(f,u)$ can be different from ${}^rL(f,v)$ if $u \neq v$. A simple example to demonstrate

this is $E = \mathbb{R}$, $f(n) = \frac{1}{n}$ for all $n \in \mathbb{N}$, $u = 1$, $v = 0$, then ${}^{r}L(f,u) = 0$, however ${}^{r}L(f,v) = \phi$.
Note that in any Riesz space E we have ${}^{r}L(f,0) \neq \phi$ for $f \in$ seq(E) if and only if f is eventually constant in E.

7.7. Definition (cf. e.g. Luxemburg and Zaanen [1971, def. 39.1, def. 39.3 and def. 42.1]).
If $u \in E^+$ *and* $f \in$ seq(E) *such that for all* $\varepsilon > 0$ *there exists an* $N_\varepsilon \in \mathbb{N}$ *such that for all* $n,m \geq N_\varepsilon$ *holds that* $|f(m) - f(n)| \leq \varepsilon u$, *then* f *is called a* u-*Cauchy sequence in* E.
$f \in$ seq(E) *is called a regulator Cauchy sequence in* E *if* f *is a* u-*Cauchy sequence in* E *for some* $u \in E^+$.
E *is called* u-*complete if for every* u-*Cauchy sequence* f *in* E *holds that* ${}^{r}L(f,u) \neq \phi$. E *is called regulator complete if* E *is* u-*complete for every* $u \in E^+$.

7.8. Theorem (cf. Luxemburg and Zaanen [1971, p. 281 (ii)]).
For every Riesz space E *the following assertions are equivalent*
(a) E *is Dedekind* σ-*complete*
(b) E *has* PPP *and* E *is regulator complete.*

7.9. Theorem (cf. Luxemburg and Zaanen [1971, p. 281 (ii)]).
For every Riesz space E *the following assertions are equivalent*
(a) E *is Dedekind complete*
(b) E *has* PP *and* E *is regulator complete.*

7.10. Theorem (cf. Luxemburg and Zaanen [1971, thm 43.1]).
The Riesz space C(X) *is regulator complete for any topological space* X.

8. Characteristic convergence

The third notion of convergence we define here is the notion of characteristic convergence, which seems to be new. In this section E is an arbitrary Riesz space.

Let ${}^cL$ be the mapping, called characteristic limit, from seq(E) to $\Pi(E)$, which assigns to $f \in$ seq(E) the element of $\Pi(E)$ consisting of all $x \in E$ such that there exists a (countable) subset $\{P_n;\ n \in \mathbb{N}\}$ of the Boolean algebra $\mathbb{P}(E)$ of all polars of E such that $f(n) - x \in P_n$ for all $n \in \mathbb{N}$, $P_{n+1} \subset P_n$ for all $n \in \mathbb{N}$ and $\cap\{P_n;\ n \in \mathbb{N}\} = \{0\}$. f is called characteristic convergent if ${}^cLf \neq \phi$. If ${}^cLf = \{x\}$ for some $x \in E$ then we shall write ${}^cLf = x$.

<u>8.1. Lemma.</u> *If* $\{P_n;\ n \in \mathbb{N}\}$ *and* $\{Q_n;\ n \in \mathbb{N}\}$ *are (countable) subsets of* $\mathbb{P}(E)$ *such that* $P_{n+1} \subset P_n$, $Q_{n+1} \subset Q_n$ *for all* $n \in \mathbb{N}$ *and* $\cap\{P_n;\ n \in \mathbb{N}\} = \cap\{P_n;\ n \in \mathbb{N}\} = \cap\{Q_n;\ n \in \mathbb{N}\} = \{0\}$, *then* $\cap\{(P_n + Q_n)^{\perp\perp};\ n \in \mathbb{N}\} = \{0\}$.

Proof: We shall denote suprema and infima in $\mathbb{P}(E)$ by ∇ and Δ respectively. Now we have for $P, Q \in P(E)$ that $P\Delta Q = P \cap Q$, $P\nabla Q = (P \cup Q)^{\perp\perp} = (P + Q)^{\perp\perp}$ (cf. e.g. Bernau [1965 a, proof of thm 1]).

Let $N \in \mathbb{N}$. It follows from Bigard, Keimel and Wolfenstein [1977, prop. 3.2.16] that $\cap\{(P_n + Q_n)^{\perp\perp};\ n \in \mathbb{N}\} = \cap\{(P_n + Q_n)^{\perp\perp};\ n \in \mathbb{N},\ n \geqslant N\} \subset \cap\{(P_N + Q_n)^{\perp\perp};\ n \in \mathbb{N}\} = \Delta\{P_N \nabla Q_n;\ n \in \mathbb{N}\} = P_N \nabla(\Delta\{Q_n;\ n \in \mathbb{N}\}) = P_N \nabla\{0\} = P_N$. It follows that $\cap\{(P_n + Q_n)^{\perp\perp};\ n \in \mathbb{N}\} \subset \cap\{P_N;\ N \in \mathbb{N}\} = \{0\}$, hence $\cap\{(P_n + Q_n)^{\perp\perp};\ n \in \mathbb{N}\} = \{0\}$.

<u>8.2. Theorem.</u> *If* $f \in$ seq(E) *is characteristic convergent then* ${}^cLf = x$ *for some* $x \in E$.

Proof: Suppose $x, y \in {}^cLf$, then there exist (countable) subsets $\{P_n;\ n \in \mathbb{N}\}$ and $\{Q_n;\ n \in \mathbb{N}\}$ of $\mathbb{P}(E)$ such that $f(n) - x \in P_n$, $f(n) - y \in Q_n$, $P_{n+1} \subset P_n$ and $Q_{n+1} \subset Q_n$ for all $n \in \mathbb{N}$ and $\cap\{P_n;\ n \in \mathbb{N}\} = \cap\{Q_n;\ n \in \mathbb{N}\} = \{0\}$. Now we have $x - y = f(n) - y - (f(n) - x) \in P_n + Q_n \subset (P_n + Q_n)^{\perp\perp}$ for all $n \in \mathbb{N}$. From the foregoing lemma it follows that $x = y$.

<u>8.3. Theorem.</u> *If* $x, y \in E$ *and* $f, g \in$ seq(E) *are such that* $x = {}^cLf$ *and* $y = {}^cLg$, *then*

(a) ${}^cL(\lambda f + \mu g) = \lambda x + \mu y$

(b) ${}^cL(f \vee g) = x \vee y$ *and* ${}^cL(f \wedge g) = x \wedge y$

(c) $x = {}^cLf'$ *for every subsequence* f' *of* f.

(d) ${}^cLg = 0$ *if* $0 \leqslant g \leqslant f$ *and* ${}^cLf = 0$.

Proof: Let $\{P_n;\ n \in \mathbb{N}\}$ be a (countable) subset of $\mathbf{P}(E)$ such that $f(n) - x \in P_n$, $g(n) - y \in Q_n$, $P_{n+1} \subset P_n$ and $Q_{n+1} \subset Q_n$ for all $n \in \mathbb{N}$, $\cap \{P_n;\ n \in \mathbb{N}\} = \cap \{Q_n;\ n \in \mathbb{N}\} = \{0\}$.

(a) $|\lambda f + \mu g - (\lambda x + \mu y)| \leqslant |\lambda(f - x)| + |\mu(g - y)|$, hence for all $n \in \mathbb{N}$ $|\lambda f(n) + \mu g(n) - (\lambda x + \mu y)| \in P_n + Q_n \subset (P_n + Q_n)^{\perp\perp}$. From $(P_{n+1} + Q_{n+1})^{\perp\perp} \subset (P_n + Q_n)^{\perp\perp}$ for all $n \in \mathbb{N}$ and lemma 8.1 it follows that ${}^{c}L(\lambda f + \mu g) = \lambda x + \mu y$

(b) From Birkhoff's identity (thm 3.3 *(i)*) and *(a)* it follows that for all $n \in \mathbb{N}$ we have $|(f(n) - g(n))^+ - (x - y)^+| \leqslant |f(n) - g(n) - (x - y)| \in (P_n + Q_n)^{\perp\perp}$, hence ${}^{c}L(f - g)^+ = (x - y)^+$. One more application of *(a)* gives ${}^{c}L((f - g)^+ + g) = (x - y)^+ + y$, hence ${}^{c}L(f \vee g) = x \vee y$. ${}^{c}L(f \wedge g) = x \wedge y$ follows similarly.

(c) If $\sigma\colon \mathbb{N} \to \mathbb{N}$ is a strictly increasing function, then $\cap \{P_{\sigma(n)};\ n \in \mathbb{N}\} = \{0\}$, $P_{\sigma(n+1)} \subset P_{\sigma(n)}$ for all $n \in \mathbb{N}$ and $(f \circ \sigma)(n) - x \in P_{\sigma(n)}$ for all $n \in \mathbb{N}$, hence ${}^{c}L(f \circ \sigma) = x$.

(d) For all $n \in \mathbb{N}$ we have $0 \leqslant g(n) \leqslant f(n) \in P_n$, hence ${}^{c}Lg = 0$.

It follows from this theorem that the class of all characteristic convergent sequences in E is a Riesz subspace of seq(E), moreover the class of all sequences which are characteristic convergent to 0 is an ideal of seq(E).

9. Comparison of convergences of sequences

In this section we compare the foregoing three types of convergence.

In this section E is an arbitrary Riesz space.

The relation between order convergence and regulator convergence for sequences has been studied in Luxemburg and Zaanen [1971, §16]. Their main results are the following.

9.1. Definition. *A Riesz space* E *is called order convergence stable if for any* $f \in \mathrm{seq}(E)$ *with* $f \downarrow 0$ *there exists a* $0 \leqslant a \in \mathrm{seq}(\mathbb{R})$*, such that* $\uparrow a$*,* $\{a(n);\ n \in \mathbb{N}\}$ *is not majorized and* ${}^{o}L(af) = 0$.

9.2. Theorem. *An Archimedean Riesz space* E *is order convergence stable if and only if every order convergent sequence in* E *is also regulator convergent.*

9.3. Theorem. *Every regulator convergent sequence* f *in an Archimedean Riesz space* E *is also order convergent, moreover* ${}^{r}Lf = {}^{o}Lf$.

9.4. Example. The condition that E is Archimedean cannot be omitted in the foregoing theorem, because in example 7.3 we have ${}^{r}L(f,u) = IS(E)$ for some $f \in$ seq(E) and $u \in E^{+}$, while IS(E) is not a singleton.

9.5. Theorem. *If for* $f \in$ seq(E) *and* $x,y \in E$ *holds that* ${}^{c}Lf = x$ *and* ${}^{o}Lf = y$ *then* $x = y$.

Proof: It is sufficient to show that if ${}^{c}Lf = 0$ and ${}^{o}Lf = x$ then $x = 0$.
There exists a $g \in$ seq(E) such that $|f - x| \leq g$ and $g \downarrow 0$, hence $f(n) \in [x - g(n), x + g(n)]$ for all $n \in \mathbb{N}$.
For all $n \in \mathbb{N}$ we have $[x - g(n+1), x + g(n+1)] \subset [x - g(n), x + g(n)]$ and $\cap [x - g(n), x + g(n)] = \{x\}$.
On the other hand $f(n) \in P_n$ with $P_n \in \mathbf{P}(E)$ such that $P_{n+1} \subset P_n$ for all $n \in \mathbb{N}$ and $\cap\{P_n;\ n \in \mathbb{N}\} = \{0\}$.
It follows now that $x = 0$.

9.6. Example. Not every characteristic convergent sequence f in a Riesz space E is automatically order convergent. If $E = C[0,1]$ and $f \in$ seq(E) is such that $(f(n))(t) = -n^2t + n$ if $t \in [0,n^{-1}]$ and $(f(n))(t) = 0$ if $t \in [n^{-1},1]$ for all $n \in \mathbb{N}$, then $\cap\{f(n)^{\perp\perp};\ n \in \mathbb{N}\} = \{0\}$, $f(n) \in f(n)^{\perp\perp}$ and $f(n+1)^{\perp\perp} \subset f(n)^{\perp\perp}$ for all $n \in \mathbb{N}$, hence ${}^{c}Lf = 0$.
But ${}^{o}Lf = \phi$ because $\{f(n);\ n \in \mathbb{N}\}$ is not bounded.

Next we show that regulator convergence and characteristic convergence do not imply each other.
If in an arbitrary Archimedean Riesz space $E \neq \{0\}$ we have $x > 0$, then the sequence f in E with $f(n) = n^{-1}x$ for all $n \in \mathbb{N}$ is regulator convergent to $\{0\}$ (regulator x).
Suppose that f is characteristic convergent, then f is characteristic convergent to $\{0\}$ by thms 9.3 and 9.5.
However, $f(n)^{\perp\perp} = (\frac{1}{n}x)^{\perp\perp} = x^{\perp\perp}$ for all $n \in \mathbb{N}$, hence f is not characteristic convergent to $\{0\}$, contradiction.

Conversely, if E is the (Dedekind complete) Riesz space of all sequences x of real numbers with componentwise linear operations and componentwise partial ordering, such that $|x|$ is bounded by a real multiple of $r = (1,2,3,\dots)$ and f is the sequence in E such that $f(n) = (1,2,3,\dots,n,0,0,0,\dots)$ for all $n \in \mathbb{N}$, then, if $g = f - r$, we have $f(n) - r \in g(n)^{\perp\perp}$, $g(n+1)^{\perp\perp} \subset g\,(n)^{\perp\perp}$ for all $n \in \mathbb{N}$ and $\cap\{g(n)^{\perp\perp};\ n \in \mathbb{N}\} = \{0\}$, hence ${}^{c}Lf = r$.
Suppose f is regulator convergent, then ${}^{r}Lf = r$ by thms 9.3 and 9.5. If u is the corresponding regulator, say $u \leq \lambda r$, then certainly λr is regulator, but then also r is regulator.
It follows that for all $\varepsilon > 0$ there exists a $N_\varepsilon \in \mathbb{N}$ such that for all $n \geq N_\varepsilon$ it holds that $|f(n) - r| \leq \varepsilon r$; however, if we take $\varepsilon = \frac{1}{2}$, then there does not exist a $N \in \mathbb{N}$ such that for all $n \geq N$ holds that $|f(n) - r| \leq \frac{1}{2}r$, because this would imply that $f(n) \geq \frac{1}{2}r$ for $n \geq N$, contradiction.
Hence, f is not regulator convergent.

From the foregoing it follows that order convergence does not imply characteristic convergence.

10. Some notions of ideals

In the sequel we need some carefully chosen notions of ideals, which are defined below.

In this section E is an arbitrary Riesz space.

For an element $x \in E$ we write $I_x = \{y \in E;\ |y| \leq \lambda x \text{ for some } \lambda \in \mathbb{R}\}$ (sect. 3).

10.1. Definition. *A d-ideal of a Riesz space* E *is an ideal* J *of* E *with the property that* $x^{\perp\perp} = y^{\perp\perp}$ *and* $x \in J$, $y \in E$ *imply* $y \in J$.

The notion of d-ideal seems to have been introduced by Ball [1975] under the name full convex ℓ-subgroup.

10.2. Proposition. (cf. Ball [1975, thm 1.1 (ii)]) *An ideal* J *of a Riesz space* E *is a d-ideal if and only if* $x^{\perp\perp} \subset J$ *for every* $x \in J$.

10.3. Proposition. *Every band* B *of a Riesz space* E *is a d-ideal.*
Proof: if $x \in B$, $y \in E$ and $x^{\perp\perp} = y^{\perp\perp}$, then by Luxemburg and Zaanen [1971, thm 24.7 (ii)] it holds that $|y| = \sup\{|y| \wedge n|x|;\ n \in \mathbb{N}\}$. For all $n \in \mathbb{N}$ we have $|y| \wedge n|x| \in B$, hence also $|y| \in B$, so $y \in B$.

10.4. Example. Not every d-ideal of a Riesz space is a band. Let E be the Archimedean Riesz space of all continuous functions x on a locally compact Hausdorff space S, which is not compact. If J is the ideal of E consisting of all $x \in E$ which have a compact support, then J is a d-ideal of E, because if $y \in x^{\perp\perp}$ for $x \in J$, then the support of y is contained in the support of x, hence is compact. But J is not a band because J is not a polar, for $J^{\perp\perp} = E$ and $J \neq E$.

10.5. Definition. *An ideal* J *of a Riesz space* E *is called*
(a) an o-ideal if for all $0 \leq x \in J$ *holds that* $y \in J$ *whenever* $y \in {}^{o}Lf$ *for some* $0 \leq f \in \mathrm{seq}(I_x)$ *such that* $\uparrow f$.
(b) a r-ideal if for all $0 \leq x \in J$ *holds that* $y \in J$ *whenever* $y \in {}^{r}Lf$ *for some* $0 \leq f \in \mathrm{seq}(I_x)$ *such that* $\uparrow f$.
(c) a c-ideal if for all $0 \leq x \in J$ *holds that* $y \in J$ *whenever* $y \in {}^{r}Lf$ *for some* $0 \leq f \in \mathrm{seq}(I_x)$ *such that* $\uparrow f$.
(${}^{o}L$, ${}^{r}L$ *and* ${}^{c}L$ *are taken in* E*).*

We note that there is a close relation between r-ideals defined here, and z-ideals, which are defined by Huijsmans and De Pagter [1980].

10.6. Proposition. *Every d-ideal* J *of a Riesz space* E *is an o-ideal.*
Proof: if $0 \leq x \in J$, $0 \leq f \in \mathrm{seq}(I_x)$, $\uparrow f$ and $y = {}^{o}Lf$, then it follows from thm 6.1*(b)* that $f \uparrow y$. For all $n \in \mathbb{N}$ we have that $f(n) \in I_x \subset x^{\perp\perp}$. From $x^{\perp\perp}$ is a band it follows now that $y \in x^{\perp\perp}$, hence $y \in J$.

A detailed discussion of the mutual connections between o-ideals, r-ideals and c-ideals is planned for the near future.

11. Continuity of linear operators

In this section we study continuity of linear operators with respect to the three types of convergence defined above. E and F are arbitrary Riesz spaces in this section. Every linear operator T from E to F induces a linear operator, also denoted by T, form seq(E) to seq(F) by (Tf)(n) = T(f(n)) for all $n \in \mathbb{N}$.

11.1. Definition. *A linear operator* T *from* E *to* F *is called*
(a) (sequentially) order continuous (or an integral operator) if for every $f \in \text{seq}(E)$ *holds that* ${}^{o}L(Tf) = 0$ *whenever* ${}^{o}Lf = 0$.
(b) (sequentially) regulator continuous if for every $f \in \text{seq}(E)$ *holds that* ${}^{r}L(Tf) = IS(F)$ *whenever* ${}^{r}Lf = IS(E)$.
(c) (sequentially) characteristic continuous if for every $f \in \text{seq}(E)$ *holds that* ${}^{c}L(Tf) = 0$ *whenever* ${}^{c}Lf = 0$.

11.2. Example. Not every positive linear operator T from a Riesz space E to a Riesz space F is order continuous, becuase if T is the linear operator from C [0,1] to $\mathbb{R}$ which assigns to x the value x(0), then T is positive, however, if $f \in \text{seq}(C[0,1])$ is such that for all $n \in \mathbb{N}$ we have $(f(n))(t) = 0$ if $t \in [n^{-1},1]$ and $(f(n))(t) = 1 - nt$ if $t \in [0,n^{-1}]$, then $f \downarrow 0$, hence ${}^{o}Lf = 0$. However, ${}^{o}L(Tf) = 1$ because $(Tf)(n) = 1$ for all $n \in \mathbb{N}$.

11.3. Theorem. (compare Vulikh [1967, thm VIII 1.2]). *Every Jordan operator* T *from a Riesz space* E *to a Riesz space* F *is regulator continuous.*

Proof: It is sufficient to give a proof for positive T only.
If ${}^{r}Lf = IS(E)$ then $0 \in {}^{r}Lf$, hence there exists a $u \in E^{+}$ such that for all $\varepsilon > 0$ there exists a $N_{\varepsilon} \in \mathbb{N}$ such that for all $n \geq N_{\varepsilon}$ holds that $|f(n)| \leq \varepsilon u$. But then also $|Tf(n)| \leq T|f(n)| \leq \varepsilon Tu$ holds for all $n \geq N_{\varepsilon}$, hence $0 \in {}^{r}L(Tf,Tu) \subset {}^{r}L(Tf)$. By prop. 7.4 we have now that ${}^{r}L(Tf) = IS(F)$.

11.4. Example. Not every positive linear operator T from a Riesz space E to a Riesz space F is characteristic continuous, because if T is the canonical embedding operator from C [0,1] into the Riesz space F of all real functions on [0,1], then T is a positive linear operator.

If $f \in seq(E)$ is such that for all $n \in \mathbb{N}$ and $t \in [0,1]$ we have $(f(n))(t) = 0$ if $t \in [n^{-1},1]$ and $(f(n))(t) = -nt + 1$ if $t \in [0,n^{-1}]$, then ${}^{c}Lf = 0$, because $f(n) \in f(n)^{\perp\perp}$ for all $n \in \mathbb{N}$, $f(n+1)^{\perp\perp} \subset f(n)^{\perp\perp}$ for all $n \in \mathbb{N}$ and $\cap\{f(n)^{\perp\perp};\ n \in \mathbb{N}\} = \{0\}$. But $(Tf(n))^{\perp\perp} \supset x^{\perp\perp}$ for all $n \in \mathbb{N}$ where $x \in F$ is such that $x(0) = 1$ and $x(t) = 0$ for $t \in (0,1]$, hence ${}^{c}L(Tf) \neq 0$.

It may be questioned whether every integral operator is order bounded. For a rather extensive class of integral operators (not exhaustive) this question is answered by Peressini [1967, prop. I. 5.15, prop. I. 5.13b], where it is proved that every integral operator from an Archimedean Riesz space to an Archimedean countably bounded Riesz space is order bounded.

<u>11.5. Example.</u> Not every positive linear operator T from a Riesz space E to a Riesz space F is an integral operator. Let $E = C[0,1]$, $F = \mathbb{R}$, $T: E \to F$ such that $Tx = x(0)$, then T is a positive linear operator. If $f \in seq(E)$ is such that for all $n \in \mathbb{N}$ we have $(f(n))(t) = 0$ if $t \in [n^{-1},1]$ and $(f(n))(t) = 1 - nt$ if $t \in [0,n^{-1}]$, then ${}^{o}Lf = 0$, but $Tf(n) = 1$ for all $n \in \mathbb{N}$, hence ${}^{o}L(Tf) = 1$.

<u>11.6. Theorem.</u> *If E and F are Archimedean Riesz spaces and E is order convergence stable, then every order bounded linear operator T from E to F is an integral operator.*

Proof: If for $f \in seq(E)$ holds that ${}^{o}Lf = 0$, then there exists a $g \in seq(E)$ such that $|f| \leq g$ and $g \downarrow 0$. E is order convergence stable, hence there exists a $0 \leq a \in seq(\mathbb{R})$ such that $\uparrow a$, $\{a(n);\ n \in \mathbb{N}\}$ is not majorized and ${}^{o}L(ag) = 0$. The latter implies the existence of $x \in E$ such that $|ag| \leq x$. Now we have that $\{a(n)\ f(n);\ n \in \mathbb{N}\}$ is order bounded, because $-x \leq -ag \leq af \leq ag \leq x$. But then also $T(\{a(n)\ f(n);\ n \in \mathbb{N}\}$ is order bounded in F, say $-y \leq T(a(n)\ f(n)) \leq y$ for all $n \in \mathbb{N}$. It follows that $|T(f(n))| \leq (a(n))^{-1}y$ for all $n \in \mathbb{N}$ such that $a(n) \neq 0$. Since F is Archimedean, we have ${}^{o}L(Tf) = 0$ by thm 6.2 *(b)*.

<u>11.7. Example.</u> If an Archimedean Riesz space E has the property that every order bounded linear operator T from E to an arbitrary Archimedean Riesz space F is an integral operator, then E is not necessary order convergence stable, even if E is universally complete.

To demonstrate this we use a Riesz space which appears in Tucker [1974]. If E is the Riesz space (with pointwise linear operations and pointwise ordering) of all realvalued functions on the set S of all $0 \leqslant a \in \mathrm{seq}(\mathbb{R})$ such that $\uparrow a$, $a(1) > 0$ and $\{a(n);\, n \in \mathbb{N}\}$ is not majorized, then E is universally complete. It follows from Fremlin [1975, cor. 1.13] that every order bounded linear operator from E to an arbitrary Archimedean Riesz space F is an integral operator. However, E is not order convergence stable, because, if $f \in \mathrm{seq}(E)$ is such that $(f(n))(a) = (a(n))^{-1}$, then $f \downarrow 0$ because $\inf\{(f(n))(a);\, n \in \mathbb{N}\} = 0$ for all $a \in S$, as $(f(n))(a) = (a(n))^{-1}$ for all $n \in \mathbb{N}$. But if $0 \leqslant a$, $\uparrow a$, $a(1) > 0$ and $\{a(n);\, n \in \mathbb{N}\}$ is not majorized, then for all $n \in \mathbb{N}$ we have that $a(n)(f(n))(a) = 1$, hence ${}^{o}L(af) \neq 0$. This implies that E is not order convergence stable.

Chapter III
DISJUNCTIVE LINEAR OPERATORS

In this chapter we study disjunctive linear operators, especially orthomorphisms and disjunctive linear functionals. Further we give some examples of unbounded orthomorphisms.

12. Orthomorphisms.

A notion of rather recent origin is the notion of disjunctive linear operator. The study of some special types of disjunctive linear operators such as Riesz homomorphisms, is much older.

12.1. Definition. (cf. Cristescu [1976, p. 186]). *A linear operator* T *from a Riesz space* E *to a Riesz space* F *is called a disjunctive linear operator if for all* $x, y \in E$ *holds that* $Tx \perp Ty$ *whenever* $x \perp y$.

12.2. Theorem. *For a linear operator* T *from a Riesz space* E *to a Riesz space* F *the following assertions are equivalent*
(a) T *is a disjunctive linear operator*
(b) $|T|x|| = |Tx|$ *for all* $x \in E$
Proof: *(a)* $\rightarrow$ *(b)*: $x^+ \perp x^-$, so $Tx^+ \perp Tx^-$. Now $|Tx| = |Tx^+ - Tx^-|$
$= |Tx^+ + Tx^-| = |T|x||$ by thm 3.3 *(h)*.
(b) $\rightarrow$ *(a)*: if $x \perp y$ then $|x| \wedge |y| = 0$. Now $|T|x| - T|y|| = |T(||x| - |y||)|$
$= |T(|x| - |y|)^+ + T(|x| - |y|)^-| = |T|x| + T|y||$, because $(|x| - |y|)^+$
$= |x|$ and $(|x| - |y|)^- = |y|$ by thm 3.3 *(g)*. By thm 3.3 *(h)* we have
$T|x| \perp T|y|$, hence $|T|x|| \perp |T|y||$, so $|Tx| \perp |Ty|$, consequently $Tx \perp Ty$.

The most important disjunctive linear operators are the positive ones, called Riesz homomorphisms. A bijective Riesz homomorphism is called a Riesz isomorphism; a Riesz space E is called Riesz isomorphic to a Riesz space F if there exists a Riesz isomorphism from E to F.

With the aid of Riesz homomorphisms factor spaces of Riesz spaces can be defined, which are Riesz spaces themselves (cf. e.g. Luxemburg and Zaanen [1971, §18]). This is a consequence of the preservation of the lattice operations by a Riesz homomorphism, a fact which is stated below.

12.3. Theorem. *For a linear operator* T *from a Riesz space* E *to a Riesz space* F *the following assertions are equivalent*

(a) T *is a Riesz homomorphism*

(b) $T(x \vee y) = Tx \vee Ty$ *for all* $x, y \in E$

(c) $T(x \wedge y) = Tx \wedge Ty$ *for all* $x, y \in E$

(d) $|Tx| = T|x|$ *for all* $x \in E$.

(compare Schaefer [1974, II, prop. 2.5] where similar statements are proved; Riesz homomorphism are called lattice homomorphisms there).

There exists an important relation between realvalued Riesz homomorphisms on a Riesz space E and maximal ideals of E. This relationship is expressed in the following theorem.

12.4. Theorem. (compare e.g. Luxemburg and Zaanen [1971, thm 27.3 (i)] and Schaefer [1974, cor. of II. prop. 3.4]). *If* ϕ *is a realvalued Riesz homomorphism on a Riesz space* E, *then the nullspace* $N(\phi)$ *of* ϕ *is a maximal ideal of* E. *If* M *is a maximal ideal of* E *and* $x \in E^+$ *is arbitrary such that* $x \notin M$, *then there exists exactly one realvalued Riesz homomorphism* ϕ *on* E *such that* $\phi(M) = \{0\}$ *and* $\phi(x) = 1$.

12.5. Definition. *A realvalued Riesz homomorphism* ϕ *on a Riesz space* E *with strong order unit* e *is called standard if* $\phi(e) = 1$. *The set of all standard realvalued Riesz homomorphisms on a Riesz space* E *is denoted by* $R(E)$, *or simply by* R *if there is no ambiguity.*

12.6. Theorem. *For an Archimedean Riesz space* E *with strong order unit* e *the set* R *is total, i.e.* R *is not empty and if* $\phi(x) = 0$ *for certain* $x \in E$ *and all* $\phi \in R$ *then* $x = 0$.

Proof: By thm 3.13 the set of all maximal ideals of E is not empty. Let M be a maximal ideal of E, then $e \notin M$. Now by thm 12.4 there exists a standard realvalued Riesz homomorphism on E, hence $R \neq \emptyset$. If $\phi(x) = 0$ for all $\phi \in R$, then by thm 12.4 we have that $x \in M$ for every maximal ideal M of E, hence by thm 3.13 we have that $x = 0$.

12.7. Example. (cf. Meyer [1979, Ex. 1.4]). Let E be the Riesz subspace of $C[0,1]$ consisting of all $x \in C[0,1]$ such that the right differential quotient of x in the point $t = \frac{1}{2}$ exists as a real number $x'(\frac{1}{2})$.

Let $T: E \to \mathbb{R}$ be the linear operator such that $Tx = x'(\frac{1}{2})$. If $x \perp y$ in E, then $x'(\frac{1}{2}) = 0$ or $y'(\frac{1}{2}) = 0$, hence $Tx \perp Ty$, so T is a disjunctive linear operator. Note that neither T nor $-T$ are Riesz homomorphisms.

12.8. Theorem. (cf. Meyer [1979, Thm I. 6]). *A disjunctive linear operator is a Jordan operator if an only if it is order bounded.*

12.9. Definition. *A linear operator* T *from a Riesz space* E *to itself is called a stabilizer on* E *if* T *preserves orthogonality in the following strong sense: if* $x \perp y$ *then also* $Tx \perp y$.

12.10. Theorem. *For a linear operator* T *from a Riesz space* E *to itself, the following assertions are equivalent*

(a) T *is a stabilizer on* E

(b) T *is polar preserving, i.e.* $T(P) \subset P$ *for every polar* P *of* E

(c) $Tx \in x^{\perp\perp}$ *for every* $x \in E$.

Proof: *(a)* $\to$ *(b)*: if $x \in P$, then $x \perp y$ for all $y \in P^{\perp}$, hence $Tx \perp y$ for all $y \in P^{\perp}$, or $Tx \perp P^{\perp}$, hence $Tx \in P^{\perp\perp} = P$

(b) $\to$ *(c)*: evident

(c) $\to$ *(a)*: Suppose $x \perp y$. $Tx \in x^{\perp\perp}$, hence $Tx \perp y$.

The linear subspace of $\mathcal{L}(E,E)$ consisting of all stabilizers on E is denoted by Stab(E). In section 4 it was observed that $\mathcal{L}(E,E)$ is a partially ordered algebra if we suppose on $\mathcal{L}(E,E)$ the operator ordering and if multiplication is the composition of mappings. Stab(E) is a partially ordered subalgebra of $\mathcal{L}(E,E)$ because $x, y \in E$, $x \perp y$ and $S, T \in$ Stab(E) imply $Sx \perp y$, hence $TSx \perp y$.

Until recently it was unknown whether every stabilizer is also a Jordan operator. A negative answer to this question was given independently by Meyer [1979] and Bernau [1979]. Their counterexamples are essentially the same and in fact a modification of a well known (norm) unbounded linear operator, namely the differential operator in an appropriate Hilbert space.

Really surprising is the fact that there exist also unbounded stabilizers in some universally complete Riesz spaces. This was proved by Wickstead [1979] by a kind of Hahn-Banach proof for the existence of an (unbounded) extension of the Meyer-Bernau stabilizer to the universal completion of

the underlying Riesz space. Abramovič, Veksler and Koldunov [1979] assert that there exists a bijective unbounded stabilizer on the Riesz space $M([0,1], \mu, \leq)$ where μ is Lebesque measure (example 3.4 *(g)*). We give more examples of unbounded stabilizers in section 14.
The fact that there exist unbounded stabilizers implies that Stab(E) for a Riesz space E is in general not a Riesz space, because for every Riesz space E holds that $E = E^+ - E^+$ (every element of a Riesz space can be written as the difference of two positive elements, section 3).

12.11. Definition. *A linear operator on a Riesz space* E *which is the difference of two positive stabilizers on* E *is called an orthomorphism on* E.

Note that every positive orthomorphism is a Riesz homomorphism.

The linear subspace of $\mathcal{L}(E,E)$ consisting of all orthomorphisms on E is denoted by Orth(E). Orth(E) is a partially ordered subalgebra of Stab(E).

In contrast to Stab(E) we have that Orth(E) is a Riesz space in general, whenever E is Archimedean. This was independently proved by Bigard and Keimel [1969] and Conrad and Diem [1971]. A direct proof was given by Bernau [1979]. However the last proof is rather complicated and not very transparent.

A linear operator T from a Riesz space E to itself is called a *centre operator* on E if T is bounded in the operator ordering of $\mathcal{L}(E,E)$ by two multiples of the identity operator I_E on E, i.e. if there exist $\lambda, \mu \in \mathbb{R}$ such that $\lambda I \leq T \leq \mu I$.
Every centre operator T on a Riesz space E is an orthomorphism, because if $\lambda I \leq T \leq \mu I$ for $\lambda, \mu \in \mathbb{R}$ then T is the difference of $|\mu| I$ and $|\mu| I - T$. $|\mu| I$ is positive and a stabilizer because $x \perp y$ implies $|\mu|\, x \perp y$. Also $|\mu| I - T$ is positive because $T \leq \mu T \leq |\mu| I$ and a stabilizer because if $x \perp y$ then $|\mu|\,|x| \wedge |y| = 0$, hence $(|\mu| I - T)|x| \wedge |y| = 0$, so certainly $|(|\mu| I - T)x| \wedge |y| = 0$, hence $(|\mu| I - T)x \perp y$.

The linear subspace of $\mathcal{L}(E,E)$ consisting of all centre operators on E is denoted by Z(E). We note that Z(E) is a partially ordered subalgebra of Orth(E). It follows immediately from the definitions that Z(E) is a Riesz subspace of Orth(E), whenever E is an Archimedean Riesz space.

12.12. Example. If E is the partially ordered linear space of all functions x from an arbitrary non-empty set S to $\mathbb{R}$, with pointwise linear operations and pointwise partial ordering, then E is a universally complete Riesz space. If we define an algebra structure on E by pointwise multiplication, then E is a partially ordered algebra with multiplicative unit e, the function constant 1 on S.

If for $z \in E$ we define the linear operator $\mathcal{R}_z$ by $\mathcal{R}_z x = xz$, then $\mathcal{R}_z$ is a stabilizer, because if $x \perp y$ in E, then $x(s)\,y(s) = 0$ for all $s \in S$, hence $x(s)\ z(s)\ y(s) = 0$ and this implies $xz \perp y$, or $\mathcal{R}_z x \perp y$. In fact, $\mathcal{R}_z$ is an orthomorphism, because $\mathcal{R}_z = \mathcal{R}_{z^+} - \mathcal{R}_{z^-}$ and $\mathcal{R}_{z^+}$ and $\mathcal{R}_{z^-}$ are positive linear operators.

Every stabilizer can be written conversely as an operator $\mathcal{R}_z$ for some $z \in E$, hence is an orthomorphism. This can be seen as follows. If T is a stabilizer on E and $z = Te$, then for every $x \in E$ and every $s \in S$ we have $(x - x(s)e)(s) = 0$, hence $x - x(s)e \perp \chi_s$ where $\chi_s(t) = 0$ if $t \neq s$, $\chi_s(s) = 1$.

It follows that also $T(x - x(s)e) \perp \chi_s$, hence $(Tx - x(s)z)(s) = 0$, which implies that $(Tx)(s) = x(s)z(s)$. Hence $T = \mathcal{R}_z$.

It is one of our purposes to find a description of orthomorphisms as multiplication operators in an arbitrary Archimedean Riesz space. While not every Archimedean Riesz space can be provided with an appropriate multiplication, we shall deal in the sequel with multiplications in Riesz spaces, which are only partial in a certain sense. In order to develop an independent theory, we shall make no use of the fact that Orth(E) is a Riesz space whenever E is an Archimedean Riesz space.

12.13. Proposition. (compare Conrad and Diem [1971, Prop. 2.1])
If T *is a positive linear operator from a Riesz space* E *to itself, then*
(a) T *is an orthomorphism whenever* I + T *is a Riesz homomorphism*
(b) I + T *is an orthomorphism whenever* T *is an orthomorphism.*
Proof:
(a) if I + T is a Riesz homomorphism and $x \perp y$ in E then
$(I + T)|x| \wedge (I + T)|y| = 0$. From $|Tx| \leqslant T|x| \leqslant (I + T)|x|$ and

$|y| \leqslant (I + T)|y|$ it follows now that $|Tx| \wedge |y| = 0$, hence $Tx \perp y$.

(b) if T is an orthomorphism and $x \perp y$ then we have $Tx \perp y$ and $x \perp y$, hence $x + Tx \perp y$, or $(I + T)x \perp y$.

12.14. Proposition. *If* T *is a positive orthomorphism on a Riesz space* E *then* I + T *is an injective orthomorphism on* E.

Proof: By the foregoing proposition I + T is an orthomorphism on E. If $x, y \geqslant 0$ in E, then from $(I + T)x = (I + T)y$ it follows that $y - x = T(x - y)$, hence $(y - x)^+ = (T(x - y))^+ = T(x - y)^+ \in (x - y)^{+\perp\perp}$. But also $(y - x)^+ = (x - y)^- \in (x - y)^{-\perp\perp}$. It follows that $(y - x)^+ \in (x - y)^{+\perp\perp} \cap (x - y)^{-\perp\perp} = \{0\}$, hence $x \geqslant y$. By symmetry also $y \geqslant x$, hence $x = y$. If $x, y \in E$ are arbitrary, then from $(I + T)x = (I + T)y$ it follows that $(I + T)x^+ = ((I + T)x)^+ = ((I + T)y)^+ = (I + T)y^+$, hence by the foregoing $x^+ = y^+$, and similarly $x^- = y^-$, so $x = y$.

The following theorem is a direct consequence of Luxemburg and Schep [1978, thm 1.3].

12.15. Theorem. *Every orthomorphism* T *on an Archimedean Riesz space* E *is order continuous.*

In this section 14 examples are given which show that thm 12.15 does not hold for arbitrary stabilizers.

12.16. Theorem. *Every stabilizer* T *on a Riesz space* E *is characteristic continuous.*

Proof: if ${}^cLf = 0$ for some $f \in seq(E)$, then there exists a (countable) subset $\{P_n;\ n \in \mathbb{N}\}$ of the Boolean algebra $\mathbb{P}(E)$ of all polars of E such that $P_{n+1} \subset P_n$ for all $n \in \mathbb{N}$, $f(n) \in P_n$ for all $n \in \mathbb{N}$ and $\cap \{P_n;\ n \in \mathbb{N}\} = \{0\}$. Now by thm 12.10 we have that $TP_n \subset P_n$ for every $n \in \mathbb{N}$, hence also $(TP_n)^{\perp\perp} \subset P_n$ for all $n \in \mathbb{N}$.

It follows that $Tf(n) \in (TP_n)^{\perp\perp}$ for all $n \in \mathbb{N}$, $(TP_{n+1})^{\perp\perp} \subset (TP_n)^{\perp\perp}$ for all $n \in \mathbb{N}$ and $\cap\{(TP_n)^{\perp\perp};\ n \in \mathbb{N}\} \subset \cap\{P_n^{\perp\perp};\ n \in \mathbb{N}\} = \{0\}$, hence $\cap \{(TP_n)^{\perp\perp};\ n \in \mathbb{N}\} = \{0\}$, so ${}^cL(Tf) = 0$.

12.17. Theorem. *Every orthomorphism* T *on an Archimedean Riesz space* E *preserves all σ-ideals of* E, *i.e.* $TJ \subset J$ *for every σ-ideal* J *of* E.

Proof: It is sufficient to give a proof for positive T only.

Let $0 \leqslant x \in J$. Define $f \in \mathrm{seq}(I_x)$ by $f(n) = Tx \wedge nx$ for all $n \in \mathbb{N}$, then $0 \leqslant f$ and $\uparrow f$. Further we have $|f(n) - Tx| = (Tx - nx)^+$ for all $n \in \mathbb{N}$. If we define $P_n = (Tx - nx)^{+\perp\perp}$ for all $n \in \mathbb{N}$, then $f(n) - Tx \in P_n$ for all $n \in \mathbb{N}$. Also $P_{n+1} \subset P_n$ for all $n \in \mathbb{N}$. $\cap \{P_n;\ n \in \mathbb{N}\} = \{0\}$, because if for $0 \leqslant v \in E$ holds that $0 \leqslant v \in (Tx - nx)^{+\perp\perp}$ for all $n \in \mathbb{N}$, then $v \perp (Tx - nx)^-$ for all $n \in \mathbb{N}$, hence $v \perp (n^{-1}Tx - x)^-$, so $v \perp (x - n^{-1}Tx)^+$ for all $n \in \mathbb{N}$. E Archimedean implies that $x = \sup\{(x - \frac{1}{n}Tx)^+;\ n \in \mathbb{N}\}$, hence by thm 3.5 it follows that $v \perp x$, so $v \perp Tx$. But then certainly $v \perp (Tx - nx)^+$, hence $v \in (Tx - nx)^{+\perp}$ for all $n \in \mathbb{N}$. It follows that $v = 0$. Hence, ${}^{c}Lf = Tx$. ${}^{c}Lf \in J$ because J is a c-ideal, thus $Tx \in J$; we conclude that $TJ \subset J$.

12.18. Theorem. *Every orthomorphism* T *on an Archimedean Riesz space* E *preserves all o-ideals of* E, *i.e.* $TJ \subset J$ *for every o-ideal* J *of* E.
Proof: It is sufficient to give a proof for positive T only. Let $0 \leqslant x \in J$. Define $f \in \mathrm{seq}(I_x)$ by $f(n) = Tx \wedge nx$ for all $n \in \mathbb{N}$, then $0 \leqslant f$ and $\uparrow f$. By thm 12.10 we have $Tx \in x^{\perp\perp}$, hence $Tx = \sup\{f(n);\ n \in \mathbb{N}\}$. From thm 6.1 *(a)* it follows now that $Tx = {}^{o}Lf$, hence $Tx \in J$, so $TJ \subset J$.

12.19. Remark. Not every orthomorphism T on a Riesz space E preserves all ideals J of E. If E is the Riesz space s of all sequences of real numbers and J is the ideal of all sequences converging to 0, $e = (1, 1, 1, \ldots) \in E$, $y = (1, \frac{1}{2}, \frac{1}{3}, \ldots) \in J$ and $T \in \mathrm{Orth}(E)$ is such that $(Tx)(n) = nx(n)$ for all $n \in \mathbb{N}$, then $Ty = e \notin J$, hence $TJ \not\subset J$.

12.20. Theorem. (Bigard, Keimel and Wolfenstein [1977, thm 12.2.7]) *If* T *is an orthomorphism on an Archimedean Riesz space* E, *then the nullspace* N(T) *of* T *is equal to* $T(E)^{\perp}$.
We note that the proof is by elementary means.

A direct consequence of this theorem is

12.21. Theorem. (compare Bigard [1972]) *If* S *and* T *are orthomorphisms on an Archimedean Riesz space* E *which coincide on a subset* X *for which* $X^{\perp\perp} = X$, *then* S = T.
Proof: $X \subset N(S - T)$, hence $E = X^{\perp\perp} \subset N(S - T)^{\perp\perp} = N(S - T)$, so S = T

Orthomorphisms on Archimedean Riesz spaces with strong order unit have special properties. Here we shall prove two of them.

12.22. Theorem. *If* E *is an Archimedean Riesz space with strong order unit* e*, then* $TM \subset M$ *for every* $T \in \mathrm{Orth}(E)$ *and every maximal ideal* M *of* E.
Proof: It is no restriction to give a proof for positive T only. R is the set of all standard realvalued Riesz homomorphisms on E. Let M be an arbitrary maximal ideal of E. By thm 12.4 there exists a $\phi \in R$ such that $\phi(M) = \{0\}$. Let λ be the real number $\phi(Te)$. Without loss of generality we may assume that $\lambda \leq \frac{1}{2}$, otherwise take the orthomorphism $\frac{1}{4}\lambda^{-1}T$ instead of T. $\lambda \geq 0$ by the positivity of $\phi \circ T$.
We shall prove now that if $0 \leq x \in M$ then $Tx \in M$. We have by the positivity of $\phi \circ T$ that $\mu \geq 0$ if $\mu = \phi(Tx)$. $I + T$ is an orthomorphism by proposition 12.13 *(b)*, by the positivity of $I + T$ we conclude that $I + T$ is also a Riesz homomorphism.
It follows now with thm 12.3 that
$0 \leq (\mu(1 - \lambda)e - x)^+ \leq (I + T)(\mu(1 - \lambda)e - x)^+$
$= (\mu(1 - \lambda)Te - Tx + \mu(1 - \lambda)e - x)^+ \in M$, because
$\phi((\mu(1 - \lambda)Te - Tx + \mu(1 - \lambda)e - x)^+) = (\phi(\mu(1 - \lambda)Te - Tx + \mu(1 - \lambda)e - x))^+$
$= (\mu(1 - \lambda)\lambda - \mu + \mu(1 - \lambda) - 0)^+ = (\mu\lambda - \mu\lambda^2 - \mu + \mu - \mu\lambda)^+ = (-\mu\lambda^2)^+ = 0.$
By the ideal property of M we have now
$(\mu(1 - \lambda)e - x)^+ \in M$, hence $\phi((\mu(1 - \lambda)e - x)^+) = 0$, hence
$(\mu(1 - \lambda) - \phi(x))^+ = 0$, so $\mu(1 - \lambda) \leq 0$. If $\mu > 0$, then necessarily $1 - \lambda \leq 0$, so $\lambda \geq 1$, contradiction. It follows that $\mu = 0$, hence $Tx \in M$.

12.23. Theorem. *If* E *is an Archimedean Riesz space with strong order unit* e*, then* $\mathrm{Orth}(E) = Z(E)$.
Proof: It is sufficient to prove that every positive orthomorphism T on E is a centre operator. There exists a $\lambda \in \mathbb{R}$ such that $Te \leq \lambda e$, because e is a strong order unit. We shall prove now that $Tx \leq \lambda x$ for all $x \geq 0$.
If $x \geq 0$ and M is an arbitrary maximal ideal of E, then let $\phi: E \to \mathbb{R}$ be the unique Riesz homomorphism such that $\phi(M) = \{0\}$ and $\phi(e) = 1$. Now we have $x - \phi(x)e \in M$. With the foregoing theorem it follows that
$T(x - \phi(x)e) = Tx - \phi(x)Te \in M$, hence also $(Tx - \phi(x)Te)^+ \in M$, but then certainly $(Tx - \phi(x)\lambda e)^+ \in M$.
Further we have that $\phi(x)\lambda e - \lambda x \in M$, because $\phi(\phi(x)\lambda e - \lambda x)$
$= \lambda\phi(x) - \lambda\phi(x) = 0$, hence also $(\phi(x)\lambda e - \lambda x)^+ \in M$.
Because $0 \leq (Tx - \lambda x)^+ \leq (Tx - \phi(x)\lambda e)^+ + (\phi(x)\lambda e - \lambda x)^+ \in M$ we also have $(Tx - \lambda x)^+ \in M$. So by thm 3.13 we have $(Tx - \lambda x)^+ = 0$, hence $Tx \leq \lambda x$.
Consequently $0 \leq T \leq \lambda I$, so T is a centre operator.

13. Nullspaces of disjunctive linear functionals

In this section we determine the nullspaces of disjunctive linear functionals on a non-trivial Riesz space E.
It appears that the forms of these nullspaces resemble the notions of prime ideals and maximal ideals, which we have already encountered in section 3.
It will be necessary to apply the notion of primeness to subspaces of a more general type than ideal.
In this section E is an arbitrary Riesz space. $\mathfrak{R}(E)$ and $\mathfrak{I}(E)$ are abbreviated to $\mathfrak{R}$ and $\mathfrak{I}$ respectively.

13.1. Definition. *A linear subspace* L *of* E *is said to be prime if it follows from* $x,y \in E$ *and* $x \wedge y \in L$ *that at least one of* x *and* y *is an element of* L.

13.2. Example. If $E = C[0,1]$ and $L = \{x \in E; x(0) = x(1) = 0\}$ then L is not prime because if $x(t) = t$ and $e(t) = 1$ for all $t \in [0,1]$, then $x \wedge (e - x) \in L$, however $x \notin L$ and $e - x \notin L$.
If M is the linear subspace of all polynomials in E, then M is prime, because if $x,y \in M$ then $x \wedge y \in M$ if and only if x and y are comparable, i.e. $x \wedge y = x$ or $x \wedge y = y$. It follows that $x \in M$ or $y \in M$. Note that M is not an ideal of E.

13.3. Lemma. *For a linear subspace* L *of* E *the following assertions are equivalent.*

(a) L *is a prime linear subspace*
(b) if $x \vee y \in L$ *then* $x \in L$ *or* $y \in L$
(c) if $x \vee y = 0$ *then* $x \in L$ *or* $y \in L$
(d) if $x \wedge y = 0$ *then* $x \in L$ *or* $y \in L$

Proof: *(a)* → *(b)*: $x \vee y \in L$ implies $-(x \vee y) = (-x) \wedge (-y) \in L$, hence $x \in L$ or $y \in L$.
(b) → *(c)* : evident
(c) → *(d)* : $x \wedge y = 0$ implies $-(x \wedge y) = (-x) \vee (-y) = 0$, hence $x \in L$ or $y \in L$
(d) → *(a)* : if $x \wedge y = h \in L$ then $(x - h) \wedge (y - h) = 0$, so $x - h \in L$ or $y - h \in L$, hence $x \in L$ or $y \in L$.

As a consequence of *(d)* we have: if for linear subspaces K and L of E holds that $K \supset L$ and L is prime, then K is prime.

If X is a subset of E, then the intersection of all linear subspaces L of E such that $L \supset X$ is also a linear subspace, which we call the linear subspace of E generated by X, in formula Lss(X).
Completely similar, by thm 3.7, we can define the Riesz subspace, respectively ideal, band, polar of E generated by X as the intersection of all Riesz subspaces, respectively ideals, bands, polars of E which contain X, in formula Rss(X), respectively Id(X), Band(X), Polar(X).
In the following we abbreviate $Rss(X \cup \{x\})$ to Rss(X,x) and $Id(X \cup \{x\})$ to Id(X,x) for $X \subset E$, $x \in E$.
In this section we are especially interested in Rss(X) for a given $X \subset E$.
In general, it is difficult to give a closed description of the elements of Rss(X) for $X \subset E$ arbitrary.
In the following we shall meet two exceptions to the rule, the first in thm 13.4, where we consider the Riesz subspace generated by a linear subspace, the second in thm 13.22 where we consider the Riesz subspace generated by a subset consisting of a prime Riesz subspace and an arbitrary element of E.

<u>13.4. Theorem.</u> *For a linear subspace* L *of* E *the Riesz subspace* Rss(L) *of* E *consists of all finite infima of all elements of* E *which are finite suprema of elements of* L, *in formula*

$$Rss(L) = \{ \bigwedge_{j \in J} \bigvee_{k \in K} x_{jk};\ x_{jk} \in L \text{ for all } j \in J, k \in K, \text{ with}$$

$$J \text{ and } K \text{ arbitrary finite index sets}\}.$$

Proof: If R(L) is the right hand set, then it is evident that $Rss(L) \supset R(L)$.
We are done if R(L) is a Riesz subspace of E.
If $x,y \in R(L)$ then we have also $x \wedge y \in R(L)$.
By Bigard, Keimel and Wolfenstein [1977, cor. 1.2.15] we have for all finite indexsets I,J and K and elements x_{jki} ($i \in I$, $j \in J$, $k \in K$) that

$$\bigvee_{i \in I} \bigwedge_{j \in J} \bigvee_{k \in K} x_{jki} = \bigwedge_{\sigma \in J^I} \bigvee_{\substack{j \in J \\ k \in K}} x_{\sigma(i)ki} ;$$

this implies that if $x,y \in R(L)$ then we have also $x \vee y \in R(L)$.
Further, if $x = \bigwedge_{j \in J} \bigvee_{k \in K} x_{jk}$ and $y = \bigwedge_{i \in I} \bigvee_{m \in M} y_{im}$,

then $x + y = \bigwedge_{\substack{j \in J \\ i \in I}} \{ \bigvee_{k \in K} x_{jk} + \bigvee_{m \in M} y_{im} \} =$

$\bigwedge_{\substack{j \in J \\ i \in I}} \bigvee_{\substack{k \in K \\ m \in M}} \{x_{jk} + y_{im}\} \in R(L).$

If $\lambda \geq 0$, $x \in R(L)$ then also $\lambda x \in R(L)$.

Finally, $-(\bigwedge_{j \in J} \bigvee_{k \in K} x_{jk}) = \bigvee_{j \in J} \bigwedge_{k \in K} (-x_{jk}) =$

$= \bigwedge_{\sigma \in K^J} \bigvee_{j \in J} (-x_{j\sigma(j)}) \in R(L).$

It follows that $Rss(L) = R(L)$.

<u>13.5. Lemma.</u> *If* $x \in E^+$ *then* $Rss(\{x\}) = \{\lambda x;\ \lambda \in \mathbb{R}\}$

Proof: evident

<u>13.6. Theorem.</u> *For any Riesz subspace* S *of a Riesz space* E *and* $x \in E$ *such that* $x \notin S$, *there exists a Riesz subspace* R *of* E *with the following properties*

(a) $x \notin R$

(b) $R \supset S$

(c) if for certain Riesz subspace T *of* E *holds that* $x \notin T$ *and* $T \supset R$ *then* $T = R$.

Proof: Let $\mathbf{R}_{x,S} = \{U \in \mathbf{R};\ x \notin U,\ U \supset S\}$ be partially ordered by inclusion, then $\mathbf{R}_{x,S} \neq \phi$, because $S \in \mathbf{R}_{x,S}$.

If Σ is an arbitrary chain in $\mathbf{R}_{x,S}$, then $Rss(\cup\Sigma) \in \mathbf{R}$ and $Rss(\cup\Sigma) \supset S$.

If $x \in Rss(\cup\Sigma)$, then $x = \bigwedge_{j \in J} \bigvee_{k \in K} x_{jk}$ for certain finite indexsets J and K, $x_{jk} \in \cup\,\Sigma$; as a consequence of the finite cardinality of the set $\{x_{jk};\ j \in J,\ k \in K\}$ there exists a W in the chain Σ such that $\{x_{jk};\ j \in J,\ k \in K\} \subset W$, so $x \in W$. Since $x \notin W$ it follows that $x \notin Rss(\cup\Sigma)$.

Hence, $Rss(\cup\Sigma) \in \mathbf{R}_{x,S}$.

Now, Zorn's lemma can be applied, and the desired result follows.

<u>13.7. Definition.</u> *A Riesz subspace* R *with the properties listed in thm* 13.6 *is called a Riesz subspace maximal in* $\mathbf{R}$ *with respect to the property*

of containing S *and not having* x *as element, abbreviated to* R *is* S,x-*maximal in* $\mathbf{R}$. *A Riesz subspace* R *is called* x-*maximal in* $\mathbf{R}$ *if* R *is* S,x-*maximal in* $\mathbf{R}$ *for some* S. R *is called an* $\mathbf{R}$-*relative maximal Riesz subspace if* R *is* S,x-*maximal in* $\mathbf{R}$ *for some* S *and* $x \notin S$.

<u>13.8. Proposition.</u> *For Riesz subspaces* R *and* S *of* E *such that* $R \supset S$ *and* $x \in E$ *such that* $x \notin S$ *the following assertions are equivalent*

(a) R *is* S,x-*maximal in* $\mathbf{R}$

(b) R *is* $\{0\}$,x-*maximal in* $\mathbf{R}$

Proof: evident

<u>13.9. Theorem.</u> (cf. e.g. Luxemburg and Zaanen [1971, thm 33.5]).
For any ideal J *of a Riesz space* E *and* $x \in E$ *such that* $x \notin J$, *there exists an ideal* M *of* E *with the following properties.*

(a) $x \notin M$

(b) $M \supset J$

(c) if for certain ideal N *of* E *holds that* $x \notin N$ *and* $N \supset J$, *then* $N = J$.

<u>13.10. Definition.</u> *An ideal* M *with the properties listed in thm* 13.9 *is called an ideal maximal in* $\mathbf{I}$ *with respect to the property of containing* J *and not having* x *as element, abbreviated to* M *is* J,x-*maximal in* $\mathbf{I}$.
An ideal M *is called* x-*maximal in* $\mathbf{I}$ *if* M *is* J,x-*maximal in* $\mathbf{I}$ *for some* J.
M *is called an* $\mathbf{I}$-*relative maximal ideal if* M *is* J,x-*maximal in* $\mathbf{I}$ *for some* J *and* $x \notin J$.

<u>13.11. Proposition.</u> *For ideals* M *and* J *of* E *such that* $M \supset J$ *and* $x \in E$ *such that* $x \notin J$ *the following assertions are equivalent*

(a) M *is* J,x-*maximal in* $\mathbf{I}$.

(b) M *is* $\{0\}$,x-*maximal in* $\mathbf{I}$.

Proof: evident

In Luxemburg and Zaanen [1971, thm 33.4] it is proved that every $\mathbf{I}$-relative maximal ideal is prime. Here we give a proof based on the distributivity of the lattice $\mathbf{I}$, in which sup and inf are denoted by ∇ and Δ respectively.

<u>13.12. Theorem.</u> *Every* $\mathbf{I}$-*relative maximal ideal* M *is prime.*

Proof: if M is x-maximal in $\mathbf{I}$ and $y,z \in E$ are such that neither y nor z is an element of M, then since M is x-maximal in $\mathbf{I}$ we have $x \in Id(M,y)$ and $x \in Id(M,z)$, hence $x \in Id(M,y) \cap Id(M,z) = Id(M,I_y) \cap Id(M,I_z) = (M \nabla I_y) \Delta (M \nabla I_z) = M \nabla (I_y \Delta I_z) = M \nabla \{0\} = M$. Since $x \notin M$ it follows that M is prime.

In thm 6.12 the similarity between $\mathbf{R}$-relative maximal Riesz subspaces and $\mathbf{I}$-relative maximal ideals breaks down, because, as a consequence of the fact that $\mathbf{R}$ is not distributive in general, for a Riesz subspace to be prime and to be $\mathbf{R}$-relative maximal are independent properties. This will be shown in the following example.

13.13. Example. Let E be the Riesz space as defined in ex. 12.7. If $x \in E$ is such that $x(t) = t$ for all $t \in [0,1]$, and S is the Riesz subspace of E consisting of all $y \in E$ such that $y(0) = y(1)$, then S is x-maximal in $\mathbf{R}$, because if S is strictly contained in a Riesz subspace R, then there exists a $z \in R$ such that $z(0) \neq z(1)$. Now for $r \in E$ with $r(t) = z(0) + (z(1) - z(0))t$ we have $r - z \in S$, hence $r - z \in R$, but then also $r \in R$, hence $s \in R$ if $s(t) = (z(1) - z(0))t$ for all $t \in [0,1]$, thus $(z(1) - z(0))^{-1}s = x \in R$.

However S is not prime, because if $e(t) = 1$ for all $t \in [0,1]$, then $x \Delta (e - x) \in S$, but $x \notin S$, $e - x \notin S$.

If $W = \{x \in E;\ x(\frac{1}{2}) = x'(\frac{1}{2}) = 0\}$, then W is a prime Riesz subspace. But W is not x-maximal in $\mathbf{R}$ for some $x \in E^+$. For, suppose $x'(\frac{1}{2}) \neq 0$, then for $T = \{y \in E;\ y'(\frac{1}{2}) = 0\}$ holds that $x \notin T$, but W is strictly contained in T. If $x'(\frac{1}{2}) = 0$ and $x(\frac{1}{2}) \neq 0$ then for $V = \{y \in E; y(\frac{1}{2}) = 0\}$ holds that $x \notin T$, but W is strictly contained in V, hence W is not a $\mathbf{R}$-relative maximal Riesz subspace.

13.14. Example. An ideal J in a Riesz space E can be x-maximal in $\mathbf{I}$ for certain $x \in E$ without being x-maximal in $\mathbf{R}$. If E is the Riesz subspace in the Riesz space of all real sequences, generated by c_{00} (all real sequences which are eventually 0), $e = (1,1,1,\ldots)$ and $r = (1,2,3,\ldots)$, then c_{00} is e-maximal in I, but $e \notin Rss(c_{00},r)$, hence c_{00} is not e-maximal in $\mathbf{R}$.

13.15. Theorem. *Every proper Riesz subspace* R *is the intersection of all Riesz subspaces* S *such that* $R \subset S$ *and* S *is* $\mathbf{R}$*-relative maximal.*

Proof: Let $\mathbf{R}_R$ be the set of all those Riesz subspaces S such that $R \subset S$

and S is $\mathbf{R}$-relative maximal. It is evident that $R \subset \cap \mathbf{R}_R$. Suppose $x \in \cap \mathbf{R}_R$, $x \notin R$. Then there is a $S \in \mathbf{R}_R$ such that S is x-maximal in $\mathbf{R}$, hence $x \notin S$, contradiction. It follows that $R = \cap \mathbf{R}_R$.

We note that the situation in $\mathbf{I}$ is completely similar. Often, that theorem is given in a milder form, namely that every proper ideal is the intersection of all prime ideals which contain it (cf. e.g. Luxemburg and Zaanen [1971, thm 33.5]). It is surprising that just the analogue of this milder form does not hold in $\mathbf{R}$, as the following example shows.

<u>13.16. Example.</u> If $E = C[0,1]$ and $R = \{x \in E;\ x(0) = x(1)\}$ then R is a proper Riesz subspace of E. R is not prime, because if $e(t) = 1$ and $y(t) = t$ for all $t \in [0,1]$ then $y \wedge (e - y) = 0$, however $y \notin R$, $e - y \notin R$. The only Riesz subspace which contains R properly is E. Hence, the intersection of all prime Riesz subspaces containing R is equal to E.

It is well known (cf. e.g. Schaefer [1974, III thm. 2.2]) that $\{M \in \mathbf{I};\ M \supset J\}$ is a chain in $\mathbf{I}$ whenever J is a prime ideal. Here another disagreement appears; the following example will demonstrate this.

<u>13.17. Example.</u> Let E be the Riesz space as defined in ex. 12.7. Then $P = \{x \in E;\ x(\frac{1}{2}) = x'(\frac{1}{2}) = 0\}$ is a prime Riesz subspace of E. The Riesz subspaces $R = \{x \in E;\ x(\frac{1}{2}) = 0\}$ and $S = \{x \in E;\ x'(\frac{1}{2}) = 0\}$ both contain P strictly, but $R \not\subset S$ and $S \not\subset R$.

<u>13.18. Definition.</u> *A prime Riesz subspace* R *is called minimal in* $\mathbf{R}$ *with respect to a Riesz subspace* S *if* $R \supset S$ *and if it follows from* $S \subset T \subset R$ *for certain prime Riesz subspace* T *that* $T = R$. *We abbreviate this by saying that* R *is a prime Riesz subspace which is* S*-minimal in* $\mathbf{R}$. R *is called* $\mathbf{R}$*-relative minimal if* R *is* S*-minimal in* $\mathbf{R}$ *for some* S.

<u>13.19. Example.</u> If a prime Riesz subspace is R-minimal in $\mathbf{R}$ for some Riesz subspace R then it is not necessarily $\{0\}$-minimal in $\mathbf{R}$ because if $E = C[0,1]$ and R is as in example 13.16 then it follows from this example that E is R-minimal in $\mathbf{R}$. But E is not $\{0\}$-minimal in $\mathbf{R}$, because for $J = \{x \in E;\ x(0) = 0\}$ it holds that J is a prime ideal and $J \neq E$.

13.20. Theorem. *For every Riesz subspace* R *there exists a prime Riesz subspace* S *which is* R*-minimal in* $\mathbf{R}$.
Proof: Let $\mathbf{R}_R = \{T \in \mathbf{R};\ T \supset R$ and T prime$\}$ be ordered by anti-inclusion. $\mathbf{R}_R \neq \phi$, because $E \in \mathbf{R}_R$. Let Σ be a chain in $\mathbf{R}_R$. Then by thm 3.7 $\cap\Sigma \in \mathbf{R}$, also $\cap\Sigma \supset \mathbf{R}$. If $x \wedge y = 0$ for $x,y \in E$ and $x \notin \cap\,\Sigma$, $y \notin \cap\Sigma$ then there exist $K, L \in \Sigma$ such that $x \notin K$, $y \notin L$. Without loss of generality we may suppose $K \subset L$. Then $x \notin K$ and $y \notin K$. Hence $\cap\,\Sigma$ is prime. It follows that $\cap\,\Sigma$ is an upper bound of Σ in $\mathbf{R}_R$. By Zorn's lemma there exists a Riesz subspace which is R-minimal in $\mathbf{R}$.

In the light of example 13.16 we cannot expect that every Riesz subspace R is the intersection of all Riesz subspaces which are R-minimal in $\mathbf{R}$, although it is correct if R is an ideal, as we prove in the following theorem.

13.21. Theorem. *Every ideal* J *is equal to the intersection of all prime Riesz subspaces which are* J*-minimal in* $\mathbf{R}$. *In particular, the intersection of all prime Riesz subspaces which are* $\{0\}$*-minimal in* $\mathbf{R}$ *is equal to* $\{0\}$.
Proof: By Luxemburg and Zaanen [1971, thm 33.6] J is equal to the intersection of all prime ideals M such that $M \supset J$. For every prime ideal M such that $M \supset J$ there exists a prime Riesz subspace R such that $R \subset M$ and R is J-minimal in $\mathbf{R}$. But then certainly J is equal to the intersection of all prime Riesz subspaces which are J-minimal in $\mathbf{R}$.

13.22. Theorem. *For every prime Riesz subspace* R *and all* $x \in E$ *we have that* $\mathrm{Rss}(R,x) = \mathrm{Lss}(R,x) = \{r + \lambda x;\ r \in R,\ \lambda \in \mathbb{R}\}$
Proof: Let $K = \{r + \lambda x;\ r \in R,\ \lambda \in \mathbb{R}\}$. From $\mathrm{Rss}(R,x) \supset K$ it follows that it is sufficient to prove that K is already a Riesz subspace of E. Therefore we shall prove that $(r + \lambda x) \vee (s + \mu x) \in K$ for $r,s \in R$ and $\lambda,\mu \in \mathbb{R}$. R is a prime Riesz subspace,
$(s - r + (\mu - \lambda)x)^+ \wedge (r - s + (\lambda - \mu)x)^+ = 0$, hence
$(s - r + (\mu - \lambda)x)^+ \in R$ or $(r - s + (\lambda - \mu)x)^+ \in R$.
Since $r,s \in R$ it follows that
$(s - r + (\mu - \lambda)x)^+ + r \in R$ or $(r - s + (\lambda - \mu)x)^+ + s \in R$, hence
$(s + (\mu - \lambda)x) \vee r \in R$ or $(r + (\lambda - \mu)x) \vee s \in R$. This implies that
$(s + \mu x) \vee (r + \lambda x) - \lambda x \in R$ or $(r + \lambda x) \vee (s + \mu x) - \mu x \in R$, hence,
there exist $t,u \in R$ such that

$(r + \lambda x) \vee (s + \mu x) = t + \lambda x$ or $(r + \lambda x) \vee (s + \mu x) = u + \mu x$, so $(r + \lambda x) \vee (s + \mu x) \in K$.

13.23. Definition. *A Riesz subspace* R *is called strongly compact in* $\mathbf{R}$ *if it follows from* $R = \cap\Gamma$ *where* $\Gamma \subset \mathbf{R}$ *that* $R = S$ *for some* $S \in \Gamma$.

13.24. Theorem. *A Riesz subspace is* $\mathbf{R}$*-relative maximal if and only if it is strongly compact in* $\mathbf{R}$.

Proof: If R is a Riesz subspace, x-maximal in $\mathbf{R}$ and Γ is a collection of Riesz subspaces such that $R = \cap\Gamma$, then there exists a $S \in \Gamma$ such that $x \notin S$. Together with $S \supset R$ this implies $S = R$.
Conversely, if R is a strongly compact Riesz subspace, then let R^* be the Riesz subspace which is the intersection of all Riesz subspaces S such that $R \subset S$ properly.
Then R^* also properly contains R, because $R^* = R$ would imply that R is equal to a Riesz subspace which properly contains R, a contradiction.
If $x \in R^* \setminus R$ is arbitrary, then R is x-maximal, because if $S \supset R$, and $S \neq R$, then the intersection is also taken over S, hence $x \in S$ because $x \in R^*$.

13.25. Theorem. *A prime Riesz subspace* $R \neq E$ *is* $\mathbf{R}$*-relative maximal if and only if there is a* $x \in E$ *such that* $Rss(R,x) = E$.

Proof: Suppose R is $\mathbf{R}$-relative maximal and there exists no $x \in E$ such that $Rss(R,x) = E$.
Let $y \in E\setminus R$ be arbitrary. Then $Rss(R,y) \neq E$, hence there exists a $z \in E\setminus R$ such that $z \notin Rss(R,y)$.
By the foregoing theorem we have $Rss(R,y) = \{r + \lambda y;\ r \in R,\ \lambda \in \mathbb{R}\}$.
Because $z \notin Rss(R,y)$ we have $z \neq r + \lambda y$ for all $r \in R$ and $\lambda \in \mathbb{R}$.
$y \in Lss(R,z)$ would imply $y = s + \mu z$ for certain $s \in R$ and $\mu \in \mathbb{R}$, $\mu \neq 0$ because $y \notin R$, hence $z = \mu^{-1}(s - y)$, so $z \in Lss(R,y)$, contradiction, hence $y \notin Lss(R,z)$, so by the foregoing theorem $y \notin Rss(R,z)$.
It follows now that $\cap\{Rss(R,y);\ y \notin R\} = R$; by the strong compactness of R we have now $R = Rss(R,y)$ for some $y \notin R$, contradiction.
Conversely, if $Rss(R,x) = E$ for some $x \in E$ and $R \subset S$, $R \neq S$ form some Riesz subspace S, then for an arbitrary $y \in S\setminus R$ we have that $y \in Rss(R,x) = Lss(R,x)$, so $y = r + \lambda x$ for some $r \in R$ and $\lambda \neq 0$.
But then $x \in Lss(R,y) = Rss(R,y) \subset S$, hence $x \in S$. It follows that R is

x-maximal in $\mathbb{R}$.

<u>13.26. Theorem.</u> *If* R *is a prime Riesz subspace which is* $\mathbb{R}$*-relative maximal, then* $\mathrm{Lss}(R,y) = E$ *for every* $y \in E\setminus R$.

Proof: By the foregoing theorem there exists an $x \in E$ such that $\mathrm{Rss}(R,x) = E$.

Let $y \in E\setminus R$ be arbitrary, then there exists a $\lambda \neq 0$ such that $y = r + \lambda x$ for some $r \in R$. Hence $x = -\lambda^{-1}r + \lambda^{-1}y$. If $z \in E$ then $z = t + \mu x$, where $t \in R$, so $z = (t - \lambda^{-1}\mu r) + \lambda^{-1}\mu y \in \mathrm{Lss}(R,y)$.

It follows that $\mathrm{Lss}(R,y) = E$.

<u>13.27. Definition.</u> *A Riesz subspace* R *is called (absolute) maximal in* $\mathbb{R}$ *if the only Riesz subspace which contains* R *properly is* E.

It is evident that every Riesz subspace which is (absolute) maximal in $\mathbb{R}$ is $\mathbb{R}$-relative maximal. The converse holds under the additional assumption that the Riesz subspace is prime, as we show in the following theorem.

<u>13.28. Theorem.</u> *Every prime Riesz subspace which is* $\mathbb{R}$*-relative maximal is (absolute) maximal in* $\mathbb{R}$.

Proof: this is an immediate consequence of thm. 13.26.

<u>13.29. Theorem.</u> *If* R *is a prime Riesz subspace which is* x*-maximal in* $\mathbb{R}$ *for some* $x \in E$*, then for every* $y \in E$ *there exists precisely one* $\lambda_y \in \mathbb{R}$ *such that* $y - \lambda_y x \in R$.

Proof: By thm 13.26 we have $\mathrm{Lss}(R,x) = E$.

If $y \in E$ then $y = r + \lambda_y x$ for certain $r \in R$, $\lambda_y \in \mathbb{R}$, hence $y - \lambda_y x \in R$. If also $y - \lambda x \in R$ for certain $\lambda \in \mathbb{R}$, say $y - \lambda x = s$, then $s - r = (\lambda_y - \lambda)x \in R$. From $x \notin R$ it follows now that $\lambda = \lambda_y$.

In the following theorem we determine the nullspaces of disjunctive linear functionals on E.

<u>13.30. Theorem.</u> *The nullspaces of non-trivial disjunctive linear functionals are precisely the prime Riesz subspaces which are* $\mathbb{R}$*-relative maximal.*

Proof: Let T be a non-trivial disjunctive linear functional on E and

denote the nullspace $N(T)$ of T by N.
$x \in N$ implies $|Tx| = 0$, hence $|T|x|| = 0$ by thm 12.2, hence $|x| \in N$. Together with N a linear subspace this implies that N is a Riesz subspace.
Furthermore, N is prime, because if $x \wedge y = 0$ then $Tx \perp Ty$, hence $Tx = 0$ or $Ty = 0$, so $x \in N$ or $y \in N$.
Because T is non-trivial there exists a $x \in E \backslash N$ such that $Tx = 1$.
Now we have that N is x-maximal in $\mathbf{R}$, because, if $R \supset N$ and $R \neq N$ then there exists a $y \in R$ with $Ty = 1$. Now $x - y \in N$, hence $x - y \in R$, but then also $x \in R$.
Conversely, let N be a prime Riesz subspace which is x-maximal in $\mathbf{R}$.
By thm 13.29 there exists a map T_x from E to $\mathbb{R}$, which assigns to $y \in E$ the real number λ_y such that $y - \lambda_y x \in N$.
We shall prove now that T_x is a disjunctive linear functional with nullspace N. T_x is linear because if $y,z \in E$ then $y + z - (\lambda_y + \lambda_z)x \in N$, hence $T_x(y + z) = T_x y + T_x z$ and for $\lambda \in \mathbb{R}$ we have $\lambda y - \lambda\lambda_y x \in N$, hence $T_x(\lambda y) = \lambda T_x y$.
For $y \in E$ we have by definition $y - (T_x y)x \in N$ and $|y| - (T_x|y|)x \in N$, hence
$|y| - (T_x|y|)x + (y - (T_x y)x) = 2y^+ - (T_x|y| + T_x y)x \in N$ and
$|y| - (T_x|y|)x - (y - (T_x y)x) = 2y^- - (T_x|y| - T_x y)x \in N$.
$2y^+ \wedge 2y^- = 0$, hence $2y^+ \in N$ or $2y^- \in N$ because N is prime.
It follows now that also
$(T_x|y| + T_x y)x \in N$ or $(T_x|y| - T_x y)x \in N$. By the fact that $x \notin N$ we have $T_x|y| + T_x y = 0$ or $T_x|y| - T_x y = 0$, hence (in both cases) $|T_x|y|| = |T_x y|$, so T_x is a disjunctive linear functional.
$N(T_x) = N$ because $T_x y = 0$ for $y \in E$ implies $y = y - 0x \in N$ and $y \in N$ implies $y - 0x \in N$, hence $T_x y = 0$.

13.31. Theorem. *For every* $0 \neq x \in E$ *there exists an* x*-maximal prime Riesz subspace.*
Proof: It follows from thm 13.9 that there exists an ideal J maximal in $\mathbf{I}$ with respect to the property $J \supset \{0\}$ and $x \notin J$. J is prime by thm 13.12. Now by thm 13.6 there exists a Riesz subspace R maximal in $\mathbf{R}$ with respect to the property $R \supset J$ and $x \notin R$.
From $R \supset J$ and J prime it follows that R is prime.

13.32. Corollary. *For every Riesz space* $E \neq \{0\}$ *the intersection of all prime Riesz subspaces which are* $\mathbb{R}$*-relative maximal is equal to* $\{0\}$.

13.33. Theorem. *The set of all disjunctive linear functionals on an arbitrary Riesz space* E *is total, i.e. not empty and such that* $x = 0$ *if* $Tx = 0$ *for all disjunctive linear functionals* T *on* E.

Proof: From thm 13.31 and thm 13.30 it follows that the set of all disjunctive linear functionals is not empty. If $Tx = 0$ for all disjunctive linear functionals T on E then x is an element of the intersection of all nullspaces of disjunctive linear functionals, hence, by thm 13.30 and cor. 13.32 we have $x = 0$.

It was conjectured by Krull that for every Archimedean Riesz space E there exists an injective Riesz homomorphism T from E to a Riesz space of all realvalued functions on an appropriate non-empty set S. A counterexample to this conjecture was given by Jaffard [1955/56] and many others. Note that if E is an Archimedean Riesz space and $T: E \to \mathbb{R}^S$ is an injective Riesz homomorphism, then $\phi \circ T$ is a realvalued Riesz homomorphism for every point-evaluation in $\mathbb{R}^S$, hence in E there exist maximal ideals. But there exist Archimedean Riesz spaces without any maximal ideals (cf. e.g. Schaefer [1974, III, exc.6]).

A general representation theory for Archimedean Riesz spaces was given by Bernau [1965a], who proved that every Archimedean Riesz space can be imbedded (by an injective Riesz homomorphism) in a Riesz space of extended realvalued continuous functions on an appropriate compact Hausdorff space.

Related to this is a theorem of Brown and Nakano [1966] which states that every Archimedean Riesz space is a factor space of an appropriate Riesz space $\mathbb{R}^S$.

With the help of nullspaces of disjunctive linear functionals we can prove the Krull-type conjecture for a T disjunctive linear operator instead of a Riesz homomorphism.

13.34. Theorem. *For every Archimedean Riesz space* E *there exists an injective disjunctive linear operator* T *from* E *to a Riesz space of all real valued functions on an appropriate non-empty set* S.

Proof: Let $\mathcal{D}$ be the set of all disjunctive linear functionals on E and

let F be the Riesz space of all realvalued functions on $\mathcal{D}$ with pointwise linear operations and pointwise partial ordering.
Define the linear operator T: $E \to F$ by $(Tx)(S) = Sx$ for all $S \in \mathcal{D}$.
$x \perp y$ for $x,y \in E$ implies $Sx \perp Sy$ for all $S \in \mathcal{D}$, hence $Tx \perp Ty$ in F, so T is a disjunctive linear operator.
If $Tx = 0$ for all $x \in E$ then $Sx = 0$ for all $S \in \mathcal{D}$, hence by thm 13.33 we have that $x = 0$, so T is injective.

14. Examples of unbounded stabilizers

In this section we give some new examples of unbounded stabilizers and we give some counterexamples for stabilizers of statements which hold for orthomorphisms.

14.1. Example. Let E be the Archimedean Riesz space of all functions x on $[1,2]$ which are piecewise polynomials, i.e. to every $x \in E$ there exist real numbers $\tau_i(x)$ such that $1 = \tau_0(x) < \ldots < \tau_{n+1}(x) = 2$ such that x coincides with a polynomial x_i on every $[\tau_i(x), \tau_{i+1}(x))$ for $i = 0,\ldots,n-1$ and a polynomial x_n on $[\tau_n(x), \tau_{n+1}(x)]$.

To any $t \in [1,2)$ and any $x \in E$ there is an $i(t,x)$ such that $t \in [\tau_{i(t,x)}(x), \tau_{i(t,x)+1}(x))$, let $i(2,x) = n$. Then define

T_t: $E \to \mathbb{R}$ by $T_t x = x_{i(t,x)}(0)$.

If we define a ring structure on E by pointwise multiplication, then for all $t \in [1,2]$ the operator T_t is a ring homomorphism such that $T_t e = 1$ (where e is the function constant 1 on $[1,2]$ and $N(T_t) = J_t$, where J_t is the maximal ring ideal of all functions x such that $x_{i(t,x)}(0) = 0$.

But T_t is for $t \in [1,2]$ not only a ring homomorphism, but also a disjunctive linear functional, because if $x \perp y$ in E then $x_{i(t,x)}(0) = 0$ or $y_{i(t,y)}(0) = 0$, hence $T_t x = 0$ or $T_t y = 0$, so $T_t x \perp T_t y$.

Now we define a linear operator T from E to itself by $(Tx)(t) = T_t x (t \in [1,2])$.
If $x \perp y$ in E then $x(t) = 0$ or $y(t) = 0$ for all $t \in [1,2]$, so if $x_i \neq 0$

for some $i = 0,\dots,n$ then $y_{i(t,y)}(t) = 0$ for $t \in [\tau_{i(t,x)}(x), \tau_{i(t,x)+1}(x))$, so $x \perp Ty$, hence T is a stabilizer on E.

T is unbounded, because $T([0,e])$ is unbounded in E. Namely, if $f \in \mathrm{seq}(E)$ is such that $(f(n))(t) = -nt + n + n^{-1}$ if $t \in [1,1 + n^{-2}]$ and $(f(n))(t) = 0$ if $t \in [1 + n^{-2}, 2]$, then for all $n \geq 2$ we have $0 \leq f(n) \leq e$, however $\{T(f(n));\ n \geq 2\}$ is unbounded, because $T(f(n))(0) = n + n^{-1}$ for all $n \in \mathbb{N}$. T is not regular continuous, because for all $n \in \mathbb{N}$ we have $0 \leq f(n) \leq n^{-1}e$, hence ${}^{r}Lf = 0$, but Tf is not even order convergent, because by the foregoing $\{T(f(n));\ n \geq 2\}$ is unbounded. Hence T is also not order continuous.

Another example of an unbounded stabilizer on E is the following.
If for every $t \in [1,2)$ and $x \in E$ we write $x'_{i(t,x)}(t)$ for the right derivate of the polynomial $x_{i(t,x)}$ in the point t and $x_{i(2,x)}(2)$ for the left limit of $x'_{i(t,x)}(t)$ for $t \to 2$, then define for every $t \in [1,2]$ a linear operator S_t from E to $\mathbb{R}$ by $S_t x = x'_{i(t,x)}(t)$.
Now again S_t is a disjunctive linear functional for every $t \in [1,2]$.

Define a linear operator S from E to itself by $(Sx)(t) = S_t x$ $(t \in [1,2])$, then with a similar proof as for T we can show that S is a stabilizer; with the same order interval $[0,e]$ and the same $f \in \mathrm{seq}(E)$ we can show that also S is unbounded.

Also with the same $f \in \mathrm{seq}(E)$ we can show that S is not regulator continuous and not order continuous.

If we compare S and T, then the following differences are conspicuous.
We have $Te = e$, but $Se = 0$, further $T^2 = T$, but for S holds that for every $x \in E$ there exists a $n_x \in \mathbb{N}$ such that $S^{n_x}x = 0$ (take n_x the maximum of all degrees of the polynomials x_i $(i = 0,\dots,n)$).

In general, orthomorphisms on an Archimedean Riesz space commute (Bernau [1979]). However, arbitrary stabilizers on an Archimedean Riesz space need not commute.
We give an example in the above Riesz space E.
Let R be the orthomorphism on E which is the right multiplication with z where $z(t) = t$ for all $t \in [1,2]$.
Then $RS \neq SR$ because $RSz = Re = z$, but $SRz = Sz^2 = 2z$.

Chapter IV

f-ALGEBRAS

In this chapter Riesz spaces are studied which also have a ring structure, which is connected with the Riesz space structure via a very strong compability condition, the socalled f-algebras. Relations between ring ideals and (order) ideals are discussed, just as relations between ring homomorphisms and Riesz homomorphisms. A variant is given of a theorem of Ellis [1964] and Phelps [1963].

15. Introduction to f-algebras

f-algebras were introduced in Nakano [1950], Amemiya [1953] and Birkhoff and Pierce [1956]. Before giving the definition of f-algebra, we give the definition of Riesz algebra, which is less restrictive.

15.1. Definition. *A Riesz algebra is a quadrupel* $(E, +, \leq, .)$ *where* $(E, +, \leq)$ *is a Riesz space and* $(E, +, .)$ *is an algebra, such that* $x \, . \, y \in E^+$ *for all* $x, y \in E^+$.

In the sequel for fixed x, $\leq$ and . we abbreviate $(E, +, \leq, .)$ to E. Unless otherwise stated we indicate a product of two elements of a Riesz algebra by juxtaposition.

15.2. Definition. *An f-algebra is a Riesz algebra* E *in which for every* $x, y, z \in E^+$ *holds that* $y \wedge z = 0$ *implies* $xy \wedge z = 0$ *and* $yx \wedge z = 0$.

15.3. Theorem. (cf. Bernau [1965b] for positive elements) *A Riesz algebra* E *is an f-algebra if and only if for all* $x, y \in E$ *holds that* $xy \in x^{\perp\perp} \cap y^{\perp\perp}$.
Proof: $\rightarrow$: if $x, y \in E^+$ then $x \wedge z = 0$ for some $z \in E$ implies $xy \wedge z = 0$, hence $xy \in x^{\perp\perp}$. Likewise $xy \in y^{\perp\perp}$. If $x, y \in E$ are arbitrary, then by the foregoing we have $x^+y^+ \in (x^+)^{\perp\perp}$, $x^+y^- \in (x^+)^{\perp\perp}$, $x^-y^+ \in (x^-)^{\perp\perp}$ and $x^-y^- \in (x^-)^{\perp\perp}$. It follows that $xy = (x^+ - x^-)(y^+ - y^-) =$ $x^+y^+ - x^+y^- - x^-y^+ + x^-y^- \in (x^+)^{\perp\perp} + (x^-)^{\perp\perp} \subset |x|^{\perp\perp} = x^{\perp\perp}$. Likewise $xy \in y^{\perp\perp}$, hence $xy \in x^{\perp\perp} \cap y^{\perp\perp}$.

$\leftarrow$: if $y \wedge z = 0$ for some $z \in E$, then for all $x \in E^+$ we have $0 \leq xy \in y^{\perp\perp}$, so $xy \wedge z = 0$. Likewise $yx \wedge z = 0$.

15.4. Definition. *An f-multiplication on a Riesz space* $(E, +, \leq)$ *is a mapping* $. : E \times E \to E$ *such that* $(E, +, \leq, .)$ *is an f-algebra.*

We note that every Riesz space can be given the structure of an f-algebra, namely by the zero-multiplication. Although this f-multiplication seems to be of no value at a first glance, sometimes it is important however; for an example see Keimel [1971, Introduction]. Such an f-algebra is called a zero-algebra.
In our case f-algebras with the property that many products vanish can give rise to pathology. A simple extra assumption can avoid this. We call f-algebras with this property faithful. They will be introduced in the following section.

15.5. Definition. *For an element* x *of an f-algebra* F, *left multiplication by* x *is denoted by* $\mathcal{L}_x$, *and right multiplication by* $\mathcal{R}_x$.

15.6. Lemma. *For each f-algebra* F *and each* $x \in F$ *the mappings* $\mathcal{L}_x$ *and* $\mathcal{R}_x$ *are orthomorphisms on* F. *Moreover, they are positive in the operator ordering if* $x \geq 0$.
Proof: straightforward.

15.7. Examples.

(a) $\mathbb{R}^2$ with componentwise linear operations, componentwise partial ordering and componentwise multiplication. Note that there exists a multiplicative unit, namely $e = (1,1)$.

(b) The componentwise multiplication on $(\mathbb{R}^2, \text{lex})$ is not an f-multiplication because $(1,-1) \geq 0$, $(0,1) \geq 0$, but $(1,-1)(0,1) = (0,-1) \not\geq 0$. Note that $(x_1,x_2)(y_1,y_2) = (x_1y_1,x_2y_1)$ is an f-multiplication on $(\mathbb{R}^2, \text{lex})$, but is not commutative

(c) The Riesz space $C(X)$ where X is a topological space, provided with pointwise multiplication is an f-algebra.

(d) The Riesz space c_{00} of all sequences of real numbers which are eventually zero, provided with componentwise multiplication is an f-algebra.

15.8. Remark. Not every Archimedean f-algebra with a strong order unit is Riesz isomorphic to a Riesz space $C(X)$ with X a (pseudocompact)

Hausdorff space (for a definiton of pseudocompact see Gillman and Jerison [1976, 1.4]). If the Riesz space FR of all sequences of real numbers with a finite range is given an f-multiplication structure by componentwise multiplication, then FR is an Archimedean f-algebra with strong order unit $e = (1,1,1, \dots)$. However FR is not regulator complete, because FR has PP but is not Dedekind complete (Aliprantis and Burkinshaw [1978, Ex. 2.13(3)] and thm 7.9). Hence FR is not Riesz isomorphic to any Riesz space $C(X)$ by thm 7.10.

<u>15.9. Theorem.</u> *In every f-algebra* F *we have for all* $x, y, z \in F$ *that*

(a) $z \geq 0$ *implies* $z(x \vee y) = zx \vee zy$ *and* $(x \vee y)z = xz \vee yz$.

(b) $|xy| = |x||y|$.

(c) $x \perp y$ *implies* $xy = 0$.

(d) $x^2 \geq 0$.

(e) $0 \leq z \leq x$ *and* $yx \geq 0$ *implies* $yz \geq 0$ *(y not necessarily positive).*

(f) if $x, y \geq 0$, $xy = yx$ *and* $(x \vee y)(x \wedge y) = (x \wedge y)(x \vee y)$ *then* $(x \wedge y)^2 = x^2 \wedge y^2$, $(x \vee y)^2 = x^2 \vee y^2$.

(g) if $xy = yx$, $x^-y^+ = y^+x^-$ *and* $(x \vee y)(x \wedge y) = (x \wedge y)(x \vee y)$ *then* $(x \vee y)(x \wedge y) = xy$.

<u>Proof:</u> *(a), (b), (c)* and *(d)* are given by Bigard, Keimel and Wolfenstein [1977, prop.9.1.10].

(e) $yx \geq 0$, so $y^+x - y^-x \geq 0$. Together with $y^+x \wedge y^-x = 0$ this implies that $y^-x = 0$. But then also $y^-z = 0$, hence $yz = y^+z - y^-z = y^+z \geq 0$.

(f) $(x \vee y)(x \wedge y) = (x \vee y)x \wedge (x \vee y)y = (x^2 \vee yx) \wedge (xy \vee y^2)$ $= (x^2 \vee xy) \wedge (xy \vee y^2) = (x^2 \wedge y^2) \vee xy \geq xy$. On the other hand we have $(x \vee y)(x \wedge y) = (x \wedge y)(x \vee y) = (x \wedge y)x \vee (x \wedge y)y =$ $(x^2 \wedge yx) \vee (xy \wedge y^2) = (x^2 \wedge xy) \vee (xy \wedge y^2) = (x^2 \vee y^2) \wedge xy \leq xy$, hence $(x \vee y)(x \wedge y) = xy$. At the same time we see that $(x^2 \wedge y^2) \vee xy = xy$, so $x^2 \wedge y^2 \leq xy$ and $(x^2 \vee y^2) \wedge xy = xy$, so $x^2 \vee y^2 \geq xy$. It follows now that $(x \wedge y)^2 = (x \wedge y)x \wedge (x \wedge y)y$ $= x^2 \wedge yx \wedge xy \wedge y^2 = x^2 \wedge y^2$ and $(x \vee y)^2 = (x \vee y)x \vee (x \vee y)y$ $= x^2 \vee yx \vee xy \vee y^2 = x^2 \vee y^2$.

(g) in *(f)* the statement is already proved for $x, y \geq 0$. By *(f)* and *(c)* we have now for arbitrary $x, y \in F$ that $(x \vee y)(x \vee y)$

$= ((x \vee y)^+ - (x \vee y)^-)((x \wedge y)^+ - (x \wedge y)^-) =$

$= (x^+ \vee y^+ + (-x^- \vee -y^-))(x^+ \wedge y^+ + (-x^- \wedge -y^-)) =$

$= x^+y^+ + (x^+ \vee y^+)(-x^- \wedge -y^-) + (-x^- \vee -y^-)(x^+ \wedge y^+) + x^-y^-$

$= x^+y^+ + x^+(-x^- \wedge -y^-) \vee y^+(-x^- \wedge -y^-) + x^-y^-$
$= x^+y^+ + (0 \wedge -x^+y^-) + (-y^+x^- \wedge 0) + x^-y^- = x^+y^+ - x^+y^- - y^+x^- + x^-y^-$
$= x^+y^+ - x^+y^- - x^-y^+ + x^-y^- = (x^+ - x^-)(y^+ - y^-) = xy.$

A remarkable property of Archimedean f-algebras is the following theorem, for which there exist two entirely different proofs, one by Zaanen [1975] by means of thm 12.20. Another proof is given by Bernau [1965b].

15.10. Theorem. *In every Archimedean f-algebra the multiplication is commutative.*

The following theorem shows that f-multiplication in an Archimedean f-algebra is sequential continuous with respect to each of the three types of convergence, defined in Chapter II.

If f, g $\in$ seq(F) for an f-algebra F, then the sequence h with $h(n) = f(n)\,g(n)$ for all $n \in \mathbb{N}$ is denoted by fg. If $x \in F$ then xf, fx $\in$ seq(F) are defined by $(xf)(n) = xf(n)$, $(fx)(n) = f(n)x$ for all $n \in \mathbb{N}$.

15.11. Theorem. *If* f *and* g *are sequences in an Archimedean f-algebra* F *and*
(a) ${}^{o}L f = x$, ${}^{o}L g = y$ *then* ${}^{o}L fg = xy$
(b) ${}^{r}L f = x$, ${}^{r}L g = y$ *then* ${}^{r}L fg = xy$
(c) ${}^{c}L f = x$, ${}^{c}L g = y$ *then* ${}^{c}L fg = xy$

Proof:
(a) if p, q $\in$ seq (F) are such that
$|f - x| \leq p \downarrow 0$ and $|g - y| \leq q \downarrow 0$, then
$|fg - xy| \leq |fg - fy| + |fy - xy|$
$= |f|\,|g - y| + |f - x|\,|y| \leq (|f - x| + |x|)\,|g - y| + |f - x|\,|y|$
$\leq (p(1) + |x|)q + p|y|$.
$\mathcal{L}_{p(1)+|x|}$ and $\mathcal{R}_{|y|}$ are orthomorphisms in F, hence by thm 12.15 sequential order continuous. It follows that $(p(1) + |x|)q + p|y| \downarrow 0$, hence ${}^{o}L fg = xy$.
(b) if ${}^{r}L(f,u) = x$, ${}^{r}L(g,w) = y$ for $u, w \in F^+$, then for every $\varepsilon > 0$ there exists a $N_\varepsilon \in \mathbb{N}$ such that for all $n \in \mathbb{N}$ holds that $|f(n) - x| \leq \varepsilon u$ and $|g(n) - y| \leq \varepsilon w$ if $n \geq N_\varepsilon$.

Let $0 < \varepsilon < 1$, then we have for all $n \in \mathbb{N}$ with $n \geqslant \max(N_1, N_\varepsilon)$ that
$|f(n)\ g(n) - xy| \leqslant |f(n)\ g(n) - f(n)y| + |f(n)y - xy|$
$\leqslant (|f(n) - x| + |x|)\ |g(n) - y| + |f(n) - x||y|$
$\leqslant (u + |x|)\ \varepsilon w + \varepsilon u|y| = \varepsilon(u + |x|w + u|y|)$, hence ${}^{r}Lfg = xy$.

(c) if for all $n \in \mathbb{N}$ we have that P_n and Q_n are polars in F such that $f(n) - x \in P_n$, $g(n) - y \in Q_n$ for all $n \in \mathbb{N}$ and $P_{n+1} \subset P_n$, $Q_{n+1} \subset Q_n$ for all $n \in \mathbb{N}$ and $\cap \{P_n;\ n \in \mathbb{N}\} = \cap \{Q_n;\ n \in \mathbb{N}\} = \{0\}$, then for all $n \in \mathbb{N}$ we have that $|f(n)\ g(n) - xy| \leqslant |f(n)\ g(n) - f(n)y| + |f(n)y - xy| = |f(n)||g(n) - y| + |f(n) - x||y| \in Q_n + P_n \subset (P_n + Q_n)^{\perp\perp}$ by thm 15.3, hence for all $n \in \mathbb{N}$ we have that $f(n)\ g(n) - xy \in (P_n + Q_n)^{\perp\perp}$. With lemma 8.1 and thm 8.2 it follows now that ${}^{c}Lfg = xy$.

16. Faithful f-algebras

f-algebras for which also holds the converse of thm 15.9 (c), the socalled faithful f-algebras, are interesting. They have already been studied by Bigard, Keimel and Wolfenstein [1977] under the name "f-anneau réduit" and by Cristescu [1976, 4.3.3.] under the name "normal lattice ordered algebra". In the latter the underlying Riesz space is supposed to be Dedekind σ-complete. Here we give another account.

16.1. Definition. *An f-algebra* F *for which holds that* $xy = 0$ *implies* $x \perp y$ $(x, y \in F)$ *is called a faithful f-algebra.*

16.2. Theorem. *For an Archimedean f-algebra* F *the following statements are equivalent*

(a) F *is faithful*

(b) $x > 0$ *implies* $x^2 > 0$ *in* F

(c) the nilradical of F *is equal to* $\{0\}$.

Proof:

(a)→(b) $x^2 = 0$ implies $x \perp x$, so $x = 0$

(b)→(c) if $n \in \mathbb{N}$ is arbitrary and $x \neq 0$ then $|x| > 0$; repeated application of (b) gives $|x|^{2^n} > 0$, hence $x^{2^n} \neq 0$, but then certainly $x^n \neq 0$.

(c)→(a) it is sufficient to prove that for $x, y \geqslant 0$ in F holds that $x \wedge y > 0$ implies $xy > 0$. $x \wedge y > 0$ implies $(x \wedge y)^2 > 0$, but then it is sure that $(x \wedge y)(x \wedge y) > 0$, hence, by thm 15.9(g), $xy > 0$.

16.3. Lemma. *If* x *and* y *are positive elements of a faithful Archimedean f-algebra* F, *then* $x^2 \geq y^2$ *holds if and only if* $x \geq y$. *As a consequence* $x^2 = y^2$ *holds if and only if* $x = y$.

Proof: If $x \geq y$, then $x - y \geq 0$. Together with $x \geq 0$ and $y \geq 0$ this implies $x^2 - xy \geq 0$, so $x^2 \geq xy$ and $xy - y^2 \geq 0$, so $xy \geq y^2$. Now we have $x^2 \geq y^2$.

Conversely, if $x^2 \geq y^2$, then we have $(x \vee y - x)^2 \leq (x \vee y - x)(x \vee y + x) = (x \vee y)^2 - x^2$.

By thm 15.9 *(f)* we have $(x \vee y)^2 = x^2 \vee y^2$, and this implies $(x \vee y - x)^2 \leq x^2 \vee y^2 - x^2 = x^2 - x^2 = 0$.

By thm 16.2 *(a)*,*(b)* we have now $x \vee y - x = 0$, hence $x \geq y$.

16.4. Definition. *A* Φ-*algebra is an f-algebra* F *in which there exists a multiplicative unit* e, *i.e.* $xe = ex = x$ *for all* $x \in F$.

From now on the symbol e (or e_F) is always reserved for the multiplicative unit of a Φ-algebra F.

The following lemma will be used frequently.

16.5. Lemma. *If* x *and* y *are arbitrary elements of an f-algebra* F *and* $y \geq 0$ *then for every* $n \in \mathbb{N}$ *we have* $x^2y - nxy + n^2y \geq 0$ *and* $x^2y - nyx + n^2y \geq 0$.

Proof: if F is a Φ-algebra, then by thm 15.9*(d)* we have $(x - ne)^2 \geq 0$, hence $(x - ne)^2y \geq 0$ and $y(x - ne)^2 \geq 0$, but then certainly $x^2y - nxy + n^2y \geq 0$ and $yx^2 - nyx + n^2y \geq 0$.

If F is an arbitrary f-algebra, we have $(ny - xy)^+ \wedge (xy - ny)^+ = 0$, so by definition $(xy - \frac{1}{n}x^2y)^+ \wedge (xy - ny)^+ = 0$, which implies $[(xy - \frac{1}{n}x^2y) \wedge (xy - ny)]^+ = 0$, so $[xy - (\frac{1}{n}x^2y \wedge ny)]^+ = 0$, but then certainly $(xy - \frac{1}{n}x^2y - ny)^+ = 0$, hence $x^2y - nxy + n^2y \geq 0$.

The following theorem is proved by Bigard, Keimel and Wolfenstein [1977, cor. 12.3.9]. Here we give a direct proof.

16.6. Theorem. *Every Archimedean* Φ-*algebra* F *is faithful.*

Proof: by thm 16.2 it is sufficient to prove that $x^2 > 0$ for every $0 < x \in F$.

If $0 < x \in F$ then since F is Archimedean there exists a $n \in \mathbb{N}$ such that $(nx - e)^+ > 0$. By thm 15.9*(d)* we have $(nx - e)^2 \geq 0$, hence $(-n^2x^2 + 2nx - e)^+ = 0$. Now $(n^2x^2 - nx)^+ = (n^2x^2 - nx)^+ + (-n^2x^2 + 2nx - e)^+$ $\geq (n^2x^2 - nx - n^2x^2 + 2nx - e)^+ = (nx - e)^+ > 0$.
So $(nx^2 - x)^+ > 0$ and this implies $x^2 > 0$.

16.7. Lemma. *The multiplicative unit* e *in a* Φ*-algebra* F *is a weak order unit.*
Proof: $e \geq 0$ because $e = e^2 \geq 0$.
$e \wedge x = 0$ implies $ex = x = 0$, hence e is a weak order unit.

Although every Riesz space can be given an f-multiplication structure, there are Riesz spaces, even with weak order units, which cannot be given a Φ-multiplication structure. We come back to this question in Chapter VI.

In Archimedean Φ-algebras also the converse of lemma 15.6 holds.
In fact they are characterized by this property.

16.8. Theorem. (cf. Zaanen [1975, thm 3]). *If* T *is an orthomorphism on an Archimedean* Φ*-algebra* F*, then there exists a* $z \in F$ *such that* $T = \mathcal{L}_z$.
Proof: if $z = Te$, then T and $\mathcal{L}_z$ coincide on $\{e\}$. Because $e^{\perp\perp} = F$ we have $T = \mathcal{L}_z$ by thm 12.20.

It is a direct consequence of this theorem that z is the unique element of F such that $T = \mathcal{L}_z$, because if also $T = \mathcal{L}_{z'}$ for $z' \in F$, then $\mathcal{L}_{z - z'} = \mathcal{L}_z - \mathcal{L}_{z'} = 0$, hence $(z - z')e = 0$, so $z = z'$.
If $T \geq 0$, then $z \geq 0$ because $z = Te$.

16.9. Theorem. *For every Archimedean* Φ*-algebra* F *it holds that* F *is f-algebra isomorphic to* Orth(F).
Proof: if A is the mapping from F to Orth(F) which assigns to z the element $\mathcal{L}_z$ of Orth(F), then A is a linear operator, bijective by thm 16.8 and lemma 15.6.
A^{-1} is positive by the foregoing.
It follows from Luxemburg and Zaanen [1971, thm 18.5] that T is a Riesz isomorphism.
Further for all $x,y \in F$ we have $A(xy) = \mathcal{L}_{xy} = \mathcal{L}_x\mathcal{L}_y = (Ax)(Ay)$, since multiplication in Orth(F) is composition.
Hence, T is also a ring isomorphism.

17. Ideals and ring ideals

From the last theorem of the foregoing section a fundamental relation can be derived between orthomorphisms on an Archimedean Φ-algebra F and the Φ-multiplication structure on F.
We use the word ideal for order ideal and ring ideal for a twosided ringideal.

17.1. Theorem. *Every orthomorphism on an Archimedean Φ-algebra* F *preserves all ring ideals of* F.
Proof: if $T \in \mathrm{Orth}(F)$, then by thm 16.8 there exists a $z \in F$ such that $T = \mathcal{L}_z$.
For every ring ideal J of F we have now $Tx = zx \in J$ for all $x \in J$.

Also in faithful Archimedean f-algebras and even in a certain sense in every Archimedean Riesz space this phenomenon plays an important role.

It is well known (cf. e.g. Birkhoff [1967, XVII.5, theorem of Fuchs]) that a Riesz algebra is an f-algebra if and only if every polar is a ring ideal. In an Archimedean f-algebra even every o-ideal is a ring ideal.

17.2. Theorem. *Every o-ideal* J *of an Archimedean f-algebra* F *is a ring ideal.*
Proof: it is sufficient to show that $0 \leq x \in J$ and $0 \leq y \in F$ implies $xy \in J$. If $f \in \mathrm{seq}\,(I_x)$ is such that $f(n) = xy \wedge nx$ for all $n \in \mathbb{N}$ then $0 \leq f$ and $\uparrow f$. By thm 15.3 we have $xy \in x^{\perp\perp}$; from Luxemburg and Zaanen [1971, thm 24.7] it follows now that $f \uparrow xy$, hence by thm 6.1*(a)* $xy = {}^{o}Lf$, so $xy \in J$.

17.3. Theorem. *Every r-ideal* J *in an f-algebra* F *is a ring ideal.*
Proof: it is sufficient to prove that $0 \leq x \in J$ and $0 \leq y \in F$ implies $xy \in J$ and $yx \in J$. By lemma 16.5 we have $yx - nx \leq \frac{1}{n}y^2x$ and thus $(yx - nx)^+ \leq \frac{1}{n}y^2x$ for all $n \in \mathbb{N}$.
Define $f \in \mathrm{seq}\,(I_x)$ by $f(n) = yx \wedge nx$ for all $n \in \mathbb{N}$, then $0 \leq f$ and $\uparrow f$.
For all $n \in \mathbb{N}$ we have now $|f(n) - yx| = |0 \wedge (nx - yx)| = (yx - nx)^+ \leq \frac{1}{n}y^2x$, hence $yx \in {}^{r}Lf$, so $yx \in J$.
It can be proved similarly that $xy \in J$; then xy^2 can be taken as a regulator.

17.4. Theorem. *Every c-ideal* J *in an Archimedean f-algebra* F *is a ring ideal.*

Proof: it is sufficient to show that $0 \leq x \in J$ and $0 \leq y \in F$ implies $yx \in J$.

If $f \in \text{seq}\,(I_x)$ is such that $f(n) = yx \wedge nx$ for all $n \in \mathbb{N}$, then $0 \leq f$ and $\uparrow f$. Now we have that $f(n) - yx = 0 \wedge (nx - yx) = -(yx - nx)^+ \in (yx - nx)^{+\perp\perp}$. For all $n \in \mathbb{N}$ it holds that $(yx - (n+1)x)^{+\perp\perp} \subset (yx - nx)^{+\perp\perp}$.

If for $0 \leq v \in F$ holds that $v \in (yx - nx)^{+\perp\perp}$ then $v \perp (yx - nx)^-$ for all $n \in \mathbb{N}$. Since F is Archimedean we have $y = \sup\{(y - \frac{1}{n}yx)^+;\ n \in \mathbb{N}\}$, by thm 3.5 it follows now that $v \perp y$, hence $v \perp yx$, but then for all $n \in \mathbb{N}$ we have $v \perp (yx - nx)^+$, hence $v \in (yx - nx)^{+\perp}$. It follows that $v = 0$, so $\cap\{(yx - nx)^{+\perp\perp};\ n \in N\} = \{0\}$. Hence $yx = {}^{o}Lf$, so $yx \in J$.

17.5. Remark. Not every ideal J in a f-algebra F is a ring ideal. If F is the f-algebra of all sequences of real numbers where the f-multiplication is the componentwise multiplication, and J is the ideal of all sequences converging to 0, $e = (1,1,1,\ldots) \in F$ and $y = (1,\frac{1}{2},\frac{1}{3},\ldots) \in J$, $z = (1,2,3,\ldots) \in F$ then $yz = e \notin J$.

In some f-algebras we are in the pleasant situation that every ideal is a ring ideal. By Luxemburg and Zaanen [1971, p.210, 211] it is observed that in an f-algebra C(X) where X is a compact Hausdorff space (pointwise multiplication on C(X)) this is the case indeed.

Their arguments in fact are sufficient to show that every Φ-algebra in which the multiplicative unit is a strong order unit, is of this type.

17.6. Theorem. *Every ideal* J *in a* Φ*-algebra* F *in which the multiplicative unit is a strong order unit, is a ring ideal.*

Proof: if $x \in J$, $y \in F$ then there exists a $\lambda \in \mathbb{R}$ such that $|y| \leq \lambda e$. Now $|xy| = |x||y| \leq |x|.\lambda e = \lambda|x| \in J$, hence $xy \in J$. Similarly $yx \in J$.

17.7. Remark. The assumption that the multiplicative unit e of F is a strong order unit of F is necessary in thm 17.6. In the f-algebra C(0,1) (pointwise multiplication) the element e with $e(t) = 1$ for all $t \in (0,1)$ is multiplicative unit, but not a strong unit. If $x(t) = t^{-1}$ for all $t \in (0,1)$, then the ideal I_x is not a ring ideal, because $e = xy$ if $y(t) = t$ for all $t \in (0,1)$.

In Luxemburg and Zaanen [1971, p. 211] it is shown that in C[0,1] not every prime ideal is a ring prime ideal.

17.8. Theorem. (Fremlin's theorem, Luxemburg and Zaanen [1971, p.211]). *In every Φ-algebra* C(X) *where* X *is an arbitrary topological space (pointwise multiplication) every ring prime ideal is a prime ideal.*

We not that in the given reference a proof is given under the condition that X is compact Hausdorff, which is not used.

In more general Φ-algebras thm 17.8 does not hold, as the following example shows.

17.9. Example. Let F be the Φ-algebra of all continuous functions on $[1,\infty)$, which are eventually polynomials (pointwise linear operations, partial ordering and multiplication). e with $e(t) = 1$ for all $t \in [1,\infty)$ is the multiplicative unit of F.
If J is the linear subspace of F consisting of all $x \in F$ such that x is eventually polynomial with constant term equal to 0, then J is a ring prime ideal.
But J is not a prime ideal, not even an ideal, because for x and y such that $x(t) = 2t$ and $y(t) = t + 1$ for all $t \in [1,\infty)$ it holds that $0 \leq y \leq x \in J$, $y \notin J$.
Note that the mapping $T: F \to \mathbb{R}$ which assigns to $x \in F$ the constant term of the eventual polynomial of x, not only is a ring homomorphism, but also a disjunctive linear functional. T is not order bounded, because, if $z(t) = t^2$ for all $t \in [1,\infty)$, then $T[0,z]$ is not bounded in $\mathbb{R}$. Let therefore $z_n = z \wedge x_n$ where $x_n(t) = t + n$ for all $n \in \mathbb{N}$.
Now we have $z_n \in [0,z]$ for all $n \in \mathbb{N}$, but $Tz_n = n$ for all $n \in \mathbb{N}$, hence T is not order bounded.

17.10. Theorem. *In every f-algebra* F *every ring prime ideal* J *is a Riesz subspace.*
Proof: it is sufficient to show that $x \in J$ implies $|x| \in J$ for every $x \in F$.
If $x \in J$, then $x^2 \in J$. By thm 15.9*(b)* and *(d)* $x^2 = |x^2| = |x|^2$, hence $|x| \in J$, because J is ring prime.

17.11. Remark. The foregoing theorem does not hold if we take an arbitrary ring ideal. If $F = C[0,1]$, $x(t) = t - \frac{1}{2}$ for all $t \in [0,1]$ and $J = \{xy;\ y \in F\}$ then J is a ring ideal; however $|x| \notin J$, because if $|x| = xy$ for some $y \in F$, then $y(t) = -1$ for all $t \in [0,\frac{1}{2}]$ and $y(t) = 1$ for all $t \in (\frac{1}{2},1]$.

It is a rather natural question to ask for necessary and sufficient conditions for an ideal of an f-algebra to be a ring ideal.
In the light of thms 17.2, 17.3 and 17.4 we can ask whether every ring ideal of an f-algebra is necessarily an o-ideal or a r-ideal or a c-ideal. The following example gives an answer to this question.

17.12. Example. If F is the f-algebra of bounded real sequences with pointwise linear operations, partial ordering and multiplication and J is the ideal of all sequences converging to 0, then J is a ring ideal.
If $e = (1,1,1,..)$, $x = (1,\frac{1}{2},\frac{1}{3},...) \in J$ and e_n is the sequence such that $e_n(m) = 1$ if $m \leq n$ and $e_n(m) = 0$ if $m > n$ ($m \in \mathbb{N}$), then for $f \in \text{seq}\,(I_x)$ with $f(n) = e_n$ for all $n \in \mathbb{N}$, it holds that $0 \leq f$ and $\uparrow f$. Further we have $e = {}^{o}Lf = {}^{r}Lf = {}^{c}Lf$, but $e \notin J$. Hence, J is not an o-ideal, nor a r-ideal, nor a c-ideal.

18. Ringhomomorphisms in Archimedean f-algebras

In this section we study connections between Riesz homomorphisms and ring homomorphisms from an Archimedean f-algebra E to an Archimedean f-algebra F. Further a variant of a recent theorem of Kutateladze is proved, as is a variant of the Ellis-Phelps theorem.

18.1. Theorem. *In every Archimedean Φ-algebra F in which the multiplicative unit is a strong order unit, for a linear operator $\phi: F \to \mathbb{R}$ holds that for the assertions*

(a) ϕ is a Riesz homomorphism with $\phi(e) = 1$ or $\phi(e) = 0$.

(b) ϕ is a ring homomorphism

(c) $\phi(x^2) = (\phi(x))^2$ for all $x \in F$

we have (a) implies (b), (b) implies (c), (c) implies (b).
In the case F *is a* Φ*-algebra* $C(X)$ *(with pointwise linear operations, partial ordering and multiplication) and* X *is pseudocompact and Hausdorff, all assertions are equivalent.*

Proof:

(a) $\rightarrow$ *(b)*: if $\phi(e) = 0$ then the positivity of ϕ implies that $\phi = 0$, hence ϕ is a ringhomomorphism. If $\phi(e) = 1$, then the nullspace J of ϕ is a maximal ideal of E. The following is a modification of a lemma of Papert [1962]. Let $x, y \in F^+$ be such that $\phi(x) = \phi(y) = 0$. If $c = \phi(xy)$, then $c \geqslant 0$. Hence $\phi((ce - x)(e + y)) = 0$. Now we have $0 \leqslant (ce - x)^+ \leqslant (ce - x)^+(e + y) = ((ce - x)(e + y))^+ \in J$, which implies $(ce - x)^+ \in J$, hence $\phi((ce - x)^+) = 0$, so $(\phi(ce - x))^+ = 0$.
Now $\phi(ce - x) \leqslant 0$, hence $c \leqslant \phi(x) = 0$. Together with $c \geqslant 0$ this implies $c = 0$, hence $\phi(xy) = \phi(x)\ \phi(y)$. If $x, y \in F$ are arbitrary such that $\phi(x) = \phi(y) = 0$ then $\phi(x^+) = \phi(x^-) = \phi(y^+) = \phi(y^-) = 0$, hence by the foregoing $\phi(xy) = \phi(x^+y^+) + \phi(x^-y^-) - \phi(x^+y^-) - \phi(x^-y^+) = 0$.
For the general case $x, y \in F$ let $a = \phi(x)$, $b = \phi(y)$ and $c = \phi(xy)$. Then $\phi(ae - x) = \phi(be - y) = 0$. By the foregoing we have $0 = \phi((ae - x)(be - y)) = c - ab$, hence $\phi(xy) = \phi(x)\ \phi(y)$. So, ϕ is a ring homomorphism and J is a maximal ring ideal, even a hyper-real ideal in the sense of Gillman and Jerison [1976, 5.6].

(b) $\rightarrow$ *(c)* : evident.

(c) $\rightarrow$ *(b)* : for $x, y \in F$ we have $\phi(xy) = \frac{1}{2}(\phi((x + y)^2) - \phi(x^2) - \phi(y^2))$

$= \frac{1}{2}((\phi(x + y))^2 - (\phi(x))^2 - (\phi(y))^2) = \phi(x)\ \phi(y)$.

If F is a Φ-algebra $C(X)$ with X pseudocompact and Hausdorff, then e is a strong order unit of F, so the only implication we have to show yet, is *(c)* $\rightarrow$ *(a)*. $\phi(e) = \phi(e^2) = (\phi(e))^2$, hence $\phi(e) = 1$ or $\phi(e) = 0$. ϕ is positive, because if $x \geqslant 0$, then $\phi(x) = \phi((\sqrt{x})^2) = (\phi(\sqrt{x}))^2 \geqslant 0$, where $\sqrt{x}$ is the pointwise square root of x. Now ϕ is a Riesz homomorphism, because for $x \in F$ we have $\phi(|x|)^2 = \phi(|x|^2) = \phi(x^2) = (\phi(x))^2$, which implies, together with the positivity of y, that $\phi(|x|) = |\phi(x)|$.

As a consequence of this theorem, every maximal ideal is a maximal ring ideal.

18.2. Example. For an arbitrary Φ-algebra with strong order unit the implication *(c)* $\to$ *(a)* in the foregoing theorem does not hold in general. If F is the Archimedean Φ-algebra of all piecewise polynomials on $[1,2]$ (ex. 14.1) and $e \in F$ is such that $e(t) = 1$ for all $t \in [1,2]$, then e is a strong order unit of F and at the same time the multiplicative unit of F.
If $\phi: F \to \mathbb{R}$ is the linear operator which assigns to $x \in F$ the constant term $x_0(0)$ of the polynomial x_0 (ex. 14.1), then ϕ is a ring homomorphism, but ϕ is not a Riesz homomorphism, for ϕ is not positive, because $\phi(x) = -1 < 0$ for x such that $x(t) = t - 1$.

In the foregoing example it appears that the ring homomorphism is at the same time a disjunctive linear functional. The following theorem shows that this is a particular case of a more general phenomenon.

18.3. Theorem. *Every ring homomorphism ϕ from an Archimedean f-algebra E to a faithful Archimedean f-algebra F is a disjunctive linear operator.*
Proof: For all $x \in E$ we have $|x|^2 = x^2$, so $\phi(|x|^2) = \phi(x^2)$, hence $(\phi(|x|))^2 = (\phi(x))^2$, or $(|\phi(|x|)|)^2 = (|\phi(x)|)^2$. Lemma 16.3 implies $|\phi(|x|)| = |\phi(x)|$, hence ϕ is a disjunctive linear operator.

18.4. Example. The converse of thm 18.3 does not hold in general, even if $F = \mathbb{R}$. If E is the Archimedean f-algebra of all continuous functions x on $[0,1]$ such that $x'(0)$ (i.e. the right differential quotient of x in the point $t = 0$) exists, then $\phi: E \to \mathbb{R}$ with $\phi(x) = x'(0)$ is a disjunctive linear functional, for let $x \in E$, if $(x(t) - x(0))t^{-1}$ has not a fixed sign for $t \in [0,\delta)$ for every $\delta > 0$, then it follows from the existence of $x'(0)$ that $x(0) = 0$ and $x'(0) = 0$. For $|x|$ similar arguments show that $(|x|)(0) = 0$ and $(|x|')(0) = 0$. This implies $|\phi(|x|)| = |\phi(x)|$. If there exists a $\delta > 0$ such that $(x(t) - x(0))t^{-1}$ has a fixed sign on $[0,\delta)$, then $x(t) \leq x(0)$ on $[0,\delta)$ or $x(t) \geq x(0)$ on $[0,\delta)$. In the case $x(t) \leq x(0)$ on $[0,\delta)$ we consider two subcases, namely $x(0) \leq 0$ and $x(0) > 0$. In the first $(|x|)(t) = -x(t)$ on $[0,\delta)$, hence

$$|\phi(|x|)| = \left|\lim_{t\downarrow 0} \frac{|x|(t) - |x|(0)}{t}\right| = \left|\lim_{t\downarrow 0} \frac{-x(t) + x(0)}{t}\right| = |-\phi(x)| = |\phi(x)|.$$

In the second, by the continuity of x there exists a $\delta_1 \in [0,\delta)$ such that $x(t) \geq 0$ on $[0,\delta_1)$. Now we have $(|x|)(t) = x(t)$ on $[0,\delta_1)$, hence $|\phi(|x|)| = |\phi(|x|)|$. The case $x(t) \geq x(0)$ on $[0,\delta)$ goes similarly. ϕ is not a ring homomorphism, because $\phi(e) = 0$ and $\phi \neq 0$.

Next, we give a theorem which is a variant of a theorem of Kutateladze (Kutateladze [1979, thm 2.5.4]) and a theorem of Meyer (Meyer [1979, Thm 3.3]). The statement in the theorem of Kutateladze is that if S and T are linear operators from a Riesz space E to a Dedekind complete Riesz space F and $0 \leq S \leq T$ and T a Riesz homomorphism then there exists a linear operator $\pi: F \to F$ such that $0 \leq \pi \leq I_F$ and $S = \pi \circ T$ No proof is given. A proof for this theorem in the case E is Archimedean is given by Luxemburg and Schep [1978, Thm 4.3]. In this proof the advantage is used that $\mathfrak{L}^b(E,F)$ is a Dedekind complete Riesz space. Meyer's theorem states that if E is an Archimedean Riesz space and F a regulator complete Riesz space and S and T are linear operators as above, then there exists a linear operator $\pi: J \to J$ such that $0 \leq \pi \leq I_J$ and $S = \pi \circ T$ The proof of this theorem makes use of representation theory.

<u>18.5. Theorem.</u> *If* S *and* T *are linear operators from a Riesz space* E *to an Archimedean* Φ*-algebra* F *in which the multiplicative unit* e *is a strong order unit, and if* $0 \leq S \leq T$ *and* T *is a Riesz homomorphism, then* $TeSx = TxSe$ *holds for all* $x \in E$.

Proof: Let $x \in E$ be arbitrary, ϕ an arbitrary Riesz homomorphism from F to $\mathbb{R}$ with $\phi(e) = 1$ (such a ϕ exists by thm 12.6).

If $y = \phi(Te)x - \phi(Tx)e$, then $(\phi\circ T)(y) = \phi(Te)\phi(Tx) - \phi(Tx)\phi(Te) = 0$. $\phi\circ T$ is a Riesz homomorphism from E to $\mathbb{R}$, hence also $(\phi\circ T)(y^+) = (\phi\circ T)(y^-) = 0$. We have $0 \leq \phi\circ S \leq \phi\circ T$, so $(\phi\circ S)(y^+) = (\phi\circ S)(y^-) = 0$, which implies $(\phi\circ S)(y) = 0$.

Now $0 = (\phi\circ S)(y) = \phi(Te)\phi(Sx) - \phi(Tx)\phi(Se)$, hence $\phi(Te)\phi(Sx) = \phi(Tx)\phi(Se)$. By thm 18.1 ϕ is a ring homomorphism, hence $\phi(TeSx) = \phi(TxSe)$. By thm 12.6 it follows now that $TeSx = TxSe$.

Note that the form of the foregoing theorem is not entirely the same as the form of Kutateladze's theorem. In the following chapter methods are developed which can bring this theorem in the same form.

Finally, we prove a theorem which is a variant of the Ellis-Phelps theorem (Schaefer [1974, III. prop.9.1]). First we have to establish some preliminaries.

18.6. Definition. *A subset* C *of a linear space* L *is called convex if the conditions* $c_1, c_2 \in C$ *and* $0 \leq \lambda \leq 1$ *imply* $\lambda c_1 + (1 - \lambda)c_2 \in C$. $c \in L$ *is called an extremal point of* C *if the conditions* $c = \lambda c_1 + (1 - \lambda)c_2$, $c_1, c_2 \in C$ *and* $0 < \lambda < 1$ *imply* $c = c_1$.

18.7. Lemma. (cf. e.g. Semadeni [1971, lemma 4.4.3]) *If* C *is a convex set of a linear space* L *and* c *and* c' *are elements of* C, *then* c *is an extremal point of* C *if and only if* $c + c' \in C$ *together with* $c - c' \in C$ *implies* $c' = 0$.

18.8. Theorem. *For a positive linear operator* T *from an Archimedean* Φ*-algebra* E *to an Archimedean* Φ*-algebra* F *such that* $Te_E = e_F$, *the following assertions are equivalent*

(a) T *is an extreme point of* $C = \{R \in \mathcal{L}(E,F);\ R \geq 0 \text{ and } Re_E = e_F\}$

(b) T *is a ring homomorphism satisfying* $Te_E = e_F$

(c) T *is a Riesz homomorphism satisfying* $Te_E = e_F$.

Proof:

(a) → *(b)*: The first part of this implication is similar to Semadeni [1971, thm 4.5.3], although there $E = C(X)$ and $F = C(Y)$ with X and Y topological spaces.

Let y_0 be a fixed element of E, such that $0 \leq y_0 \leq e_E$. Then $0 \leq Ty_0 \leq Te_E = e_F$ by the positivity of T. Let $S: E \to F$ be the linear operator such that $Sx = T(xy_0) - TxTy_0$. Now we show that $T + S \in C$ and $T - S \in C$. $(T + S)e_E = e_F + Ty_0 - Ty_0 = e_F$, similarly $(T - S)e_E = e_F$. Further, if $0 \leq x \in E$, then

$(T + S)x = Tx + T(xy_0) - TxTy_0 = Tx(e_F - Ty_0) + T(xy_0) \geq 0$ and

$(T - S)x = Tx - T(xy_0) + TxTy_0 = T(x(e_E - y_0)) + TxTy_0 \geq 0$, hence $T + S$ and $T - S$ are positive.

By lemma 18.7 we have now $S = 0$, hence $T(xy_0) = TxTy_0$ for every $x \in E$.

Now let y_0 be a bounded element of E^+, say $0 \leq y_0 \leq \lambda e_E$ for some $0 < \lambda \in \mathbb{R}$. Then by the preceding argument $T(x\lambda^{-1}y_0) = TxT(\lambda^{-1}y_0)$, hence $T(xy_0) = TxTy_0$.

Let $y \in E^+$ be arbitrary. Define $f \in \text{seq}(E)$ by $f(n) = y \wedge ne_E$ for all $n \in \mathbb{N}$. By lemma 16.5 we have $y^2 - ny + n^2e_E \geq 0$, so $ny - n^2e_E \leq y^2$, hence $(y - ne_E)^+ \leq \frac{1}{n}y^2$, or $|f(n) - y| \leq \frac{1}{n}y^2$.
It follows that ${}^r\mathcal{L}f = y$. Thm 15.11*(b)* gives now that ${}^r\mathcal{L}(xf) = xy$.
By thm 11.3 this implies that ${}^r\mathcal{L}T(xf) = T(xy)$. By the foregoing for every $n \in \mathbb{N}$ holds that $T(xf(n)) = TxTf(n)$, hence also ${}^r\mathcal{L}T(xf) = {}^r\mathcal{L}TxTf = Tx{}^r\mathcal{L}Tf = TxTy$. From the uniqueness of the regulator limit it follows that $T(xy) = TxTy$.
Finally, let $y \in E$ be arbitrary. From $T(xy) = T(xy^+ - xy^-) = T(xy^+) - T(xy^-) = TxT(y^+) - TxT(y^-) = TxT(y^+ - y^-) = TxTy$ it follows now that T is a ring homomorphism.

(b) → *(c)*: F is faithfull because F is a Φ-algebra. Thm 18.3 implies now that T is a disjunctive linear operator. Now, by the positivity of T we have that T is a Riesz homomorphism.

(c) → *(a)*: Let $\bar{E} = \{x \in E;\ |x| \leq \lambda e_E$ for some $\lambda \in \mathbb{R}\}$, $\bar{F} = \{y \in F;\ |y| \leq \mu e_F$ for some $\mu \in \mathbb{R}\}$. Then $T(\bar{E}) \subset \bar{F}$ because, if $x \in \bar{E}$, say $|x| \leq \lambda e_E$ for some $\lambda \in \mathbb{R}$, then $|Tx| = T|x| \leq \lambda Te_E = \lambda e_F$, so $Tx \in \bar{F}$.
Let $\bar{T}: \bar{E} \to \bar{F}$ be defined by $\bar{T}x = Tx$ for all $x \in \bar{E}$, then $\bar{T}$ is a Riesz homomorphism.
If S is a linear operator from E to F such that $0 \leq S \leq T$ then we can define in a similar way a linear operator $\bar{S}: \bar{E} \to \bar{F}$ satisfying $\bar{S}x = Sx$ for all $x \in \bar{E}$.
By thm 18.5 we have now $\bar{S}x = \bar{T}xSe_E$ for all $x \in \bar{E}$. If π from $\bar{F}$ to $\bar{F}$ is the multiplication operator R_{Se_E} then $\pi \in \text{Orth}(\bar{F})$ and $\bar{S} = \pi \circ \bar{T}$.
Suppose now that $T = \lambda T_1 + (1 - \lambda)T_2$ with $T_1, T_2 \in C$, $\lambda \in (0,1)$, then $0 \leq \lambda T_1 \leq T$ and $0 \leq (1 - \lambda)T_2 \leq T$. By the foregoing, there exist $\pi_1, \pi_2 \in \text{Orth}(\bar{F})$ such that $\lambda\bar{T}_1 = \pi_1 \circ \bar{T}$ and $(1 - \lambda)\bar{T}_2 = \pi_2 \circ \bar{T}$.
If $\psi_1 = \lambda^{-1}\pi_1$ and $\psi_2 = (1 - \lambda)^{-1}\pi_2$, then $\bar{T}_1 = \psi_1 \circ \bar{T}$, $\bar{T}_2 = \psi_2 \circ \bar{T}$. From $\bar{T}_1e_E = \bar{T}_2e_E = e_F$ we have $\psi_1e_F = \psi_2e_F = e_F$, so ψ_1 and ψ_2 coincide on the subset $\{e_F\}$ of the Archimedean Riesz space $\bar{F}$. By thm 12.21 it follows now that $\psi_1 = \psi_2$, so $\bar{T}_1 = \bar{T}_2$, hence T_1 and T_2 coincide on $\bar{E}$.
Let $x \in E^+$ be arbitrary. If $f \in \text{seq}(E)$ is defined by $f(n) = x \wedge ne$ for all $n \in \mathbb{N}$, then as in the proof of *(a)* → *(b)* we have that ${}^r\mathcal{L}f = x$.
Now by thm 11.3 we have that ${}^r\mathcal{L}(T_1 - T_2)f = (T_1 - T_2)x$, hence $(T_1 - T_2)x = 0$, so $T_1 = T_2$.

18.9. Remark. The positiveness of T is essentially used in the implication *(b)* $\to$ *(c)* (compare ex. 18.2). However, if E is supposed to be regulator complete, then this assumption can be omitted, as we shall see in thm 22.1.

19. Normalizers and local multipliers on f-algebras

In this section F is an f-algebra.

19.1. Definition. *A linear operator* $A : F \to F$ *is called a normalizer on* F *if for all* $x \in F$ *holds that* $A \circ \mathcal{L}_x = \mathcal{L}_{Ax}$.

19.2. Definition. *A linear operator* $B : F \to F$ *is called a local multiplier on* F *if there exists a mapping* $\alpha: F \to F$ *such that for* $x \in F$ *holds that* $\mathcal{L}_x \circ B = \mathcal{L}_{\alpha(x)}$.

19.3. Theorem. *Every normalizer* A *on* F *is a Jordan operator.*
Proof: $A = A \circ \mathcal{L}_e = \mathcal{L}_{Ae} = \mathcal{L}_{(Ae)^+} - \mathcal{L}_{(Ae)^-}$. By lemma 15.6 $\mathcal{L}_{(Ae)^+}$ and $\mathcal{L}_{(Ae)^-}$ are positive linear operators.

In a general (Archimedean) f-algebra thm 19.3 is false, because on a zero-algebra F every linear operator A is a normalizer, since $A \circ \mathcal{L}_x = 0 = \mathcal{L}_{Ax}$ for all $x \in F$. But it is well known that not every linear operator on a Riesz space is a Jordan operator (cf. Meyer [1979, ex. 2.6.]).

19.4. Theorem. *Every orthomorphism* T *on an Archimedean faithful f-algebra* F *is a normalizer.*
Proof: Let $x,y \in F$.
First, suppose that $Tx = 0$. By thm 15.3 we have $xy \in x^{\perp\perp}$. It is a consequence of thm 12.20 that the nullspace $N(T)$ of T is a band in F, hence $x^{\perp\perp} \subset N(T)$ and thus $xy \in N(T)$, so $T(xy) = 0$.
In this case we have therefore $T(xy) = (Tx)y$.
In the general case, let $S = \mathcal{R}_x \circ T - \mathcal{L}_{Tx}$, then S also is an orthomorphism, because S is the difference of two orthomorphisms.
We have now $Sx = \mathcal{R}_x(Tx) - (Tx)x = 0$.

With the foregoing it follows that $S(xy) = (Sx)y$, hence $(T(xy))x = (Tx)xy$, and since F is commutative $(T(xy))x = (Tx)yx$, hence $(T(xy) - (Tx)y)x = 0$. Since F is faithful $T(xy) - (Tx)y \perp x$. On the other hand we have $Tx \in x^{\perp\perp}$, hence $(Tx)y \in x^{\perp\perp}$; $xy \in x^{\perp\perp}$, hence $T(xy) \in x^{\perp\perp}$. It follows that also $T(xy) - (Tx)y \in x^{\perp\perp}$, hence $T(xy) = (Tx)y$, so $T \circ \mathcal{L}_x = \mathcal{L}_{Tx}$ for all $x \in F$.

19.5. Theorem. *Every normalizer* A *on a faithful f-algebra* F *is a stabilizer.*
Proof: if $x \perp y$ then $xy = 0$, hence $A(xy) = 0$, so $(Ax)y = 0$. Since F is faithful we have $Ax \perp y$, hence A is a stabilizer.

19.6. Theorem. *Every orthomorphism* T *on an Archimedean f-algebra* F *is a local multiplier.*
Proof: Let $x \in F$ be arbitrary, then $\mathcal{L}_x \circ T$ is an orthomorphism and $N(\mathcal{L}_x \circ T) \supset x^{\perp}$.
For $X = \{x, x^{\perp}\}$ holds that $X^{\perp\perp} = F$.
$(\mathcal{L}_{Tx})x = (Tx)x = x(Tx) = (\mathcal{L}_x \circ T)x$.
If $y \in x^{\perp}$, then $(\mathcal{L}_{Tx})y = (Tx)y = 0$ and $(\mathcal{L}_x \circ T)y = x(Ty) = 0$ hence $(\mathcal{L}_{Tx})y = (\mathcal{L}_x \circ T)y$.
It follows now by thm 12.21 that $\mathcal{L}_x \circ T = \mathcal{L}_{Tx}$, so T is a local multiplier.

19.7. Theorem. *Every local multiplier* B *on a faithful Archimedean f-algebra* F *which is a Jordan operator, is an orthomorphism.*
Proof: it is no restriction to take $B \geqslant 0$.
Let $x, y \in F$ be such that $x \wedge y = 0$.
There exists a mapping $\alpha: F \to F$ such that $\mathcal{L}_x \circ B = \mathcal{L}_{\alpha(x)}$, hence $(\mathcal{L}_x \circ B)y = \mathcal{L}_{\alpha(x)}y$.

We may assume $\alpha(x) \geqslant 0$, for otherwise replace $\alpha(x)$ by $\alpha(x)^+$ (because $((By)x)^+ = (\alpha(x)y)^+$, so $(B(y))x = \alpha(x)^+ y)$.
$x \wedge y = 0$ and $\alpha(x) \geqslant 0$ imply $x \wedge \alpha(x)y = 0$, so $x \wedge x(By) = 0$.
Suppose $x \wedge By \neq 0$, then, because F is faithful, $xBy \neq 0$, hence $(xBy)^2 \neq 0$, so, since F is commutative, $x^2(By)^2 \neq 0$, but then certainly $x^2By \neq 0$, hence, since F is faithful, $x \wedge xBy \neq 0$, contradiction. It follows that $x \wedge By = 0$, so B is an orthomorphism.

The following theorem follows directly from the foregoing theorems of this section.

<u>19.8. Theorem.</u> *For a Jordan operator* T *on a faithful Archimedean f-algebra* F *the following assertions are equivalent.*

(a) T *is a normalizer*

(b) T *is a local multiplier*

(c) T *is an orthomorphism*

We remark that in Brainerd [1962] the class of all linear operators on an f-algebra F which are normalizer and local multiplier at the same time, is studied under the name "the normalizer of F". It is proved there by elementary means that the normalizer of an f-algebra without non-zero left or right annihilators is an f-algebra. Since every faithful Archimedean f-algebra satisfies this condition, we have by thm 19.8 the following theorem.

<u>19.9. Theorem.</u> *For every faithful Archimedean f-algebra* F *the space* Orth(F) *is an Archimedean f-algebra under the operator ordering and composition as multiplication.*

Chapter V

INVERSIONS AND SQUARE ROOTS IN ARCHIMEDEAN Φ-ALGEBRAS

An Archimedean Φ-algebra can be closed more or less under taking inverses and square roots. For the case of inversion three closure property conditions are given; each of them seems to be natural. For the case of square roots only one closure property condition is given.
As a corollary of this theory a generalization of a theorem of Ellis [1963] and Phelps [1964] is given.

20. Inversions in Archimedean Φ-algebras.

In this section F is a given Archimedean Φ-algebra, so in particular F is faithful by thm 16.6 and commutative by thm 15.10.

20.1. Definition. $x \in F$ *is called invertible if both the following conditions are satisfied.*
(a) the band B_x *generated by* x *is a projection band*
(b) there exist a $y \in B_x$ *such that* $xy = P_x e$*, where* P_x *is the projection on the band* B_x *(*P_x *exists by virtue of (a)).*
In that case y *is called an inverse of* x*.*

20.2. Theorem. *Any* $x \in F$ *has at most one inverse.*
Proof: if y_1 and y_2 are inverses of x then by definition $y_1, y_2 \in B_x$ and $xy_1 = xy_2 = P_x e \in B_x$, hence $y_1 - y_2 \in B_x$ and $x(y_1 - y_2) = 0$. F is faithful, hence the latter implies $y_1 - y_2 \in B_x^{\perp}$.
It follows that $y_1 = y_2$.
The inverse of x is denoted by x^{-1}.

It follows directly from the definition that x^{-1} is invertible and $(x^{-1})^{-1} = x$.

20.3. Lemma. *For every* $x \in F$ *it holds that* $(x^2)^{-1}$ *exists if and only if* x^{-1} *exists; in that case* $(x^{-1})^2 = (x^2)^{-1}$.
Proof: By thm 15.3 we have $B_{x^2} \subset B_x$.
To prove the converse, let $0 \leq z \in B_x$. It is sufficient to show that $s \wedge x^2 = 0$ for $s \in F$ implies $s \wedge z = 0$. If $s \wedge x^2 = 0$, then also $xs \wedge x^2 = 0$,

hence $x(s\wedge x) = 0$, so since F is faithful $x \perp s\wedge x$, so $s\wedge x = 0$, hence $s\wedge z = 0$. It follows that $B_x = B_{x^2}$ and $P_x = P_{x^2}$. Further we have that $P_x e P_x e = (e - (e - P_x e))P_x e = (e - (I - P_x)e)P_x e = P_x e$ by thm 15.3. If $(x^2)^{-1} = y$, then $x^2 y = P_{x^2} e = P_x e$, hence $x(xy) = P_x e$, so x^{-1} exists and is equal to xy. If x^{-1} exists, then $x^2(x^{-1})^2 = xx^{-1}xx^{-1}$ $= P_x e P_x e = P_x e = P_{x^2} e$. It follows that $(x^{-1})^2 = (x^2)^{-1}$.

20.4. Theorem. (compare Vulikh [1948] and Rice [1968]). *If* x *and* y *are invertible in* F, *then*

(a) $|x|$ *is invertible and* $|x|^{-1} = |x^{-1}|$

(b) $x \geq 0$ *if and only if* $x^{-1} \geq 0$

(c) $x \perp y$ *implies* $x + y$ *invertible and in that case* $(x + y)^{-1} = x^{-1} + y^{-1}$

Proof: *(a)* From thm 15.9 *(b)* it follows that $P_{|x|}e = P_x e = |P_x e| = |x||x^{-1}|$, $B_x = B_{|x|}$ and $|x^{-1}| \in B_{x^{-1}} \subset B_x$, hence $|x|$ is invertible and $|x|^{-1} = |x^{-1}|$.
(b) it is sufficient to show that $x \geq 0$ implies $x^{-1} \geq 0$. $x^{-1} = x^{-1}P_x e$ $= x^{-1}x^{-1}x = (x^{-1})^2 x \geq 0$ by thm 15.9 *(d)*.
(c) from $B_x \cap B_y = \{0\}$ it follows that $B_x + B_y = B_{x+y}$ and B_{x+y} a projection band. Now we have $P_{x+y}e = P_x e + P_y e$ and from $y^{-1} \in B_y$, $x^{-1} \in B_x$ it follows that $xy^{-1} = 0$, $yx^{-1} = 0$, hence $(x + y)(x^{-1} + y^{-1}) = xx^{-1} + xy^{-1} + yx^{-1} + yy^{-1}$ $= P_x e + P_y e$.

Now the announced three closure property conditions for taking inverses are given. Note that the conditions are arranged according to increasing strength.

20.5. Definition. F *is called*

(a) closed under bounded inversion, abbreviated to BI, if every $x \geq e$ *is invertible*

(b) weakly closed under inversion, abbreviated to WI, if every weak order unit is invertible

(c) completely invertible, abbreviated to CI, if every element of F *is invertible.*

Note that for every weak order unit $x \in F$ condition *(a)* of definiton 20.1 is fulfilled automatically.
BI was defined by Henriksen and Johnson [1961]. They proved by means of representation theory that regulator completeness is a

sufficient condition for F to be *BI*. Here we give a direct proof of this fact and an example to show that this condition is not necessary.
CI is already defined by Vulikh [1948] in Dedekind complete Riesz spaces with respect to a given weak order unit, and was studied by Rice [1968]. Their results depend on Freudenthal's spectral theorem (cf. e.g. Luxemburg and Zaanen [1971, Ch. 6]) which holds in Dedekind complete Riesz spaces.

20.6. Theorem. *Every regulator complete Archimedean Φ-algebra* F *is BI.*
Proof: Let $x \in F$, $x \geq e$.
Define $f, g \in seq(F)$ by $f(n) = x \wedge ne$ and $g(n) = (e - \frac{1}{n}x)^+$ for all $n \in \mathbb{N}$. Then $0 \leq g(n) \leq (e - \frac{1}{n}e)^+ = \frac{n-1}{n} e$ for all $n \in \mathbb{N}$. Let for all $n \in \mathbb{N}$ a sequence s_n in F be defined by $s_n(m) = e + g(n) + \ldots + g(n)^m$, then s_n is a regulator Cauchy sequence for all $n \in \mathbb{N}$, because for all $\varepsilon > 0$ there exists an $N_\varepsilon^n \in \mathbb{N}$ such that $(\frac{n-1}{n})^{N_\varepsilon^n} \leq \frac{\varepsilon}{n}$; if $m_1 \geq m_2 \geq N_\varepsilon^n$ then
$|s_n(m_1) - s_n(m_2)| = g(n)^{m_2+1} + \ldots + g(n)^{m_1} = g(n)^{m_2}(e + g(n) + \ldots$
$+ g(n)^{m_1-m_2+1}) \leq (\frac{n-1}{n}e)^{m_2}(e + (\frac{n-1}{n})e + \ldots + (\frac{n-1}{n})^{m_1-m_2+1} e)$
$\leq (\frac{n-1}{n})^{N_\varepsilon^n} (\sum_{i=1}^{\infty} (\frac{n-1}{n})^i) e \leq \frac{\varepsilon}{n} . ne = \varepsilon e.$
Now we can define a sequence s in F by $s(n) = {}^rLs_n$.

Using $f(n) = ne - ng(n)$ for all $n \in \mathbb{N}$ it follows by induction that $f(n)s_n(m) = n(e - g(n)^{m+1})$ holds for all $m \in \mathbb{N}$. If for all $n \in \mathbb{N}$ a sequence h_n in F is defined by $h_n(m) = g(n)^{m+1}$ $(m \in \mathbb{N})$, then ${}^rL(h_n,e) = 0$ for all $n \in \mathbb{N}$, hence, if $t \in seq(F)$ is defined by $t(n) = n^{-1}s(n)$ for all $n \in \mathbb{N}$, then we have $f(n)t(n) = {}^rLf(n)n^{-1}s_n = n^{-1}\, {}^rLn(e - h_n) = {}^rL(e - h_n) = e$. Hence, $t(n)$ is the inverse of $f(n)$ for all $n \in \mathbb{N}$. We shall prove now that t is a x^2-Cauchy sequence in F. Let $\varepsilon > 0$. If $N \in \mathbb{N}$ is such that $N \geq \frac{1}{\varepsilon}$, then for all $n, m \in \mathbb{N}$ with $m \geq n \geq N$ we have $|t(m) - t(n)| = t(n) - t(m)$ $\leq (t(n) - t(m))f(m) = t(n) f(m) - e = t(n)(x \wedge me) - e = (t(n)x \wedge t(n)me) - e$ $= (t(n)x - e) \wedge (t(n)m - e) \leq n^{-1}x^2 \leq N^{-1}x^2 \leq \varepsilon x^2$. From the regulator completeness of F it follows that $y = {}^rL(t,x^2)$ exists in F. It can be proved similarly as in thm 18.8 *(a) → (b)* that ${}^rLf = x$. Now by thm 15.11 *(b)* we have $xy = {}^rLf\, {}^rLt = {}^rLft = e$.

20.7. Example. FR, the Archimedean Φ-algebra of all sequences of real numbers with a finite range (with componentwise multiplication) is not regulator complete (remark 15.8), but FR is *BI*, even *CI*. Hence, regulator completeness is not necessary in the foregoing theorem.

20.8. Example.

(a) If F_1 is the Dedekind complete Φ-algebra of all bounded sequences of real numbers (componentwise multiplication), then $x = (1, \frac{1}{2}, \frac{1}{3}, \ldots)$ is a weak order unit in F_1, however x^{-1} does not exist.

(b) If F_2 is the Archimedean Φ-algebra of all functions x on $\mathbb{R}$ such that for all $t \in \mathbb{R}$ there exists a $\varepsilon > 0$ such that x is constant on $[t, t + \varepsilon)$ (pointwise multiplication), then F_2 is lateral complete and not Dedekind complete. (section 3). Observe that F_2 is *CI*.

A Φ-algebra which is *CI* has PPP by definition, however PP is not automatically fulfilled, as the following example shows.

20.9. Example. If F is the Riesz subspace of the Φ-algebras s of all sequences of real numbers with componentwise multiplication, generated by c_{00} and all elements r^k $(k \in \mathbb{Z})$ with $r = (1, 2, 3, \ldots)$, then F has PPP, but F does not have PP because the band of all elements of F with odd components all equal to 0, is not a projection band in F. Provided with the Φ-multiplication of E, F is itself a Φ-algebra. It is easy to see that F is *CI*.

20.10. Theorem. *If* F *is an Archimedean* Φ*-algebra which is* *WI*, *then for every weak order unit* $u \in F$ *there exists an* f*-multiplication* $*$ *on the underlying Riesz space such that* F *provided with this multiplication* $*$ *is a* Φ*-algebra with multiplicative unit* u.

Proof: Let $*$ be defined by $x * y = xyu^{-1}$ for all $x, y \in F$. Then $(F,*)$ is a Riesz algebra by the positivity of u^{-1}.

If for $x, y, z \in E^+$ holds that $y \wedge z = 0$, then $xyu^{-1} \wedge z = 0$, hence $x * y \wedge z = 0$, hence $(F,*)$ is an f-algebra. For all $x \in F$ we have that $x * u = xuu^{-1} = xe = x$, so u is the multiplicative unit of $(F,*)$.

21. Square roots in Archimedean Φ-algebras

In this section we prove that for an Archimedean Φ-algebra to be closed under taking square roots, its regulator completeness is a sufficient condition. Some results can be generalized to arbitrary f-algebras, but here we restrict ourself to Archimedean Φ-algebras. In this section F is a given Archimedean Φ-algebra, so in particular F is faithful by thm 16.6 and commutative by thm 15.10.

21.1. Definition. *If for* $x, y \in F^+$ *holds that* $y^2 = x$ *then* y *is called a square root of* x.

n th roots are defined by Rice [1966], [1968] in Dedekind complete Riesz spaces with respect to a given weak order unit and by Cristescu [1976, 4.3.2] in Dedekind σ-complete Riesz spaces.

21.2. Theorem. *Any* $x \in F^+$ *has at most one square root.*
Proof: if for $y_1, y_2 \geqslant 0$ holds that $y_1^2 = x$ and $y_2^2 = x$ then $y_1^2 = y_2^2$; since F is faithful we may apply lemma 16.3, hence $y_1 = y_2$.

The square root of x is denoted by $\sqrt{x}$ or $x^{\frac{1}{2}}$.

21.3. Lemma. *If* $\sqrt{x}$ *and* $\sqrt{y}$ exist for $x, y \in F^+$, *then also* $\sqrt{xy}$ *exists and* $\sqrt{xy} = \sqrt{x}\,\sqrt{y}$.
Proof: if $x = w^2$, $y = z^2$ for certain $w, z \in F^+$, then $xy = w^2z^2 = (wz)^2$ and $wz \geqslant 0$, hence $\sqrt{xy}$ exists and is equal to $\sqrt{x}\,\sqrt{y}$.

21.4. Theorem. *If* $\sqrt{x}$ *for* $0 \leqslant x \in F$ *exists then* $\sqrt{x} \in B_x$.
Proof: it is sufficient to prove that $\sqrt{x} \wedge z = 0$ for all $z \in F$ such that $x \wedge z = 0$. From $x \wedge z = 0$ it follows that $\sqrt{x}(\sqrt{x} \wedge z) = x \wedge \sqrt{x}\,z = 0$. Since F is faithful this implies that $\sqrt{x} \perp \sqrt{x} \wedge z$, hence $\sqrt{x} \wedge z = 0$ because $0 \leqslant \sqrt{x} \wedge z \leqslant \sqrt{x}$.

21.5. Theorem. *If* $\sqrt{x}$ *and* $\sqrt{y}$ *exist for* $x, y \in F^+$ *then* $x \perp y$ *implies that* $\sqrt{x+y}$ *exists and* $\sqrt{x+y} = \sqrt{x} + \sqrt{y}$.
Proof: by the foregoing theorem we have that $\sqrt{x} \in B_x$ and $\sqrt{y} \in B_y$, hence

$\sqrt{x} \perp \sqrt{y}$, so $\sqrt{x}\sqrt{y} = 0$. It follows that $(\sqrt{x} + \sqrt{y})^2 = x + y + 2\sqrt{x}\sqrt{y} = x + y$. Together with $0 \leqslant \sqrt{x} + \sqrt{y}$ this implies that $\sqrt{x + y}$ exists and $\sqrt{x + y} = \sqrt{x} + \sqrt{y}$.

In general not every positive element of an Archimedean Φ-algebra has a square root; e.g. in ex. 17.9 the element z such that $z(t) = t$ has no square root.

21.6. Definition. F *is called closed under taking square roots, abbreviated to SR if* $\sqrt{x}$ *exists for every* $x \in F^+$.

Similar to the case of bounded inversion (sect. 19), a sufficient condition for an Archimedean Φ-algebra F to be *SR* is regulator completeness of F. That this condition is not necessary is clear if we take FR, as in ex. 20.7. FR is *SR*, but not regulator complete.

21.7. Theorem. *Every regulator complete Archimedean* Φ*-algebra* F *is SR.*
Proof: The proof is by means of an approximation procedure, which is a generalization to regulator complete Archimedean Φ-algebras of the well known Newton method for approximating zeros of a real- or complex valued function on $\mathbb{R}$. For the proof that there exists a limit we have to be more careful than in the real case, where in general the proof is based on the Dedekind completeness of $\mathbb{R}$.
The proof that our approximating sequence is monotone is similar to the proof of Visser [1937] (see also Luxemburg and Zaanen [1971, p. 377]). However, in the sequel of that proof a property of the ordered vector space of Hermitean operators on a Hilbert space is used (Luxemburg and Zaanen [1971, Thm 53.4]), which is analogous to Dedekind completeness of a Riesz space and is not available in our case, so we have to proceed differently. First we prove that $\sqrt{x}$ exists for every $x \in [\delta e, e]$, where $\delta \in (0,1]$.

Let $f \in seq(F)$ be defined inductively by $f(1) = e$, $f(n + 1) = f(n) + \frac{1}{2}(x - f(n)^2)$ for all $n \in \mathbb{N}$. If $z = e - x$ and $g = e - f$, then $z \in [0, (1 - \delta)e]$, $g(1) = 0$ and $g(n + 1) = \frac{1}{2}(z + g(n)^2)$ for all $n \in \mathbb{N}$.

Now we prove by induction that $g(n)$ and $g(n+1) - g(n)$ are polynomials in z with non-negative coefficients for all $n \in \mathbb{N}$. For $n = 1$ the statement is true, because $g(1) = 0$ and $g(2) - g(1) = \frac{1}{2}z$. If for certain $k \in \mathbb{N}$, $k \geq 2$, holds that $g(k-1)$ and $g(k) - g(k-1)$ are polynomials in z with non-negative coefficients, then also $g(k)$ is such a polynomial, but then also $g(k) + g(k-1)$. Now we have that $g(k+1) - g(k)$ $= \frac{1}{2}(g(k)^2 - g(k-1)^2) = \frac{1}{2}(g(k) + g(k-1))(g(k) - g(k-1))$ is a polynomial in z with non-negative coefficients. It follows that $0 \leq g$ and $\uparrow g$.

In order to prove that f is an e-Cauchy sequence in F we show that $0 \leq g \leq (1-\delta)e$. $0 \leq g(1) \leq (1-\delta)e$ because $g(1) = 0$. Suppose $0 \leq g(k) \leq (1-\delta)e$, then $g(k+1) = \frac{1}{2}(z + g(k)^2) \leq \frac{1}{2}((1-\delta)e + (1-\delta)^2 e)$ $\leq \frac{1}{2}((1-\delta)e + (1-\delta)e) = (1-\delta)e$. Now by induction we have $0 \leq g \leq (1-\delta)e$. It follows that $g(n+1) - g(n) = \frac{1}{2}(g(n) + g(n-1))(g(n) - g(n-1))$ $\leq (1-\delta)(g(n) - g(n-1))$ for all $n \in \mathbb{N}$. Now $f(n) - f(n+1)$ $= g(n+1) - g(n) \leq (1-\delta)(g(n) - g(n-1)) = (1-\delta)(f(n-1) - f(n))$ holds for all $n \in \mathbb{N}$. Further $0 \leq f(1) - f(2) = -\frac{1}{2}(x - f(1)^2)$ $= -\frac{1}{2}(x - e) = \frac{1}{2}e - \frac{1}{2}x \leq \frac{1}{2}e$. This implies that $|f(n) - f(n+1)| \leq \frac{1}{2}(1-\delta)^n e$ holds for all $n \in \mathbb{N}$.
Let $\varepsilon > 0$. If $N \in \mathbb{N}$ is such that $\frac{1}{2}(1-\delta)^{N+1}\delta^{-1} \leq \varepsilon$, then for all $n, m \in \mathbb{N}$ such that $n \geq m \geq N$ holds that $|f(n) - f(m)| = f(m) - f(n)$

$$= (f(m) - f(m+1)) + (f(m+1) - f(m+2)) + \dots (f(n-1) - f(n))$$
$$\leq \frac{1}{2}(1-\delta)^m e + \frac{1}{2}(1-\delta)^{m+1} e + \dots + \frac{1}{2}(1-\delta)^{n-1} e$$
$$\leq \frac{1}{2}(1-\delta)^m \sum_{k=1}^{\infty} (1-\delta)^k e \leq \frac{1}{2}(1-\delta)^{N+1}\delta^{-1} e \leq \varepsilon e.$$

It follows that f is an e-Cauchy sequence in F; if $y = {}^r L f$, then by thm 15.11 *(b)* we have that $y^2 = {}^r L(f^2)$

But also ${}^r L(f^2) = x$, because if $\varepsilon > 0$, then take $N \in \mathbb{N}$ such that $(1-\delta)^N \leq \varepsilon$. Then for all $n \in \mathbb{N}$ such that $n \geq N$ we have $|x - f(n)^2|$ $= 2(f(n+1) - f(n)) \leq 2 \cdot \frac{1}{2}(1-\delta)^n e \leq (1-\delta)^N \leq \varepsilon e$. Now by the uniqueness of the regulator limit in F we have $x = y^2$.

Suppose now $x \in [0, e]$ arbitrary. By the foregoing $\sqrt{x \vee n^{-1} e}$ exists for every $n \in \mathbb{N}$. Let $h \in \mathrm{seq}(F)$ be defined by $h(n) = \sqrt{x \vee n^{-1} e}$ for all $n \in \mathbb{N}$.

We show that h is an e-Cauchy sequence in F. Let $\varepsilon > 0$. Take $N \in \mathbb{N}$ such that $N \geq 2\varepsilon^{-2}$. For all $n, m \in \mathbb{N}$ such that $n \geq m \geq N$ we have that $\sqrt{(x\vee m^{-1}e)(x\vee n^{-1}e)}$ exists.

Further $(N^{-1} - m^{-1})e \leq N^{-1}e \leq \frac{1}{2}\varepsilon^2 e$. Now we have $(x - N^{-1}e)^+ \leq (x - m^{-1}e)^+$, hence $(x\vee N^{-1}e) - (x\vee m^{-1}e) \leq (N^{-1} - m^{-1})e \leq \frac{1}{2}\varepsilon^2 e$.

This implies that $2(x\vee N^{-1}e) - 2\sqrt{(x\vee m^{-1}e)(x\vee n^{-1}e)} \leq \varepsilon^2 e$, so $(x\vee m^{-1}e) + (x\vee n^{-1}e) - 2\sqrt{(x\vee m^{-1}e)(x\vee n^{-1}e)} \leq \varepsilon^2 e$, hence $\left(\sqrt{x\vee m^{-1}e} - \sqrt{x\vee n^{-1}e}\right)^2 \leq (\varepsilon e)^2$. It follows from thm 16.6 that $\sqrt{x\vee m^{-1}e} - \sqrt{x\vee n^{-1}e} \leq \varepsilon e$, so h is an e-Cauchy sequence in F.

Let $w = {}^r\mathcal{L}h$. For all $n \in \mathbb{N}$ it holds that $0 \leq (\frac{1}{n}e - x)^+ \leq \frac{1}{n}e$; this implies (thm 7.1 *(e)*) that ${}^r\mathcal{L}p = 0$ if $p(n) = (\frac{1}{n}e - x)^+$ for all $n \in \mathbb{N}$. It follows that for $q \in \mathrm{seq}(F)$ with $q(n) = x\vee n^{-1}e$ holds that ${}^r\mathcal{L}q = x$.

By thm 15.11 *(b)* we have $w^2 = {}^r\mathcal{L}h^2 = {}^r\mathcal{L}q = x$.

Finally, let $x \in F^+$ be arbitrary. Then $\sqrt{x\wedge e}$ exists by the foregoing. By thm 20.6 we have that $(x\vee e)^{-1}$ exists and $(x\vee e)^{-1} = (x\vee e)^{-1}e \leq (x\vee e)^{-1}(x\vee e) = e$, hence by the foregoing we have that $\sqrt{(x\vee e)^{-1}}$ exists. $(x\vee e)^{-1} = \left(\sqrt{(x\vee e)^{-1}}\right)^2$ is invertible, so $\left(\left(\sqrt{(x\vee e)^{-1}}\right)^2\right)^{-1}$ exists and is equal to $x\vee e$.

By lemma 20.3 also $\left(\sqrt{(x\vee e)^{-1}}\right)^{-1}$ exists and it holds that $\left(\left(\sqrt{(x\vee e)^{-1}}\right)^{-1}\right)^2 = \left(\left(\sqrt{(x\vee e)^{-1}}\right)^2\right)^{-1} = x\vee e$, so $\sqrt{x\vee e}$ exists. It follows by lemma 21.3 that $\sqrt{x} = \sqrt{(x\wedge e)(x\vee e)} = \sqrt{x\wedge e}\,\sqrt{x\vee e}$ exists.

22. A generalization of the Ellis-Phelps theorem

The following theorem is a variant of thm 18.8 and a generalization of the Ellis-Phelps theorem (Schaefer [1974, III thm 9.1]).

22.1. Theorem. *If* E *and* F *are Archimedean* Φ*-algebras and moreover* E *is regulator complete then for* $T \in \mathcal{L}(E,F)$ *the following assertions are equivalent* *(a)* T *is an extreme point of* $C = \{R \in \mathcal{L}(E,F);\ R \geq 0$ and $Re_E = e_F\}$

(b) T *is a ring homomorphism satisfying* $Te_E = e_F$

(c) T *is a Riesz homomorphism satisfying* $Te_E = e_F$.

Proof: The proof is essentially the same as the proof of thm 18.8 except *(b)* $\to$ *(c)* where we have to prove that $T \geq 0$.

By thm 21.7 E is *SR*. Now we have for every $x \in E^+$ that $Tx = T((\sqrt{x})^2) = (T(\sqrt{x}))^2 \geq 0$, hence $T \geq 0$.

22.2. Remark. It is obvious that assertion (d) of Schaefer [1974, III thm 9.1] cannot be taken into account here.

Chapter VI

PARTIAL f-MULTIPLICATIONS IN ARCHIMEDEAN RIESZ SPACES

In this chapter a mapping reminiscent of f-multiplication is defined on Archimedean Riesz spaces with a strong order unit, and a connection with orthomorphisms is demonstrated.

23. Introduction and definitions

In section 15 it was shown that every Riesz space can be made into an f-algebra by the zero-multiplication, but it was remarked that for our purposes this multiplication is not useful.
It is a natural question to ask whether every Archimedean Riesz space E can be provided with an f-multiplication (def. 15.4) such that a previously given weak order unit e of E is the unit of multiplication. The answer is no. If $E = c_0$ and $e = (1, \frac{1}{2}, \frac{1}{3}, \ldots)$ then Orth (E) is Riesz isomorphic to b (Zaanen [1975, ex.vi]). But b is not Riesz isomorphic to c_0, because b has a strong order unit and c_0 has not, hence by thm 16.9 there exists no f-multiplication on c_0 such that e is the unit of multiplication.
Hager and Robertson [1977] ask for necessary and sufficient conditions for the existence of such an f-multiplication, but the authors remark that in general they had little success in identifying such "rings in disguise". In section 24 this question is answered for Archimedean Riesz spaces with a given strong order unit.
From Conrad [1974] it can be deduced that in fact the ring structure of an Archimedean Φ-algebra is determined by the Riesz space structure of E, because he proved that if E is an Archimedean Φ-algebra with f-multiplication #, and * is another f-multiplication on E, then $x * y = x \# y \# z$ for all $x,y \in E$ and some fixed $z \in E^+$. Hence, if there exists an f-multiplication on a given Archimedean Riesz space E with a weak order unit e such that e is the multiplicative unit, then there exists no other f-multiplication on E with the same multiplicative unit.

The foregoing is a reason to study Archimedean Riesz spaces on which is defined a partial f-multiplication or a partial Φ-multiplication, of which the definitions are given below. To avoid the situation that too many products vanish, we require a kind of faithfulness of the partial f-multiplication.

23.1. Definition (cf. Veksler [1967]) *A pair* $(E,*)$ *is called an Archimedean Riesz space with a partial f-multiplication (abbreviated to Archimedean pf-algebra) if* E *is an Archimedean Riesz space and* $*$ *is a mapping from* $E\times E$ *to* $E\cup\{\phi\}$ *such that for all* $x,y,z \in E$ *holds that (we write* $x*y$ *in stead of* $*(x,y)$*)*

(PF1) $x*y = y*x$

(PF2) *if* $x*y \neq \phi$ *and* $y*z \neq \phi$, *then* $x*(y*z) = (x*y)*z$

(PF3) *if* $x*y \neq \phi$, $x*z \neq \phi$, *then* $x*(y+z) = x*y + x*z$

(PF4) *if* $x,y \geq 0$ *and* $x*y \neq \phi$ *then* $x*y \geq 0$

(PF5) *if* $x*y \neq \phi$ *then* $(\lambda x)*y = \lambda(x*y)$ *for all* $\lambda \in \mathbb{R}$

(PF6) $x*y = 0$ *if and only if* $x \perp y$.

For fixed $*$ we abbreviate $(E,*)$ to E.
$*$ is called a pf-multiplication on E.

23.2. Definition. *An Archimedean pf-algebra* $(E,*)$ *is called multiplication complete if* $x*y \neq \phi$ *for all* $x,y \in E$.

Note that if $*$ is a pf-multiplication on an Archimedean Riesz space E, then it follows from (PF6) that $*^{-1}(E) \supset \{(x,y) \in E\times E;\ x\perp y\}$.

23.3. Theorem. *On every Archimedean Riesz space* E *a pf-multiplication can be defined.*

Proof: Let $*$ be the mapping from $E\times E$ to $E\cup\{\phi\}$ which assigns to $(x,y) \in E\times E$ the element 0 if $x\perp y$ and ϕ otherwise. Then it follows from the elementary properties of orthogonality that $*$ is a pf-multiplication on E.

23.4. Theorem. *Every multiplication complete Archimedean pf- algebra* $(E,*)$ *is a faithful Archimedean f-algebra and conversely.*

Proof: $\rightarrow$: if $x,y,z \in E^+$ such that $y\wedge z = 0$, then $y*z = 0$, hence $x\perp y*z$, so $x*(y*z) = 0$.

It follows that $(x*y)*z = 0$, hence $x*y\perp z$.

Together with $x*y \geq 0$ and $z \geq 0$ this implies $x*y\wedge z = 0$. The other equality follows from the fact that $(x*y)*z = 0$ implies $(y*x)*z = 0$, so $(y*x)\perp z$, and the positiveness of $y*x$ and z. $(E,*)$ is faithful by (PF6).

$\leftarrow$:(PF1) follows from thm 15.10, (PF4) follows from the fact that E is a Riesz algebra, (PF6) follows from the faithfulness of E. The remaining PF's are easily verified.

Now we state a result similar to thm 15.3.

<u>23.5. Theorem.</u> *If* x *and* y *are elements of an Archimedean pf-algebra* E *such that* $x*y \neq \phi$, *then* $x*y \in x^{\perp\perp} \cap y^{\perp\perp}$.
Proof: if $x*y \notin x^{\perp\perp} \cap y^{\perp\perp}$, assume $x*y \notin x^{\perp\perp}$, then there exists a $z \in x^{\perp}$ such that $x*y \perp z$ does not hold. If $w = |x*y| \wedge |z|$ then $w > 0$, $w \in (x*y)^{\perp\perp}$ and $w \in x^{\perp}$, so $w*x = 0$, hence $(w*x)*y = 0$ so $w*(x*y) = 0$, consequently $w \perp (x*y)$, contradiction. It follows that $x*y \in x^{\perp\perp} \cap y^{\perp\perp}$.

Veksler [1967] gives an interesting list of properties an Archimedean pf-algebra can have. We give this list here and we show that none of these properties follows from the axioms for an Archimedean pf-algebra. Our notation is slightly different from Veksler's.

<u>23.6. Definition.</u> *An Archimedean pf-algebra* E *is said to have property*

A. *(normality of the multiplication) if* $x*y \neq \phi \rightarrow x_1*y_1 \neq \phi$ *for every* $x_1, y_1 \in E$ *such that* $|x_1| \leqslant |x|$ *and* $|y_1| \leqslant |y|$.
B. *(monotony of the multiplication) if* $|x_1| \leqslant |x|$, $|y_1| \leqslant |y|$ *and* $x*y \neq \phi$, $x_1*y_1 \neq \phi \rightarrow |x_1*y_1| \leqslant |x*y|$.
C. *if for certain* $z \in E$ *holds that the band* B_z *is a projection band and* $x*y \neq \phi \rightarrow P_z x * P_z y \neq \phi$.
D. *if* $x*y \neq \phi \rightarrow |x|*|y| = |x*y|$
E. *if* $x*y \neq \phi \rightarrow x*|y| \neq \phi$
F. *if* $x*y \neq \phi \rightarrow (x \vee y)*(x \wedge y) = x*y$
G. *(Rule of signs)* if $(xy)^{+\perp\perp} \subset ((x^+ \wedge y^+) \vee (x^- \wedge y^-))^{\perp\perp}$.

<u>23.7. Examples.</u>

(a) Let E be the Riesz space of all piecewise linear continuous functions x on [0.1], i.e. to every $x \in E$ there exist real numbers $\tau_i(x)$ such that $0 = \tau_0(x) < \ldots < \tau_{n+1}(x) = 1$ such that x coincides with a linear function x_i on every $[\tau_i(x), \tau_{i+1}(x))$ $(i = 0, \ldots, n-1)$ and with a linear function x_n on $[\tau_n(x), \tau_{n+1}(x)]$.
Let $e(t) = 1$ for all $t \in [0,1]$.
Define $x*y = z$ for x, y, $z \in E$ if for all $t \in [0,1]$ holds that $x(t)\, y(t) = z(t)$; if for x, $y \in E$ there exists no $z \in E$ such that $x*y = z$, then define $x*y = \phi$. Then $(E,*)$ is an Archimedean pf-algebra,

but does not have property A because $e*e \neq \phi$, but $x*x = \phi$ if $x(t) = t$ for al $t \in [0,1]$, although $|x| \leqslant |e|$.

(b) Let E be the Riesz space $(\mathbb{R}^2, \leqslant)$ where $\leqslant$ is the componentwise partial ordering.
Define * from $E \times E$ to $E \cup \{\phi\}$ by $x*y = 0$ if $x \perp y$, $x*y = (\frac{1}{2}\lambda\mu, \lambda\mu)$ if $x = (2\lambda, \lambda)$ and $y = (2\mu, \mu)$ $(\lambda, \mu \in \mathbb{R})$, $x*y = (\lambda\mu, \lambda\mu)$ if $x = (\lambda, \lambda)$ and $y = (\mu, \mu)$ $(\lambda, \mu \in \mathbb{R})$ and $x*y = \phi$ otherwise.
Then $(E,*)$ is an Archimedean pf-algebra. $(E,*)$ does not have property B, because if $x = y = (2,1)$ and $x_1 = y_1 = (1,1)$ then $|x_1| \leqslant |x|$ and $|y_1| \leqslant |y|$, however $x*y = (\frac{1}{2},0)$ and $x_1*y_1 = (1,0)$.

(c) Let E be the Riesz space $(\mathbb{R}^2, \leqslant)$ where $\leqslant$ is the componentwise partial ordering.
Define * from $E \times E$ to $E \cup \{\phi\}$ by $x*y = 0$ if $x \perp y$, $x*y = (\lambda\mu, \lambda\mu)$ if $x = (\lambda,\lambda)$ and $y = (\mu,\mu)$ $(\lambda,\mu \in \mathbb{R})$, otherwise $x*y = \phi$.
Then $(E,*)$ is an Archimedean pf-algebra which has PPP. If $z = (1,0)$, $x = y = (1,1)$ then $x*y = (1,1) \in E$, $P_z x = P_z y = (1,0)$ and $(P_z x)*(P_z y) = \phi$, hence $(E,*)$ does not have property C.

(d) Let E be the Riesz space s of all sequences of real numbers. We define $x*y$ to be the componentwise product of x and y if $|\sum_{i=1}^{\infty} x_i y_i| < \infty$, otherwise $x*y = \phi$, then $(E,*)$ is an Archimedean pf-algebra. If $x = (1, \frac{1}{2}, \frac{1}{3}, \ldots)$ and $y = (1,-1,1,-1,\ldots)$ then $x*y \in E$, however $|x|*|y| = \phi$ and $|x*y| \neq \phi$ so $(E,*)$ does not have property D

(e) The example (d) shows at the same time that $(E,*)$ does not have property E. However, there exists an example of an Archimedean pf-algebra which does have property D, but not property E. This example is given by Veksler [1967]. Let E be the set of all pairs $[f,g]$ where f and g are continuous functions from $[0,1]$ to the two-point compactification $\bar{\mathbb{R}}$ of $\mathbb{R}$, such that there exist $\lambda,\mu,\nu \in \mathbb{R}$ and $\phi,\psi \in C[0,1]$ (all depending on $[f,g]$ such that for all $t \in [0,1]$, $t \neq \frac{1}{2}$ holds that

$$f(t) = \phi(t) + \frac{\lambda}{(t-\frac{1}{2})^2} + \frac{\mu}{|t-\frac{1}{2}|} \qquad \text{and}$$

$$g(t) = \psi(t) + \frac{\lambda}{(t-\frac{1}{2})^2} + \frac{\nu}{|t-\frac{1}{2}|}$$

E can be made into an Archimedean Riesz space by componentwise pointwise operations. Let $[f_1,g_1]*[f_2,g_2]$ be the componentwise pointwise product (where by definition $-\infty.0 = \infty.0 = 0.-\infty = 0.\infty = 0$) only if this product is again a member of E, otherwise $[f_1,g_1]*[f_2,g_2] = \phi$.
Then (E,*) is an Archimedean pf-algebra.
If $f: [0,1] \to \bar{\mathbb{R}}$ is such that $f(t) = \frac{1}{|t-\frac{1}{2}|}$ for $t \neq \frac{1}{2}$, $f(\frac{1}{2}) = \infty$,
then, if $x = y = [f,-f]$, we have $x*y = [f^2,f^2] \in E$, but $x*|y| = \phi$.
We remark that $[f^2,f^2]$ is a strong order unit of E.
Note that the pf-multiplication used here is rather natural; this seems to be an indication that the conditions given by Bernau [1965a] and Papert [1962] for a Riesz space to be an Archimedean pf-algebra are too strong.

(f) (Veksler [1967, ex.2]) Let E be the Riesz space of all sequences of real numbers. Define x*y to be the componentwise product of x and y only if $\min(\sup\{x(n);\, n \in N\}, \sup\{y(n);\, n \in N\}) < \infty$, where $N = \{n \in \mathbb{N};\, x_n y_n \neq 0\}$, and $x*y = \phi$ otherwise.
If $x = (-1,2,-3,4,\ldots)$ and $y = (1,1,1,\ldots)$ then $x*y \neq \phi$, but $(x \vee y)*(x \wedge y) = \phi$, hence (E,*) does not have property *F*.

(g) Let E be the Riesz space $(\mathbb{R}^2, \leq)$ where $\leq$ is the componentwise partial ordering. Define $x*y = 0$ if $x \perp y$, $x*y = (-\lambda\mu, -\lambda\mu)$ if $x = (\lambda,-\lambda)$ and $y = (\mu,-\mu)$ $(\lambda,\mu \in \mathbb{R})$, otherwise $x*y = \phi$.
Then (E,*) is an Archimedean pf-algebra.
If $x = (1,-1)$, $y = (-1,1)$, then $(x^+ \wedge y^+) \vee (x^- \wedge y^-) = 0$, hence $((x^+ \wedge y^+) \vee (x^- \wedge y^-))^{\perp\perp} = \{0\}$. However, $(x*y)^+ = (1,1)$, hence $(x*y)^{+\perp\perp} = E$.
This implies that (E,*) does not have property *G*.

<u>23.8. Theorem.</u> *Every Archimedean pf-algebra* E *which is multiplication complete has properties A,B,C,D,E,F and G.*
Proof: *A*,*C* and *E* are evidently fulfilled. (E,*) is an Archimedean f-algebra by thm 23.4. Hence (E,*) does have property *D* by thm 15.9 *(b)*.
(E,*) does have property *B*, because if $|x_1| \leq |x|$ and $|y_1| \leq |y|$ then by (PF4) we have $(|x| - |x_1|)*|y_1| \geq 0$ and $|x|*(|y| - |y_1|) \geq 0$, hence by property *D* we have $|x_1*y_1| = |x_1|*|y_1| \leq |x|*|y_1| \leq |x|*|y| = |x*y|$.

$(E,*)$ does have property F by thm 15.9 *(g)*.
It was proved by Veksler [1967, prop. 4, prop. 7] that properties D and F together imply property G, hence $(E,*)$ does have property G.

23.9. Definition. *For every* x *in an Archimedean pf-algebra* $(E,*)$ *let*

$$V_x = \{y \in E;\ x*y \neq \phi\} \text{ and}$$
$$W_x = \{y \in E;\ x*z \neq \phi \text{ for all } z \in E \text{ with } |z| \leq |y|\}$$

Let $K(E) = \{x \in E;\ x*y \neq \phi \text{ for all } y \in E\}$; $K(E)$ *is called the kernel of* $(E,*)$.
From ex. 23.7 (e) it follows that V_x is not always a Riesz subspace of E.
However, we have the following proposition.

23.10. Proposition. *In an Archimedean pf-algebra* E *which has property* E *every* $V_x (x \in E)$ *is a Riesz subspace of* E.
Proof: By (PF3) and (PF5) we have that V_x is a linear subspace of E. If $y \in V_x$, then $x*y \neq \phi$, hence, by property E, $x*|y| \neq \phi$, so $|y| \in V_x$.
It follows that V_x is a Riesz subspace of E.

23.11. Proposition. *In an Archimedean pf-algebra* E *every* $W_x (x \in E)$ *is an ideal of* E.
Proof: From the definition of W_x it follows that $|y_1| \leq |y_2|$ and $y_2 \in W_x$ imply $y_1 \in W_x$.
It is sufficient to show now that $y_1, y_2 \in W_x$ and $\lambda_1, \lambda_2 \in \mathbb{R}$ imply that $\lambda_1 y_1 + \lambda_2 y_2 \in W_x$. Let $z \in E$ be such that $|z| \leq |\lambda_1 y_1 + \lambda_2 y_2|$, then certainly $|z| \leq (|\lambda_1| + 1)|y_1| + (|\lambda_2| + 1)|y_2|$, hence
$z^+ \leq (|\lambda_1| + 1)|y_1| + (|\lambda_2| + 1)|y_2|$.
With thm 3.4 *(a)* it follows now that
$z^+ = z_+^1 + z_+^2$ such that $0 \leq z_+^i \leq (|\lambda_i| + 1)|y_i|$, hence
$0 \leq (|\lambda_i| + 1)^{-1} z_+^i \leq |y_i| \quad (i = 1,2)$.
It follows that $x*(|\lambda_i| + 1)^{-1} z_+^i \neq \phi$, but then also $x*z_+^i \neq \phi$ $(i = 1,2)$.
With (PF3) we have now $x*z^+ \neq \phi$.
Similarly $x*z^- \neq \phi$, hence $x*z \neq \phi$; this implies that $\lambda_1 y_1 + \lambda_2 y_2 \in W_x$.

23.12. Theorem. *In every Archimedean pf-algebra* $(E,*)$ *which has property* E *the set* $K(E)$ *is a Riesz subspace of* E. *With the induced pf-algebra structure* $K(E)$ *is a faithful Archimedean f-algebra.*

Proof: $K(E) = \cap\{V_y;\ y \in E\}$ because if $x \in K(E)$ then $x*y \in E$ for all $y \in E$, hence $x \in V_y$ for all $y \in E$. Conversely if $x \in V_y$ for all $y \in E$ then $x*y \neq \phi$ for all $y \in E$, hence $x \in K(E)$.
By thm 3.7 and thm 23.10 it follows that $K(E)$ is a Riesz subspace of E.
Hence, $K(E)$ is, with the induced Riesz space structure, an Archimedean Riesz space.
Although the product of an element of $K(E)$ with an element of E may fall outside $K(E)$, we have that $K(E)*K(E) \subset K(E)$, because if $x,y \in K(E)$ then $x*(y*z) \neq \phi$ for all $z \in E$, hence, by (PF2) we have $(x*y)*z \neq \phi$ for all $z \in E$, so $x*y \in K(E)$. If $*_K$ is the pf-multiplication $*$ restricted to $K(E)$, then $(K(E), *_K)$ satisfies (PF1), (PF2), (PF3), (PF4), (PF5) and (PF6), hence $(K(E), *_K)$ is an Archimedean pf-algebra.
But $(K(E), *_K)$ is multiplication complete, hence, by thm 23.4 we have that $(K(E), *_K)$ is a faithful Archimedean f-algebra.

23.13. Definition. *A linear subspace* J *of an Archimedean pf-algebra* $(E,*)$ *is called a pf-ideal if* $x \in J$, $y \in E$ *and* $x*y \neq \phi$ imply $x*y \in J$.

In Archimedean pf-algebras which are multiplication complete the pf-ideals are precisely the ring ideals.

23.14. Definition. *A tripel* $(E,e,*)$ *is called an Archimedean Riesz space with a partial* Φ*-multiplication (abbreviated to Archimedean* $p\Phi$*-algebra) if* $(E,*)$ *is an Archimedean pf-algebra and* e *is a weak order unit of* E, *such that for all* $x \in E$ *holds*

($P\Phi$) $\quad x*e = x$.

Then there is only one such an e, *which is called the* $p\Phi$*-unit of* $(E,e,*)$.
For fixed e and $*$ we abbreviate $(E,e,*)$ to E.
$*$ is called the $p\Phi$-multiplication of E.

23.15. Theorem. *Every multiplication complete Archimedean* $p\Phi$*-algebra* $(E,e,*)$ *is an Archimedean* Φ*-algebra and conversely.*
Proof: $\leftarrow$: By thm 23.4 we have that E is an f-algebra.
From ($P\Phi$) it follows that the $p\Phi$-unit of E is the multiplicative unit of E.
$\leftarrow$: it follows from thm 16.6 that E is a faithful Archimedean f-algebra, so E is an Archimedean pf-algebra. From lemma 16.7 it follows that e is a weak order unit of E.

Of particular interest are the pΦ-multiplications studied by Vulikh [1948] and Rice [1968], which are defined intrinsically on a Dedekind complete Riesz space with respect to a given weak order unit. In connection with this subject we mention Tucker [1972].

In the following section we provide another class of Riesz spaces with a pΦ-multiplication, namely the class of Archimedean Riesz spaces with a strong order unit. A pleasant accidental circumstance there is that property E is always fulfilled.

24. Intrinsically defined pΦ-multiplication on Archimedean Riesz spaces with a strong order unit

In this section E is an Archimedean Riesz space and e is a fixed strong order unit of E.
Let R be the set of all standard Riesz homomorphisms ϕ: E $\to \mathbb{R}$.
By thm 12.6 we have that R is non-empty, and in fact there is a 1-1 correspondence between R and the set J of all maximal ideals of E, and for the last set it holds that $\cap J = \{0\}$. Hence, it follows from $\phi x = 0$ for all $\phi \in R$ that $x = 0$.

We define a mapping : ExE $\to$ E$\cup\{\phi\}$ by $*(x,y) = z$ for $x,y,z \in E$ if for all $\phi \in R$ holds that $\phi(x)\phi(y) = \phi(z)$. In that case z is unique by the foregoing. $x*y = \phi$ if no $z \in E$ exists with the abovementioned property. In the sequel we write $x*y$ in stead of $*(x,y)$.

24.1. Theorem. *If* E *is an Archimedean Riesz space with a strong order unit* e, *and* * *is the mapping as defined above, then* (E,e,*) *is an Archimedean* pΦ-*algebra.*
Proof:
(PF1) if $x*y = z$ for $x,y,z \in E$, then for all $\phi \in R$ we have that $\phi(x)\phi(y) = \phi(z)$ or $\phi(y)\phi(x) = \phi(z)$, hence $y*x = z$.
If $x*y = \phi$, then by the foregoing also $y*x = \phi$.
(PF2) if $x*y$, $y*z$ and $x*(y*z)$ are in E, then for all $\phi \in R$ we have $\phi(x*(y*z))$ $= \phi(x)\phi(y*z) = \phi(x)\phi(y)\phi(z) = \phi(x*y)\phi(z)$, hence $(x*y)*z = x*(y*z)$.

If $x*(y*z) = \phi$, then by the foregoing also $(x*y)*z = \phi$.

(PF3) for all $\phi \in R$ we have $\phi(x*y) = \phi(x)\phi(y)$ and $\phi(x*z) = \phi(x)\phi(z)$, hence $\phi(x*y + x*z) = \phi(x)\{\phi(y + z)\}$. This implies $x*(y + z) = x*y + x*z$.

(PF4) for all $\phi \in R$ we have $\phi(x*y) = \phi(x)\phi(y) \geqslant 0$ by the positivity of ϕ. Hence $\phi(x*y \wedge 0) = \phi(x*y) \wedge 0 = 0$, so $x*y \wedge 0 = 0$, so $x*y \geqslant 0$.

(PF5) if $\lambda \neq 0$ and $x*y \neq \phi$, then for all $\phi \in R$ we have $\phi(x*y) = \phi(x)\phi(y)$, hence $\phi(\lambda(x*y)) = \phi(\lambda x)\phi(y)$, so $(\lambda x)*y = \lambda(x*y)$.

(PF6) suppose $x*y = 0$, then $\phi(x)\phi(y) = 0$ for all $\phi \in R$. Because $\mathbb{R}$ is a faithful Archimedean f-algebra it follows that $\phi(x) \perp \phi(y)$, so $|\phi(x)| \wedge |\phi(y)| = 0$, hence $\phi(|x|) \wedge \phi(|y|) = 0$, so $\phi(|x| \wedge |y|) = 0$ for all $\phi \in R$; this implies $|x| \wedge |y| = 0$, so $x \perp y$.

On the other hand $x \perp y$ implies that for all $\phi \in R$ we have $\phi(x) \perp \phi(y)$, so $\phi(x)\phi(y) = 0$, hence $x*y = 0$.

(PΦ) for all $x \in E$ and $\phi \in R$ we have $\phi(x) = \phi(x)\phi(e)$, so $x*e = x$.

<u>24.2. Definition.</u> *The mapping *, defined above is called the intrinsically defined pΦ-multiplication on* E *(relative to* e*).*

In the sequel the symbol * is reserved for this intrinsically defined pΦ-multiplication on E.

Now we shall examine which properties of Veksler's list (def. 23.6) are fulfilled in the case of our intrinsically defined pΦ-multiplication.

<u>24.3. Proposition.</u> *Property A is not fulfilled in general.*

Proof: We take the Riesz space E of ex. 23.7(a).
As a consequence of the compactness of $[0,1]$ the only realvalued Riesz homomorphisms ϕ with $\phi(e) = 1$ are the point-evaluations ϕ_t ($t \in [0,1]$), i.e. $\phi_t(x) = x(t)$ for $x \in E$ (cf. e.g. Schaefer [1974, III. 1 Ex. 1]).
Hence, * is exactly the pf-multiplication as defined in ex. 23.7 (a), hence also (E,*) does not satisfy property *A*.

<u>24.4. Proposition.</u> *Property B is always fulfilled.*

Proof: Let $\phi \in R$ be arbitrary. By the positivity of ϕ we have
$\phi(|x_1*y_1|) = |\phi(x_1*y_1)| = |\phi(x_1)\phi(y_1)| = |\phi(x_1)||\phi(y_1)| =$
$\phi(|x_1|)\phi(|y_1|) \leqslant \phi(|x|)\phi(|y|) = |\phi(x)||\phi(y)| = |\phi(x*y)| = \phi(|x*y|)$.
Hence $|x_1*y_1| \leqslant |x*y|$.

24.5. Proposition. *Property C is always fulfilled.*

Proof: Suppose $x,y,z \in E$ such that $x*y \neq \phi$ and B_z is a projection band. If for certain $s,t \in E$ we have that $P_t s = \sup\{s \wedge n|t|\}$ exists in E, then we shall write s_t in stead of $P_t s$.

For $w \in E$ we have $w \perp z$ if and only if $w \perp e_z$ because $w \perp z$ implies $w \perp n|z|$ for all $n \in \mathbb{N}$, hence $w \perp n|z| \wedge e$ for all $n \in \mathbb{N}$, so $w \perp e_z$. Conversely $w \perp e_z$ implies $|w| \wedge (|z| \wedge e) = 0$, hence $(|w| \wedge |z|) \wedge e = 0$, so $|w| \wedge |z| = 0$, hence $w \perp z$. This implies $B_z = B_{e_z}$ and $P_z = P_{e_z}$.

By elementary projection properties we have $e_z \wedge (e - e_z) = 0$ hence $e_z * (e_z - e) = 0$, and $I - P_{e_z} = P_{e-e_z}$

Further we have $e_z * e = e_z$.

With (PF3) it follows that $e_z * e_z = e_z$, hence for all $\phi \in R$ we have $\phi(e_z)\phi(e_z) = \phi(e_z)$, so $\phi(e_z)$ is 0 or 1.

This implies $\phi(e_z)\phi(x) = \begin{cases} 0 \text{ if } \phi(e_z) = 0 \\ \phi(x) \text{ if } \phi(e_z) = 1 \end{cases}$

We have $\phi(P_z x) = \phi(P_{e_z} x) = \phi(\sup\{x \wedge ne_z; n \in \mathbb{N}\}) \geq \sup\{\phi(x \wedge ne_z); n \in \mathbb{N}\} = \sup\{\phi(x) \wedge n\phi(e_z); n \in \mathbb{N}\}$

$$= \begin{cases} 0 \text{ if } \phi(e_z) = 0 \\ \phi(x) \text{ if } \phi(e_z) = 1 \end{cases} = \phi(e_z)\phi(x) \qquad \ldots (\P)$$

Now we shall prove that in (¶) in fact equality holds.

We do that by observing that

$\phi((I - P_z)x) = \phi((I - P_{e_z})x) = \phi(P_{e-e_z} x)$

$= \phi(\sup\{x \wedge n(e - e_z); n \in \mathbb{N}\}) \geq$

$\sup\{\phi(x \wedge n(e - e_z); n \in \mathbb{N}\} = \sup\{\phi(x) \wedge n\phi(e - e_z); n \in \mathbb{N}\} =$

$$= \begin{cases} 0 \text{ if } \phi(e - e_z) = 0 \\ \phi(x) \text{ if } \phi(e - e_z) = 1 \end{cases} = \begin{cases} 0 \text{ if } \phi(e_z) = 1 \\ \phi(x) \text{ if } \phi(e_z) = 0 \end{cases} \qquad \ldots (\P\P)$$

(¶) and (¶¶) give now that $\phi(x) = \phi(P_z x + (I - P_z)x) = \phi(P_z x) + \phi((I - P_z)x) \geq \phi(x)$ for all $\phi \in R$, so in (¶) equality holds.

Now it follows that $e_z*x = P_zx$.
Similarly can be proved that $e_z*y = P_zy$ and $e_z*(x*y) = P_z(x*y)$.

Finally, $P_z(x*y) = e_z*(x*y) = (e_z*e_z)*(x*y) = e_z*(e_z*(x*y)) =$
$= e_z*((e_z*x)*y) = e_z*(y*(e_z*x)) = (e_z*y)*(e_z*x) = (e_z*x)*(e_z*y) = P_zx*P_zy$.

24.6. Proposition. *Property D is always fulfilled.*
Proof: If $x*y \in E$ then for all $\phi \in R$ we have $\phi(|x*y|) = |\phi(x*y)| =$
$|\phi(x)\phi(y)| = |\phi(x)||\phi(y)| = \phi(|x|)\phi(|y|)$. Hence $|x|*|y| = |x*y|$.

24.7. Proposition. *Property E is always fulfilled.*
Proof: Let $x,y \in E$ be such that $x*y \in E$. e is a strong order unit of E, hence there exists a $\lambda \in \mathbb{R}$ such that $|x| \leqslant \lambda e$, so $-x \leqslant \lambda e$, so $x + \lambda e \geqslant 0$.
$\lambda e*y \in E$ by (PΦ) and (PF5). So by (PF3)$(x + \lambda e)*y \in E$.
The foregoing proposition now gives that $|x + \lambda e|*|y| \in E$, so $(x + \lambda e)*|y| \in E$.
$(-\lambda e)*|y| \in E$; it follows by (PF3) that also $x*|y| \in E$.

24.8. Proposition. *Property F is always fulfilled.*
Proof: For all $\phi \in R$ is $\phi(x\vee y)\phi(x\wedge y) = \{\phi(x)\vee\phi(y)\}\{\phi(x)\wedge\phi(y)\} = \phi(x)\phi(y) =$
$\phi(x*y)$, hence $(x\vee y)*(x\wedge y) = x*y$.

24.9. Proposition. *Property G is always fulfilled.*
Proof: In Veksler [1967, prop. 4, prop. 7] it is proved that properties *D* and *F* together imply property *G*, hence (E,*) does have property *G*.

24.10. Theorem. *If* E *is an Archimedean Riesz space with a strong order unit* e, *and* E *is moreover a Φ-algebra such that* e *is the multiplicative unit of* E, *then the intrinsically defined* pΦ*-multiplication* * *on* E *(relative to* e*) coincides with the Φ-multiplication, hence in particular* (E,*) *is multiplication complete.*
Proof: Let # be the Φ-multiplication on E. If $x,y,z \in E$ are such that $z = x \# y$, then by thm 18.1 we have for all $\phi \in R$ that $\phi(z) = \phi(x)\phi(y)$, hence $z = x*y$.

24.11. Theorem. *An Archimedean Riesz space* E *with a given strong order unit* e *can be provided with an f-multiplication such that* e *is the multiplicative unit if and only if* (E,*) *is multiplication complete, where* * *is the intrinsically defined* pΦ*-multiplication on* E *(relative to* e*).*

Proof: if E is a Φ-algebra with multiplicative unit e, then by thm 24.10 multiplication * coincides with the Φ-multiplication, hence $(E,*)$ is multiplication complete.
Conversely, if $(E,*)$ is multiplication complete, then, by thm 23.4, * is an f-multiplication on E. e is the pΦ-unit, hence is the multiplicative unit.

25. Orth(E) for E an Archimedean Riesz space with a strong order unit

In this section, we prove, independently of Bernau [1979], that Orth(E) is an Archimedean f-algebra if E is an Archimedean Riesz space with a strong order unit e.

Let * be the intrinsically defined pΦ-multiplication on E (relative to e) (section 24); let $K = K(E)$ be the kernel of $(E,*)$(section 23), then by thm 23.12 we have that K with the induced ordering and multiplication is an Archimedean f-algebra. Let $Z = Z(E)$ be the class of all centre operators on E (section 12), then provided with the operator ordering and composition as multiplication, Z is a partially ordered algebra.

25.1. Theorem. *For every* $x \in K$ *the mapping* $\mathcal{L}_x$ *from* E *to* E *which assigns to* $y \in E$ *the element* $x * y$ *of* E, *is an element of* Z; *conversely, for every* $T \in Z$ *there exists an* $x \in K$ *such that* $T = \mathcal{L}_x$.
Proof: $\rightarrow$: if $x \in K$, then on account of the fact that e is a strong order unit of E there exists a $\lambda \in \mathbb{R}$ such that $|x| \leq \lambda e$. For all $y \in E^+$ we have now by prop 24.4 that $|\mathcal{L}_x y| = |x * y| \leq |\lambda e * y| = \lambda y$, hence $\mathcal{L}_x$ is a centre operator.
Conversely, if $T \in Z$, then there exists a $\lambda \in \mathbb{R}$ such that $|Tx| \leq \lambda x$ for all $x \in E^+$. We shall prove now that $T = \mathcal{L}_{Te}$. Therefore, we have to show that $Te \in K$. If $x \in E$ and $\phi \in \mathcal{R}$ are arbitrary and if $y = x - \phi(x)e$, then $\phi(y) = \phi(x) - \phi(x) = 0$. $N(\phi)$ is an ideal of E, hence also $|y| \in N(\phi)$, but then also $y^+ \in N(\phi)$, $y^- \in N(\phi)$. Further it holds that $Ty = Tx - \phi(x)Te$
Now we have that
$0 \leq \phi(T(y^+)) \leq \phi(\lambda y^+) = \lambda\phi(y^+) = 0$ and $0 \leq \phi(T(y^-)) \leq \phi(\lambda y^-) = \lambda\phi(y^-) = 0$, hence $\phi(Ty) = \phi(T(y^+)) - \phi(T(y^-)) = 0$; together with $Ty = Tx - \phi(x)Te$ this implies $0 = \phi(Tx) - \phi(x)\phi(Te)$, so $\phi(Tx) = \phi(Te)\phi(x)$, hence $(Te) * x = Tx$.

consequently $Te \in K$ and $T = \mathcal{L}_{Te}$.
Next we prove that in fact K can be identified with Z, i.e. there exists an order isomorphism A from K to Z.

<u>25.2. Theorem.</u> *The linear operator* A *from* K *to* Z *which assigns to* $x \in K$ *the element* $\mathcal{L}_x$ *of* Z *is an order isomorphism.*
Proof: A is injective, because $\mathcal{L}_x = \mathcal{L}_y$ for $x, y \in K$ implies $\mathcal{L}_x e = \mathcal{L}_y e$, hence $x = y$. A is surjective by thm 25.1. A is positive because $x \geqslant 0$ implies $\mathcal{L}_x \geqslant 0$, A^{-1} is positive because $\mathcal{L}_x \geqslant 0$ implies $\mathcal{L}_x e \geqslant 0$, hence $x \geqslant 0$. It follows that T is an order isomorphism.

<u>25.3. Theorem.</u> *For every Archimedean Riesz space* E *with strong order unit* e *the space* Orth(E) *is an Archimedean f-algebra under the operator ordering and composition as multiplication.*
Proof: by thm 12.23 we have Orth(E) = Z(E). K is an Archimedean Riesz space by thm 23.12, hence, by thm 25.2 Z(E) is also an Archimedean Riesz space. A is not only an order isomorphism, but also an algebra isomorphism, because, if $x, y \in K$, then $A(x * y) = \mathcal{L}_{x*y} = \mathcal{L}_x \circ \mathcal{L}_y = (Ax)(Ay)$.
It follows that Z is an Archimedean f-algebra.

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SAMENVATTING

In dit proefschrift worden lineaire operatoren op Riesz ruimten bestudeerd, in het bijzonder disjunctieve lineaire operatoren. Enkele voorbeelden van niet orde begrensde disjunctieve lineaire operatoren van een Riesz ruimte op zichzelf worden gegeven, terwijl de orde begrensde disjunctieve lineaire operatoren in verband worden gebracht met een partieel gedefinieerde vermenigvuldigingsoperatie in de Riesz ruimte.

Hoofdstuk I geeft een inleiding in de theorie van Riesz ruimten. In hoofdstuk II worden enkele typen van convergentie bestudeerd in verband met lineaire operatoren. Disjunctieve lineaire operatoren worden in hoofdstuk III bestudeerd. In het bijzonder wordt aandacht geschonken aan disjunctieve lineaire functionalen en orde begrensde disjunctieve lineaire operatoren van een Archimedische Riesz ruimte met een sterke orde eenheid op zichzelf. In hoofdstuk IV worden Riesz ruimten bestudeerd die voorzien zijn van een vrij star gedefinieerde vermenigvuldigingsstructuur, de zogenaamde f-algebra's. Enige overeenkomsten en verschillen tussen Riesz homomorfismen en ring homomorfismen worden aangegeven en een variant van de stelling van Ellis-Phelps wordt afgeleid. Met behulp van de in hoofdstuk V ontwikkelde theorie van inverteren en worteltrekken in regulator complete Φ-algebra's wordt een generalisatie verkregen van bovengenoemde stelling. In hoofdstuk VI worden partiële vermenigvuldigingsoperaties op Archimedische Riesz ruimten met een zwakke orde eenheid axiomatisch ingevoerd. Op iedere Archimedische Riesz ruimte met sterke orde eenheid wordt tenslotte een intrinsieke partiële vermenigvuldiging aangegeven en in verband gebracht met de ruimte van alle orde begrensde disjunctieve lineaire operatoren van de Riesz ruimte op zichzelf.

CURRICULUM VITAE

Brand van Putten werd geboren op 15 juni 1954 te Zwolle.
Mei 1971 legde hij het HBS-B examen af aan het Ichthus College te Enschede en werd in datzelfde jaar ingeschreven als student aan de Rijksuniversiteit te Utrecht. Juli 1974 legde hij het kandidaatsexamen wiskunde met bijvak natuurkunde af.
Als specialisatie koos hij de numerieke wiskunde en werd daarin onderwezen door Professor Dr. A. van der Sluis. Juni 1976 legde hij het doctoraal examen af met uitgebreid hoofdvak wiskunde en klein bijvak muziektheorie.
In oktober 1976 werd hij benoemd tot wetenschappelijk ambtenaar aan de Landbouwhogeschool te Wageningen, sectie zuivere en toegepaste wiskunde.
Per 15 september 1977 werd hij in staat gesteld om als wetenschappelijk assistent het onderzoek te verrichten, dat tot dit proefschrift heeft geleid. Sinds 1 september 1980 is hij als wetenschappelijk ambtenaar verbonden aan de Landbouwhogeschool te Wageningen, sectie statistiek.

ERRATA to Disjunctive linear operators and partial multiplications in Riesz spaces - B. van Putten.

4^6 : $x,y \in E^+$ should read $x,y \geq 0$

$4^{9,10,11}$: $\sup(x,-x)$ should read $\sup\{x,-x\}$

$8_{2,1}$: *(f) a polar* P *of* E *is called a principal polar if there exists a* $z \in E$ *such that* $P = z^{\perp\perp}$

$9^{2,3}$: should be deleted

$9^{6,7}$: The polar $z^{\perp\perp}$ is called the principal polar generated by z, and is denoted by P_z.

11 : Def. 3.14 should read *The intersection of all bands which contain an element* z *of a Riesz space* E *(which is a band by thm 3.7) is called the principal band generated by* z. *Every band of this form is called a principal band.* E *is said to have the projection property (abbreviated to* PP*) if every band of* E *is a projection band.* E *is said to have the principal projection property (abbreviated to* PPP*) if every principal band of* E *is a projection band.*

13^{12} : S should read M

16^2 : $= \left(T^+ (\lambda e + \sum_{\substack{i=1 \\ i \neq N}}^{\infty} \lambda_i e_i - \lambda e_N)\right)_M \geq 0$

16_7 : T^+x should read $(T^+x)_M$

28^3 : a Riesz space should read an Archimedean Riesz space

28_{11} : $y \in {}^r\mathcal{L}f$ should read $y \in {}^c\mathcal{L}f$

35_5 : $T \leq \mu T \leq |\mu| I$ should read $T \leq \mu I \leq |\mu| I$

37_{15} : In this section 14 should read In section 14

38_3 : $X^{\perp\perp} = X$ should read $X^{\perp\perp} = E$

40_{16-13} : should read In ex.13.13 the Riesz subspace T is prime. Note that T is not an ideal.

41_3 : $\bigwedge_{\sigma \in J^I} \bigvee_{\substack{i \in I \\ k \in K}} x_{\sigma(i)ki}$

43^{15} : $N \supset J$, *then* $N=J$ should read $N \supset M$, *then* $N=M$

44^{5} : thm 6.12 should read thm 13.12

44_{14} : $x \in E^{+}$ should read $x \in E$

44_{5} : I should read $\mathbf{I}$

45^{11} : $y \wedge (e-y) = 0$ should read $y \wedge (e-y) \in R$

46^{5} : $\cap\, \Sigma \supset \mathbf{R}$ should read $\cap\, \Sigma \supset R$

47_{11} : By the foregoing theorem should read By thm 13.22

51^{6} : for all $x \in E$ should read for some $x \in E$

52^{6} : $T(f(n))(0)$ should read $T(f(n))(1)$

52_{9} : n_x the maximum should read n_x greater then the maximum

58_{13} : $x^2y - nyx + n^2y$ should read $yx^2 - nyx + n^2y$

64_{3} : y should read ϕ

66^{16} : $S = \pi \circ T$ if $J = Id(T(E))$

66_{13} : $TzSx = TxSz$ holds for all $x,z \in E$.

66_{10-1} : e should read z

69^{12} : on F should read on a Φ-algebra F

72_{16} : (for a definition of P_x see Luxemburg and Zaanen [1971, thm 24.5]).

78^{17} : $\leq \frac{1}{2}(1-\delta)^{n}e$ should read $\leq \frac{1}{2}(1-\delta)^{n-1}e$

78_{13} : $\frac{1}{2}(1-\delta)^{N+1}\,\delta^{-1}$ should read $\frac{1}{2}(1-\delta)^{N}\delta^{-1}$

78_{9} : $\leq \frac{1}{2}(1-\delta)^{m} \sum_{k=0}^{\infty} (1-\delta)^{k}e \leq \frac{1}{2}(1-\delta)^{N}\delta^{-1}e \leq \varepsilon e$

79^{11} : thm 7.1*(e)* should read thm 7.1*(d)*

$83^{17,18}$: *C. if* B_z *projection band* $(z \in E)$, $x * y \neq \phi \rightarrow P_z x * P_z y \neq \phi$

85^{15} : $\min\,(\sup\{|x(n)|;\ n \in N\},\ \sup\{|y(n)|;\ n \in N\})$

90^{3} : $P_t s = \sup\,\{s^{+} \wedge n|t|;\ n \in \mathbb{N}\} - \sup\,\{s^{-} \wedge n|t|;\ n \in \mathbb{N}\}$

90^{16} : We have should read Assume first $x \geq 0$, then we have

91^{1} : Now it follows that $e_z * x = P_z x$ for all $x \geq 0$, hence for all $x \in E$ we have $P_z x = P_z x^{+} - P_z x^{-} = e_z * x^{+} - e_z * x^{-} = e_z * x$.

92_{3} : $0 \leq \phi((Ty)^{+}) \leq \phi(|Ty|) \leq \lambda\phi(|y|) = 0$, similarly $\phi((Ty)^{-}) = 0$