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Weierstrass Transformations and Cubic Surfaces

by

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1. Introduction

It is wellknown that, given a nonsingular curve of genus 1, say Γ, and a point on it $A\in\Gamma$, there is a birational transformation from Γ to a Weierstrass cubic (a plane cubic curve defined by a Weierstrass equation):

$$E:\ y^2=x^3+px+q \tag{1}$$

sending the given point A to the point at infinity O: $(x:y:1)=(0:1:0)$. We assume that the ground field K is arbitrary field of characteristic different from 2 and 3.

This is an easy consequence of the Riemann-Roch theorem for Γ. The point is to consider the space of rational functions on Γ having poles only at A and of order upto n, which has dimension n by Riemann-Roch. Taking a basis of this space for $n=2$ and 3 and considering the relations in the space for $n=6$, one gets a Weierstrass equation.

Now consider the case where Γ is given as a plane cubic. Then such a birational transformation from Γ to E can be made explicit and it extends to a birational transformation $\boldsymbol{P}^2\to\boldsymbol{P}^2$ of the ambient projective planes. We propose to call such a transformation of $(\Gamma\subset\boldsymbol{P}^2, A)$ to $(E\subset\boldsymbol{P}^2, O)$ a *Weierstrass transformation.*

The Weierstrass transformations are not *linear* transformations of $\boldsymbol{P}^2$ in general, but they are *cubic* transformations. This makes the question of putting a given elliptic curve into a Weierstrass form not completely trivial, and gives room for a rather unexpected application to global blowing-up process of $\boldsymbol{P}^2$. As an illustration, we will see a neat application to the classical topic of the cubic surfaces as blowing-up of 6 points in a general position of $\boldsymbol{P}^2$. Also we obtain an important supplement to the construction theorem for Mordell-Weil lattices of type E_6 treated in [S2].

2. Weierstrass transformations

Given a plane cubic Γ of genus 1 and a point A on it, we may first take a co-ordinate system (X, Y, Z) of the ambient plane $\boldsymbol{P}^2$ so that A is given by $(0:1:0)$. Then the defining equation of Γ is of the form:

$$s_1 Y^2 + s_2 Y + s_3 = 0\,, \tag{2}$$

where $s_i = s_i(X, Z)$ is a homogeneous polynomial of degree i in X, Z.

The linear form s_1 is not zero, since otherwise the curve would be a rational curve. The line $s_1 = 0$ is the tangent line to Γ at A. Choosing it as the line at infinity and passing to the affine coordinates x, y, we have

$$y^2 + s_2 y + s_3 = 0\,, \tag{3}$$

where s_2, s_3 are now polynomials in x of degree at most 2 or 3. If s_2 has degree less than 2, then (3) can be easily transformed to the Weierstrass form (1) by linear change of coordinates.

So assume that s_2 has degree 2. Then, by replacing y by $y - s_2/2$, (3) becomes

$$y^2 = a_0 x^4 + a_1 x^3 + a_2 x^2 + a_3 x + a_4\,. \tag{4}$$

Here the coefficients a_i belong to the ground field K (over which Γ and A are defined), and a_0 is a nonzero square in K: $a_0 = (b/2)^2$ if $b \in K$ denotes the coefficient of x^2 in s_2.

By linear change of coordinates again, we can reduce (4) to the case where $a_0 = 1$ and $a_1 = 0$, that is:

$$y^2 = x^4 + kx^2 + lx + m\,. \tag{5}$$

So far we used only very simple coordinate change. Finally we need to transform (5) to a Weierstrass cubic, and this requires a nonlinear (cubic) transformation. Let

$$x' = \frac{1}{2}\left(-y + x^2 + \frac{k}{6}\right), \qquad -y' = x\left(x' + \frac{k}{6}\right) + \frac{l}{8}\,. \tag{6}$$

Then this gives a tansformation of (5) to the following Weierstrass cubic

$$y'^2 = x'^3 + px' + q\,, \tag{7}$$

where p and q are the basic invariants of the binary quartic defined by the right-hand side of (5). Explicitly they are given by the formula:

$$p = -\frac{k^2}{48} - \frac{m}{4}\,, \qquad q = \frac{k^3}{864} + \frac{l^2}{64} - \frac{km}{24}\,. \tag{8}$$

The verification is straightforward.

The above argument has been adapted from Mordell's book [Mo], p. 139, where he discusses the original proof of Mordell(-Weil) theorem for a cubic curve.

Observe that if we regard x, y independent variables over K, then (6) defines a birational transformation $(x, y) \to (x', y')$, i.e. $K(x, y) = K(x', y')$. In this paper, we call a *Weierstrass transformation* the birational transformation $w: \boldsymbol{P}^2 \to \boldsymbol{P}^2$ defined by $(X: Y: Z) \mapsto (x', y')$, or its variant $w: \boldsymbol{P}^2 \to S$ defined by $(x, y) \mapsto (x', y', t)$ when we have $K = k(t)$, $K(x, y) = k(x, y)$ as in the next section and (x', y', t) is viewed as a generic point of some rational surface S over k.

REMARK. Our main interest is in the case where the given point A is a K-rational point of Γ. In this case, the Weierstrass transformation induces an isomorphism of (Γ, A) to (E, O) as elliptic curves (i.e. preserving the group laws with the origin A or O). On the other hand, in case the curve Γ has no K-rational points, there is a related problem to construct its Jacobian variety as an elliptic curve over K, and more explicitly, to realize the latter as a Weierstrass cubic. Weil treated this question using the method of classical invariant theory following Cayley and Hadamard (see [W]). It should be remarked that our viewpoint dealing with rational points (in connection with Mordell-Weil lattices) forces us to look at not only the transformed curve but also the trasformation itself.

3. Linear pencils of cubic curves

Let us consider a linear pencil of cubic curves in $\boldsymbol{P}^2$:

$$\Gamma_t: tg(X, Y, Z) + h(X, Y, Z) = 0 \qquad (t \in \boldsymbol{P}^1),$$

where g, h are cubic forms with coefficients in a field k. The base points of the pencil are the 9 points (counting multiplicity) defined by $g(X, Y, Z) = h(X, Y, Z) = 0$; suppose that they are k-rational points. The generic member of the pencil is a plane cubic $\Gamma = \Gamma_t$ defined over $K = k(t)$, t being a variable over k, and it has K-rational points given by the base points.

Choosing one of them as the origin A and applying a Weierstrass transformation, (Γ, A) is mapped to an elliptic curve (E, O) defined over K in Weierstrass form. Let

$$f: S \to \boldsymbol{P}^1$$

be the elliptic surface associated with E/K (Kodaira-Néron model); each rational point $P \in E(K)$ corresponds to a section (P) of the fibration f and conversely. The Weierstrass transformation gives a birational transformation from $\boldsymbol{P}^2$ to S, in which distinct base points of the pencil are transformed into disjoint sections in S. In other words, the Weierstrass transformation from $\boldsymbol{P}^2$ to S blows up the base points of the given pencil.

Naturally this subject is related to the theory of del Pezzo surfaces, in particular to the classical topics of the 27 lines on a smooth cubic surface (see §6 and Manin's book [Ma], Ch. IV). With application to it in mind, we consider the following linear pencil.

First let

$$g(X, Y, Z)=X^3-Y^2Z. \tag{9}$$

This defines a cuspidal cubic curve with the cusp $A=(0:0:1)$. Its smooth part is isomorphic to the additive group $\boldsymbol{G}_a$ with the group parameter $u=X/Y$. Each u corresponds to the point $Q(u)=(u^{-2}:u^{-3}:1)$, and three points $Q(u_i)$ lie on a line if and only if $u_1+u_2+u_3=0$ (cf. [S4]).

Take six points $Q_i=Q(u_i)$ $(1\leq i\leq 6)$ on the cuspidal cubic, satisfying the condition

$$(\#)\qquad u_i\neq u_j\ (i\neq j)\,,\quad u_i+u_j+u_k\neq 0\ (i,j,k:\text{distinct})\,,\quad \sum_{i=1}^{6} u_i\neq 0\,. \tag{10}$$

We note that it is equivalent to the condition of being in a general position (cf. §6). Letting

$$c_i=(-1)^i\cdot(i\text{-th elementary symmetric function of } u_1,\cdots,u_6)\,, \tag{11}$$

we set

$$h(X, Y, Z)=YZ^2+c_1X^2Z+c_2XYZ+c_3Y^2Z+c_4X^2Y+c_5XY^2+c_6Y^3\,. \tag{12}$$

Now we consider the linear pencil

$$\Gamma_t: tg(X, Y, Z)+h(X, Y, Z)=0 \qquad (t\in\boldsymbol{P}^1)\,. \tag{13}$$

LEMMA 1. *The base points of this pencil are* $Q_1,\cdots,Q_6$, *and* A (*counted with multiplicity* 3).

Proof. Indeed, by substituting $(X:Y:Z)=(u^{-2}:u^{-3}:1)$ into $tg+h$, we find

$$\begin{aligned} h(u^{-2}, u^{-3}, 1)&=u^{-9}(u^6+c_1u^5+c_2u^4+c_3u^3+c_4u^2+c_5u+c_6)\\ &=u^{-9}\prod_{i=1}^{6}(u-u_i)\,,\end{aligned}$$

which clearly shows the assertion.

PROPOSITION 2. *Let* $\Gamma=\Gamma_t$ *be the generic member of the pencil, and let*

$$w:\ (\Gamma\subset\boldsymbol{P}^2, A)\to(E\subset\boldsymbol{P}^2, O)$$

be the Weierstrass transformation $w:(X:Y:Z)\mapsto(x, y)$ *as defined in* §2. *Noting that* $A=(0:0:1)$, *we take the inhomogeneous coordinates* $x_1=X/Y$, $y_1=Z/Y$ *in the first* $\boldsymbol{P}^2$. *Then the map* $w:(x_1, y_1)\mapsto(x, y)$ *is explicitly given as follows*:

$$x=(c_2^2-4c_1c_3-4c_4+4c_1t-12tx_1-12c_1y_1)/48 \tag{14}$$

$$y=(c_1c_5-c_3t+t^2+2c_1c_4x_1-c_2tx_1+2c_1tx_1^2+c_1c_2y_1-2ty_1+2c_1^2x_1y_1)/16\,. \tag{15}$$

Further E has the Weierstrass equation

$$y^2=x^3+px+q\,, \tag{16}$$

where p, q are polynomials of degrees 2 *and* 4 *in t of the form*:

$$p=\left(\frac{c_2}{32}-\frac{c_1^2}{48}\right)t^2+\cdots \tag{17}$$

$$q=\frac{1}{256}t^4+\left(-\frac{c_3}{128}-\frac{c_1c_2}{384}+\frac{c_1^3}{864}\right)t^3+\cdots, \tag{18}$$

the unwritten part $\cdots$ *having polynomial expressions in* $c_1, \cdots, c_6$, t *with rational coefficients.*

Proof. In terms of the coordinates x_1, y_1, Γ has the equation:

$$y_1^2+(c_1x_1^2+c_2x_1+c_3-t)y_1+tx_1^3+c_4x_1^2+c_5x_1+c_6=0\,.$$

Following the prescription (3) to (6) indicated in §2, we set

$$y_2=y_1+(c_3-t+c_2x_1+c_1x_1^2)/2$$
$$x_2=x_1(c_1/2)$$
$$y_3=y_2(c_1/2)$$
$$x_3=x_2+(c_2-2t/c_1)/4$$
$$x=(-y_3+x_3^2+k/6)/2$$
$$-y=x_3(x+k/6)+l/8\,.$$

Then, composing the above, we obtain the formulae (14), (15). Further the expressions (17), (18) for p, q follow from (8).

4. Mordell-Weil lattice of type E_6

Let us change the parameter t of the linear pencil by

$$t=4s-\left(\frac{2c_1^3}{27}-\frac{c_1c_2}{6}-\frac{c_3}{2}\right). \tag{19}$$

Then the above p, q are of the form

$$p=p_0+p_1s+p_2s^2\,,\qquad q=q_0+q_1s+q_2s^2+s^4\,, \tag{20}$$

where p_i, q_j are certain polynomials in $c_1, \cdots, c_6$ with rational coefficients.

Observe that, if we rewrite the letter s by t, the above Weierstrass equation (16) becomes

$$E\colon\ y^2=x^3+(p_0+p_1t+p_2t^2)x+q_0+q_1t+q_2t^2+t^4\,, \tag{21}$$

and this is exactly the same as what we studied as the case (E_6) in [S2], §10. (Indeed, our choice of the linear pencil (13) was made in anticipating this fact.)

According to it, under the condition (#′) that the elliptic surface $f: S\to \boldsymbol{P}^1$ has no other reducible fibres than $f^{-1}(\infty)$ (in particular, for general p_i, q_j), the Mordell-Weil lattice $E(k(t))$ is of rank 6 and it is isomorphic to the dual lattice E_6^* of the root lattice E_6. (We shall see below that the two conditions (♯) and (♯′) are actually equivalent.) Such a lattice has 54 minimal vectors of norm 4/3, which are of the form $\pm P=(x, \pm y)$:

$$x=at+b\,, \qquad y=t^2+dt+e\,. \tag{22}$$

The coefficient a is the most important parameter of P, which is a root of the algebraic equation of degree 27 with coefficients in the ring $\boldsymbol{Z}[p_i, q_j]$ whose Galois group is, for generic p_i, q_j, the Weyl group $W(E_6)$ of order 51840. The coefficients d, e are uniquely determined by a, b:

$$d=(a^3+ap_2)/2\,, \qquad e=(3a^2b-d^2+ap_1+bp_2+q_2)/2\,. \tag{23}$$

We have shown in [S2] that b is a rational function of a with coefficients in $\boldsymbol{Q}(p_i, q_j)$, which is also a polynomial in $\boldsymbol{Q}[a_1, \cdots, a_6]$ for suitable a_i, but its explicit formula was not given there due to the length and complexity. In the present paper, we can improve this by finding an explicit, yet relatively simple expression for b with the help of the Weierstrass transformation discussed in the previous section. (Compare [Se], Lemma 3.1.)

Let us first see that the base points Q_i of the linear pencil, viewed as rational points on Γ, are transformed to six points of the above form (22) under the Weierstrass transformation. Of course, the point $A=(0:0:1)$ is transformed to the origin $O=(0:1:0)\in E$. As before, we let $K=k(t)$, k being an algebraically closed field containing u_i (and hence p_i, q_j).

PROPOSITION 3. *Let $P_i \in E(K)$ $(1\le i\le 6)$ denote the 6 rational points corresponding to $Q_i=(u_i^{-2}:u_i^{-3}:1)\in\Gamma(K)$ under the Weierstrass transformation $w:\Gamma\to E$. Then each point P_i has the form* (22) *in which a, b are given as follows.*

$$a=a_i:=\frac{c_1}{3}-u_i \tag{24}$$

$$b=\frac{-8c_1^4+18c_1^2c_2+27c_2^2-54c_1c_3-108c_4}{1296}+\frac{4c_1^3-9c_1c_2-27c_3}{216}u_i-\frac{c_1}{4}u_i^3\,. \tag{25}$$

Proof. For the point $Q_i=(u^{-2}:u^{-3}:1)$ $(u=u_i)$, we have $x_1=u$, $y_1=u^3$. Then a direct computation using the formula (14) gives the above value of a, b for $P_i=w(Q_i)$, in view of (19). The formula (15) implies that $y=t^2/16+\cdots=s^2+\cdots$ (in the old notation), so P_i is of the form (22).

PROPOSITION 4. *Under the condition* (♯′) *mentioned above, we have*

$$\langle P_i, P_j\rangle=\delta_{ij}+\frac{1}{3}\,. \tag{26}$$

Hence the 6 *points* $P_1, \cdots, P_6$ *are linearly independent and generate a subgroup of index* 3 *in* $E(K)$.

Proof. The height pairing formula ([S1]) takes the following form under (♯′):

$$\langle P_i, P_j\rangle = 1 + (P_iO) + (P_jO) - (P_iP_j) - \mathrm{Const}_\infty(P_i, P_j)\,. \tag{27}$$

Here we have $(P_iO)=(P_jO)=0$ and $(P_iP_j)=0$ $(i\neq j)$ since the sections (P_i), (P_j), (O) are disjoint, and $\mathrm{Const}_\infty(P_i, P_j)=2/3$ since (P_i), (P_j) hit the same non-identity component of the singular fibre $f^{-1}(\infty)$ of type IV (cf. §7 below). This proves (26).

Then we see easily that $\det(\langle P_i, P_j\rangle)=3$, which implies first that $P_1, \cdots, P_6$ are linearly independent and second that they generate an index 3 subgroup in $E(K)\simeq E_6^*$ since $\det E_6^*=1/3$.

At this point, we recall some basic facts on the lattice E_6^*.

LEMMA 5. *Suppose that the symbols* $a_1, \cdots, a_6$ *denote* 6 *minimal vectors in the lattice* E_6^* *such that* $\langle a_i, a_j\rangle = \delta_{ij} + 1/3$. *If we set* $r = \frac{1}{3}\sum_{i=1}^{6} a_i$, *then* r *is a root in* E_6: $\langle r, r\rangle = 2$. *Any* 5 *among the* 6 a_i *and* r *generate the lattice* E_6^*. *Let*

$$a_i' = a_i - r \ (1\leq i\leq 6)\,, \quad a_{ij}'' = r - a_i - a_j \ (i<j)\,. \tag{28}$$

Then $\{a_i, a_j', a_{ij}''\}$ *is a system of* 27 *minimal vectors in* E_6^* *on which the Weyl group* $W(E_6)$ *acts transitively.*

Furthermore the 36 *elements*

$$r\,, \quad a_i - a_j\ (i<j)\,, \quad a_i + a_j + a_k - r\ (i<j<k) \tag{29}$$

and their minus give all the 72 *roots of the root lattice* E_6.

Proof. This should be wellknown and also the verification is easy (cf.[B], [CS], or [S2], Appendix).

Going back to the Mordell-Weil lattice $E(K)\simeq E_6^*$, we set

$$P_i' = P_i - R \ (1\leq i\leq 6)\,, \quad P_{ij}'' = R - P_i - P_j \ (i<j)\,, \tag{30}$$

where

$$R = \frac{1}{3}\sum_{i=1}^{6} P_i \in E(K)^0 \simeq E_6\,. \tag{31}$$

Then all the points of the form (22) are given by the 27 rational points $P_i, P_j', P_{ij}'' \in E(K)$.

We recall one more important fact that the map

$$P = (at+b,\ t^2+dt+e) \mapsto a$$

extends to a group homomorphism $\sigma: E(K)\to k$ (it is a specialization homomorphism, up to a constant; cf. [S2], p. 714).

LEMMA 6. *We have*

$$a_i := \sigma(P_i) = c_1/3 - u_i \tag{32}$$

$$a'_i := \sigma(P'_i) == -2c_1/3 - u_i \tag{33}$$

$$a''_{ij} := \sigma(P''_{ij}) = c_1/3 + u_i + u_j\,, \tag{34}$$

where $c_1 = -(u_1 + \cdots + u_6)$ *as before.*

Proof. We have $a_i = c_1/3 - u_i$ by (24). Thus $r = \sum_{i=1}^{6} a_i/3 = c_1$. Since σ is a homomorphism, we have $a'_i = a_i - r$, which implies the formula for a'_i. The rest is similar.

THEOREM 7. *The following conditions on* $u_1, \cdots, u_6$ *are equivalent to each other.*

(i) *The condition* (♯), *i.e.*

$$u_i \neq u_j\ (i \neq j)\,, \quad u_i + u_j + u_k \neq 0\ (i, j, k : distinct)\,, \quad \sum_{i=1}^{6} u_i \neq 0\,.$$

(ii) *The condition* (♯′) *that the elliptic surface* $f: S \to \boldsymbol{P}^1$ *has no other reducible fibres than* $f^{-1}(\infty)$.

(iii) *The Mordell-Weil lattice is nondegenerate, i.e.* $E(K) \simeq E_6^*$.

(iv) *The affine surface defined by* (21) *in the* 3-*space with coordinates* x, y, t *is smooth.*

(v) *There are no "vanishing roots" under the specialization* $\tilde{u}_i \to u_i$, *where* $\tilde{u}_i$ *denotes the generic parameter.*

Proof. The 36 "roots" in (29) now take the following form:

$$r = c_1 = -\sum_{i=1}^{6} u_i\,, \quad a_i - a_j = -(u_i - u_j)\,, \quad a_i + a_j + a_k - r = -(u_i + u_j + u_k)\,.$$

The condition (♯) says that none of them are zero. Thus (i) and (v) are equivalent. That (ii), (iii), (iv), (v) are mutually equivalent is known in our previous work; cf. [S2], §10.

THEOREM 8. *Let* ε_n *be the n-th elementary symmetric function of the* 27 *linear forms in* $u_1, \cdots, u_6$:

$$a_i := \frac{c_1}{3} - u_i\,, \quad a'_i := \frac{-2c_1}{3} - u_i\,, \quad a''_{ij} := \frac{c_1}{3} + u_i + u_j\,. \tag{35}$$

Then we have

$$p_2 = \varepsilon_2/12$$

$$p_1 = \varepsilon_5/48$$

$$q_2 = (\varepsilon_6 - 168p_2^3)/96$$

$$p_0 = (\varepsilon_8 - 294p_2^4 - 528p_2q_2)/480$$

$$q_1 = (\varepsilon_9 - 1008p_1p_2^2)/1344$$

$$q_0 = (\varepsilon_{12} - 608p_1^2p_2 - 4768p_0p_2^2 - 252p_2^6 - 1200p_2^3q_2 + 1248q_2^2)/17280\,.$$

The p_i, q_j are the fundamental invariants of the Weyl group $W(E_6)$ acting on the space with coordinates $u_1, \cdots, u_6$.

Proof. This follows from Theorem 10.3 of [S2].

5. Explicit form of the 27 rational points

Let us determine the explicit formula for the rational points P'_i, P''_{ij}, as we did for P_i in Proposition 3.

For this purpose, we consider some points of interest on the cubic curve Γ. Starting from the base points $A, Q_1, \cdots, Q_6 \in \Gamma(K)$, let us define the K-rational points of Γ

$$A', \ B_i, \ Q'_i, \ Q''_{ij}$$

as follows. First the tangent line to Γ at A (i.e. $Y=0$) intersects Γ at one more point A'; in coordinates, we have $A'=(-c_1:0:t)$. Next the line $\overline{AQ_i}$ connecting the points A and Q_i meets Γ at the third point B_i, and similarly the line $\overline{A'B_i}$ meets the curve at Q'_i. Finally the line $\overline{Q_iQ_j}$ intersects Γ at the third point Q''_{ij} for $i \neq j$.

It is easy to write down the coordinates of these points, which shows that they are indeed K-rational points ($K=k(t)$).

LEMMA 9. *With respect to the linear equivalence relation $\sim$ in the divisor group of Γ, we have the following relations*:

$$A'+2A \sim H \tag{36}$$

$$Q'_i - A \sim A - B_i \sim Q_i - A' \tag{37}$$

$$Q_i + Q'_j + Q''_{ij} \sim 3A \quad (i \neq j) \tag{38}$$

$$\sum_{j=1}^{6} Q_j \sim 3A' + 3A \tag{39}$$

$$Q'_i + \sum_{j \neq i} Q_j \sim 2H. \tag{40}$$

In the above, H denotes any hyperplane (=line) section of Γ.

Proof. By definition, we have

$$A'+2A \sim H$$
$$B_i + A + Q_i \sim H$$
$$B_i + A' + Q'_i \sim H$$
$$Q_i + Q_j + Q''_{ij} \sim H \quad (i \neq j)$$
$$3A + \sum_{j=1}^{6} Q_j \sim 2H.$$

(The last relation follows from Lemma 1.) Using these relations, it is immediate to verify the assertion.

PROPOSITION 10. *Under the Weierstrass transformation* $w: \Gamma \to E$ *sending* A *to* O *and* Q_i *to* P_i, *the above points* A', Q'_i, $Q''_{ij} \in \Gamma$ *are transformed to the points* R, P'_i, $P''_{ij} \in E(K)$ *defined by* (30), (31). *In particular, we have*

$$P_i + P'_j + P''_{ij} = O \quad (i \neq j) . \tag{41}$$

Proof. Write down the linear equivalence relations in the divisor group of the elliptic curve E corresponding to those in Lemma 9 under the isomorphism $w: \Gamma \simeq E$, and rewrite them in the group law of E by the rule: $P+Q+T=0$ if and only if $P+Q+T \sim 3O$. Thus (39) implies $\sum P_j = 3w(A')$, showing $w(A') = R$ as in (31). Further (37) implies $w(Q'_i) = -w(B_i) = P_i - R$, which shows $w(Q'_i) = P'_i$. Similarly for P''_{ij}. The relation (41) follows from (38).

Now we can write down the explicit formula for P'_i and P''_{ij}. Note that the coefficients a (and d, e) are given by (33), (34) (and (23)), we have only to give a formula of b for them.

PROPOSITION 11. *Each of the points* P'_i *and* P''_{ij} *has the form* (22) *where* a, b *are given as follows.*
For P'_i $(1 \leq i \leq 6)$, *we have* $a = a'_i = -2c_1/3 - u_i$ *and*

$$b = \frac{16c_1^4 - 36c_1^2c_2 + 27c_2^2 + 108c_1c_3 - 108c_4}{1296} + \frac{4c_1^3 + 45c_1c_2 - 27c_3}{216} u_i + \frac{c_1^2}{4} u_i^2 + \frac{c_1}{4} u_i^3 . \tag{42}$$

For P''_{ij} $(1 \leq i < j \leq 6)$, *we have* $a = a''_{ij} = c_1/3 + u_i + u_j$ *and*

$$b = (-8c_1^4 + 18c_1^2c_2 + 27c_2^2 - 54c_1c_3 + 216c_4 - 24c_1^3s_1 + 54c_1c_2s_1 + 162c_3s_1 + 324c_2s_1^2 + 324c_1s_1^3 + 324s_1^4 - 324c_2s_2 - 324c_1s_1s_2 - 648s_1^2s_2 + 324s_2^2)/1296 , \tag{43}$$

where we set $s_1 = u_i + u_j$, $s_2 = u_i \cdot u_j$.

Proof. Since we have $P'_i = w(Q'_i) = -w(B_i)$, P'_i and $w(B_i)$ have the same x-coordinate. Since B_i is determined as an intersection point of the line $\overline{AQ_i}$ with Γ, we see that $x_1 = u$, $y_1 = t - (c_3 + c_2u + c_1u^2 + u^3)(u = u_i)$, Then (14) and (19) imply (42).

As for P''_{ij}, a direct application of (14) to the point Q''_{ij} leads to a somewhat complicated expression. To avoid this, we apply the addition theorem on E to $P''_{ij} = -P_i - P'_j$; togegher with the results for P_i, P'_j, this shows (43).

In summary, we state an improved version of Theorem (E_6) in [S2], p. 679.

THEOREM 12. *Consider the elliptic curve*

$$E: \ y^2 = x^3 + (p_0 + p_1t + p_2t^2)x + q_0 + q_1t + q_2t^2 + t^4 \tag{44}$$

over $\mathscr{K}(t)$, *where* $\mathscr{K}=\boldsymbol{Q}(u_1, \cdots, u_6)$. *Then the Mordell-Weil group* $E(\mathscr{K}(t))$ *has rank* 6 *precisely under the condition* (♯). *In this case, the* 54 *minimal vectors in the Mordell-Weil lattice* $E(\mathscr{K}(t))\simeq E_6^*$ *are the* 27 *rational points* P_i, P'_i *or* P''_{ij}, *up to sign, whose explicit form are given by Propositions* 3 *and* 11. *Further the six rational points* $\{P_i\}$ *(resp.* $\{P'_i\}$*) generate a sublattice of index* 3 *in* $E(\mathscr{K}(t))$, *and any five among them, say* P_i $(i\neq j)$, *together with one more* P'_j, *generate the full Mordell-Weil group.*

6. Cubic surfaces

First let us recall the following fundamental facts on a cubic surface and the 27 lines on it (cf. [Ma], Ch. IV):

(1) Let $Q_1, \cdots, Q_6\in \boldsymbol{P}^2$ be any six points in a general position, which means that they are not on a conic and no three points among them lie on a line in $\boldsymbol{P}^2$. Let $\beta: V\to\boldsymbol{P}^2$ be the blowing up of $\boldsymbol{P}^2$ at these six points. Then V is isomorphic to a smooth cubic surface in $\boldsymbol{P}^3$, which will be identified with V. Every smooth cubic surface arises this way.

(2) The 27 lines on the cubic surface V are exactly the exceptional curves of the first kind on V, and they are given as follows:

the 6 lines L_i such that each L_i is the exceptional curve arising from blowing up Q_i: $\beta(L_i)=Q_i$.

the 6 lines L'_i such that $\beta(L'_i)$ is the conic in $\boldsymbol{P}^2$ passing through the 5 points Q_j $(j\neq i)$

the 15 lines L''_{ij} $(1\leq i<j\leq 6)$ such that each $\beta(L''_{ij})$ is the line in $\boldsymbol{P}^2$ connecting Q_i, Q_j.

The system $\{L_i, L'_i\ (1\leq i\leq 6)\}$ forms the so-called *double six*. Namely, the six lines L_i (resp. L'_i) are mutually disjoint, and L_i intersects L'_j if and only if $i\neq j$, in which case the plane containing L_i and L'_j contains one more line L''_{ij}; such a plane is called a *tritangent plane* of V.

(3) The birational transformation

$$\beta^{-1}\colon \boldsymbol{P}^2\cdots\to V\subset\boldsymbol{P}^3$$

is given as follows. The vector space of cubic forms in $\boldsymbol{P}^2$ passing through the given 6 points Q_i has 4 dimensions. Choose 4 cubic forms $h_0, \cdots, h_3$ which form a basis. Then the map $x\to(h_0(x)\colon\cdots\colon h_3(x))$ defines β^{-1}.

Now one can ask a simple-minded question:

QUESTION. Given 6 points Q_i as above, is it possible to write down the defining equation of the cubic surface V and also the equations of the 27 lines L_i, L'_i, L''_{ij} in $\boldsymbol{P}^3$ in a comprehensive way? Conversely, given 6 disjoint lines on a cubic surface, is it possible to explicitly determine the 6 points which are obtained by blowing down the 6 lines?

It seems to the author that the answer to this question has not been given in a satisfactory manner, even putting aside and difficulty in carrying out necessary computations. The latter difficulty being removed (or at least eased) by the use of computer, what lacked in the classical theory seems to be a method for adequate normalization (or for choosing natural parameters) to answer the question. (In most literature on cubic surfaces, the subject has been studied from more geometrical standpoint, and it may be the case that the above question has never been asked?)

Now, by translating the results in the previous sections based on Mordell-Weil lattices and Weierstrass transformations, we can answer the above question in the following way. As before, we state the results in characteristic 0, but the same results hold in any characteristic different from 2, 3, 5, 7.

LEMMA 13. *Given any 6 points $Q_1, \cdots, Q_6 \in \boldsymbol{P}^2$ in a general position, there is a cuspidal curve passing through them such that the cusp is different from any Q_i.*

Proof. Here is an indirect but short proof using the above facts (1), (2), (3). Let $\beta: V \to \boldsymbol{P}^2$ by the blowing up of $\boldsymbol{P}^2$ at Q_i, which is identified with a smooth cubic surface. For any point $v \in V$, let T_v denote the tangent plane to V at v. Then the intersection $V \cap T_v$ is a plane cubic with a singularity at v, and it is cuspidal cubic (with the cusp v) for some point $v \in V$ (cf. [R], Ch. 7). Taking its image $\beta(V \cap T_v)$ in $\boldsymbol{P}^2$, we obtain a cubic with the cusp $\beta(v)$ which is different from any $Q_i = \beta(L_i \cap T_v)$, This proves the lemma.

By the lemma, there is no loss of generality in assuming that the given 6 points are of the form $Q_i = (u_i^{-2} : u_i^{-3} : 1) \in \boldsymbol{P}^2$ as studied in §3. Further it is easily seen that the 6 points Q_i are in general position if and only if the 6 values u_i satisfy the condition (#).

Now we have the following results:

THEOREM 14. *Let $Q_i = (u_i^{-2} : u_i^{-3} : 1) \in \boldsymbol{P}^2$ $(1 \le i \le 6)$ be six points on the cuspidal cubic $X^3 - Y^2 Z = 0$ and assume the condition (#), i.e.*

$$(\#) \quad u_i \neq u_j \ (i \neq j), \quad u_i + u_j + u_k \neq 0 \ (i, j, k: \textit{distinct}), \quad \sum_{i=1}^{6} u_i \neq 0 .$$

Then the cubic surface $V \subset \boldsymbol{P}^3$ obtained by blowing up these points has the following defining equation

$$Y^2 W + 2YZ^2 = X^3 + X(p_0 W^2 + p_1 ZW + p_2 Z^2) + q_0 W^3 + q_1 ZW^2 + q_2 Z^2 W \quad (45)$$

where p_i, q_j are as in Theorem 8, §4. Moreover the above cubic surface V is smooth if and only if the assumption (#) is satisfied.

THEOREM 15. *The 27 lines on the cubic surface V (45) are given in the form:*

$$X = aZ + bW, \quad Y = dZ + eW, \quad (46)$$

where d, e are determind by a, b by (23), while a, b are given for each of the lines L_i, L'_i $(1 \le i \le 6)$ and L''_{ij} $(1 \le i < j < 6)$ as follows. (As before, $(-1)^i c_i$ is the i-th elementary *symmetric function of* $u_1, \cdots, u_6$).

First, for the line L_i $(1 \le i \le 6)$, *we have*

$$a = a_i := \frac{c_1}{3} - u_i$$

$$b = \frac{-8c_1^4 + 18c_1^2c_2 + 27c_2^2 - 54c_1c_3 - 108c_4}{1296} + \frac{4c_1^3 - 9c_1c_2 - 27c_3}{216} u_i - \frac{c_1}{4} u_i^3 . \quad (48)$$

Second, for the line L'_i $(1 \le i \le 6)$, *we have*

$$a = a'_i := \frac{-2c_1}{3} - u_i \quad (49)$$

$$b = \frac{16c_1^4 - 36c_1^2c_2 + 27c_2^2 + 108c_1c_3 - 108c_4}{1296} + \frac{4c_1^3 + 45c_1c_2 - 27c_3}{216} u_i + \frac{c_1^2}{4} u_1^2$$

$$+ \frac{c_1}{4} u_i^3 . \quad (50)$$

Finally, for the line L''_{ij} $(1 \le i < j \le 6)$, *we have*

$$a = a''_{ij} := \frac{c_1}{3} + u_i + u_j \quad (51)$$

$$b = (-8c_1^4 + 18c_1^2c_2 + 27c_2^2 - 54c_1c_3 + 216c_4 - 24c_1^3s_1 + 54c_1c_2s_1 + 162c_3s_1$$
$$+ 324c_2s_1^2 + 324c_1s_1^3 + 324s_1^4 - 324c_2s_2 - 324c_1s_1s_2 - 648s_1^2s_2 + 324s_2^2)/1296 , \quad (52)$$

where we set $s_1 = u_i + u_j$, $s_2 = u_i \cdot u_j$.

7. Proof of Theorems 14, 15

The proof will follow from the relationship between the cubic surface V (45) and the rational elliptic surface S associated with $E/k(t)$ (44), which can be stated as follows.

THEOREM 16. (i) *The rational surface S is obtained from* $\boldsymbol{P}^2$ *by blowing up the* 6 *points* Q_i, *the point A and* 2 *more points which are infinitely near to A* (*in other words, all the* 9 *base points of the linear pencil* (13); *cf. Lemma* 1, §3).

(ii) *The surface V is obtained from S by blowing down the* 3 *exceptional curves* (O), Θ_0, Θ_1 *in this order.* (*See the notation below.*)

(iii) *The birational morphism* $\pi: S \to V$ *is explicitly given by the map*

$$(x, y, t) \mapsto (X:Y:Z:W) = (x : y - t^2 : t : 1) . \quad (53)$$

Under π, *any section* (P) *in the elliptic surface S of the form* $P = (at + b, t^2 + dt + e)$ *is mapped to the line* $X = aZ + bW$, $Y = dZ + eW$ *on the cubic surface V. Thus the* 27 *sections* (P_i), (P'_i), (P''_{ij}) *are mapped to the* 27 *lines* L_i, L'_i, L''_{ij}.

First we recall the following facts (cf. [S2], §10). The elliptic surface S has a reducible singular fibre of Kodaira type IV ([K]) at $t=\infty$: $f^{-1}(\infty)=\Theta_0+\Theta_1+\Theta_2$, where Θ_i are 3 smooth rational curves with self-intersection number -2 meeting at a single point; we let Θ_0 be the unique component of the singular fibre meeting the zero section (O).

Let us show that Theorem 16 implies Theorems 14 and 15. It is easy to check that the map (53) gives a birational mapping from the elliptic surface S to the cubic surface V defined by the equation (45) (this was first noted in [S3], §8). Then everything in Th. 14 and 15 immediately follows, except the last statement in Th. 14 about smoothness. For this, if we denote by S' the affine surface in (x, y, t)-space defined by (44), then we have

$$S=S'\cup(O)\cup\Theta_0\cup\Theta_1\cup\Theta_2\,. \tag{54}$$

On the other hand, let V' denote the affine open set of V defined by $W\neq 0$. In terms of affine coordinates, the map $\pi\colon S'\to V'$ is given by $(x, y, t)\mapsto(x, y-t^2, t)$, which is clearly an isomorphism. By Theorem 7, S' is smooth if and only if the condition (#) holds. Further, the complement of V' in V is a cuspidal cubic curve and a simple computation shows that every point of this curve is a smooth point on the surface V. Hence we conclude that V is smooth precisely under the condition (#).

Now let us turn to the proof of Theorem 16. Remember that we have an isomorphism of elliptic curves $w\colon(\Gamma, A)\simeq(E, O)$, where Γ is the generic member of the linear pencil of cubic curves $\{\Gamma_t\}$ in $\boldsymbol{P}^2$ with the base points Q_i and $3A$. In order to make this pencil a family $\{\Gamma'_t\}$ parametrized by $t\in\boldsymbol{P}^1$, it is necessary to blow up the base points. At each point Q_i, a single blow-up separates the members of the pencil, but at $A=(0:0:1)$, we need to blow up three times to achieve this. Indeed, in terms of the local coordinates (x, y) at A, the first blow-up is given by (x, η), $\eta=y/x$, the second by (x, η'), $\eta'=\eta/x$, and the third by (x, η''), $\eta''=(\eta'+c_1)/x$, which then separates the members at A. In other words, by blowing up $\boldsymbol{P}^2$ at the base points of the linear pencil, we obtain a smooth surface $\tilde{S}$ with elliptic fibration having global sections whose generic fibre is Γ. Therefore it is the Kodaira-Néron model of $\Gamma/k(t)$ (note that it is relatively minimal since the Picard number of $\tilde{S}$ is 10).

In view of the isomorphism $w\colon(\Gamma, A)\simeq(E, O)$, the elliptic surfaces $\tilde{S}$ and S are isomorphic (over $\boldsymbol{P}^1$), and the Weierstrass transformation $w\colon\boldsymbol{P}^2\to S$ induces such an isomorphism $\tilde{S}\simeq S$. By comparison, we see that the 3 rational curves on $\tilde{S}$ arising from A, i.e. η-line, η'-line and η''-line, correspond respectively to Θ_1, Θ_0 and (O) on S. This proves the parts (i) and (ii) of Theorem 16.

It remains to show that the birational map π defined by (53) is a morphism from S to V which contracts the 3 curves (O), Θ_0, Θ_1 to the point $(0:1:0:0)\in V$. As we saw above, the restriction of π to $S'\to V'$ is an isomorphism. Thus we have only to work in a neighborhood of (O) and Θ_i. By expressing π in the local coordinates in such a neighborhood, we can verify the above claim. This completes the proof of

Theorem 16.

REMARK. Observe that the system of lines $\{L_i, L'_i\}$ in Theorem 15 forms a double six in the sense of the fact (2) in §6. In fact, the line L_i corresponding to P_i clearly comes from blowing up the point $Q_i \in \boldsymbol{P}^2$. On the other hand, the line L'_i corresponds to P'_i, which is the image of $Q'_i \in \Gamma$ under the Weierstrass transformation w. The formula (40) in Lemma 9 says that Q'_i is the 6-th point on the conic determined by the 5 points Q_j $(j \neq i)$. Thus the line L'_i has the property stated in (2). Similarly, the line L''_{ij} corresponds to P''_{ij}, which comes from the point Q''_{ij} lying on the line connecting Q_i, Q_j.

In view of the above, we may call a *double six* the system of rational points $\{P_i, P'_i\}$ in the Mordell-Weil lattice of type E_6, $E(K) \simeq E_6^*$, and more generally any system of vectors $\{a_i, a'_i\}$ in the lattice E_6^* satisfying the properties of Lemma 5.

8. Example

To show that our method is effective, we present a numerical example here. Suppose, for a moment, that we employ the notation of Lemma 5, §4. Then we can choose the six elements

$$a_1 - a_2, \cdots, a_5 - a_6, a_4 + a_5 + a_6 - r \qquad \left(r = \frac{1}{3}\sum_{i=1}^{6} a_i\right) \tag{55}$$

as simple roots of the root lattice E_6 (cf. [B]). Then the 36 elements in (29) become "positive roots" of E_6.

Now, associating the value 2 to all of the 6 simple roots is easily shown to be the same as setting, $a_i = 2(9-i)$ $(1 \leq i \leq 6)$, which is equivalent to the following choice of u_i in view of (24). The condition (♯) is satisfied since none of the "roots" in (29) vanish.

Take

$$u_i = (6i - 32)/3 \qquad (1 \leq i \leq 6). \tag{56}$$

Then the linear pencil of cubics passing through $Q_i = (u_i : 1 : u_i^3)$ has the following equation by (13):

$$\begin{aligned}\Gamma:\ & 729t(X^3 - Y^2) + 729Y + 16038X^2 + 121500XY + 107136X^2Y \\ & + 344520Y^2 - 747648XY^2 - 465920Y^3 = 0.\end{aligned} \tag{57}$$

Under the Weierstrass transformation, it is transformed to the elliptic curve

$$E:\ y^2 = x^3 + (-59475 - 78t^2)x + 2848750 + 18226t^2 + t^4, \tag{58}$$

where the old t is replaced by a new one according to (19).

The corresponding cubic surface (45) is given by the equation:

$$Y^2W + 2YZ^2 = X^3 + X(-59475W^2 - 78Z^2) + 2848750W^3 + 18226Z^2W \tag{59}$$

(cf. [S3], Example 8.6).

Now the values of a_i, a'_i, a''_{ij} are determined by (35) as follows:

$$a_i=16, 14, 12, 10, 8, 6; \mathrm{a}'_i=-6, -8, -10, -12, -14, -16 ;$$

$$a''_{ij}=-8, -6, -4, -2, 0, -4, -2, 0, 2, 0, 2, 4, 4, 6, 8 .$$

The 6 lines L_i on V corresponding to Q_i are:

$$\begin{array}{lll} L_1: & X=16Z+3515W, & Y=1424Z+207900W \\ L_2: & X=14Z+1535W, & Y=826Z+59400W \\ L_3: & X=12Z+435W, & Y=396Z+7700W \\ L_4: & X=10Z-49W, & Y=110Z-2376W \\ L_5: & X=8Z-181W, & Y=-56Z-2772W \\ L_6: & X=6Z-225W, & Y=-126Z-2200W. \end{array}$$

The other 6 lines L'_i on V forming a double six with L_i are given by

$$\begin{array}{lll} L'_1: & X=-6Z-225W, & Y=+126Z-2200W \\ L'_2: & X=-8Z-181W, & Y=+56Z-2772W \\ L'_3: & X=-10Z-49W, & Y=-110Z-2376W \\ L'_4: & X=-12Z+435W, & Y=-396Z+7700W \\ L'_5: & X=-14Z+1535W, & Y=-826Z+59400W \\ L'_6: & X=-16Z+3515W, & Y=-1424Z+207900W. \end{array}$$

Observe that the equation of L'_i is the same as that of L_{7-i} up to changing the sign of Z. This extra symmetry is a consequence of the symmetric choice of the initial data.

The remaining 15 lines L''_{ij} are as follows:

$$\begin{array}{lll} L''_{12}: & X=-8Z-85W, & Y=56Z+2700W \\ L''_{13}: & X=-6Z+15W, & Y=126Z+1400W \\ L''_{14}: & X=-4Z+275W, & Y=124Z-2700W \\ L''_{15}: & X=-2Z+575W, & Y=74Z-12600W \\ L''_{16}: & X=891W, & Y=-25636W \\ L''_{23}: & X=-4Z+35W, & Y=124Z+900W \\ L''_{24}: & X=-2Z+239W, & Y=74Z-1512W \\ L''_{25}: & X=411W, & Y=-6916W \\ L''_{26}: & X=2Z+575W, & Y=-74Z-12600W \\ L''_{34}: & X=219W, & Y=572W \\ L''_{35}: & X=2Z+239W, & Y=-74Z-1512W \\ L''_{36}: & X=4Z+275W, & Y=-124Z-2700W \\ L''_{45}: & X=4Z+35W, & Y=-124Z+900W \\ L''_{46}: & X=6Z+15W, & Y=-126Z+1400W \\ L''_{56}: & X=8Z-85W, & \mathrm{Y}=-56Z+2700W. \end{array}$$

In terms of the elliptic curve E defined over $\boldsymbol{Q}(t)$, the Mordell-Weil group $E(\boldsymbol{Q}(t))$ has rank 6 and it has the 27 rational points P_i, P_i', P_{ij}'' whose coordinates can be read off from the above equations for the lines. by Theorem 12, $\{P_1, \cdots, P_5, P_6'\}$ forms a set of generators of $E(\boldsymbol{Q}(t))$. The 36 rational points corresponding to the 36 positive roots of the root lattice E_6 can also be written down. They are all of the form

$$P=\left(\frac{1}{u^2}\,t^2+at+b, \frac{1}{u^3}\,t^3+ct^2+dt+e\right),$$

where u are the linear forms in $u_1, \cdots, u_6$ given by (29), divided by -2 (cf. [S2], Lemma 10.2). For instance, for the rational point R corresponding the root r, we have $r=(a_1+\cdots+a_6)/3=c_1=22$ and

$$R=\left(\frac{1}{121}\,t^2-265, -\frac{1}{1331}\,t^3-\frac{2204}{11}\,t\right).$$

Note, in general, that we have $R=P_i-P_i'$ for all i, and also that $R=w(A')=w(H)$ is the rational point corresponding to the hyperplane class of the plane cubic Γ.

9. The 28 bitangents to a plane quartic curve

It may be of some interest to note the related result on the 28 bitangents to a plane quartic curve which can be derived from the above in an effortless manner. It can be used, for example, to produce an explicit example of a smooth plane quartic curve defined over $\boldsymbol{Q}$ such that all the 28 bitangents are $\boldsymbol{Q}$-rational. (Noting that $\boldsymbol{Q}\subset\boldsymbol{R}$, these bitangents are all real so that we could draw their picture on a real plane!)

The right hand side of (44)

$$x^3+(p_0+p_1t+p_2t^2)x+q_0+q_1t+q_2t^2+t^4=0 \tag{60}$$

defines a plane quartic curve, say Γ. Let $(X:Z:W)=(x:t:1)$ be the homogeneous coordinates on $\boldsymbol{P}^2$. The line at infinity $W=0$ meets Γ at a single point $(1:0:0)$, which is a special flex in the sense of [S5], i.e. an intersection point of multiplicity 4. So the line $W=0$ is at the same time a bitangent.

THEOREM 17. *The plane quartic Γ is smooth precisely when the condition* (♯) *is satisfied. The* 27 *bitangents to Γ other than the line at infinity are given by*

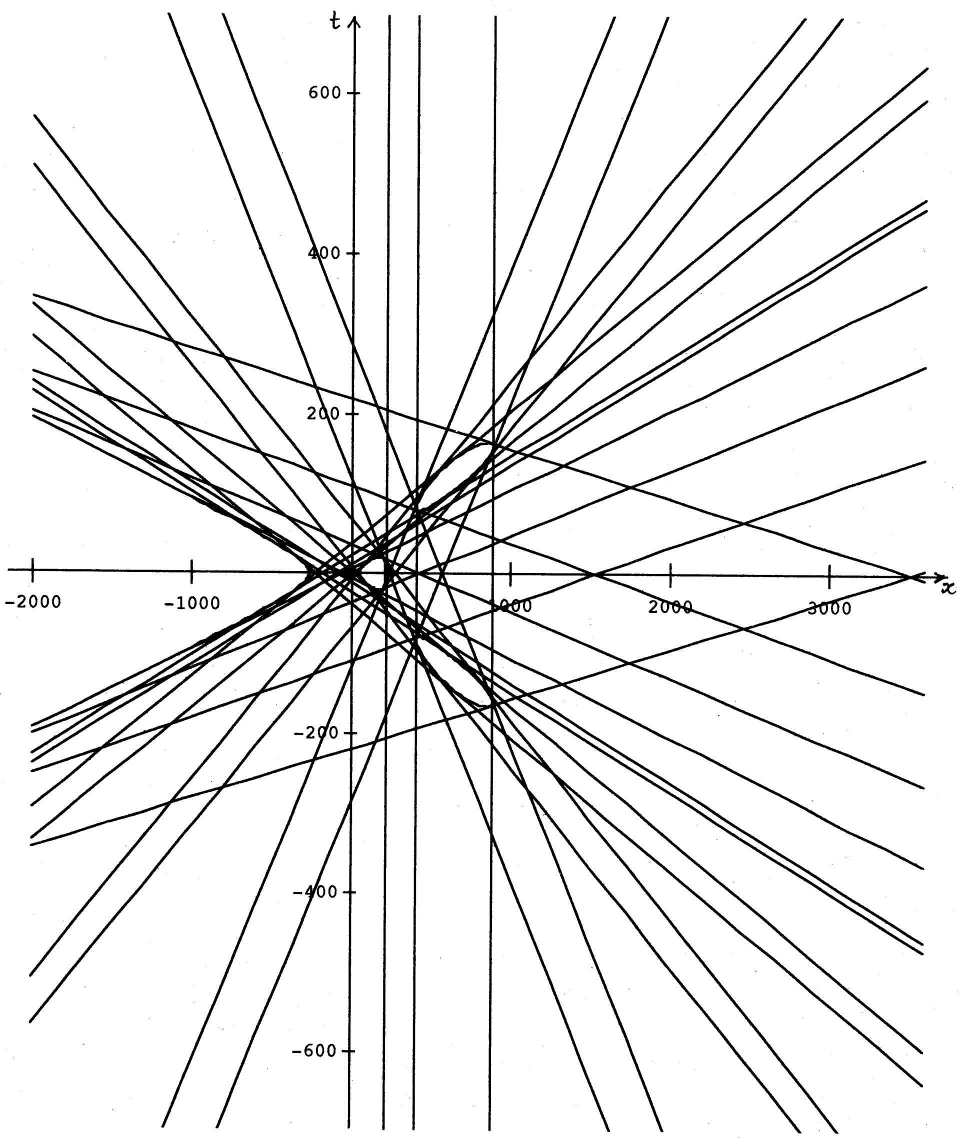
$$x=at+b,$$

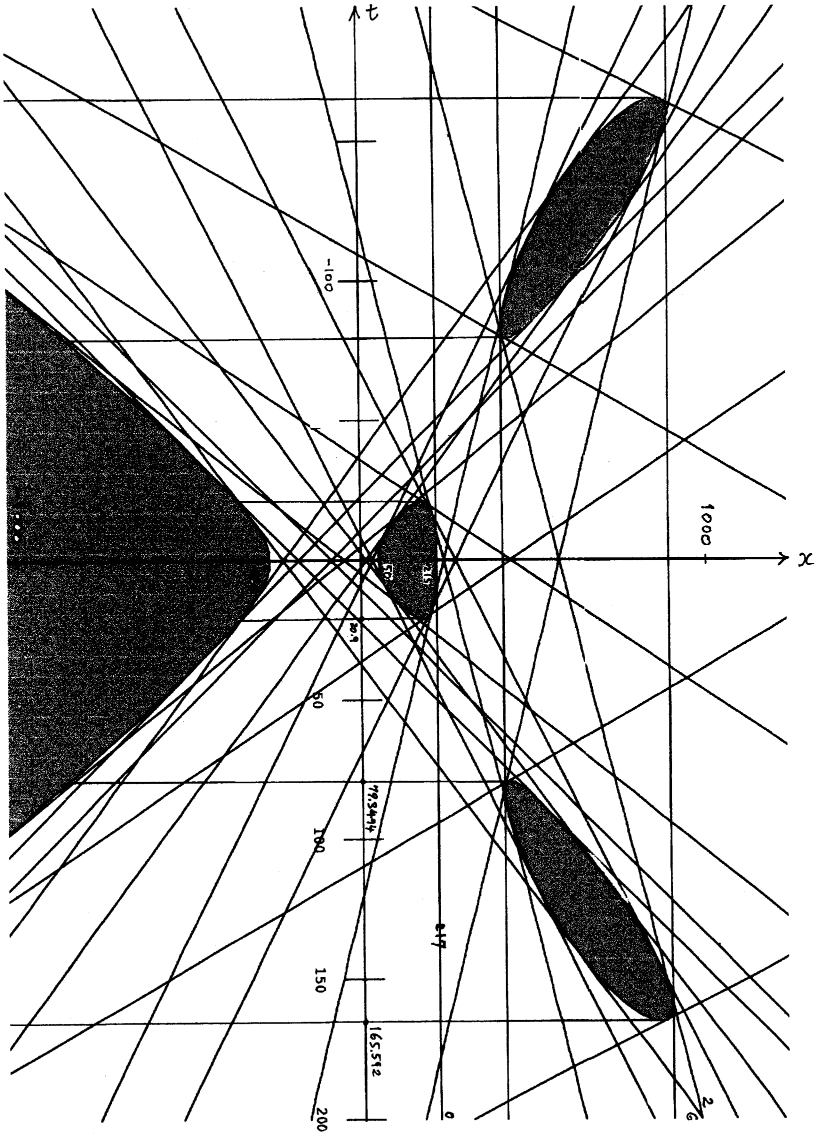
where a, b have the values as in Theorem 15.

Proof. Substitute $x=at+b$ into the left hand side of (60), and then we obtain a square in $k(t)$ exactly when $x=at+b$ is the x-coordinate of a rational point of E over $k(t)$ defined by (44). This means that the line $x=at+b$ is a bitangent to Γ for the 27 pairs of (a, b) given by Theorem 15. This proves the second assertion. The first part can be proven in the same way as Theorem 5 of [S5].

Observe that the 27 bitangents to our plane quartic Γ are obtained from the 27 lines on the cubic surface V via the projection $(X:Y:Z:W)\to(X:Z:W)$ of $\boldsymbol{P}^3$ to $\boldsymbol{P}^2$ from the point $(0:1:0:0)\in V$. Thus our result can be regarded as a concrete special case of the classical relationship of a cubic surface and a plane quartic due to Geiser (in Vol. 1 of Math. Ann. 1869).

Finally we draw the picture of the plane quartic and the 27 bitangents corresponding to the example in §8; see the attached figure.

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