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The Pure Spectrum of a *PF*-Ring

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1. Introduction

All rings considered in this paper are commutative with unity. A ring R is called a PF-ring if every principal ideal aR is a flat R-module, and it is called a PP-ring if every principal ideal aR is a projective R-module. An ideal I of a ring R is called pure if for every $x \in I$, there exists $y \in I$ such that $xy = x$.

For each proper prime ideal P of a reduced (without nonzero nilpotent elements) ring, let

$$O_P = \{x \in R : \exists y \notin P,\ xy = 0\} .$$

This is an ideal contained in P and is contained in any prime ideal contained in P. Clearly, P is minimal iff $O_P = P$.

Obviously any finite intersection of pure ideals of a ring R is pure, and it is well known that the sum of any collection of pure ideals is pure, see Borceux and Van den Bossche [3]. An ideal I is called "purely maximal" if it is maximal in the lattice of pure ideals of the ring R, and it is called "purely prime" if it satisfies: whenever I_1 and I_2 are any two pure ideals such that $I_1 \cap I_2 \subseteq I$, then either $I_1 \subseteq I$ or $I_2 \subseteq I$. Let $\mathrm{pp}(R)$ be the set of all purely prime ideals of R. For each pure ideal I of R, let

$$O^I = \{J \in \mathrm{pp}(R) : I \not\subseteq J\} .$$

It is well known that the sets of the form O^I defines a topology on the set $\mathrm{pp}(R)$, called the pure spectrum topology and it is denoted by $\mathrm{Spp}(R)$.

In Section 2 of this paper we characterize pure ideals in a PF-ring.

In Section 3 we characterize the set $\mathrm{pp}(R)$ of a PF-ring, and then we prove that there is a continuous bijection from the space of minimal prime ideals with spectral topology to $\mathrm{pp}(R)$ with the pure spectrum topology.

Finally, in Section 4, we prove that $\mathrm{pp}(R)$ with pure spectrum topology is homeomorphic to $\mathrm{Min}(R)$ with the spectral topology iff R is a PP-ring.

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2. Pure ideals in a *PF*-ring

First, we list some alternate characterizations for *PF*-rings in the following theorem.

THEOREM 2.1. *Let R be a ring. Then the following are equivalent.*

1. *R is a PF-ring.*
2. *For each* $a \in R$, *the annihilator of a*, $\text{ann}(a)$, *is a pure ideal, see Al-Ezeh* [1].
3. *For each maximal ideal M, the localization* R_M *is an integral domain, see Matlis* [5].
4. *R is reduced and every prime ideal P of R contains a unique minimal prime ideal, namely* O_P, *see Matlis* [5].
5. *For each* $a, b \in R$, $\text{ann}(a) + \text{ann}(b) = R$ *whenever* $ab = 0$, *see Aritico and Marconi* [2].

To give the complete characterization of pure ideals in a *PF*-ring, we first prove two lemmas that we will need later.

LEMMA 2.2. *Let R be a PF-ring. Then for every prime ideal of R,* O_P *is pure.*

Proof. Let $a \in O_P$, then there exists $b \notin P$ such that $ab = 0$. Since R is a *PF*-ring, by Theorem 2.1, $1 = x + y$ such that $xa = 0$ and $yb = 0$, hence $a = ay$. Since $b \notin P$, $y \in O_P$. So O_P is pure.

LEMMA 2.3. *Let R be a PF-ring. Then a prime ideal P is pure iff* $O_P = P$.

Proof. If $P = O_P$, then by Theorem 2.1 and Lemma 2.2, P is pure.

Conversely, assume P is pure. First $O_P \subseteq P$. Now, let $a \in P$, then there exists $b \in P$ such that $ab = a$. Since $1 - b \notin P$ and $a(1-b) = 0$, $a \in O_P$. Thus $P \subseteq O_P$. Consequently, $P = O_P$.

THEOREM 2.4. *Let R be a PF-ring, and I a pure ideal of R. Then* $I = \bigcap_{I \subseteq M} O_M$ *where M ranges over all maximal ideals containing I.*

Proof. Let $x \in I$. Since I is pure, there exists $y \in I$ such that $x(1-y) = 0$. For each maximal ideal M containing I, $1 - y \notin M$, and hence $x \in O_M$. Therefore $I \subseteq \bigcap_{I \subseteq M} O_M$.

Now, let $x \notin I$. Let $S = \{b \in R : bx = x\}$. Then S is a multiplicative subset of R that is disjoint from I. Hence by Zorn's lemma, there exists a prime ideal P that contains I and disjoint from S. P is contained in a maximal ideal M. We claim, $x \notin O_M$. If not then $x \in O_M$. Since O_M is pure, there exists $y \in O_M$ such that $xy = x$. Because $P \subseteq M$, we get $O_M \subseteq P$. Therefore $y \in P$. Hence $y \in P \cap S$, which is absurd. So $x \notin O_M$, and hence $x \in \bigcap_{I \subseteq M} O_M$. Consequently, $I = \bigcap_{I \subseteq M} O_M$.

THEOREM 2.5. *Let R be a PF-ring, and J a proper ideal of R. Then the ideal* $K = \bigcap_{I \subseteq M} O_M$ *is pure, where M ranges over all maximal ideals M containing J.*

Proof. Let $a \in K$. Then for each maximal ideal M containing J, there exists $b_M \notin M$ such that $ab_M = 0$. For each maximal ideal M not containing J, let $c_M \in J$ and $c_M \notin M$. Let C be the ideal generated by all b_Ms and c_Ms. Then $C = R$. So $1 = x_1 c_{M_1} + \cdots + x_k c_{M_k} + x_{k+1} b_{M_{k+1}} + \cdots + x_{k+n} b_{M_{k+n}}$. Therefore, $af = a$, where $f = x_1 c_{M_1} + \cdots + x_k c_{M_k} \in J$. Thus, $a(1-f) = a$ and $(1-f) \notin M$ for every maximal ideal M containing J. If $(1-f) = g$, then since R is a PF-ring, $\text{ann}(a) + \text{ann}(g) = R$. So, $1 = x + y$ with $xa = 0$ and $yg = 0$. Consequently, $a = ay$ and $y \in O_M$ for each maximal ideal M containing J. Therefore K is pure.

THEOREM 2.6. *A reduced ring R is a PF-ring iff every minimal prime ideal is pure.*

Proof. Assume R is a PF-ring. If P is minimal, then $P = O_P$. But O_P is pure by Lemma 2.2, so P is pure.

Conversely, let R be a reduced ring in which every minimal prime ideal is pure. Let Q be a prime ideal of R, then there exists a minimal prime ideal P contained in Q. Hence $O_Q \subseteq P = O_P$. Now, let $x \in O_P$. Since $P = O_P$ is pure there exists $y \in P \subseteq Q$ such that $xy = x$. Then $1 - y \notin Q$ and $x(1-y) = 0$. So $x \in O_Q$. Hence $P = O_Q$. Thus every O_Q is prime, and so by Theorem 2.1, R is a PF-ring.

3. The pure spectrum of a *PF*-ring

We start this section by recalling a theorem that was proved in Borceux and Van den Bossche [3].

THEOREM 3.1. *Let R be a ring. Then*

a. *every purely maximal ideal is purely prime*

b. *every proper pure ideal is contained in a purely maximal ideal.*

In the following theorem we characterize purely maximal ideals in a PF-ring.

THEOREM 3.2. *Let R be a PF-ring. Then a proper ideal Q of R is purely maximal iff $Q = O_P$ for some minimal prime ideal P of R.*

Proof. Let Q be a purely maximal ideal. Then Q is contained in a maximal ideal M. Then it is quite easy to see that $Q \subseteq O_M$ because Q is pure. But by Lemma 2.2, O_M is pure, so $Q = O_M$ because Q is purely maximal. Since R is a PF-ring, $O_M = P$ is a minimal prime ideal. So $Q = O_M = P = O_P$.

Conversely, assume $Q = O_P$ for some minimal prime ideal P. Let I be a proper pure ideal such that $O_P \subseteq I$. I is contained in a maximal ideal M. Clearly, $I \subseteq O_M$. Since $P = O_P \subseteq I \subseteq M$, then $O_M \subseteq P \subseteq I$. Thus $O_M = O_P = P = I$. So Q is purely maximal.

Our aim now is to prove that every purely prime ideal of a PF-ring is purely maximal. We need for the proof two preliminary lemmas. For any elements a in a PF-ring R, let $J_a = \bigcap_{a \in M} O_M$, where M ranges over all maximal ideals containing a. By Theorem 2.5, J_a is a pure ideal.

LEMMA 3.3. *Let R be a PF-ring. For any maximal ideal M of R, if $a \in M$, then $J_a \subseteq O_M$.*

Proof. Let a be an element of a maximal ideal M. Let $x \in J_a$. Since J_a is pure, there exist $y \in J_a$ such that $xy = x$. Since $y \in J_a$, $y \in M$. So $1 - y \notin M$ and $x(1-y) = 0$. Consequently, $x \in O_M$. So, $J_a \subseteq O_M$.

LEMMA 3.4. *Let I be a proper pure ideal of a PF-ring R. Let M be a maximal ideal containing I. Then if $a \in O_M$ implies $J_a \subseteq I$, then $I = O_M$.*

Proof. Since $I \subseteq M$ and I is pure, $I \subseteq O_M$. Let $x \in O_M$. Since O_M is pure, there exists $y \in O_M$ such that $x(1-y) = 0$. Then $x \in J_y$. By assumption, because $y \in O_M$, $J_y \subseteq I$. Hence $x \in I$. Consequently, $I = O_M$.

THEOREM 3.5. *Let R be a PF-ring. Then any purely prime ideal is purely maximal.*

Proof. Assume that I is purely prime that is not purely maximal. Hence there exists a maximal ideal M such that $I \subseteq O_M$ and $I \neq O_M$. Since $I \neq O_M$, by Lemma 3.4, there exists $a \in O_M$ such that $J_a \nsubseteq I$. Let $K = \bigcap_{\mathrm{ann}(a) \subseteq N} O_N$, where N ranges over all maximal ideals containing ann(a). Clearly, $J_a \cap K = 0$ since $J_a \cap K$ is a pure ideal contained in the Jacobson radical of R. Therefore $J_a \cap K \subseteq I$. Since I is purely prime, $K \subseteq I$.

Now, since $a \in O_M$, there exists $b \in O_M$ such that $a(1-b) = 0$ because O_M is pure. Again, since O_M is pure and $b \in O_M$, there exists $c \in O_M$ such that $b(1-c) = 0$. So $1 - b \in N$ for each maximal ideal containing ann(a). So, for all such N we get $b \notin N$. Hence $1 - c \in K$. Because $c \in M$, we get $K + M = R$. This contradicts the fact that $K \subseteq I \subseteq M$. Therefore $O_M = I$. Consequently, $I = O_P = P$ for some minimal prime ideal $P = O_M$. Thus I is purely maximal.

Now, it is clear that

$$\mathrm{pp}(R) = \{O_P : P \in \mathrm{Min}(R)\} = \{O_M : M \text{ is a maximal ideal of } R\}.$$

So $\mathrm{pp}(R) = \mathrm{Min}(R)$.

COROLLARY 3.6. *Let R be a PF-ring. Then the mapping ψ: Min(R)$\to$pp(R) defined by $\psi(P) = O_P$ is a continuous bijection, Min(R) is given the spectral topology.*

Proof. Clearly ψ is the identity mapping. Let I be a pure ideal and let $O^I \in \mathrm{Spp}(R)$. Then

$$\psi^{-1}(O^I) = \{P \in \mathrm{Min}(R) : I \nsubseteq P\} = D(I) \cap \mathrm{Min}(R)$$

which is spectrally open in Min(R).

In the next section, we prove that for a PF-ring the mapping ψ defined above is a homeomorphism iff R is a PP-ring. Of course, there are PF-rings that are not PP-rings, e.g. see Vasconcelos [6].

4. The pure spectrum of a *PP*-ring

It is well known that a ring R is a *PP*-ring iff for every $a \in R$, $\text{ann}(a)$ is generated by an idempotent. In fact, this is because the principal ideal aR is a projective R-module iff $\text{ann}(a)$ is generated by an idempotent, see Evans [4]. Clearly, every *PP*-ring is a *PF*-ring. Vasconcelos [7] proved that R is a *PP*-ring iff R is a *PF*-ring and $\text{Min}(R)$ with the spectral topology is compact. We come to our final result.

THEOREM 4. *Let R be a PF-ring. Then the mapping* $\psi: \text{Min}(R) \to \text{pp}(R)$ *defined by* $\psi(P) = O_P = P$ *is a homeomorphism iff R is a PP-ring.*

Proof. Assume ψ is a homemorphism. Then $\text{Min}(R)$ is compact because $\text{pp}(R)$ with the pure spectrum topology is compact, see Borceux and Van den Bossche [3]. Hence R is a *PF*-ring for which $\text{Min}(R)$ with the spectral topology is compact. So R is a *PP*-ring, see Vasconcelos [7].

Conversely, assume R is a *PP*-ring. To show that ψ is a homeomorphism, it is enough to show that $\psi(D^o(aR))$ is open for every $a \in R$ where $D^o(aR) = D(aR) \cap \text{Min}(R)$. Consider $\psi(D^o(aR)) = \{P \in PP(R) : a \notin P\}$. Since R is a *PP*-ring and $a \in R$, $\text{ann}(a) = eR$ with $e^2 = e$. To complete the proof, we have to show that

$$\{P \in PP(R) : a \notin P\} = \{P \in \text{pp}(R) : (1-e) \notin P\}$$

because $I = (1-e)R$ is pure and $O^I = \{P \in \text{pp}(R) : (1-e) \notin P\}$. Let P be a minimal prime ideal such that $a \notin P$. Since $ae = 0$, $e \in P$. So $1 - e \notin P$. For the other way around, assume, $a \in P$ where $P \in \text{pp}(R)$. Since P is a minimal prime ideal, there exists $b \notin P$ such that $ab = 0$. Hence $b \in \text{ann}(a) = eR$, and so $b = er$, where $r \in R$. Thus $b(1-e) = 0$, i.e. $b = be$. Consequently, $e \notin P$. Because $e(1-e) = 0$, $1 - e \in P$. Thus

$$\{P \in \text{pp}(R) : a \notin P\} = O^I, \qquad \text{where} \quad I = (1-e)R.$$

So the image of each basic open set is open. Hence ψ is a homeomorphism.

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