Thermal runaway and bistability in microwave heated slabs, cylinders, and spheres

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Thermal runaway and bistability in microwave heated slabs, cylinders, and spheres

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Proefschrift

ter verkrijging van de graad van doctor
op gezag van de rector magnificus
van de Wageningen Universiteit
Dr. C.M. Karssen,
in het openbaar te verdedigen
op vrijdag 19 mei 2000
des namiddags te één uur dertig in de Aula

CIP-gegevens Koninklijke Bibliotheek, Den Haag
ISBN 90-5808-206-7

Aan mijn ouders, Yt en de kinderen

ABSTRACT

Vriezinga, C.A., 2000. Thermal runaway and bistability in microwave heated slabs, cylinders, and spheres, Ph. D. Thesis, Wageningen University, the Netherlands, 144 pp., with English and Dutch summary, ISBN 90-5808-206-7

This study analyzes the runaway of microwave (2450 MHz) heated slabs, cylinders, and spheres in free space. It is shown that the phenomenon can be described with an S-shaped response curve of steady-state temperature versus microwave power at any position within the sample. The analysis demonstrates that the direct influence of the temperature dependent loss factor in the absorption of electromagnetic energy is canceled by the attenuation constant. Runaway can be understood from the behavior of the waves within the sample. In foodstuffs the small decrease of the phase constant with increasing temperature causes resonance at certain temperature, which can be considered as the physical origin of runaway. So characteristic dimension of food samples have to be smaller than a quarter of the temperature dependent wavelength to prevent runaway. The investigation of a slab of alumina demonstrates that thermal runaway in ceramics with a dielectric loss factor exponentially increasing with increasing temperature is caused by the strongly increasing attenuation constant at high temperatures. This means that only relatively thick ceramic objects will not be damaged by the runaway. The physical and mathematical aspects of bistability are investigated for a system with a microwave power directly proportional to time.

Keywords: microwave heating, thermal runaway, bistability, relaxation oscillations

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Voorwoord

Een proefschrift kan niet tot stand komen door de inspanning van één persoon alleen. Er is een omgeving nodig, die ruimte biedt aan de promovendus om te onderzoeken (niet al te zeer gehinderd door bestuurlijke perikelen en persoonlijke problemen), er zijn faciliteiten nodig (bijv. computers), er is stimulans nodig, er is overleg nodig, er is afstemming nodig, enz. Naast deze algemene zaken, van toepassing op iedereen die een werk verricht, is er de concrete hulp. De promovendus kan tijdens zijn onderzoek terecht komen op voor hem onbekend terrein, of hij raakt in gevecht met computer programma's, of hij wordt geconfronteerd met extreem hoge eisen van uitgevers van wetenschappelijke tijdschriften, of hij komt op een andere manier vast te zitten. Dan is er concrete hulp nodig.

Tal van mensen hebben in meer of mindere mate een bijdrage geleverd in het tot stand komen van dit proefschrift. Mijn dank gaat uit naar hen allen, maar de mensen die direct betrokken waren bij dit onderzoek, wil ik met name noemen.

Om te beginnen wil ik de begeleidingscommissie, bestaande uit Gerard Bot, Johan Grasman, Wim Jansen en Paul Bartels bedanken. Gerard, bedankt voor het creëren van de ruimte voor dit onderzoek in een bestuurlijk zeer turbulente tijd en ook bedankt voor de vele adviezen op het terrein van de warmtetransport. Johan, welk een genoegen was het om te ontdekken dat dit onderzoek nauw verwant is aan één van jouw specialiteiten, de relaxatie oscillaties. Bedankt dat je de bedding en de stimulans hebt gegeven om daar iets mee te doen. Wim, nestor van de Nederlandse microgolf wereld, bedankt voor je enthousiasme en voor de wijze waarop je mij in de internationale wetenschappelijke wereld geïntroduceerd hebt. Paul, bedankt voor je adviezen in relatie tot de praktijk van de microgolf verwarming. Met jou wil ik ook je beide collega's Eric Torringa en Eric Esveld bedanken.

De eigenlijke inhoud van dit proefschrift bestaat voor het merendeel uit artikelen, gepubliceerd in gerenommeerde wetenschappelijke tijdschriften. Iedere publicatie is beoordeeld door een referee. Vaak was het oordeel zeer positief, maar ook waren er kritische kanttekeningen, die mij dwongen bepaalde stellingen opnieuw te overwegen en zo nodig te nuanceren. Op internationale congressen werd ik aangesproken op mijn werk of zocht ikzelf de discussie op. De onbekende referees en het contact met de buitenlandse vakbroeders gaven mij het klankbord voor dit onderzoek. In dit verband wil ik drie mensen noemen. Dat zijn Gregory Kriegsmann, Lynn Johnson en Barmatz. De gesprekken met hen waren zeer stimulerend en motiveerden mij om door te gaan op de ingeslagen weg.

De volgende persoon, die ik wil noemen, is mijn oud-collega Henk van Remmen. Samen met jou heb ik de eerste stappen gezet in de microgolf wereld. Bedankt Henk voor je steun en kameraadschap.

Mijn dank gaat ook uit naar de totale fysica groep, waarin we ondanks de ogenschijnlijk verschillende onderzoeksgebieden als team opereren. Bedankt Edo Gerkema, Hennie Boshoven, Dane Bicanic, Wilko van Loon, Ies van Haneghem en Joost van Opheusden. Aan jullie denk ik bij concrete hulp op velerlei gebied. Rachel van Ooteghem en Paul van Espelo wil ik bedanken voor hun hulp bij het maken van mooie plaatjes.

Mijn speciale dank gaat uit naar Josie Zeevat voor haar volkomen belangeloze correctie van het Engels. Mijn eerste publicatie voor het Journal of Applied Physics kreeg ik retour met de opmerking:"Very interesting article, but the English needs improvement". Zonder Josie zou het nog steeds in aanmerking komen voor improvement.

Tot slot wil ik degenen bedanken, die mij het naaste staan. In de eerste plaats wil ik mijn erkenning uitspreken voor de positie die mijn vrouw Yt heeft ingenomen, namelijk naast mij, als een vriend. Ook de kinderen Sjoertje, Evert, Nelleke en Gerrit wil ik bedanken. Het was geweldig leuk om samen met jullie in de tijd te reizen, ook al geloofden jullie er soms niets van. In deze rij van naasten horen ook mijn ouders, mijn zus en mijn broer. Zij vertegenwoordigen de omgeving waarin ik ben opgegroeid. Mijn ouders hebben mij de ruimte, de faciliteiten en de stimulans gegeven om fysica te studeren, uitmondend in dit proefschrift. Aan mijn ouders, Yt en de kinderen draag ik deze thesis op.

Parts of this thesis have been published in (or submitted to) international scientific journals or conference proceedings. They appeared as:

Chapter 3: C.A. Vriezinga, Thermal runaway and bistability in microwave heated isothermal slabs, J. Appl. Phys. **79** (3), 1779 (1996)

Chapter 4: C.A. Vriezinga, S. Sánchez-Pedreño G. and J. Grasman, Thermal runaway in microwave heating: A mathematical analysis, Submitted to Appl. Math. Mod.

Chapter 5: C.A. Vriezinga, Thermal runaway in microwave heated isothermal slabs, cylinders, and spheres, J. Appl. Phys. **83** (1), 483 (1997)

Chapter 6: C.A. Vriezinga, Thermal profiles and thermal runaway in microwave heated slabs, J. Appl. Phys. **85** (7), 3774 (1999)

Chapter 8: C.A. Vriezinga, Thermal runaway in microwave heated slabs of foodstuffs and ceramics: A comparison, *Proceedings of the International Conference on Microwave and High Frequency Heating*, B1.4, 453 (Valencia, Spain, 1999)

Every chapter of this thesis has its own numbering of formulas, figures and tables.

Chapter 1

General introduction

formulation of the problem

The use of microwaves (frequency 2450 MHz) has found its way into various applications in industry. Perhaps the largest consumer of microwave power is the foods industry where it is used for cooking, tempering, drying, pasteurization, sterilization, etc. (Decareau[1], Mudgett[2]). Another application is the joining and sintering of ceramics (Metaxas[3], Sutton[4]). Both foodstuffs and ceramics have in common that, within the dynamic range of operation, a slight increase of the microwave power causes the temperature of the irradiated sample to increase rapidly. This is the so-called thermal runaway phenomenon. Because of this foodstuffs and ceramics might become overheated.

the aim of the investigation

In this study the physical-mathematical modeling of the runaway effect is the leading theme. The understanding of this process is still somewhat empirical and speculative due to its highly nonlinear character. Therefore the runaway effect cannot be predicted accurately. Insight in the physical origin and the formulation of the conditions necessary for runaway, could lead to rules to prevent this, sometimes catastrophic, phenomenon.

A second difficulty associated with the application of microwave heating is the nonuniform temperature distribution within the sample, linked to the uneven spatial dissipation of energy. Also this problem has its roots in the basic physics of the heating process and is related to the runaway problem. Therefore also the spatial distribution of energy dissipation has been studied.

starting-point of the investigation

Thermal runaway, meaning that a slight increase of the applied microwave power causes the temperature of the irradiated sample to increase rapidly, has first been reported in the microwave sintering of ceramics (Brodwin *et al.*[5]). This process especially is seriously hampered by thermal runaway. Kriegsmann[6] (1992) introduced the S-shaped response curve of a ceramic slab steady-state temperature versus microwave power as a plausible explanation of the observed runaway phenomenon. At the 4th International Conference on Microwave and High Frequency Heating (Göteborg, 1993) Stuerga *et al.*[7] and Zahreddine[8] demonstrated that the response curve of an isothermal slab of water could also be S-shaped. Based on this observation they introduced a new type of bistability. Heating an isothermal slab of water with the microwave power directly proportional to time would result in an instantaneous(!) jump of the slab temperature at a certain moment. They also claimed to have experimental evidence for their statement (Stuerga *et al.*[9]). This discovery drew much attention, because real macroscopic, bistable behavior is a remarkable phenomenon in physics. It justifies the choice of the topic for the present study.

outline of the thesis

Preceding the investigation of thermal runaway a study was made of the dissipated energy within an microwave irradiated object. This work was done in cooperation with the Agrotechnological Research Institute (ATO, Wageningen). It has led to an approximation in the form of a model from geometric optics with multiple reflections and interference of beams (Van Remmen and Vriezinga[10]). Knowledge of the interaction of electromagnetic waves with slabs, cylinders and spheres, including the implementation of equations in computer programs, was necessary for the comparison of the approximation with the exact solution based on Maxwell's equations. The theory on the microwave heating of slabs, cylinders and spheres is presented in *Chapter 2*.

For a better understanding of the concept of bistability (as introduced by Stuerga *et al.*[7]) a thorough analysis of a microwave heated isothermal slab of water is necessary. This analysis can be found in *Chapter 3*. Besides the bistability in case of a time-dependent power, the physical origin of runaway and the possibilities of avoiding runaway are investigated.

The experimental evidence of bistability, which shows a notable curve of the slab temperature versus time, including a (not instantaneous) jump, needs a theoretical foundation. In cooperation with mathematicians of Wageningen and Murcia University (Spain), this theoretical foundation has been developed and discussed. This mathematical analysis is formulated in *Chapter 4.*

The microwave heated isothermal slab can be described by a relatively simple one-dimensional model. The next question is: Are the principles based on such a simple model also applicable to other objects than the slab? This is the subject of *Chapter 5*, which analyzes the microwave heating of isothermal slabs, cylinders and spheres of water. The aim of this article is to investigate how the geometry of the irradiated object influences runaway. The physics and mathematics of irradiated cylinders and spheres are rather sizeable. Where necessary reference has been made to *Chapter 2*.

So far isothermal objects have been investigated with respect to runaway. The next step in this investigation is the analysis of real nonisothermal microwave heated samples. By doing so the other problem of microwave heating (the uneven spatial dissipation of energy within the sample) becomes part of the investigation. The non-isothermal slab of water is discussed in *Chapter 6.* Besides the description of the runaway phenomena, it contains a comparison of temperature profiles calculated with a temperature dependent and a temperature independent permittivity.

In addition to Kriegsmann[6] the nonisothermal ceramic slab is analyzed in *Chapter 7.* The aim of this investigation is to find the physical origin of thermal runaway in the case of ceramics and to understand why the isothermal approach of Kriegsmann in his explanation of runaway is justified. Also the influence of resonance within the slab has been studied.

As remarked in the beginning of this introduction, the physics of microwave heated foodstuffs and ceramics are rather similar. Therefore the runaway phenomena of both kind of materials have many aspects in common. A comparison of the thermal runaway phenomena of foodstuffs and ceramics can be found in *Chapter 8.*

General conclusions are presented at the end of this thesis in *Chapter 9*.

The physical-mathematical modeling of the runaway effect and its consequences are reported in the articles (*Ch.3-Ch.8*) and in the general conclusions (*Ch.9*). The theory of microwave heating (*Ch.2*) has been written to account for the power formulas, which are used in the articles. Therefore this chapter contains a brief introduction to the electromagnetic description of the microwave heating process, starting with Maxwell's equations and ending at the wave equation and the general relation for the absorbed power (*Section 2.1-2.4*). For the calculation of the temperature the absorbed power has to be combined with the heat conduction theory (*Section 2.10*). The reader who is familiar with the general theory of microwave heating can easily skip over *Chapter 2*, after taking notice of *Section 2.5*, because the notation in this thesis differs from the conventional one.

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Chapter 2

Theory of microwave heating

2.1. INTRODUCTION

The microwave heating of dielectrics is caused by the interaction of the electromagnetic waves with permanent and induced dipoles of the irradiated sample. The free charges (free electrons, ions etc.) are responsible for an additional heating. The oscillating electromagnetic field causes alternating currents of the free charges, which yields ohmic heating. The interaction of the field and the dipoles also causes alternating currents, but the mechanism is different. The oscillating field rotates the dipoles. It is possible to describe the interaction of an electromagnetic field and molecules in terms of a damped harmonic oscillator (Feynman[1]). Using this model the "friction" in the rotation of the dipoles can be regarded as the origin of dielectric heating.

It is not necessary to describe the details of this interaction. In this thesis the absorption of electromagnetic energy is expressed by a known complex permittivity or complex dielectric constant. The imaginary part of the dielectric constant, the so-called dielectric loss factor, expresses the fact that the dielectric absorbs electromagnetic energy. By the introduction of the complex permittivity it is possible to describe the absorption of electromagnetic energy with Maxwell's equations and the Pointing vector in the complex formulation. Maxwell's equations contain the charge and current density of the free charges. The current density becomes part of the complex permittivity. The loss factor expresses the dielectric heating as well as the ohmic heating. The impact of the free charges does not vanish in the electromagnetic boundary conditions. There will be surface charges and surface currents. The analytical description of such a system (a mixture of a dielectric and conductor) is possible, but in this chapter the theory is limited to pure dielectrics. It is assumed no free charges are present.

The temperature of the irradiated sample increases during the heating process. This is described by Fourier's differential equation, where the source term corresponds to the absorbed electromagnetic energy. The problem of microwave heating is to find simultaneous solutions of electromagnetic and thermal equations. An analytical solution is hardly possible, because of the uneven spatial absorption of energy and because the permittivity depends on temperature. However formula's are needed for a better understanding of the heating process. For this reason the calculations in this chapter are limited to isothermal objects, where an analytical description of the absorbed power is possible. These solutions, combined with a numerical analysis, yield sufficient insight in the microwave heating process.

This chapter starts with Maxwell's equations and the Pointing vector (section 2.2-2.4). The electromagnetic wave equation and the general expression for the absorbed power are developed. The general theory can be found in many textbooks (e.g. Stratton[2], Blok *et al.*[3], Reitz *et al.*[4]). A brief description of this theory, applied to an isothermal slab is found in Ayappa *et al.*[5]

The power formula's for the isothermal slab are developed in section 2.6. The electromagnetic waves and the formula for the absorbed power in an isothermal cylinder (irradiated in two different ways) are described in the next two sections 2.7. and 2.8. The power formula for the isothermal sphere is developed in section 2.9. The general theory of electromagnetic waves in cylinders and spheres was deducted from Panofski *et al.*[6] and Stratton[2].

Section 2.10. briefly outlines the heat conduction of the microwave heated sample, based on Özişik[7].

The power formula's are complicated and interpreting them in physical processes is difficult. Therefore, these formula's are simplified. This is the subject of section 2.11 of this chapter. The theorems about the special functions are from Spiegel[8].

Finally a list of symbols for this chapter is given.

2.2. MAXWELL'S EQUATIONS

Microwave heating is a macroscopic phenomenon. The theory of classical electrodynamics provides the best way describing this process. Maxwell's equations:

$$\vec{\nabla}\times\vec{H}_r = \frac{\partial\vec{D}_r}{\partial t} + \vec{J}_r \tag{1}$$

$$\vec{\nabla}\times\vec{E}_r = -\frac{\partial\vec{B}_r}{\partial t} \tag{2}$$

Supplied with compatibility equations

$$\vec{\nabla}\cdot\vec{D}_r = \rho_r \tag{3}$$

$$\vec{\nabla}\cdot\vec{B}_r = 0 \tag{4}$$

where E_r and H_r are the electric and magnetic fields, D_r electric displacement, B_r magnetic induction, ρ_r and J_r are the charge density and current flux of the external charges (the free charges).

In the case of linear, non-instantaneously reacting materials D_r, B_r and J_r obey an equation of the form

$$\vec{Q}_r(t,T) = \int_0^\infty R(t',T)\ \vec{E}_r(t-t')\ dt' \tag{5}$$

where $R(t',T)$ is a response function characteristic of the quantity D_r, B_r or J_r and the material at the temperature T. The temperature is an external parameter, depending on position and time during the dielectric heating process, but is kept constant in this definition of linear medium. Let us consider the electric displacement to illustrate the mathematical description of the dielectric heating process. In accordance with equation (5) we have

$$\vec{D}_r(\vec{r},t) = \int_0^{\infty} [\varepsilon_0 \delta(t') + \chi(\vec{r},t')] \, \vec{E}_r(\vec{r},t-t') \, dt' \tag{6}$$

where ε_0 is the permittivity in vacuum and χ the (temperature dependent) response function. Assuming the electric field is turned on at $t=0$, equation (6) becomes

$$\vec{D}_r(\vec{r},t) = \varepsilon_0 \vec{E}_r(\vec{r},t) + \int_0^{t} \chi(\vec{r},t') \, \vec{E}_r(\vec{r},t-t') \, dt' \tag{7}$$

We are interested in the response of materials to radiation of a particular (angular) frequency ω. The complex electric and magnetic field E and H are introduced according to:

$$\vec{E}_r(\vec{r},t) = \mathrm{Re}\,(\vec{E}(\vec{r})e^{-i\omega t}) \qquad \text{and} \qquad \vec{H}_r(\vec{r},t) = \mathrm{Re}\,(\vec{H}(\vec{r})e^{-i\omega t}) \tag{8}$$

Alternatively $e^{i\omega t}$ can be used to express the time dependence.

Many dielectric heating processes (e.g. Debeye's dipolar loss process) can be described by rotating dipoles returning to their equilibrium according to an exponential decay law. This process is expressed by the response function.

$$\chi(\vec{r},t',T) = a e^{-t'/\tau} \tag{9}$$

where a and the molecular relaxation time τ depends on temperature at the position of the electric field. Substituting the decay law in (7) and using complex fields, yields

$$\vec{D}(\vec{r})e^{-i\omega t} = \varepsilon_0 \vec{E}(\vec{r})e^{-i\omega t} + \int_0^{t} a e^{-t'/\tau} e^{i\omega t'} \vec{E}(\vec{r}) e^{-i\omega t} \, dt' \tag{10}$$

$$\vec{D}(\vec{r}) = \varepsilon_0 \vec{E}(\vec{r}) + \frac{a\tau}{1 - i\omega\tau}(1 - e^{i\omega t} e^{-t/\tau}) \vec{E}(\vec{r}) \tag{11}$$

When τ is small compared to t, the time dependence disappears and we are able to introduce a complex permittivity ε.

$$\vec{D}(\vec{r}) = \varepsilon \vec{E}(\vec{r}) = \varepsilon_0 (1 + \frac{a\tau/\varepsilon_0}{1 - i\omega\tau}) \vec{E}(\vec{r}) \tag{12}$$

Thus, the complex dielectric constant $\kappa = \varepsilon/\varepsilon_0$ can be written as

$$\kappa = \kappa_\infty + \frac{\kappa_0 - \kappa_\infty}{1 - i\omega\tau} \tag{13}$$

where κ_0 (κ_∞) is the dielectric constant at frequency 0 (∞). This expression corresponds to the formula of the dielectric constant for water. See eq.(2.5) in chapter 4. The $(1 + i\omega\tau)$ in eq.(2.5) is caused by the $e^{i\omega t}$ choice in eq.(8) for the time dependence of the fields.

This example illustrates the frequency dependence and the complex character of the permittivity, and the fact that many materials react instantaneously in microwave heating processes. For linear, isotropic and instantaneously reacting materials the following constitutive relations apply.

$$\vec{D} = \varepsilon \vec{E} \qquad \vec{B} = \mu \vec{H} \qquad \vec{J} = \sigma \vec{E} \tag{14}$$

where ε is the permittivity, μ the permeability, and σ the electric conductivity of the free charges within the medium. These quantities are temperature dependent. During the microwave heating process the temperature will change (in the case of runaway very rapidly). Therefore D, B and J are time dependent. "The change in temperature acts as an extra generator of fields". In the complex formulation Maxwell's and compatibility equations read

$$\vec{\nabla} \times \vec{E} = i\omega\mu_n \vec{H} \tag{15}$$

$$\vec{\nabla} \times \vec{H} = -i\omega\varepsilon_n \vec{E} \tag{16}$$

$$\vec{\nabla} \cdot \vec{D} = \rho \tag{17}$$

$$\vec{\nabla} \cdot \vec{B} = 0 \tag{18}$$

with

$$\mu_n = \mu + \frac{i}{\omega}\frac{\partial\mu}{\partial t} \quad \text{and} \quad \varepsilon_n = \varepsilon + \frac{i\sigma}{\omega} + \frac{i}{\omega}\frac{\partial\varepsilon}{\partial t} \tag{19}$$

With the condition of electro neutrality of the materials considered [which implies $\vec{\nabla}(\vec{\nabla}\cdot\vec{E})=0$] eq.(16) can be inserted in eq.(15) to obtain the classical wave equation

$$\nabla^2\vec{E} + k^2\vec{E} = 0 \tag{20}$$

where $$k^2 = \omega^2 \mu_n \varepsilon_n \tag{21}$$

k is called the complex wave number or propagation constant (in this thesis). The main problem in microwave heating is finding the solution of eq.(20).

The complex permittivity is usually written as the sum or the difference of a real part ε' and an imaginary part ε'', where ε' and ε'' are positive numbers.

$$\varepsilon = \varepsilon' \pm i\varepsilon'' \tag{22}$$

In this section we have to choose the sum to express the lossy character of the dielectric. Let us neglect the $(\partial\varepsilon/\partial t)=(\partial\varepsilon/\partial T)\ (\partial T/\partial t)$ term in eq.(19) and assume that the electric conductivity σ is a real number, then eq.(16) can be written as

$$\vec{\nabla} \times \vec{H} = (-i\omega\varepsilon + \sigma)\vec{E} = -i\omega\varepsilon^{\prime}\vec{E} + (\omega\varepsilon^{\prime\prime} + \sigma)\vec{E} \tag{23}$$

Thus, by choosing the plus sign in eq.(22) the dielectric conductivity σ is increased by the dipolar relaxation $\omega\varepsilon''$, expressing the extra absorption of electromagnetic energy. By choosing the $e^{i\omega t}$ in eq.(8) for the time dependence of the fields, one is forced to choose the minus sign in eq.(22) to express the lossy character of the dielectric.

2.3. POWER DISSIPATION

The power flux associated with a propagating electromagnetic wave is represented by the Pointing vector $\vec{S}$. The time average flux for harmonic fields is given by

$$<\vec{S}> = \frac{1}{4}[(\vec{E}\times\vec{H}^{*}) + (\vec{E}^{*}\times\vec{H})] \tag{24}$$

The average power dissipated per unit volume is

$$D = -\vec{\nabla}\cdot<\vec{S}> \tag{25}$$

Inserting Maxwell's equations (15) and (16) in (24) and (25), yields

$$D = \frac{i\omega}{4}[(\varepsilon_n^{*}-\varepsilon_n)|\vec{E}|^2 + (\mu_n^{*}-\mu_n)|\vec{H}|^2] \tag{26}$$

Writing ε_n and μ_n as the sum of a real and imaginary part

$$\varepsilon_n = \varepsilon_n^{\prime} + i\varepsilon_n^{\prime\prime} \qquad \text{and} \qquad \mu_n = \mu_n^{\prime} + i\mu_n^{\prime\prime} \tag{27}$$

gives the general expression for the absorbed power per unit volume

$$D = \frac{1}{2}\omega[\varepsilon_n^{//}|\vec{E}|^2 + \mu_n^{//}|\vec{H}|^2] \tag{28}$$

The $e^{i\omega t}$ convention in eq.(8) yields eq.(26) multiplied by a minus sign, but the convention obliges to choose eq.(27) with minus signs instead of plus signs, and the result (28) is the same.

2.4. DIELECTRIC PROPERTIES

In food systems, as well as in ceramics, the permeability μ is well approximated by its value μ_0 in vacuum. This means that the magnetic part in the expression of the absorbed power (28) disappears. It is also assumed that there are no free charges. This study describes the behavior of pure dielectrics. The oscillating electromagnetic field rotates the permanent and induced dipoles of the material specimen. The specific character of this process is expressed by the complex permittivity ε. This ε depends on the temperature of the irradiated object. Because the temperature increases with increasing time during the heating process, the time dependence of the permittivity has to be taken into account. Investigations have shown that the influence of the term $\partial(i\varepsilon)/\partial(\omega t)$ of eq.(19) can be neglected (see chapter 6). With these considerations the main equations of the heating process reduce to

$$\nabla^2\vec{E} + k^2\vec{E} = 0 \tag{29}$$

$$k^2 = \omega^2\mu_0\,\varepsilon \tag{30}$$

$$D = \frac{1}{2}\omega\varepsilon^{//}|\vec{E}|^2 = \frac{1}{2}\omega\varepsilon_0\kappa^{//}|\vec{E}|^2 \tag{31}$$

The permittivity is usually replaced by the dielectric constant κ, which equals the permittivity divided by the permittivity of vacuum ε_0. The dielectric loss factor κ''

is the imaginary part of the dielectric constant. Representing the dielectric properties of materials in tabulating the values of κ' and κ'' is customary. The propagation constant k is represented as a complex quantity

$$k = \alpha + i\beta \tag{32}$$

where the phase constant (or real wavenumber) α and the attenuation constant β are related to the dielectric properties of the medium and frequency of radiation by

$$\alpha = \frac{\omega}{c}\sqrt{\frac{\kappa'\left(\sqrt{1+\tan^2\delta}+1\right)}{2}} \tag{33}$$

$$\beta = \frac{\omega}{c}\sqrt{\frac{\kappa'\left(\sqrt{1+\tan^2\delta}-1\right)}{2}} \tag{34}$$

$$\tan\delta = \frac{\kappa''}{\kappa'} \tag{35}$$

where c $(=1/\sqrt{(\varepsilon_0\,\mu_0)}$ is the speed of light, so ω/c is the propagation constant in vacuum. The use of α and β, instead of κ' and κ'', gives more insight in the microwave heating process.

2.5. ABOUT THE NOTATION

The notation used in this thesis differs from the conventional one. Therefore a comparison is given.

Conventional	This thesis
$\vec{E}_r = \mathrm{Re}\,(\vec{E}\, e^{+i\omega t})$	$\vec{E}_r = \mathrm{Re}\,(\vec{E}\, e^{-i\omega t})$
$\vec{\nabla}^2 \vec{E} - \gamma^2 \vec{E} = 0$	$\vec{\nabla}^2 \vec{E} + k^2 \vec{E} = 0$
γ = propagation constant	k = propagation constant
$\gamma = \alpha + i\beta$	$k = \alpha + i\beta$
α = attenuation constant	α = phase constant
β = phase constant	β = attenuation constant
$\gamma^2 = -\omega^2(\varepsilon - \dfrac{i\sigma}{\omega})\mu$	$k^2 = \omega^2(\varepsilon + \dfrac{i\sigma}{\omega})\mu$
ε = permittivity	ε = permittivity
$\varepsilon = \varepsilon_0(\varepsilon_r^{/} - i\varepsilon_r^{//})$	$\varepsilon = \varepsilon^{/} + i\varepsilon^{//}$
$\varepsilon_r = \varepsilon_r^{/} - i\varepsilon_r^{//}$	$\kappa = \kappa^{/} + i\kappa^{//}$
ε_r = relative permittivity	κ = complex dielectric constant
$\varepsilon_r^{/}$ = dielectric constant	$\kappa^{/}$ = dielectric constant
$\varepsilon_r^{//}$ = loss factor	$\kappa^{//}$ = loss factor

Many authors leave out the subscript r in the conventional symbols for dielectric constant and loss factor.

Most of the calculations in this thesis are expressed in terms of the phase and attenuation constant, so by changing the meaning of α and β compared to the conventional meaning, this thesis can easily be read.

2.6. THE POWER FORMULA OF A SLAB

Consider a slab in free space, irradiated from one side by microwave radiation. The wave is a plane, harmonic one and impinges normally upon the material (Fig. 1).

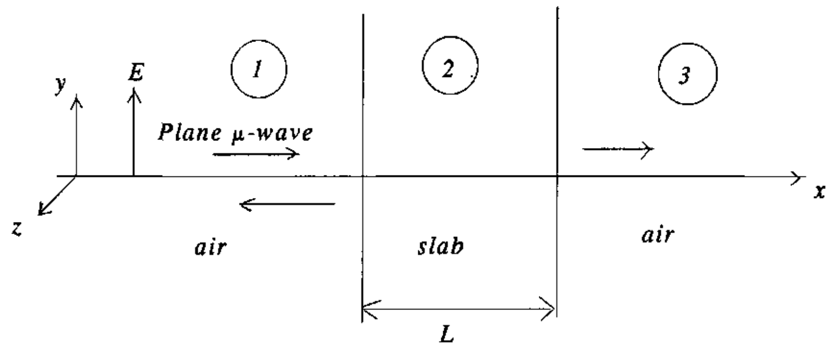

Figure 1. A slab, being irradiated from one side, in an echofree cavity.

It is assumed that the slab is isothermal continuously. The field equation (29) becomes a one-dimensional differential equation.

$$\frac{d^2E}{dx^2} + k^2E = 0 \tag{36}$$

with the solution

$$E_1 = A_1 e^{ik_1x} + B_1 e^{-ik_1x} \tag{37}$$

$$E_2 = A_2 e^{ik_2x} + B_2 e^{-ik_2x} \tag{38}$$

$$E_3 = A_3 e^{ik_1x} \tag{39}$$

The subscript 1 refers to the space where the microwave source has been located, the subscript 2 belongs to the irradiated slab, and 3 refers to the free space at the rear side of the slab. The integration constants follow from the electromagnetic boundary conditions.

$$E_1 = E_2 \ ; \ \frac{\mathrm{d}E_1}{\mathrm{d}x} = \frac{\mathrm{d}E_2}{\mathrm{d}x} \text{ at } x=0 \quad \text{and} \quad E_2 = E_3 \ ; \ \frac{\mathrm{d}E_2}{\mathrm{d}x} = \frac{\mathrm{d}E_3}{\mathrm{d}x} \text{ at } x=L \tag{40}$$

The thickness of the slab is L. Introducing the reflection and transmission coefficient R_{12} and T_{12} according to,

$$R_{12} = |R_{12}| e^{i\delta_{12}} = \frac{k_1 - k_2}{k_1 + k_2} \quad ; \quad T_{12} = |T_{12}| e^{i\tau_{12}} = \frac{2k_1}{k_1 + k_2} \tag{41}$$

makes it possible to write the electric field within the medium as

$$E_2 = T_{12} \frac{e^{i k_2 x} - R_{12} e^{2 i k_2 L} e^{-i k_2 x}}{1 - R_{12}^2 e^{2 i k_2 L}} A_1 \tag{42}$$

Adding the waves, going back and forwards within the slab, gives the same result. The sum of all the waves is a simple arimethic sum. The power flux of the incident wave (the microwave power) P equals

$$P = \frac{1}{2} c \, \varepsilon_0 \, A_1^2 \tag{43}$$

The absorbed power as a function of the position x within the slab is

$$D = P \frac{\omega}{c} \kappa_2'' |T_{12}|^2 \frac{e^{-2\beta_2 x} - 2|R_{12}| e^{-2\beta_2 L} \cos(2\alpha_2 L - 2\alpha_2 x + \delta_{12}) + |R_{12}|^2 e^{-4\beta_2 L} e^{2\beta_2 x}}{1 - 2|R_{12}|^2 e^{-2\beta_2 L} \cos(2\alpha_2 L + 2\delta_{12}) + |R_{12}|^4 e^{-4\beta_2 L}} \tag{44}$$

Integration over the slab yields the total amount of absorbed power

$$D_{tot} = P\,\frac{\omega}{c}\,\kappa_2''\,|T_{12}|^2\,\frac{1}{2\beta_2} \times$$

$$\frac{(1-e^{-2\beta_2 L})\,(1+|R_{12}|^2 e^{-2\beta_2 L}) - \dfrac{4\beta_2}{\alpha_2}|R_{12}|e^{-2\beta_2 L}\sin(\alpha_2 L)\cos(\alpha_2 L+\delta_{12})}{1-2|R_{12}|^2 e^{-2\beta_2 L}\cos(2\alpha_2 L+2\delta_{12})+|R_{12}|^4 e^{-4\beta_2 L}} \tag{45}$$

In case of a slab, irradiated from two sides symmetrically (at the surface of the slab the incident waves are in phase), the same kind of calculation yields

$$D = P\,\frac{\omega}{c}\,\kappa_2''\,|T_{12}|^2 \times$$

$$\frac{[1-2|R_{12}|e^{-\beta_2 L}\cos(\alpha_2 L+\delta_{12})+|R_{12}|^2 e^{-2\beta_2 L}][e^{-2\beta_2 x}+e^{-2\beta_2(L-x)}+2e^{-\beta_2 L}\cos\alpha_2(L-2x)]}{1-2|R_{12}|^2 e^{-2\beta_2 L}\cos(2\alpha_2 L+2\delta_{12})+|R_{12}|^4 e^{-4\beta_2 L}} \tag{46}$$

and

$$D_{tot} = P\,\frac{\omega}{c}\,\kappa_2''\,|T_{12}|^2\,\frac{1}{\beta_2} \times \tag{47}$$

$$\frac{[1-2|R_{12}|e^{-\beta_2 L}\cos(\alpha_2 L+\delta_{12})+|R_{12}|^2 e^{-2\beta_2 L}]\;[1-e^{-2\beta_2 L}+\dfrac{2\beta_2}{\alpha_2}e^{-\beta_2 L}\sin(\alpha_2 L)]}{1-2|R_{12}|^2 e^{-2\beta_2 L}\cos(2\alpha_2 L+2\delta_{12})+|R_{12}|^4 e^{-4\beta_2 L}}$$

Because of the symmetry there is always a hot spot at the center of the nonisothermal slab.

2.7. THE POWER FORMULA OF A CYLINDER (A)

Consider an isothermal cylinder in free space, irradiated from one side by a plane, harmonic wave. The electric field vector E_z of the incident field is parallel to the central symmetry axis of the cylinder, while it propagates along the x-axis (Fig.2).

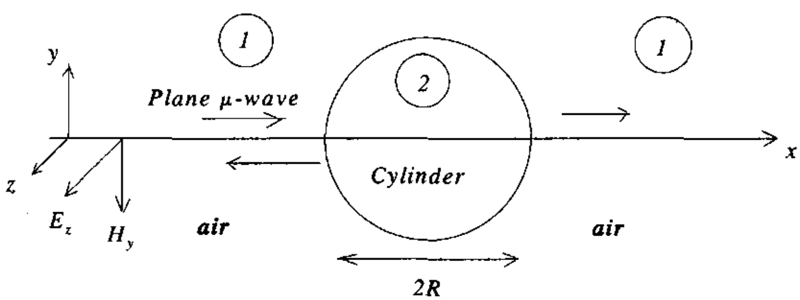

Figure 2. A cylinder, being irradiated by a plane (E_z , H_y) wave from one side, in an echofree cavity.

Expressing the wave equation (29) in polar coordinates r and ϕ is convenient.

$$\frac{1}{r^2}\frac{\partial^2 E_z}{\partial \phi^2} + \frac{\partial^2 E_z}{\partial r^2} + \frac{1}{r}\frac{\partial E_z}{\partial r} + k^2\, E_z = 0 \tag{48}$$

Substituting $E_z(x,y) = R(r)\,\Phi(\phi)$ splits the wave equation in two other equations.

$$\frac{d^2\Phi}{d\phi^2} + n^2\Phi = 0 \tag{49}$$

with the solution $\Phi = c\, e^{in\phi} + d\, e^{-in\phi}$ (50)

where $n=0, \pm 1, \pm 2, \ldots$, because $\Phi(\phi) = \Phi(\phi + 2\pi)$.

The second equation is the differential equation of Bessel.

$$k^2 r^2 \frac{d^2 R}{d(kr)^2} + kr \frac{dR}{d(kr)} + (k^2 r^2 - n^2) R = 0 \tag{51}$$

so $\quad R\,(kr) = A\, J_n\,(kr) + B\, N_n\,(kr) \qquad (52)$

where J_n is the Bessel function of the first kind of order n, and N_n is the Bessel function of the second kind (Neumann's or Weber's function).

The incident field equals $A_1\, e^{ikx}$. Using the generating function for J_n

$$e^{x(t-\frac{1}{t})/2} = \sum_{n=-\infty}^{+\infty} J_n(x)\, t^n \tag{53}$$

the incident field becomes

$$A_1\, e^{i k_1 x} = A_1\, e^{i k_1 r \cos \phi} = A_1 \sum_{n=-\infty}^{+\infty} J_n\,(k_1\, r)\,(i)^n\, e^{i n \phi} \tag{54}$$

The scattered field should read $e^{ikr}/\sqrt{(r)}$ for large values of r. Therefore, the scattered field is written as

$$A_1 \sum_{n=-\infty}^{+\infty} a_n\, H_n\,(k_1\, r)\, e^{i n \phi} \tag{55}$$

where $H_n = J_n + i\, N_n$ is the Hankel function of the first kind.
The total field outside the cylinder is the sum of the incident and scattered field.

$$E_z^{(1)} = A_1 \sum_{n=-\infty}^{+\infty} J_n(k_1 r)\,(i)^n\, e^{i n \phi} + A_1 \sum_{n=-\infty}^{+\infty} a_n\, H_n(k_1 r)\, e^{i n \phi} \tag{56}$$

Because the Neumann function becomes $-\infty$ for $r \to 0$, this function should be dropped in the description of the field within the cylinder.

$$E_z^{(2)} = A_1 \sum_{n=-\infty}^{+\infty} c_n \, J_n(k_2 r) \, e^{i n \phi} \tag{57}$$

At the surface of the cylinder one has the conditions

$$E_z^{(1)} = E_z^{(2)} \quad ; \quad H_r^{(1)} = H_r^{(2)} \quad ; \quad H_\phi^{(1)} = H_\phi^{(2)} \tag{58}$$

where

$$i \omega \mu_0 H_r = \frac{1}{r} \frac{\partial E_z}{\partial \phi} \qquad \text{and} \qquad -i \omega \mu_0 H_\phi = \frac{\partial E_z}{\partial r} \tag{59}$$

Applying these boundary conditions, generates the a_n and c_n.

$$a_n = \frac{[k_1 \, J_n(k_2 R) \, J_n'(k_1 R) \, - \, k_2 \, J_n(k_1 R) \, J_n'(k_2 R)] \, (i)^n}{k_2 \, H_n(k_1 R) \, J_n'(k_2 R) \, - \, k_1 \, J_n(k_2 R) \, H_n'(k_1 R)} \tag{60}$$

$$c_n = \frac{k_1 \, [J_n'(k_1 R) \, H_n(k_1 R) \, - \, J_n(k_1 R) \, H_n'(k_1 R)] \, (i)^n}{k_2 \, J_n'(k_2 R) \, H_n(k_1 R) \, - \, k_1 \, J_n(k_2 R) \, H_n'(k_1 R)} \tag{61}$$

The radius of the cylinder is R. The prime refers to the derivative with respect to $k_1 r$ or $k_2 r$. The numerator of c_n can be simplified by using recurrence formulas.

$$c_n = \frac{-2 \, (i)^{n+1}}{\pi \, R \, [\, k_2 \, J_n'(k_2 R) \, H_n(k_1 R) \, - \, k_1 \, J_n(k_2 R) \, H_n'(k_1 R) \,]} \tag{62}$$

According to eq.(31) the absorbed power as function of the position (r,ϕ) is

$$D = \frac{1}{2}\omega\varepsilon_0\kappa_2^{//}A_1^2 \;|\sum_{n=-\infty}^{+\infty} c_n J_n(k_2 r)e^{in\phi}|^2 \tag{63}$$

For the total amount of absorbed power one has

$$D_{tot} = \frac{1}{2}\omega\varepsilon_0\kappa_2^{//}A_1^2 \int_{\phi=0}^{2\pi}\int_{r=0}^{R} |\sum_{n=-\infty}^{+\infty} c_n J_n(k_2 r)e^{in\phi}|^2 \; r \;\, \mathrm{d}\phi\, \mathrm{d}r = \tag{64}$$

$$\frac{1}{2}\omega\varepsilon_0\kappa_2^{//}A_1^2 \; 2\pi \int_{r=0}^{R} [\; |c_0|^2 \; |J_0(k_2 r)|^2 + 2\sum_{n=1}^{\infty} |c_n|^2 \; |J_n(k_2 r)|^2 \;]r \,\mathrm{d}r$$

and

$$\int_{r=0}^{R} |J_n(k_2 r)|^2 \; r\,\mathrm{d}r =$$

$$\frac{k_2 \; R \; J_n^{*}(k_2R)J_n^{/}(k_2R) \;-\; k_2^{*} \; R \; J_n(k_2R) \; J_n^{/*}(k_2R)}{-4 \; i \; \alpha_2 \; \beta_2} = \tag{65}$$

$$\frac{\beta_2 \; R \; Re(J_n^{*}J_n^{/}) \;+\; \alpha_2 \; R \; Im(J_n^{*}J_n^{/})}{-2 \; \alpha_2 \; \beta_2}$$

and $|c_n|^2 =$

$$\frac{4/(\pi^2 R^2)}{|k_2 J_n^{/}|^2 |H_n|^2 + k_1^2 |J_n|^2 |H_n^{/}|^2 - k_1 (k_2 J_n^{/} H_n J_n^{*} H_n^{/*} + k_2^{*} J_n^{/*} H_n^{*} J_n H_n^{/*})} \tag{66}$$

2.8. THE POWER FORMULA OF A CYLINDER (B)

This section describes the irradiation of the cylinder by a plane, harmonic wave (E_y, H_z) instead of the wave (E_z, H_y) as formulated above (Fig.3).

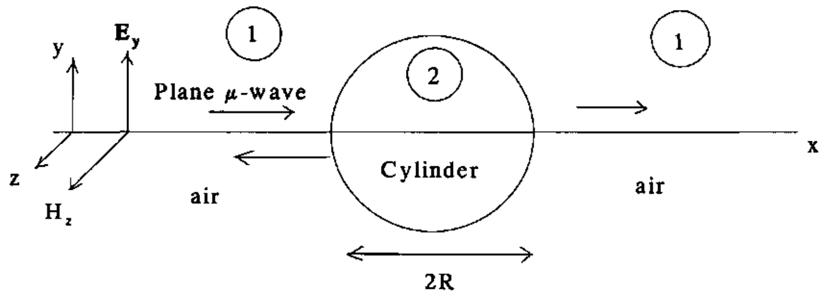

Figure 3. A cylinder, being irradiated by a plane (E_y , H_z) wave from one side, in an echofree cavity.

This means that the role of the electric and magnetic components has been changed. On the analogy to the former calculations the magnetic field will be

$$H_z^{(1)} = A_H \sum_{n=-\infty}^{+\infty} J_n(k_1 r)\ (i)^n\ e^{i n \phi} + A_H \sum_{n=-\infty}^{+\infty} a_n\ H_n(k_1 r)\ e^{i n \phi} \tag{67}$$

$$H_z^{(2)} = A_H \sum_{n=-\infty}^{+\infty} c_n\ J_n(k_2 r)\ e^{i n \phi} \tag{68}$$

where A_H is the amplitude of the magnetic component of the incident field. The boundary conditions read

$$H_z^{(1)} = H_z^{(2)} \quad ; \quad \varepsilon_1 E_r^{(1)} = \varepsilon_2 E_r^{(2)} \quad ; \quad E_\phi^{(1)} = E_\phi^{(2)} \tag{69}$$

$$-i\omega\varepsilon E_r = \frac{1}{r}\frac{\partial H_z}{\partial\phi} \qquad \text{and} \qquad i\omega\varepsilon E_\phi = \frac{\partial H_z}{\partial r} \tag{70}$$

These conditions yield

$$a_n = \frac{[k_2\ J_n(k_2R)\ J_n'(k_1R)\ -\ k_1\ J_n(k_1R)\ J_n'(k_2R)]\ (i)^n}{k_1\ H_n(k_1R)\ J_n'(k_2R)\ -\ k_2\ J_n(k_2R)\ H_n'(k_1R)} \tag{71}$$

$$c_n = \frac{k_2\ [J_n'(k_1R)\ H_n(k_1R)\ -\ J_n(k_1R)\ H_n'(k_1R)]\ (i)^n}{k_1\ J_n'(k_2R)\ H_n(k_1R)\ -\ k_2\ J_n(k_2R)\ H_n'(k_1R)} \tag{72}$$

or

$$c_n = \frac{-2k_2\ (i)^{n+1}}{\pi k_1 R\ [\ k_1 J_n'(k_2R) H_n(k_1R)\ -\ k_2 J_n(k_2R) H_n'(k_1R)\ ]} \tag{73}$$

According to eq.(31) the absorbed power equals

$$D = \frac{1}{2}\omega\varepsilon_0\kappa_2''\ (\ |E_x^{(2)}|^2 + |E_y^{(2)}|^2\ ) \tag{74}$$

In polar coordinates

$$D = \frac{1}{2}\omega\varepsilon_0\kappa_2''\ \frac{1}{\omega^2|\varepsilon_2|^2}\ (\ |\frac{\partial H_z^{(2)}}{\partial r}|^2 + |\frac{1}{r}\frac{\partial H_z^{(2)}}{\partial\phi}|^2\ ) \tag{75}$$

Using recurrence formulas for Bessel functions this can be written as

$$D = \frac{1}{2}\omega\varepsilon_0\kappa_2^{//}\,\frac{|k_2|^2 A_H^2}{\omega^2|\varepsilon_2|^2} \quad \times$$

$$[\ |-c_0 J_1 + \sum_{n=1}^{+\infty} c_n\ (J_{n-1} - J_{n+1})\ \cos(n\phi)|^2 + |\sum_{n=1}^{+\infty} c_n\ (J_{n+1} + J_{n-1})\ \sin(n\phi)|^2\ ] \tag{76}$$

The incident wave is a plane, harmonic one. This means that $A_H{}^2 = \varepsilon_0\, A_1{}^2/\mu_0$, where A_1 is the amplitude of the electric component E_y of the incident field. With eq.(30) the factor of (75) before the brackets becomes

$$\frac{1}{2}\ \omega\ \varepsilon_0\ \kappa_2^{//}\ A_1^2\ \frac{1}{\sqrt{(\kappa_2^{/})^2 + (\kappa_2^{//})^2}} \tag{77}$$

Integrated over ϕ

$$D_{tot} = \frac{1}{2}\omega\varepsilon_0\kappa_2^{//}\ A_1^2\frac{1}{|\kappa_2|}\ 2\pi \quad \times$$

$$\int_{r=0}^{R} [\ |c_0 J_1|^2 + \sum_{n=1}^{+\infty} |c_n|^2\ (\,|J_{n-1}|^2 + |J_{n+1}|^2)\ ]\, r\, dr \tag{78}$$

$$|c_n|^2 = \frac{4\,|k_2|^2/(\pi^2 k_1^2 R^2)}{|k_2 J_n|^2 |H_n^{/}|^2 + k_1^2 |J_n^{/}|^2 |H_n|^2 - k_1 (k_2^{*} J_n^{/} H_n J_n^{*} H_n^{/*} - k_2 J_n^{/*} H_n^{*} J_n H_n^{/})} \tag{79}$$

The integral over r is known, see eq. (65).

2.9. THE POWER FORMULA OF A SPHERE

An isothermal sphere is irradiated from one side by a plane harmonic wave in free space (Fig.4).

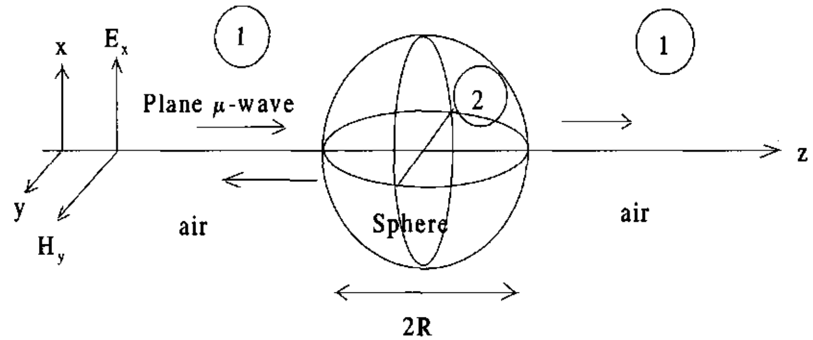

Figure 4. A sphere, being irradiated by a plane (E_x , H_y) wave from one side, in an echofree cavity.

In spherical coordinates (r,θ,ϕ) the wave equation reads

$$\frac{1}{r^2}\frac{\partial}{\partial r}(r^2\frac{\partial \vec{E}}{\partial r}) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\frac{\partial\vec{E}}{\partial\theta}) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\vec{E}}{\partial\phi^2} + k^2\vec{E} = 0 \tag{80}$$

This is a vector equation. The solution for one component ψ of the electric field (the scalar equation) is well known. The substitution $\psi = f(r)Y(\theta,\phi)$, where $Y=\Phi(\phi)\,T(\theta)$, generates three other equations. The first one is the Bessel equation

$$x^2\frac{d^2g}{dx^2} + x\frac{dg}{dx} + [x^2 - l(l+1) - \frac{1}{4}]g = 0 \tag{81}$$

where $x=kr$ and $f=g/\sqrt{(kr)}$. So $f(r)$ equals $Z_{l+½}(kr)\,/\sqrt{(kr)}$, where $Z_{l+½}$ refers to the Bessel, Neumann or Hankel function of order $l+½$. The second equation reads

$$\frac{d^2\Phi}{d\phi^2} + m^2\Phi = 0 \tag{82}$$

with the solution $\Phi = a\cos(m\phi) + b\sin(m\phi)$ and $m = 0, \pm 1, \pm 2, \ldots$ The third equation is Legendre's associated differential equation.

$$(1-x^2)\frac{d^2T}{dx^2} - 2x\frac{dT}{dx} + [l(l+1) - \frac{m^2}{1-x^2}]\, T = 0 \tag{83}$$

where $x = \cos(\theta)$. The solution is a combination of the associated Legendre functions of the first (P_l^m) and second (Q_l^m) kind. Summarizing

$$\psi = \frac{1}{\sqrt{kr}} Z_{l+\frac{1}{2}}(kr)\; Y_l^m(\theta,\phi) \tag{84}$$

The function Y_l^m is the spherical harmonic of order (l,m) ; l is a positive integer and $m = l,\ l\text{-}1,\ l\text{-}2,\ \ldots,\ \text{-}l$. If ψ is a solution of the scalar equation, then $\vec{\nabla}\psi$ and $\vec{r} \times \vec{\nabla}\psi$ are solutions of the vector equation. Applying this theorem, results in two set of equations: the electric and magnetic multipole fields. The electric multipole fields (E-type) are

$$\begin{aligned}
E_r &= \frac{l(l+1)}{r} z_l(kr) Y_l^m(\theta,\phi) \quad ; \quad E_\theta = \frac{1}{r}\frac{d}{dr}[r z_l(kr)]\frac{\partial}{\partial\theta}[Y_l^m(\theta,\phi)] \\
E_\phi &= \frac{im}{r\sin\theta}\frac{d}{dr}[r z_l(kr)] Y_l^m(\theta,\phi) \quad ; \quad H_r = 0 \\
H_\theta &= \sqrt{\frac{\varepsilon}{\mu}}\frac{km}{\sin\theta} z_l(kr) Y_l^m(\theta,\phi) \quad ; \quad H_\phi = ik\sqrt{\frac{\varepsilon}{\mu}} z_l(kr)\frac{\partial}{\partial\theta}[Y_l^m(\theta,\phi)]
\end{aligned} \tag{85}$$

The magnetic multipole fields (M-type) are

$$E_r = 0 \qquad ; \quad E_\theta = \frac{-km}{\sin\theta}\, z_l(kr)\; Y_l^m(\theta,\phi)$$

$$E_\phi = -ikz_l(kr)\, \frac{\partial}{\partial\theta}[Y_l^m(\theta,\varphi)] \quad ; \quad H_r = \sqrt{\frac{\varepsilon}{\mu}}\, \frac{l(l+1)}{r} z_l(kr)\; Y_l^m(\theta,\phi) \tag{86}$$

$$H_\theta = \sqrt{\frac{\varepsilon}{\mu}}\, \frac{1}{r}\, \frac{\mathrm{d}}{\mathrm{d}r}[rz_l(kr)]\, \frac{\partial}{\partial\theta}[Y_l^m(\theta,\phi)]$$

$$H_\phi = \sqrt{\frac{\varepsilon}{\mu}}\, \frac{im}{r\,\sin\theta}\, \frac{\mathrm{d}}{\mathrm{d}r}[rz_l(kr)] Y_l^m(\theta,\phi)$$

where

$$z_l(kr) = \sqrt{\frac{\pi}{2kr}}\, Z_{l+1/2}(kr) \tag{87}$$

Introducing even and odd functions is customary. The even function has a factor $\cos(m\phi)$, while the odd function contains a factor $\sin(m\phi)$. The aim is to write the incident, scattered and transmitted field in terms of the multipole fields. The incident field is a plane harmonic wave (E_x, H_y), propagating in the direction of the $+z$-axis.

$$E_x = A_1 e^{ik_1 z}\, \vec{i}_x = A_1 e^{ik_1 r\cos\theta}(\sin\theta\,\cos\varphi\; \vec{i}_r + \cos\theta\,\cos\phi\; \vec{i}_\theta - \sin\phi\; \vec{i}_\phi) \tag{88}$$

with the property

$$e^{ik_1 r\cos\theta} = \sum_{l=1}^{\infty} i^l\,(2l+1)\; j_l(k_1 r) P_l(\cos\theta) \tag{89}$$

Comparing (88), (89), and the corresponding expressions for H_y , to the multipole fields, results in

$$E_x = A_1 \sum_{l=1}^{\infty} \frac{i^{l-1}}{k_1} \frac{2l+1}{l(l+1)} \left[\vec{E}_M(l,1;\text{odd}) + \vec{E}_E(l,1;\text{even}) \right]$$
$$H_y = -A_1 \sum_{l=1}^{\infty} \frac{i^{l+1}}{k_1} \frac{2l+1}{l(l+1)} \left[\vec{H}_M(l,1;\text{odd}) + \vec{H}_E(l,1;\text{even}) \right] \tag{90}$$

The $z_l(kr)$ corresponds to the Bessel function $j_l(kr)$ in this expansion.
The scattered field should be described with the Hankel function, because for large r we have

$$h_l(kr) = j_l(kr) + i n_l(kr) \approx (-i)^{l+1} \frac{e^{ikr}}{kr} \tag{91}$$

Corresponding reasoning yields the scattered field.

$$\vec{E}_s = A_1 \sum_{l=1}^{\infty} \frac{i^{l-1}}{k_1} \frac{2l+1}{l(l+1)} \left[a_l \vec{E}_M(l,1;\text{odd}) + b_l \vec{E}_E(l,1;\text{even}) \right] \tag{92}$$

together with the corresponding formula for $\vec{H}_s$. The field outside the sphere is the sum of the incident and scattered field. The field within the sphere reads

$$\vec{E}^{(2)} = A_1 \sum_{l=1}^{\infty} \frac{i^{l-1}}{k_2} \frac{2l+1}{l(l+1)} \left[c_l \vec{E}_M(l,1;\text{odd}) + d_l \vec{E}_E(l,1;\text{even}) \right]$$
$$\vec{H}^{(2)} = -A_1 \sum_{l=1}^{\infty} \frac{i^{l+1}}{k_2} \frac{2l+1}{l(l+1)} \left[c_l \vec{H}_M(l,1;\text{odd}) + d_l \vec{H}_E(l,1;\text{even}) \right] \tag{93}$$

In spherical components

$$E_r^{(2)} = A_1 \sum_{n=1}^{\infty} i^{n-1} \frac{2n+1}{k_2 r} \sqrt{\frac{\pi}{2k_2 r}} \, b_n \, J_{n+1/2}(k_2 r) \, \frac{dP_n(\cos\theta)}{d(\cos\theta)} \cos\phi \, \sin\theta \tag{94}$$

$$\begin{aligned} E_\theta^{(2)} = A_1 \sum_{n=1}^{\infty} i^n \frac{2n+1}{n(n+1)} \sqrt{\frac{\pi}{2k_2 r}} \, \Big[a_n \, J_{n+1/2}(k_2 r) \, \frac{dP_n(\cos\theta)}{d(\cos\theta)} - \\ - i b_n \Big\{ \frac{n+1}{k_2 r} J_{n+1/2}(k_2 r) - J_{n+3/2}(k_2 r) \Big\} \frac{dP_n^1(\cos\theta)}{d\theta} \Big] \cos\phi \end{aligned} \tag{95}$$

$$\begin{aligned} E_\phi^{(2)} = A_1 \sum_{n=1}^{\infty} i^n \frac{2n+1}{n(n+1)} \sqrt{\frac{\pi}{2k_2 r}} \, \Big[-a_n \, J_{n+1/2}(k_2 r) \, \frac{dP_n^1(\cos\theta)}{d\theta} + \\ + i b_n \Big\{ \frac{n+1}{k_2 r} J_{n+1/2}(k_2 r) - J_{n+3/2}(k_2 r) \Big\} \frac{dP_n(\cos\theta)}{d(\cos\theta)} \Big] \sin\phi \end{aligned} \tag{96}$$

where P_n are the Legendre polynomials and P_n^1 correspond to the associated Legendre functions of the first kind.

At the surface of the sphere one has the following boundary conditions.

$$\begin{aligned} \varepsilon_1 \, E_r^{(1)} = \varepsilon_2 \, E_r^{(2)} \quad &; \quad \vec{E}_\phi^{(1)} + \vec{E}_\theta^{(1)} = \vec{E}_\phi^{(2)} + \vec{E}_\theta^{(2)} \\ H_r^{(1)} = H_r^{(2)} \quad &; \quad \vec{H}_\phi^{(1)} + \vec{H}_\theta^{(1)} = \vec{H}_\phi^{(2)} + \vec{H}_\theta^{(2)} \end{aligned} \tag{97}$$

The coefficients a_n and b_n can be found by applying these conditions.

$$a_n = k_1\sqrt{k_1 k_2}\ [\ J_{n+1/2}(k_1R)\ H_{n+3/2}(k_1R)\ -\ H_{n+1/2}(k_1R)\ J_{n+3/2}(k_1R)\ ]\ /$$

$$[\ \frac{k_2-k_1}{R}\ (n+1)\ J_{n+1/2}(k_2R)\ H_{n+1/2}(k_1R)\ - \tag{98}$$

$$-\ k_2^2\ J_{n+3/2}(k_2R)\ H_{n+1/2}(k_1R)\ +\ k_1^2\ J_{n+1/2}(k_2R)\ H_{n+3/2}(k_1R)\ ]$$

$$b_n = k_1\sqrt{k_1 k_2}\ [\ J_{n+1/2}(k_1R)\ H_{n+3/2}(k_1R)\ -\ H_{n+1/2}(k_1R)\ J_{n+3/2}(k_1R)\ ]\ /$$

$$[\ \frac{k_2-k_1}{R}\ (1-n)\ J_{n+1/2}(k_2R)\ H_{n+1/2}(k_1R)\ - \tag{99}$$

$$-\ k_1k_2\ J_{n+3/2}(k_2R)\ H_{n+1/2}(k_1R)\ +\ k_1k_2\ J_{n+1/2}(k_2R)\ H_{n+3/2}(k_1R)\ ]$$

where R is the radius of the sphere. The numerator of a_n and b_n can be written as $-2i\ (\ k_1\ k_2)^{½}\ /\ \pi\ R$ (applying the recurrence formulas for Bessel functions).
The absorbed power is

$$D = \frac{1}{2}\ \omega\ \varepsilon_0\ \kappa_2^{//}\ (|E_r|^2\ +\ |E_\theta|^2\ +\ |E_\phi|^2) \tag{100}$$

2.10. THE HEAT CONDUCTION

The absorbed power D is the source term in Fourier's differential equation (the heat balance or heat conduction equation).

$$\rho\ C_p\ \frac{\partial T(\vec{r},t)}{\partial t} = \vec{\nabla}\cdot(K\ \vec{\nabla}T(\vec{r},t))\ +\ D(\vec{r},T) \tag{101}$$

where ρ is the material density, C_p is the specific heat capacity, K is the thermal conductivity, and T is the temperature, depending on position and time.
We write an energy balance for the boundary surface S_i as

$$-K_i\ \frac{\partial T}{\partial n_i}\Big|_{S_i} = h_i(T|_{S_i}\ -\ T_a)\ +\ em_i\ \sigma\ (T^4|_{S_i}\ -\ T_a^4) \tag{102}$$

where $\partial/\partial n_i$ denotes differentiation along the outward-drawn normal at the boundary surface S_i. The symbol h_i is the heat transfer coefficient related to the convective heat loss. The radiative heat loss is expressed by the second part of the equation with em_i the emissivity of the surface, and σ the Stefan-Boltzmann constant. T_a, the ambient temperature, equals the initial temperature of the sample at $t=0$.

The slab is described with a one-dimensional model.

$$\rho\ C_p\ \frac{\partial T(x,t)}{\partial t} = \frac{\partial}{\partial x}(K\ \frac{\partial T(x,t)}{\partial x})\ +\ D(x) \tag{103}$$

$$\begin{aligned} K\ \frac{\partial T}{\partial x} &= h(T\ -\ T_a)\ +\ em\ \sigma\ (T^4\ -\ T_a^4) \qquad , x=0 \\ -K\ \frac{\partial T}{\partial x} &= h(T\ -\ T_a)\ +\ em\ \sigma\ (T^4\ -\ T_a^4) \qquad , x=L \end{aligned} \tag{104}$$

The thermal boundary conditions (102) are parts of the heat balance in case of isothermal objects. For an isothermal slab one reads

$$\rho\ C_p\ L\frac{\mathrm{d}T}{\mathrm{d}t} = -2h(T-T_a) - 2em\ \sigma(T^4 - T_a^4) + D_{tot} \tag{105}$$

The isothermal cylinder

$$\rho\ C_p\ \pi R^2\frac{\mathrm{d}T}{\mathrm{d}t} = -2\pi R[h(T-T_a) + em\ \sigma(T^4 - T_a^4)] + D_{tot} \tag{106}$$

The isothermal sphere

$$\rho\ C_p\ \frac{4}{3}\pi R^3\frac{\mathrm{d}T}{\mathrm{d}t} = -4\pi R^2[h(T-T_a) + em\ \sigma(T^4 - T_a^4)] + D_{tot} \tag{107}$$

The total ammount of absorbed power for an isothermal slab D_{tot} (45) equals the microwave power P times an absorption function D_1. In chapter 3 and 4 the effect of a time dependent microwave power $P(t)$ is studied, where D_{tot} is written as $P(t)D_1$. It should be emphasized that this is only correct, if $P(t)$ is a slowly varying function compared to the 2450 MHz oscillations of the electromagnetic field.

2.11. APPROXIMATIONS OF POWER FORMULA' S

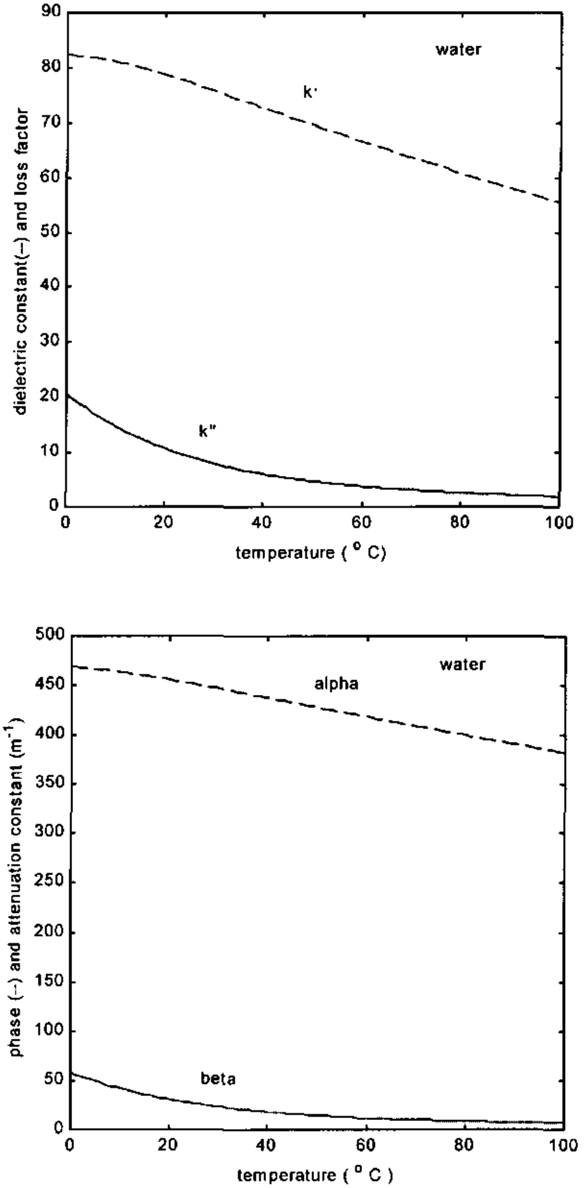

Figure 5. Dielectric properties of water at 2450 MHz.

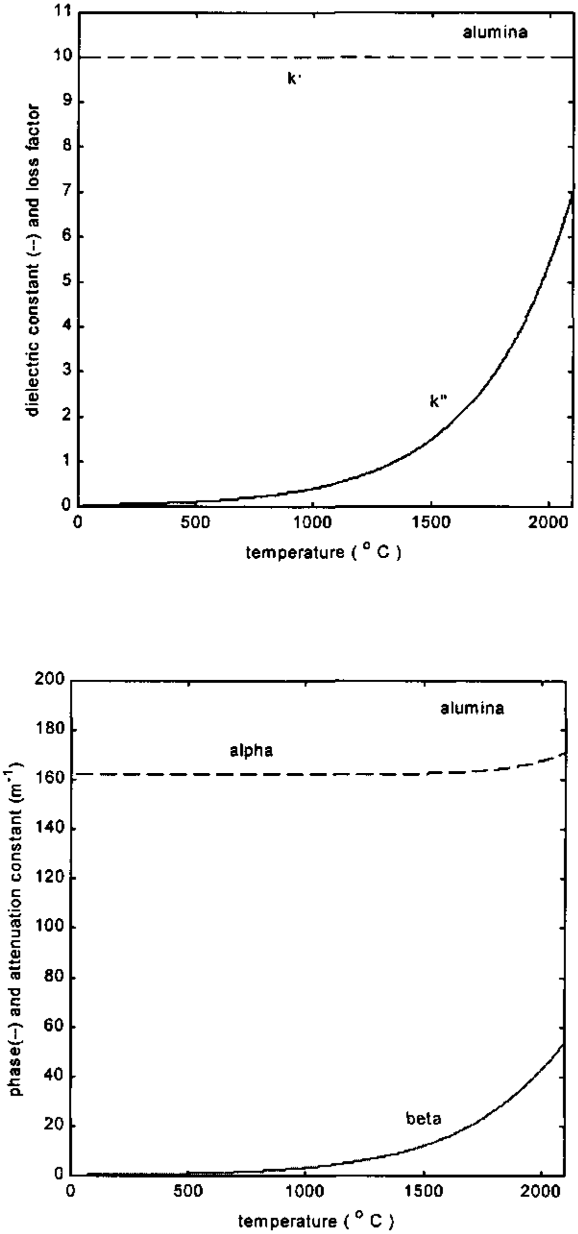

Figure 6. Dielectric properties of alumina at 2450 MHz.

The equations of the total amount of the absorbed power are complicated and interpreting them in physical processes is difficult. A simplification of the formulas is necessary. This is done for two material specimens: Water (as an example of a foodstuff) and alumina (Al_2O_3, as an example of a ceramic). The dielectric properties are shown in fig. 5 and fig. 6.

THE SLAB

The simplification of eq.(45) is easy. For both materials δ_{12} is about $-\pi$. Neglecting small terms

$$\text{water} \qquad D_{tot} \approx P\ k_1\ \kappa_2^{\prime\prime}\ |T_{12}|^2\ \frac{1}{2\beta_2}\ \frac{1}{1\ -\ 2\,|R_{12}|^2 e^{-2\beta_2 L}\cos(2\alpha_2 L)} \tag{108}$$

$$\text{alumina} \qquad D_{tot} \approx P\ k_1\ \kappa_2^{\prime\prime}\ |T_{12}|^2\ \frac{1}{2\beta_2}\ (1\ -\ e^{-2\beta_2 L}) \tag{109}$$

THE CYLINDER

The equations for the total amount of absorbed power for the (A)-cylinder (64), (65), and (66), have been formulated in terms of Bessel and Hankel functions. These are difficult to interpret. Therefore, these functions are replaced by sine and cosine functions. For x is large, we have

$$\begin{aligned} J_n(x) &\approx \sqrt{\frac{2}{\pi x}}\ \cos(x\ -\ \frac{n\pi}{2}\ -\ \frac{\pi}{4}) \\ N_n(x) &\approx \sqrt{\frac{2}{\pi x}}\ \sin(x\ -\ \frac{n\pi}{2}\ -\ \frac{\pi}{4}) \end{aligned} \tag{110}$$

The x equals $k_1 r$ ($k_1 = 51.3\ m^{-1}$) or $k_2 r$ (for water $k_2 \approx 430\ m^{-1}$, for alumina $k_2 \approx 162\ m^{-1}$). The application of these theorems generates

$$D_{tot} = P\, k_1\, \kappa_2^{//} \frac{1}{2\beta_2} \frac{1}{|k_2|} e^{2\beta_2 R} \quad \times$$

$$[\ |c_0|^2 \{\ (1 - e^{-4\beta_2 R}) - \frac{2\beta_2}{\alpha_2} e^{-2\beta_2 R} \cos(2\alpha_2 R)\ \} + \tag{111}$$

$$+2 \sum_{n=1}^{\infty} |c_n|^2 \{\ (1 - e^{-4\beta_2 R}) - \frac{2\beta_2}{\alpha_2} e^{-2\beta_2 R} \cos(2\alpha_2 R - n\pi)\ \}\ ]$$

where

$$|c_n|^2 = [4 k_1 |k_2| e^{-2\beta_2 R}] \ / \ [(k_1^2 + |k_2|^2)(1 + e^{-4\beta_2 R}) + 2\alpha_2 k_1 (1 - e^{-4\beta_2 R}) + \tag{112}$$

$$+2(k_1^2 - |k_2|^2) e^{-2\beta_2 R} \{\ \sin(2\alpha_2 R - n\pi) - \frac{2k_1\beta_2}{k_1^2 - |k_2|^2} \cos(2\alpha_2 R - n\pi)\ \}\ ]$$

Write

$$|T_{12}|^2 = \frac{4k_1^2}{k_1^2 + |k_2|^2 + 2k_1\alpha_2} \tag{113}$$

$$|R_{12}|^2 = \frac{k_1^2 + |k_2|^2 - 2k_1\alpha_2}{k_1^2 + |k_2|^2 + 2k_1\alpha_2} \quad ; \quad \tan(\delta_{12}) = \frac{-2\beta_2}{k_1^2 - |k_2|^2}$$

Neglect the factor $\exp(-4\beta_2 R)$ in the denominator of (112) and use $\delta_{12} \approx -\pi$.

$$|c_n|^2 = \frac{|k_2| e^{-2\beta_2 R} |T_{12}|^2}{1 + 2\dfrac{k_1^2 - |k_2|^2}{k_1^2 + |k_2|^2 + 2k_1\alpha_2} e^{-2\beta_2 R} \sin(2\alpha_2 R - n\pi)} \tag{114}$$

It can be demonstrated that

$$\frac{k_1^2 - |k_2|^2}{k_1^2 + |k_2|^2 + 2k_1\alpha_2} \approx -|R_{12}| \qquad \text{, where } \; k_1 < |k_2| \; \text{ and } \; \beta_2 < \alpha_2 \tag{115}$$

Finally we have

$$D_{tot} = P \; k_1 \; \kappa_2'' \; |T_{12}|^2 \; \frac{1}{2\beta_2} \; \times$$

$$\left[\frac{(1 - e^{-4\beta_2 R}) - \frac{2\beta_2}{\alpha_2}\cos(2\alpha_2 R)}{1 - 2|R_{12}| e^{-2\beta_2 R}\sin(2\alpha_2 R)} + 2\sum_{n=1}^{\infty} \frac{(1 - e^{-4\beta_2 R}) - \frac{2\beta_2}{\alpha_2} e^{-2\beta_2 R}\cos(2\alpha_2 R - n\pi)}{1 - 2|R_{12}| e^{-2\beta_2 R}\sin(2\alpha_2 R - n\pi)} \right] \tag{116}$$

The terms in this series have equal weight. In the original expression (64) the terms decrease with increasing order, where n is large. Which terms one has to take into account, depends on the radius R. For very thick cylinders ($R >$ 10 cm) the zero order term of (116) is enough. For cylinders with a radius between 0.5 and 10 cm a very good approximation (compared to the numerical analysis) has been achieved by the combination of the first and second order term. Neglecting the small factors in this sum, results in

$$\text{water} \qquad D_{tot} \approx P \; k_1 \; \kappa_2'' \; |T_{12}|^2 \; \frac{1}{2\beta_2} \; \frac{1}{1 + 2|R_{12}|^2 e^{-2\beta_2 L}\cos(2\alpha_2 L)} \tag{117}$$

$$\text{alumina} \qquad D_{tot} \approx P \; k_1 \; \kappa_2'' \; |T_{12}|^2 \; \frac{1}{2\beta_2} \; (1 - e^{-2\beta_2 L}) \tag{118}$$

where L is the diameter of the cylinder. The approximation of alumina (118) is quite rough, but this is done to emphasize the difference in the physical origin of thermal runaway in water and alumina. The development of eq. (78) and (79) of the cylinder (B) in sine and cosine functions is accordingly. The result is the same.

THE SPHERE

The integration of (100) over the volume of the sphere, with the approximation (110), yields two kinds of terms.

$$|a_n|^2 \frac{-e^{2\beta_2 R}}{4\pi|k_2|\beta_2} [(1-e^{-4\beta_2 R})\cos(2\alpha_2 R-n\pi) + \frac{\beta_2}{\alpha_2}(1+e^{-4\beta_2 R})\sin(2\alpha_2 R-n\pi)] \qquad (119)$$

$$|b_n|^2 \frac{e^{2\beta_2 R}}{4\pi|k_2|\beta_2} [(1-e^{-4\beta_2 R}) + \frac{2\beta_2}{\alpha_2}e^{-2\beta_2 R}\sin(2\alpha_2 R-n\pi)] \qquad (120)$$

The numerators of $|a_n|^2$ and $|b_n|^2$ are proportional to $4k_1|k_2|/(\pi R^2)$,^{2} while the denominators are complicated expressions with Bessel and Hankel functions. See eq. (98) and (99) for the definition of numerator and denominator. This means

$$D_{tot} \propto P\ k_1\ \kappa_2'' \frac{1}{\beta_2}\ k_1 \frac{e^{2\beta_2 R}}{\pi^3 R^2} \qquad (121)$$

In the first approximation the D_{tot} is temperature independent, because κ_2'' is divided by β_2 (See fig.5 and fig.6). The factor $\exp(2\beta_2 R)$ becomes part of the denominator of a_n and b_n in the same way as has been done to the slab and the cylinder.

2.12. LIST OF SYMBOLS

Whenever the description refers to electric or magnetic field, the complex amplitude of the field is meant. Real fields are discerned from the complex fields by the addition real.

symbol	description	unit	introduced in section
A	integration constant in $R(kr)$	-	2.7
A_1	real amplitude of incident electric field component	V/m	2.6
A_2, A_3	integration constants in field expressions	-	2.6
A_H	real amplitude of incident magnetic field component	A/m	2.8
a	integration constant	-	2.2
a	integration constant in $\Phi(\phi)$ of sphere	-	2.9
a_l	coefficient in the sum of the scattered electric field of the sphere	-	2.9
a_n	coefficient in the sum of the scattered electric field component of the cylinder	-	2.7
a_n	coefficient in the sum of the scattered magnetic field component of the cylinder	-	2.8
a_n	coefficient in the sum of the electric field components within the sphere	-	2.9
B	integration constant in $R(kr)$	-	2.7
B_1, B_2, B_3	integration constants in field expressions	-	2.6
$\vec{B}$	complex amplitude of magnetic induction	-	2.2
$\vec{B}_r$	real magnetic induction	T	2.2
b	integration constant in $\Phi(\phi)$ of sphere	-	2.9
b_l	coefficient in the sum of the scattered electric field of the sphere	-	2.9

b_n	coefficient in the sum of the electric field components within the sphere	-	2.9
c	velocity of light	m/s	2.4
c	integration constant in $\Phi(\phi)$ of cylinder	-	2.7
c_l	coefficient in the sum of the electromagnetic field within the sphere	-	2.9
c_n	coefficient in the sum of the electric field component within the cylinder	-	2.7
c_n	coefficient in the sum of the magnetic field component within the cylinder	-	2.8
C_p	specific heat capacity	J/kgK	2.10
D	time average dissipated power	W/m^3	2.3
D_{tot}	total amount of absorbed power		
	in slab per unit area	W/m^2	2.6
	in cylinder per unit distance	W/m	2.7
	in sphere	W	2.9
$\vec{D}$	complex amplitude of electric displacement	-	2.2
$\vec{D}_r$	real electric displacement	C/m^2	2.2
d	integration constant in $\Phi(\phi)$ of cylinder	-	2.7
d_l	coefficient in the sum of the electromagnetic field within the sphere	-	2.9
$\vec{E}$	complex amplitude of electric field	-	2.2
$\vec{E}_r$	real electric field	V/m	2.2
$\vec{E}_E$	electric component of electric multipole field	-	2.9
$\vec{E}_H$	electric component of magnetic multipole field	-	2.9
$\vec{E}_s$	scattered electric field of sphere	-	2.9
$\vec{E}^{(2)}$	electric field within sphere	-	2.9
E_1, E_2, E_3	component of the electric field in medium 1, 2 or 3	-	2.6
E_r, E_ϕ, E_z	polar components of the electric field	-	2.7

$E_r^{(1)}$, $E_\phi^{(1)}$, $E_z^{(1)}$	polar components of the electric field outside the cylinder	-	2.7
$E_r^{(2)}$, $E_\phi^{(2)}$, $E_z^{(2)}$	polar components of the electric field within the cylinder	-	2.7
$E_x^{(2)}$, $E_y^{(2)}$, $E_z^{(2)}$	rectangular components of the electric field within the cylinder	-	2.8
E_r, E_θ, E_ϕ	spherical components of the electric field	-	2.9
$E_r^{(1)}$, $E_\theta^{(1)}$, $E_\phi^{(1)}$	spherical components of the electric field outside the sphere	-	2.9
$E_r^{(2)}$, $E_\theta^{(2)}$, $E_\phi^{(2)}$	spherical components of the electric field within the sphere	-	2.9
em	emissivity	-	2.10
em_i	emissivity at surface i	-	2.10
$f(r)$	radius dependent function	-	2.9
g	abbreviation of $f\sqrt{(kr)}$	-	2.9
$\vec{H}$	complex amplitude of magnetic field	-	2.2
$\vec{H}_r$	real magnetic field	A/m	2.2
$\vec{H}_E$	magnetic component of electric multipole field	-	2.9
$\vec{H}_M$	magnetic component of magnetic multipole field	-	2.9
$\vec{H}_s$	scattered magnetic field of sphere	-	2.9
$\vec{H}^{(2)}$	magnetic field within sphere	-	2.9
H_r, H_ϕ, H_z	polar components of the magnetic field	-	2.7
$H_r^{(1)}$, $H_\phi^{(1)}$, $H_z^{(1)}$	polar components of the magnetic field outside the cylinder	-	2.7
$H_r^{(2)}$, $H_\phi^{(2)}$, $H_z^{(2)}$	polar components of the magnetic field within the cylinder	-	2.7
$H_x^{(2)}$, $H_y^{(2)}$, $H_z^{(2)}$	rectangular components of the magnetic field within the cylinder	-	2.8
H_r, H_θ, H_ϕ	spherical components of the magnetic field	-	2.9

$H_r^{(1)}$, $H_\theta^{(1)}$, $H_\phi^{(1)}$	spherical components of the magnetic field outside the sphere	-	2.9
$H_r^{(2)}$, $H_\theta^{(2)}$, $H_\phi^{(2)}$	spherical components of the magnetic field within the sphere	-	2.9
H_n	Hankel function of the first kind of order n	-	2.7
h	heat transfer coefficient	W/m^2K	2.10
h_i	heat transfer coefficient at surface i	W/m^2K	2.10
h_l	Hankel function in z_l	-	2.9
i	imaginary unit	-	2.2
$\vec{i}_x$	unit vector of x-axis	-	2.9
$\vec{i}_r$, $\vec{i}_\theta$, $\vec{i}_\phi$	spherical unit vectors	-	2.9
$\vec{J}$	compl.ampl.of current flux of free charges	-	2.2
$\vec{J}_r$	real current flux of free charges	A/m^2	2.2
J_n	Bessel function of the first kind of order n	-	2.7
j_l	Bessel function in z_l	-	2.9
K	thermal conductivity	W/mK	2.10
K_i	thermal conductivity at surface i	W/mK	2.10
k	complex propagation constant	m^{-1}	2.2
k_1	propagation constant in air	m^{-1}	2.6
k_2	complex propagation constant in medium 2	-	2.6
L	slab thickness	m	2.6
L	diameter of cylinder	m	2.11
N_n	Bessel function of the second kind of order n (Neumann's or Weber's function)	-	2.7
n_l	Neumann function in z_l	-	2.9
P	power flux of incident field	W/m^2	2.6
P_l, P_n	Legendre polynomial of order l or n	-	2.9
P_l^m	associated Legendre function of the first kind of order (l, m)	-	2.9
Q_l^m	associated Legendre function of the second kind of order (l, m)	-	2.9

$\vec{Q}_r(t,T)$	symbol for $\vec{D}_r$, $\vec{B}_r$ or $\vec{J}_r$	-	2.2
R	radius of the cylinder	m	2.7
R	radius of the sphere	m	2.9
R, $R(r)$	radius dependent function	-	2.7
$R(t,T)$	response function	-	2.2
R_{12}	complex reflection coefficient from medium 1 to medium 2	-	2.6
$\vec{r}$	position vector	m	2.9
S_i	surface i	m^2	2.10
$<\vec{S}>$	time average power flux	W/m^2	2.3
T	temperature	K	2.2
T, $T(\theta)$	theta dependent function	-	2.9
T_a	ambient temperature	K	2.10
T_{12}	complex transmission coefficient from medium 1 to medium 2	-	2.6
t	time	s	2.2
x	abbreviation of kr in Eq.(81)	-	2.9
x	abbreviation of $\cos(\theta)$ in Eq.(83)	-	2.9
$Y(\theta,\phi)$	theta and phi dependent function	-	2.9
Y_l^m	spherical harmonic function of order (l, m)	-	2.9
$Z_{l+1/2}$	Bessel, Neumann or Hankel function of order $l+1/2$	-	2.9
z_l	abbreviation of $Z_{l+1/2}\sqrt{(\pi/2kr)}$	-	2.9

Coordinates

x, y, z	rectangular coordinates	m	2.2
r, ϕ, z	polar coordinates	m, rad, m	2.7
r, θ, ϕ	spherical coordinates	m, rad, rad	2.9

Greek symbols

α	phase constant	m^{-1}	2.4
α_2	phase constant of medium 2	m^{-1}	2.6
β	attenuation constant	m^{-1}	2.4
β_2	attenuation constant of medium 2	m^{-1}	2.6
δ	loss angle	rad	2.4
δ_{12}	amplitude of R_{12}	rad	2.6
ε	complex permittivity	-	2.2
ε'	real part of ε	C^2/Nm^2	2.4
ε''	imaginary part of ε	C^2/Nm^2	2.4
ε_0	permittivity in vacuum	C^2/Nm^2	2.4
ε_n	time dependent complex permittivity	-	2.2
ε_n'	real part of ε_n	C^2/Nm^2	2.3
ε_n''	imaginary part of ε_n	C^2/Nm^2	2.3
κ	complex dielectric constant	-	2.4
κ_0	dielectric constant at $\omega=0\ s^{-1}$	-	2.2
κ_∞	dielectric constant at $\omega=\infty\ s^{-1}$	-	2.2
κ'	dielectric constant, real part of κ	-	2.4
κ''	dielectric loss factor, imaginary part of κ	-	2.4
κ_2'	dielectric constant of medium 2	-	2.8
κ_2''	dielectric loss factor of medium 2	-	2.6
μ	complex permeability	-	2.2
μ_0	permeability of vacuum	Tm/A	2.4
μ_n	time dependent complex permeability	-	2.2
μ_n'	real part of μ_n	Tm/A	2.3
μ_n''	imaginary part of μ_n	Tm/A	2.3
ρ	density	Kg/m^3	2.10
ρ	comp. ampl. of charge density of free charges	-	2.2
ρ_r	charge density of free charges	C/m^3	2.2
σ	Stefan-Boltzmann constant	W/m^2K^4	2.10
σ	electric conductivity of free charges	S/m	2.2

τ	molecular relaxation time	s	2.2
τ_{12}	amplitude of T_{12}	rad	2.6
$\Phi, \Phi(\phi)$	phi dependent function	-	2.7
$\chi(\vec{r}, t)$	time dependent electric susceptibility	C^2/Nm^2s	2.2
Ψ	component of electric field	-	2.9
ω	angular frequency	s^{-1}	2.2

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Chapter 3

Thermal runaway and bistability in microwave heated isothermal slabs

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Journal of Applied Physics, **79** (3), 1779-1783, 1 February 1996

(Received 3 May 1995; accepted for publication 11 October 1995)

ABSTRACT

The dissipation of electromagnetic energy within a microwave heated layer has been analyzed. It has been shown that the dissipation oscillates as a function of temperature, regardless of the material specimen. This oscillation, combined with the heat loss, is found to be responsible for thermal runaway phenomenon in isothermal slabs. Based on such an observation, a general rule to prevent thermal runaway was developed. Slab temperature analysis for time dependent microwave power indicates that the concept of bistability is not the appropriate term to describe the observed jumps in temperature.

3.1. INTRODUCTION

The application of microwave heating in, for example the food industry is seriously hampered by two problems which have their roots in the basic physics of the heating process. The first difficulty is the uneven spatial dissipation of energy within foodstuffs. The second difficulty is the catastrophic phenomenon of thermal runaway in which a slight change of microwave power causes the temperature to increase rapidly. In this article we will study the second problem. The motive for this research was the introduction of the concept of bistability in microwave physics. According to Stuerga[1], the temperature of a microwave heated slab, when irradiated by microwave power directly proportional to time, shows bistable behavior.. As usual the bistable phenomenon is accompanied by a hysteresis loop. Stuerga also claims to have found experimental evidence for this idea. However, up to now the predicted hysteresis loop has never been found.

A number of bistable phenomena exist in physics. Probably the one which is best known is the phase transition of vapor into liquid. A vapor may be compressed to a pressure well above the vapor pressure of the liquid without condensation taking place and, on the other hand, a liquid may be heated well above its boiling point without boiling. Both processes are limited to certain values of pressure above (below) at which condensation (boiling) starts. This is a typical metastable or bistable phenomenon. This bistable behavior can be described according to the Van der Waals equation, which is very remarkable because it is an equation of state related to a gas and contains nothing about phase transitions. Van der Waals law suggests bistability, but for a complete understanding the theory of phase transitions has to be added.

It seems that the same kind of situation is present in microwave physics. Fourier's differential equation suggests bistability (for time dependent microwave power), but this is insufficient for a complete understanding. The purpose of the study described in this article was to develop a kind of phase transition theory, aiming to explain and support the idea by Stuerga.

The phenomena of thermal runaway and bistability are closely related. In fact the idea of bistability was inspired by thermal runaway. This is why the first part of this article contains a study of dissipation which, combined with heat losses, might result in thermal runaway. The aim was to find the origin of the runaway process and hope that it would lead to the real explanation of bistability. A side product of this part of the investigation resulted in a rule to eventually prevent thermal runaway. In the final part of the article the concept of bistability in microwave heating is discussed.

3.2. THE ORIGIN OF THERMAL RUNAWAY IN ISOTHERMAL SLABS

Consider a layer of specimen material, irradiated from one side by microwave radiation with a frequency of 2450 MHz. The wave is a plane, harmonic one and impinges normally upon the material. In order to explain the principles of thermal runaway, the simplest possible system was conceived. The absence of reflection from the cavity in which the actual experiment was performed, is assumed. This means that the initial wave is reflected, absorbed, and transmitted (Fig.1).

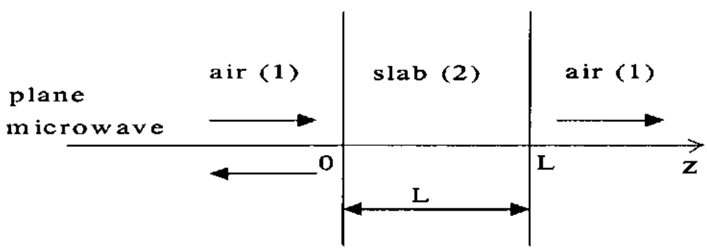

Figure 1. A layer, being irradiated from one side, in an echofree cavity

It is also assumed that the temperature throughout the layer is the same at every moment. This can be achieved by taking a liquid as the medium and mixing it in such a way that the spatial equalization of temperature is much faster than the process causing temperature increase. This process is due to the dissipation of electromagnetic energy within the slab. Under these conditions simple relationships evolve, which indicate the reasons for thermal runaway in isothermal slabs.

Initially a system without convective and radiative thermal losses is considered. The differential equation, describing the relationship between temperature T and time t, (Fourier's law) reads:

$$\rho\, C_p\, L\, \frac{dT}{dt} = D \tag{1}$$

where ρ is the density, C_p is the thermal capacity, L is the layer thickness, and D the production of heat (total amount of power per square meter generated along the z-axis, extending from $z=0$ to $z=L$). This power follows from Maxwell's equations, together with the appropriate boundary conditions at the surfaces of the slab. According to Stratton[2] or Ayappa[3] for a pure dielectric one has:

$$D = P\, \frac{\omega}{c}\, \kappa_2^{\prime\prime}\, |T_{12}|^2\, \frac{\textbf{numerator}}{\textbf{denominator}} \tag{2}$$

with

$$\frac{1}{2\beta_2}\,(1 - e^{-2\beta_2 L})(1 + |R_{12}|^2 e^{-2\beta_2 L}) - \frac{2}{\alpha_2}|R_{12}|e^{-2\beta_2 L}\,\sin\,(\alpha_2 L)\,\cos(\alpha_2 L + \delta_{12}) \tag{3}$$

as a numerator and

$$1 - 2|R_{12}|^2 e^{-2\beta_2 L}\cos(2\delta_{12} + 2\alpha_2 L) + |R_{12}|^4 e^{-4\beta_2 L} \tag{4}$$

as a denominator, where P is the intensity of the initial microwave, better known as the microwave power. The frequency has the symbol ω and c refers to the speed of light in vacuum.

The dielectric constant κ is written in the usual way as the difference between a real part κ' and an imaginary part κ''.

$$\kappa = \kappa' - i\kappa'' \tag{5}$$

κ' and κ'' are material constants, independent of the geometry of the system. They only depend on frequency and temperature. They are not very appropriate to describe the propagation of waves, however, and this is why the wavenumber (phase constant) α and the attenuation constant β are introduced. For this free layer model α and β are related to the dielectric properties of a medium and frequency of radiation by

$$\alpha = \frac{\omega}{c}\sqrt{\frac{\kappa'[\sqrt{1 + \tan^2\delta} + 1]}{2}} \tag{6}$$

$$\beta = \frac{\omega}{c}\,\frac{\sqrt{\kappa'[\sqrt{1 + \tan^2\delta} - 1]}}{2} \tag{7}$$

with

$$\tan\delta = \kappa''/\kappa' \tag{8}$$

The reflection coefficient R_{12} and the transmission coefficient T_{12} are related to α and β by

$$|R_{12}|^2 = \frac{(\alpha_1 - \alpha_2)^2 + (\beta_1 - \beta_2)^2}{(\alpha_1 + \alpha_2)^2 + (\beta_1 + \beta_2)^2} \tag{9}$$

$$|T_{12}|^2 = \frac{4(\alpha_1^2 + \beta_1^2)}{(\alpha_1 + \alpha_2)^2 + (\beta_1 + \beta_2)^2} \tag{10}$$

$$\delta_{12} = \tan^{-1}\left[\frac{-2(\alpha_1\beta_2 - \alpha_2\beta_1)}{(\alpha_1^2 - \alpha_2^2) + (\beta_1^2 - \beta_2^2)}\right] \tag{11}$$

The objective now is to find the solution to equation (1), i.e., the temperature as a function of time.

The electromagnetic dissipation is always directly proportional to κ'', regardless of product's geometry. From this it follows that in crude approximation the temperature-time graph is governed by the temperature behavior of κ''. For instance, if κ'' decreases exponentially in temperature, the temperature will exponen-

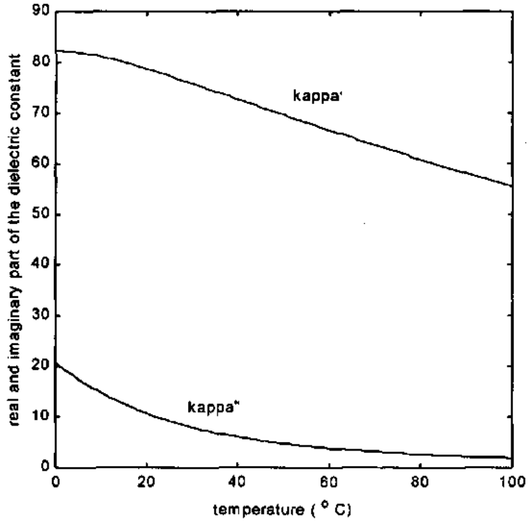

Figure 2. The real (κ') and imaginary (κ'') part of the dielectric constant of water plotted versus temperature, at 2450 MHz, according to Kaatze.

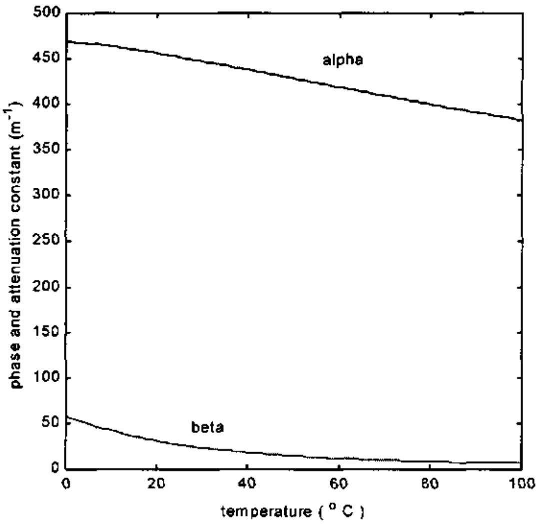

Figure 3. The phase (α) and attenuation (β) constant of water plotted versus temperature at 2450 MHz.

tially increase to a constant value in time. According to this approximation no sudden jumps in temperature or other nonlinear behavior will ever occur. So thermal runaway, if present, is caused by the interference of waves within the irradiated material. In order to investigate this process in detail the above relationships have to be simplified. Only then will it be possible to see which phenomenon is responsible for thermal runaway.

In experimental set-ups medium 1 is usually air. Air hardly absorbs electromagnetic energy, so $\beta_1 = 0$. Medium 2 is a liquid or solid, so α_1 will be smaller than α_2. From the definition of α Eq.(6) and β Eq.(7) it is obvious that β is always smaller than α. With these considerations the $2/\alpha_2$ term in the numerator can be neglected and thus the numerator is proportional to $1/2\beta_2$. For the absolute value of the transmission coefficient one roughly obtains:

$$|T_{12}| = \frac{2\alpha_1}{\alpha_2} \tag{12}$$

Without the denominator the dissipation D is roughly proportional to $\kappa_2'' \, |T_{12}|^2/2\beta_2$. From Eqs.(6) and (7) it follows that $\alpha_2\beta_2 = \omega^2 \kappa_2''/2c^2$. With this expression and equation (12) one can conclude that for a slab the dissipation, as far as its temperature behavior is concerned, is proportional to $1/\alpha_2$. In the 2450 MHz region α_2 hardly varies with temperature, so our final conclusion is that in the first approximation the dissipation in a slab is independent of temperature! The temperature dependency of κ' and κ'' is more or less canceled out and the temperature as a function of heating time will be represented as a straight line (Fig.5).

In the denominator the small temperature dependency of α_2 becomes very important, because α_2 is part of the argument of a cosine. With increasing temperature the denominator oscillates. In the temperature-time diagram this oscillation is superimposed on the straight line. Here we see the reason for "sudden" temperature jumps in isothermal slabs.

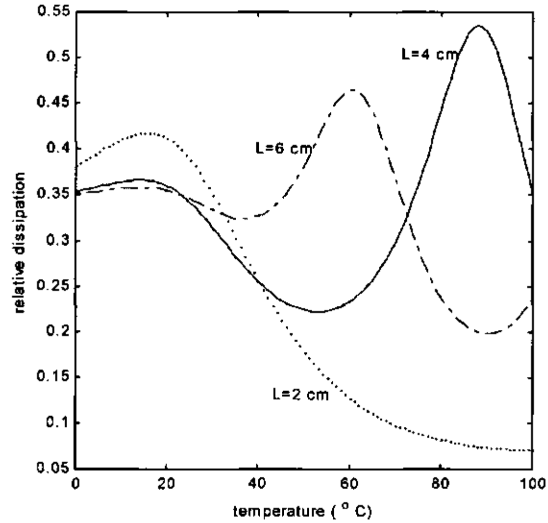

Figure 4. The relative dissipation *D/P* oscillates as a function of temperature.

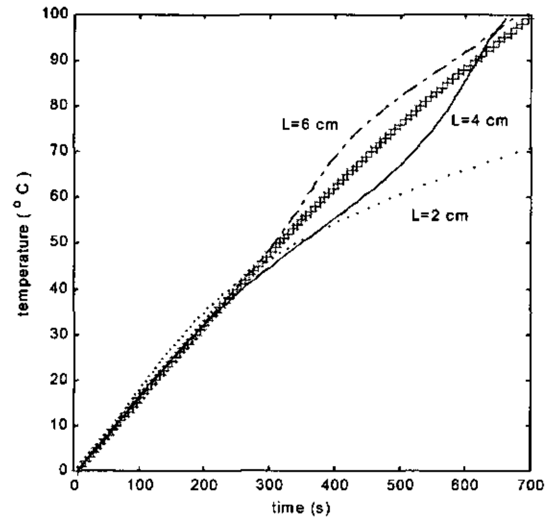

Figure 5. In the first approximation the temprature-time diagram is a straight line (+ + +curve for L=4 cm). Superimposed on this line is an oscillation of which the amplitude increases with temperature, but decreases with layer thickness (P=0.2L kW/cm^2).

To illustrate these ideas a numerical example for demineralized water is given. Water has been taken, because its dielectric constant is well-known. Kaatze[4] describes the behavior of κ_2' and κ_2'' as a function of temperature between 0 and 50 °C. In this example his formulae are extrapolated to 100 °C. Figure 2 shows the temperature behaviour of κ_2' and κ_2''. More important is the behaviour of α_2 and β_2 (Fig.3). Based on these data the relative dissipation D/P for several layer thicknesses L, is calculated and plotted in Fig. 4. The oscillating character of the dissipation is clearly seen. This oscillation is caused by the interference of waves. In a slab without damping, the wave reflected on the rear side of the layer is in phase with the initial wave if $L = n\lambda_2/2$ for $n = 1,2,3,\ldots$. This situation corresponds to maximal dissipation. The dissipation is small when reflected wave and initial wave more or less cancel each other. This is the situation for $L = (2n+1)\lambda_2/4$ and $n = 0,1,2,\ldots$. The wavelength λ_2 of water varies from 1.34 cm at 0 °C to 1.65 cm at 100 °C. For example, if $L = 4$ cm, the dissipation starts at 0 °C at a maximum ($L = 3\lambda_2$). With increasing temperature it will run to a minimum ($L = 11\lambda_2/4$), followed by a maximum ($L = 5\lambda_2/2$), and finally it falls off to a neutral situation at 100 °C (Fig.4). For a complete period of oscillation the layer thickness L must be equal to $\pi/\Delta\alpha_2$, where $\Delta\alpha_2$ is the difference between the maximum and the minimum value of α_2. For water this results in a layer thickness of 3.5 cm. Important for this oscillation is the behavior of its amplitude as a function of temperature. The amplitude is proportional to $\exp(-2\beta_2 L)$. If β_2 decreases strongly with temperature (as is the case for water), the amplitude increases strongly and the deviations from the straight line in the temperature-time diagram are larger. In such a case the oscillation becomes significant at higher temperatures. For large L the amplitude is small. Many periods of oscillation exist, but they are hardly noticeable. A small L ($< \pi/\Delta\alpha_2$) results in a large deviation from the straight line, but no complete period exists. Applied to water, the combination of all of these effects, results in a significant temperature jump for a thickness of about 4 cm. Figure 5 shows temperature-time diagrams for several values of L. In these plots the quotient P/L is constant.

3.3. THERMAL RUNAWAY AND HEAT LOSS

In this section the heat loss at the surfaces of the slab is taken into account. The convective heat loss is proportional to the temperature difference (T - T_0) to the ambient. For small temperature differences this is also the case for radiative loss.

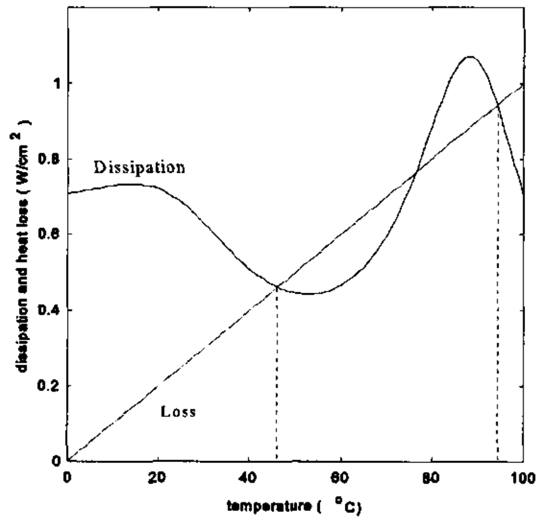

Figure 6. The marked intersection points of dissipation and heat loss correspond to stable steady-state slab temperatures (L=4 cm, P=2 W/cm^2, h=0.01 W/Kcm^2, T_0=0 °C).

Thus the total heat loss can be described by some effective heat transfer coefficient h, multiplied by the temperature difference (T - T_0). Equation (1) is then transformed into

$$\rho C_p L \frac{dT}{dt} = D - h(T - T_0) \tag{13}$$

Because the right-hand side of Eq.(13) is only dependent on temperature, the steady-state temperature T_f of the slab follows from

$$D - h(T_f - T_0) = 0 \tag{14}$$

Depending on the microwave power P and the layer thickness L one has one or more intersecting points of the functions D and $h(T - T_0)$, as shown in Fig. 6. For certain

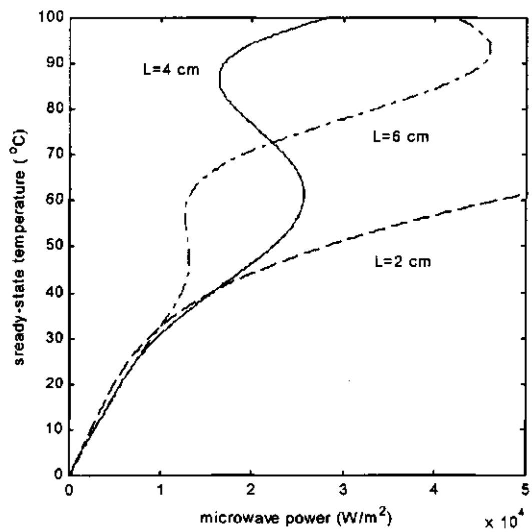

Figure 7. Only the upper and lower branch of S-shaped curves represent stable steady-state temperatures. The steady-state temperatures will be "high" or "low", depending on the initial slab temperature.

layer thicknesses L this results in an S-shaped or multi S-shaped curve in a plot showing the steady-state slab temperature versus the microwave power P (Fig. 7). Only the upper and lower branch of the S-shape represent stable final temperatures. Between these two branches an unstable temperature region exists. Depending on the initial temperature of the slab the steady-state temperature will either be "high" or "low" as a result of the microwave heating. It is seen that a small increase of the microwave power might lead to the well-known thermal runaway phenomenon, i.e. a jump from the lower to the upper branch. For more details on these phenomena see Kriegsmann[5] (for a ceramic slab) and Stuerga[6] (for water). The S-shape or multi S-shape is caused by oscillations of the dissipation. Without oscillations there is no S-shape and hence no thermal runaway. For half a period of oscillation there is only a single point of intersection of the dissipation D and the heat loss $h(T - T_0)$, which suggests the following rule: If the layer thickness is smaller than $\pi/2\Delta\alpha_2$, where $\Delta\alpha_2$ is the difference between the maximum and minimum value of the phase constant α_2, no thermal runaway will ever occur. No materials will be damaged by thermal runaway if the layer thickness is smaller than $\pi/2\Delta\alpha_2$. For water this results in a thickness of 1.75 cm. A detailed analysis gives 2.2 cm as a safety thickness. For the drying of food products this observation might be of great importance.

3.4. BISTABILITY

Besides instability and thermal runaway, the oscillation of the dissipation, combined with heat loss, causes a third problem. If the dissipation takes a minimum value with increasing temperature and this minimum value is of the same order of magnitude as the heat loss at that very same temperature, it takes a lot of time to reach a desired final temperature. This is a very inefficient situation. For a fixed L, only one degree of freedom exists to accelerate this process and that is the variation of the microwave power in time. For this reason the heat balance is written down as:

$$\rho C_p L \frac{dT}{dt} = P(t) D_1 - h(T - T_0) \tag{15}$$

where D_1 is the temperature dependent factor calculated from $D = P(t)D_1$. Kriegsmann[5] and Tian[7] suggest several functions $P(t)$ to accelerate the sintering process of a ceramic slab.

Physically very interesting is a control of heat in such a way that the temperature, as a function of time, describes the S-shape as shown in Fig. 7. This will be the case when the microwave power P of Eq.(14) is replaced by γt, where γ is a constant. This leads to the requirement:

$$\gamma t D_1 - h(T - T_0) = 0 \tag{16}$$

from which

$$\frac{\mathrm{d}T}{\mathrm{d}t} = \frac{-\gamma D_1}{\gamma t \dfrac{\mathrm{d}D_1}{\mathrm{d}T_1} - h} \tag{17}$$

Substitution of Eq.(16) and Eq.(17) into Eq.(15) yields a desired function $P(t)$:

$$P(t) = \gamma t - \frac{\rho C_p L}{t\left[\dfrac{\mathrm{d}D_1}{\mathrm{d}T} - \dfrac{D_1}{T - T_0}\right]} \tag{18}$$

With this $P(t)$ the temperature-time curve is S-shaped. Because the temperature cannot go back in time, all kinds of remarkable effects such as bistability and

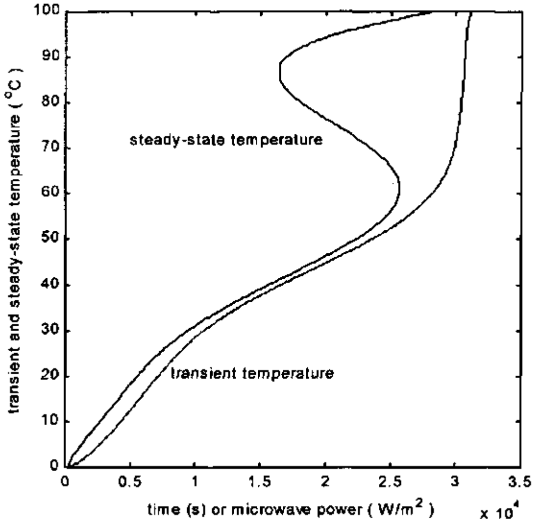

Figure 8. Plot of steady-state temperature T_f as a function of time-independent microwave power, compared to a plot of transient temperature versus time for microwave power directly proportional to time, resulting in artificial bistability (L=4 cm, P= t W/m^2, h=100 W/Km^2, T_0=0 ^{0}C).

hysteresis, occur. As explained in the introduction, bistability is a rare physical phenomenon that needs a more complete explanation. In the case considered here the explanation is very simple. The function $P(t)$ becomes negative in the unstable temperature region. One has negative irradiation. The best way to avoid this is to omit the second term in equation $P(t)$. If the second term is small in comparison with γt, the temperature-time curve will follow the S-shape, except in the unstable regions. Figure 8 shows a plot for $P(t)=t$ and $L=4$ cm. The temperature increases rapidly from the lower to the upper branch of the S-shape in the unstable region. This temperature jump has been found experimentally[8], but it is incorrect to interpret this phenomenon as evidence of bistability. As a consequence of the absence of bistability, hysteresis will never be found.

3.5. CONCLUSIONS

In the first approximation the dissipation in a slab is independent of the temperature within the slab, so that the temperature-time diagram is a straight line. An oscillation is superimposed on this straight line. This is caused by the temperature dependency of the wavenumber. For isothermal slabs the convective and radiative heat losses amplify the effects of oscillation, resulting in instability and thermal runaway. Thermal runaway never occurs for a layer thickness smaller than $\pi/2\Delta\alpha_2$, where $\Delta\alpha_2$ is the difference between the maximum and minimum value of α_2.

Finally, it has been shown that the concept of bistability is not the appropriate term to describe the temperature jumps for a system for which the microwave power is directly proportional to time.

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Chapter 4

Thermal runaway in microwave heating: A mathematical analysis[1]

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Submitted to Applied Mathematical Modelling

ABSTRACT

A study is made of the solution of a differential equation modelling the heating of a layer of material specimen by microwave radiation. Depending on the microwave power bi-stable steady-state temperatures may be expected. When changing the power, a switch from one stable branch to another one may arise. The sudden increase of temperature, known as thermal runaway, is studied from the differential equation using asymptotic methods. Such analysis reveals distinct stages in the process of thermal runaway. At the moment the solution leaves a branch, and becomes unstable a particular type of behaviour is observed (onset of runaway). The most specific element at this stage is a time shift delaying the rapid change in temperature. For this shift a simple expression in terms of the parameters of the system is given. Next it is shown that the rapid transition from one branch to the other can be put in a mathematical formula that smoothly matches the two steady

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[1] Part of this investigation has been supported by a grant of the "Subprograma de Estancias de Investigadores Españoles en Centros de Investigacion Extanjeros (SEUID, Ministerio de Educaciòn y Cultura)".

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state solutions.

keywords: microwave heating, thermal runaway, bistability, relaxation oscillations.

4.1. INTRODUCTION

The use of microwaves has found its way in various applications in industry e.g. in ceramics and in food processing. The understanding of the microwave heating process still is somewhat empirical and speculative due to its highly nonlinear character. In this study we take up this problem and analyse mathematically the dynamics of a first order differential equation model of a layer of material specimen heated by microwave radiation, see Kriegsmann [1] and Vriezinga [2]. Typical for the system is the rapid return to a steady state (relaxation time), meaning that the differential equation, when appropriately scaled, must be of the form

$$\epsilon \frac{dT}{dt} = f(T,P) \tag{1.1}$$

with T the temperature, P the microwave power and ϵ a small positive parameter. The steady state $T(P)$ satisfies the equation $f(T, P) = 0$. If in the P,T-plane the graph has an S-shape we have to deal with bistability (Stuerga *et al.*, [3]). When changing the power with time we may observe hysteresis type of phenomena.

In our mathematical analysis we consider the behaviour of a solution of (1.1) in the limit $\epsilon \to 0$ for some given $P = P(t)$. In case of instability such a limit solution $T_0(t)$ can be discontinuous, see Grasman [4]. In figure 1 we sketch the various functional relations for the differential equation derived in section 2.

The limit solution gives a clear picture of the dynamics. However, for quantitative purposes it may not be sufficiently accurate. In a next step we assume that ϵ is small and construct a refined approximation. Since ϵ multiplies the derivative, we may expect that away from the steady state the temperature T changes rapidly. In singular perturbation theory (Kevorkian and Cole, [5]) one introduces a so-called boundary layer approximation for such time interval. In order to connect this approximation to the steady state solution, a local approximation at the beginning of interval of rapid change has to be made. This solution expressed in the Airy function (Abramowitz and Stegun, [6]) reveals a new element in the onset of thermal runaway: the rapid change is delayed with a time of order $O(\epsilon^{2/3})$. For this time shift an expression is given.

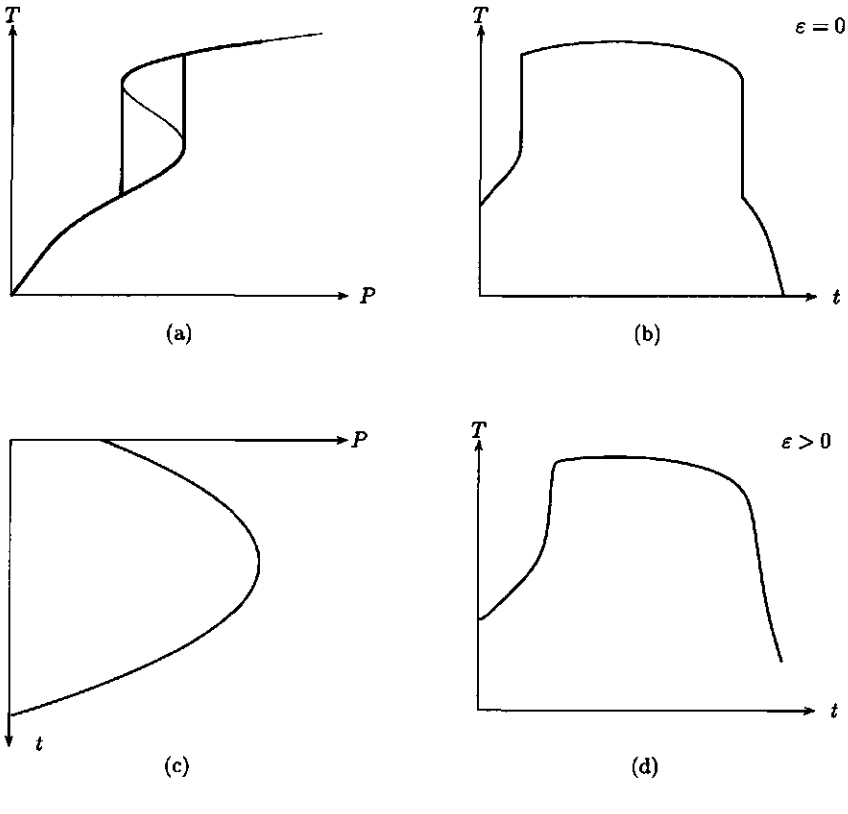

Figure 1. A solution of (1.1) as $\epsilon \to 0$ for a given function $P(t)$, see (c). The function $f(T,P)$ is from (2.10) with $L = 4$. In (d) the parameter $\epsilon = 0.167$.

In section 2 we derive the specific form of Eq. (1.1) for the problem of microwave heating of a layer. Next in section 3 we carry out the asymptotic analysis and determine the time shift at the onset of thermal runaway. Finally, in section 4 we compare the asymptotic solution with the numerical solution for parameter values that agree with an experimental set up for a layer consisting of demi-water, see Kaatze [7].

4.2. DERIVATION OF THE DIFFERENTIAL EQUATION

We consider a layer of material specimen, irradiated from one side by microwave radiation with a frequency of 2450 MHz. The wave is a plane, harmonic one and impinges normally upon the material. In order to explain the principles of thermal runaway, the simplest possible system was conceived. The slab was located in free space, so no other waves than the incident one would be involved. It is also assumed that the temperature throughout the layer is the same at every moment. Solving Maxwell's equations (see e.g. Stratton, [8] or Ayappa *et al.*, [9]) yields the classical wave equation in one dimension.

$$\frac{d^2E}{dx^2} + k^2(T)E = 0 \ , \tag{2.1}$$

where the electric field E is a function of position x and temperature T at that position. The coefficient $k(T)$ stands for the temperature dependent complex wavenumber, which is connected to the dielectric constant κ (kappa) of the medium by the following equation.

$$k^2 = \frac{\omega^2}{c^2}\left(\kappa + \frac{i}{\omega}\frac{\partial\kappa}{\partial t}\right) \ , \tag{2.2}$$

where ω is the angular frequency and c the velocity of light. Eq. (2.2) takes into account that the medium is a pure dielectric and that the permeability of the irradiated object almost equals μ_0, being the permeability in vacuum. Many materials treated by microwaves fulfil this last requirement. The time dependent term in Eq. (2.2) is very small compared to the dielectric constant and can be neglected. Eq.(2.1) has to be combined with the electromagnetic boundary conditions, which read:

$$E_1 = E_2 \quad \text{and} \quad \frac{dE_1}{dx} = \frac{dE_2}{dx} \tag{2.3}$$

at the boundaries $x=0$ and $x=L$ (L is the thickness of the slab). The subscript 1 refers to vacuum or air and the subscript 2 is related to the irradiated medium. We also need the formula of the absorbed power D :

$$D = P \frac{\omega}{c} \kappa_2^{//} \int_0^L (|E_2|^2/E_i^2) \, dx \ , \tag{2.4}$$

where P is the microwave power, E_i the amplitude of the incident wave, and κ is written as the difference of a real and an imaginary part, according to $\kappa = \kappa' - i\kappa''$. For water we have

$$\kappa_2 = \kappa_\infty + \frac{(\kappa_0 - \kappa_\infty)}{1 + i\omega\tau_2}, \tag{2.5}$$

with

$$^{10}\log(\kappa_0) = 1.94404 - 1.991 \cdot 10^{-3}(T - 273.15),$$

$$\kappa_\infty = 5.77 - 2.74 \cdot 10^{-2}(T - 273.15),$$

$$\tau_2 = 3.745 \cdot 10^{-15}((1 + 7 \cdot 10^{-5})(T - 300.65)^2)e^{2.2957 \cdot 10^{-3}/T}. \qquad (s^{-1})$$

where T is in Kelvin. Substituting the field E_2 of (2.1) in Eq.(2.4) yields

$$D = P\frac{\omega}{c}\kappa_2^{//}|T_{12}|^2 \frac{\alpha_2(1 - e^{-2\beta_2 L})(1 + |R_{12}|^2 e^{-2\beta_2 L}) - 4\beta_2 |R_{12}| e^{-2\beta_2 L} \sin(\alpha_2 L)\cos(\alpha_2 L + \delta_{12})}{2\alpha_2\beta_2(1 - 2|R_{12}|^2 e^{-2\beta_2 L}\cos(2\alpha_2 L + 2\delta_{12}) + |R_{12}|^4 e^{-4\beta_2 L})}$$

(2.6)

The dielectric constant has been replaced by the wavenumber (phase constant) α and the attenuation constant β.

$$\alpha_1 = 51.3\ \mathrm{m^{-1}} \quad \text{and} \quad \alpha_2 = \frac{\omega}{c}\{\kappa_2'(\sqrt{1+\tan^2\delta_2}+1)/2\}^{1/2} \tag{2.7}$$

$$\beta_1 = 0 \quad \mathrm{m^{-1}} \quad \text{and} \quad \beta_2 = \frac{\omega}{c}\{\kappa_2'(\sqrt{1+\tan^2\delta_2}-1)/2\}^{1/2} \tag{2.8}$$

with $\tan\delta_2 = \kappa_2''/\kappa_2'$.

The reflection coefficient R_{12} and the transmission coefficient T_{12} are related to α and β by

$$|R_{12}|^2 = \frac{(\alpha_1-\alpha_2)^2 + (\beta_1-\beta_2)^2}{(\alpha_1+\alpha_2)^2 + (\beta_1+\beta_2)^2}, \quad |T_{12}|^2 = \frac{4(\alpha_1^2+\beta_1^2)}{(\alpha_1+\alpha_2)^2 + (\beta_1+\beta_2)^2} \tag{2.9}$$

$$\delta_{12} = \tan^{-1}\left[\frac{-2(\alpha_1\beta_2-\alpha_2\beta_1)}{(\alpha_1^2-\alpha_2^2) + (\beta_1^2-\beta_2^2)}\right] \tag{2.10}$$

To calculate the temperature within the isothermal slab as a function of time Eq.(2.6) has to be combined with Fourier's law. For small temperature differences $(T\text{-}T_0)$ to the ambient one reads

$$\rho C_p L \frac{\mathrm{d}T}{\mathrm{d}t} = D - h(T-T_0)\ , \tag{2.11}$$

where ρ is the density, C_p is the thermal capacity, and h is some effective heat transfer coefficient. In this study we take

$$\rho = 1000\ \mathrm{kg/m^2}\ ,\quad C_p = 4186\ \mathrm{J/Kkg} \quad \text{and} \quad h = 100\ \mathrm{W/Km^2}.$$

It is assumed that T_0 is also the initial temperature of the slab at $t=0$.

For certain thicknesses L of the slab a plot of the steady-state temperature versus the microwave power shows an S-shaped curve. This means that (in the neighbourhood of the unstable region) a slight change of the microwave power causes the temperature to increase rapidly. This is the catastrophic phenomenon of thermal runaway. This runaway process is caused by resonance of the microwave within the slab. It is evident that in experimental tests only the upper and lower branch of the S-shaped curve will be found. This can be done by measuring the steady-state temperature for a large number of microwave powers and initial temperatures of the slab. Another much less elaborate way of testing the theory is by measuring the transient temperature of the slab as a function of time by using a time dependent microwave power. The idea was that a plot of the transient temperature versus time (choosing $P=t$) would be equal to the S-shaped curve except for the unstable region, where a jump from the lower to the upper branch was expected. This experiment has been performed [3]. The results were more or less in agreement with the idea mentioned above. The only problem was that the idea did not had a theoretical foundation.

4.3. ASYMPTOTIC APPROXIMATION OF THE ACTUAL TEMPERATURE

From figure 1d it is seen that the system (1.1) does not follow exactly the steady state if P changes with time. In the following we compute the small change in T in the regular state and the shift in the jump time during runaway. The system (2.11) is of the form (1.1):

$$\epsilon \frac{\mathrm{d}T}{\mathrm{d}t} = f(T,P(t)) \tag{3.1}$$

where the small parameter ϵ is a measure for the relaxation time of the system. Our approximation is based on the assumption that ϵ is asymptotically small. In figure 1 it is indicated how the occurrence of runaway can be deduced from the functions $P(t)$ and $f(P,T)$. In the sequel it is assumed that $P(t)$ is chosen such that runaway only occurs in generic situations. It means that $P'(t) \neq 0$ for points P_0 for which $\partial f/\partial T = 0$ including an $\epsilon^{2/3}$-neighbourhood of these points (P lies in an $\epsilon^{2/3}$-neighbourhood of P_0 if $|P-P_0| < L\,\epsilon^{2/3}$ with L an arbitrary large number independent of ϵ).

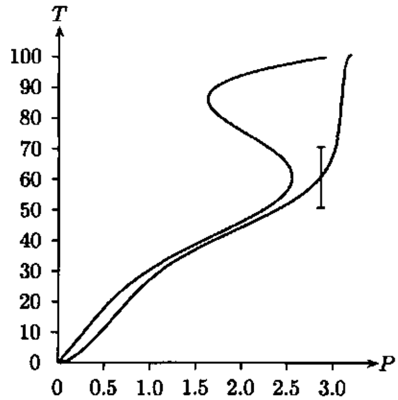

Figure 2. The graph of $f(T,P) = 0$ for $L = 4$ cm and the numerical solution of (3.1) for $P = t$ W/m^2.

For any starting value $(P(0),T(0)) = (P_0,T_0)$ the system rapidly tends (within a time interval of order ϵ) to a stable branch of the curve $f(T,P) = 0$. A branch of this curve is stable if $\partial f/\partial P < 0$. Then the temperature is approximated by

$$T(t) = Q_0(P(t)) + \epsilon\,\frac{Q_0{}'(P(t))P'(t)}{f_T(Q_0(P(t)),P(t))} + \ldots, \tag{3.2}$$

where $Q_0(P)$ satisfies $f(P,Q_0(P)) = 0$. The approximation (3.2) is found by substituting

$$T = Q_0(t) + \epsilon Q_1(t) + \ldots$$

into (3.1) and equating terms with equal powers in ϵ. When approaching the time $t=t_1$ the state (P,T) arrives in a neighbourhood of the value (P_1,T_1) where $\partial f/\partial T=$-0.

Start of runaway

In order to analyze the behaviour of the system near this point (P_1,T_1) we apply two stretching transformations:

$$T = T_1 + y\epsilon^{1/3} \text{ and } P = P_1 + x\epsilon^{2/3} . \tag{3.3ab}$$

The choice of the powers of ϵ follows from the type of degeneration of (3.1) arising after substitution of (3.3ab) and letting $\epsilon \to 0$. For the present choice of the exponent of ϵ in (3.3ab) the limiting system contains a maximum of local information; it reads

$$P'(t_1)\,\frac{dy}{dx} = \frac{1}{2}\,\frac{\partial^2 f}{\partial T^2}\,y^2 + \frac{\partial f}{\partial P}\,x \tag{3.4}$$

if one switched to P as dependent variable, which can be done in view of the condition of genericity. The general solution of (3.4) can be expressed in Airy functions. For that purpose we make the transformations

$$y = \gamma u \quad \text{and} \quad x = \beta v \tag{3.5ab}$$

with

$$\gamma = \frac{1}{P'(t_1)}\,\frac{\partial f}{\partial P}(T_1,P_1)\beta^2 \;, \qquad \beta^3 = \frac{2P'(t_1)^2}{\dfrac{\partial^2 f}{\partial T^2}(T_1,P_1)\,\dfrac{\partial f}{\partial P}(T_1,P_1)} \;, \tag{3.6ab}$$

so that

$$\frac{du}{dv} = u^2 + v \ . \tag{3.7}$$

Then by setting $u = -z'(v)/z(v)$ we obtain

$$z''(v) + vz(v) = 0 \quad \text{or} \quad z(v) = K_1\mathrm{Ai}(-v) + K_2\mathrm{Bi}(-v), \tag{3.8}$$

where $\mathrm{Ai}(\cdot)$ and $\mathrm{Bi}(\cdot)$ denote the two Airy functions, see Abramowitz and Stegun [6]. The integration constants K_1 and K_2 follow from the matching of (3.3)-(3.8) for $x \to -\infty$ with (3.2) for $t \to t_1$. Carrying out some computations one finds that (3.2) then behaves as

$$T = T_1 + [\frac{1}{2}\frac{d^2}{dT^2}Q_0^{-1}(t_1)]^{-1/2}\sqrt{P_1 - P} \tag{3.9}$$

and that (3.8) has the same behaviour if $K_2 = 0$ so that $u(v) = \mathrm{Ai}'(-v)/\mathrm{Ai}(-v)$. For $v \uparrow -\alpha$ we arrive at the first zero of the Airy function then the temperature behaves as

$$T \approx T_1 + \gamma \frac{\epsilon\beta}{P_1 + \alpha\beta\epsilon^{2/3} - P} \quad \text{for} \quad P \uparrow P_1 + \alpha\beta\epsilon^{2/3} \ . \tag{3.10}$$

Thermal runaway

Thus as P approaches $P_1 + \alpha\beta\epsilon^{2/3}$ from bellow T increases rapidly and approaches the value T_2, see figure 2, while P only changes order ϵ near $P_1 + \alpha\beta\epsilon^{2/3}$. Interchanging the dependent and independent variable and introducing

$$p = (P - P_1 - \alpha\beta\epsilon^{2/3})\epsilon^{-1} \tag{3.11}$$

we arrive at the approximating system

$$\frac{1}{P'(t)}\frac{\mathrm{d}p}{\mathrm{d}T} = \frac{1}{f(T,P_1)} \quad \text{or} \quad p(T) = \int_{T^*}^{T} \frac{1}{f(\mathcal{T},P_1)}\,\mathrm{d}\mathcal{T} + K\,, \tag{3.12}$$

where T^* is a fixed value between T_1 and T_2 and K the integration constant. Since

$$f(T,P_1) \approx \frac{1}{2} f_{TT}(T,P_1)(T - T_1)^2 \tag{3.13}$$

for $T \downarrow T_1$, we may easily check that in this limit (3.12) precisely matches (3.10) indepently of the choice of K (K represents the next order in the shift of P coming after $\alpha\beta\epsilon^{2/3}$). At the other hand for $T \uparrow T_2$ the integrand (3.12) behaves as

$$f(T,P_1) \approx f_T(T_2,P_1)(T - T_2),$$

so that

$$p(T) \approx \frac{P'(t_1)}{f_T(T_2,P_1)} \ln(T_2 - T).$$

It means that T approaches T_2 exponentially fast. Then we have arrived at a stable branch where an approximation of the type (3.2) holds.

4.4. NUMERICAL VERSUS ASYMPTOTIC APPROXIMATION

Let us consider a time dependent microwave power $P(t) = t$ in Eq. (2.11). The S-shaped curve representing the steady-state temperature shows that critical behaviour occurs for values of P of the order 10^4 W/m^2, so time is rescaled as follows

$$t = 10^4\,\tau. \tag{4.1}$$

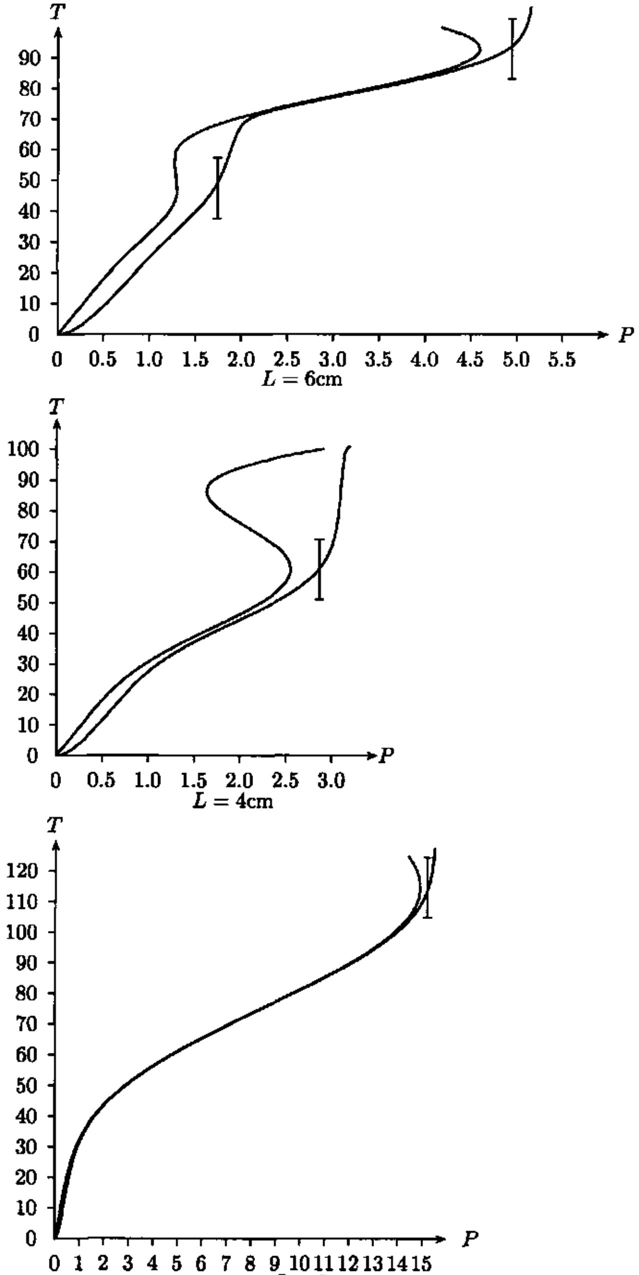

Figure 3. Numerical solutions of (4.2) for different values of L. The asymptotic shift $\alpha\beta\epsilon^{2/3}$ is indicated by a vertical line segment.

In the new time scale Eq. (2.11) changes into

$$\epsilon\frac{dT}{d\tau} = 10^2\tau D_1(T) - 10^{-2}h(T - T_0), \tag{4.2}$$

where

$$\epsilon = 10^{-6}\rho C_p L.$$

From formula (3.10) we know that runaway starts as P approaches the value $P_1 + \alpha\beta\epsilon^{2/3}$ where $-\alpha = -2.338107...$ is the first (negative) zero of the Airy function and β is given by (3.6b). In this formula the term $\partial^2 f/\partial T^2(P_1,T_1)$ is approximated numerically. In table 1 we compare the time shift $\alpha\beta\epsilon^{2/3}$ from the asymptotic solution with the difference between the moment the numerical solution crosses respectively the lines $P = P_1$ and $T = T_1$. Since we take $P = t$, the difference indicates the delay Δ in crossing the line $T = T_1$ with respect to the steady state solution. In figure 3 we give the delay based on the asymptotic expression $\alpha\beta\epsilon^{2/3}$.

Table 1. The difference between the delay in the asymptotic approximation ($\alpha\beta\epsilon^{2/3}$) and in the numerical solution (Δ).

L(cm)	ϵ	$\alpha\beta\epsilon^{2/3}$	Δ
1	$4.19.10^{-2}$	0.10	0.26
2	$8.37.10^{-2}$	0.30	0.34
4	$1.67.10^{-1}$	0.31	0.31
6	$2.51.10^{-1}$	0.43	0.41
		0.35	0.31
10	$4.19.10^{-1}$	0.43	0.50
		0.28	0.32

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Chapter 5

Thermal runaway in microwave heated isothermal slabs, cylinders, and spheres

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Journal of Applied Physics, **83** (1), 438-442, 1 January 1998

(Received 20 June 1997; accepted for publication 2 October 1997)

ABSTRACT

The absorption of electromagnetic energy within a microwave heated isothermal slab, cylinder, and sphere is analyzed and compared to each other. It is shown that the absorbed heat oscillates as a function of temperature, regardless of the geometry of the irradiated object. It is possible to formulate this behavior in a simple mathematical equation, which proves that the oscillation is basically caused by resonance of the electromagnetic waves within the object. This oscillation, combined with the heat loss, is found to be responsible for thermal runaway phenomenon in isothermal objects. Based on such an observation, a general rule to prevent thermal runaway has been developed.

5.1. INTRODUCTION

One of the difficulties associated with the application of microwave heating (frequency range 2450 MHz) is the catastrophic phenomenon of thermal runaway, in which a slight change of microwave power causes the temperature of the irradiated object to increase rapidly. The microwave sintering of ceramics especially is seriously hampered by this phenomenon. The aim of this investigation is to find the physical origin of the runaway process and hope that it leads to a general rule in preventing runaway. The simplest possible system is conceived in order to achieve this. In the first place it has been assumed that the irradiated object is situated in free space without any reflections from the ambient. The object was irradiated by a plane harmonic wave from one side. Deliberately the choice has been not to describe actual experiments, because then the characteristics of the oven combined with the irradiated object might veil the physical origin of the runaway. In the second place it has been assumed that the objects are isothermal. This does not look very realistic, but in the case of small Biot numbers it is a good approximation. In the small Biot number limit many studies have been performed on ceramics. Another category of materials which behave almost isothermal is the liquids. The temperature gradient within the liquid causes strong convection, which diminishes the temperature differences in the liquid. The study of isothermal systems might be regarded as an initial step towards a more complete understanding of the runaway phenomenon. Nonisothermal objects will be taken into account in the second stage of this investigation. The idea is to regard such a system as a multilayer of isothermal layers with different temperatures.

The principles developed are applied to demineralized water. Compared to other materials water has a number of advantages. Its physical parameters are well known[1]; as a liquid it behaves almost isothermally, and the theory can been verified experimentally with little effort. A potential application of the study of water in relation to thermal runaway is the improvement of the quality of microwave dried foodstuffs. The mathematical formulas developed below are generally true and also applicable to ceramics. However one should be very careful in generalizing the results of the calculations. When compared to ceramics the dielectric constant of water hardly depends on temperature. The dielectric loss factor in many ceramics especially is a strong function of temperature, which is a major factor in thermal runaway in these materials. On the other hand there are some remarkable similarities between the thermal behavior of water and ceramics.

The specific aim of this article is to investigate how the geometry of the irradiated object influences the runaway process. This has been done by comparing a slab, a cylinder, and a sphere to each other. In an earlier article[2] based on the work of Kriegsmann[3] and of Stuerga *et al.*[4], it was shown that the runaway effect in a slab of water is caused by resonance of the electromagnetic waves inside the irradiated object. It is a matter of standing waves. It is obvious that the same thing will happen to (infinitely long) cylinders and spheres, irradiated from all sides, with the condition that the waves at the surface are in phase. This is analogous to a slab irradiated from two sides with coherent waves. The only difference is that the "standing waves" are described by cylindrical functions (Bessel and Hankel functions) or spherical functions.

In the case of cylinders and spheres irradiated from one side there is neither cylindrical nor spherical symmetry. It is impossible to apply the basic idea of standing waves, as developed for the slab, to cylindrical or spherical waves, because these waves are not present in an elementary form. It will be demonstrated that also in this case the phenomenon of thermal runaway is still present.

Attention has to be paid to the notation. In contravention of the usual notation[5] the symbol α will be used as the phase constant and β as the attenuation constant. Both are real numbers. This notation reads easier and it fits in better with the theory of wave propagation as described in classic books. The runaway phenomenon, as described in this article, is a result of the application of the wave propagation theory.

5.2. THE ISOTHERMAL SLAB

Let us consider a layer of material specimen, irradiated from one side by microwave radiation with a frequency of 2450 MHz. The wave is a plane, harmonic one and impinges normally upon the material (Fig. 1).

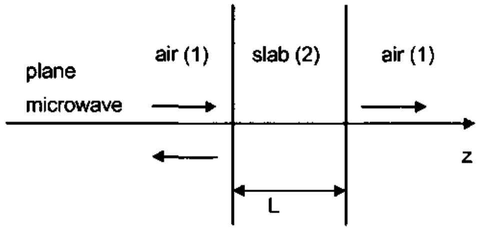

Figure 1. A layer, being irradiated from one side in an echo-free cavity.

The differential equation describing the relationship between temperature T and time t (Fourier's law) reads:

$$L \rho C_p \frac{dT}{dt} = D - 2h (T - T_0) \tag{1}$$

where ρ is the density, C_p is the thermal capacity, and L stands for the layer thickness. D is the heat production (total amount of power per square meter generated along the z-axis, extending from $z=0$ to $z=L$). The total heat loss is described by an effective heat transfer coefficient h, multiplied by the temperature difference $(T-T_0)$ between slab and ambient. The expression for the heat loss is an approximation, only valid in the case of small temperature differences. The absorbed energy D follows from Maxwell's equations, together with the appropriate boundary conditions at the surface of the slab. Approximately one obtains:

$$D \approx P \frac{4\alpha_1}{\alpha_2} [1 + 2|R_{12}|^2 e^{-2\beta_2 L} \cos(2\alpha_2 L)] \tag{2}$$

where P is the microwave power and R_{12} the reflection coefficient. This approximation is based on two requirements; first, the phase constant α_1 of air is much smaller than the phase constant α_2 of the irradiated medium. This is a very general demand and most solids and liquids answer this demand; second, α_2 should be much smaller than the attenuation factor β_2 of the medium. This is true for water.

The special thing about water as compared to ceramics is the fact that α_2 hardly depends on temperature. This means that in the first approximation the transient temperature as a function of heating time will be represented as a straight line. Superimposed on this line is an oscillation of which the amplitude increases with temperature, but decreases with layer thickness (Fig. 2).

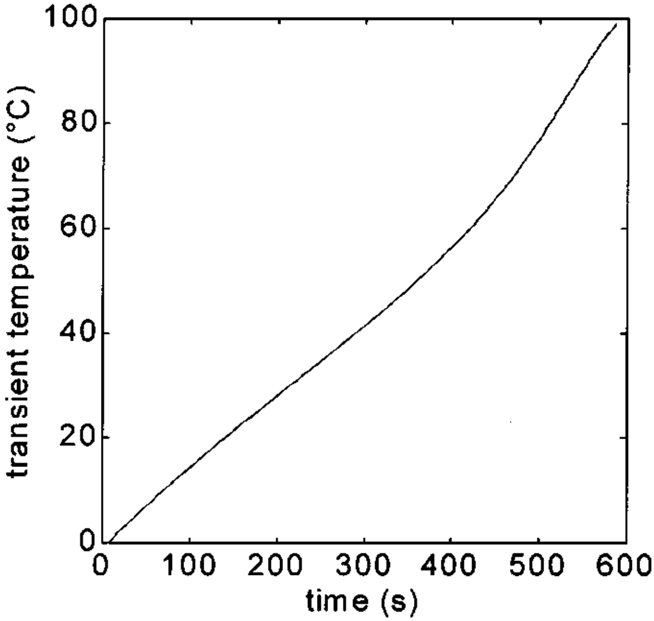

Figure 2. Transient temperature of a slab without heat loss as a function of heating time, showing that the absorbed microwave power is almost temperature independent. L=1.6 cm, P=40 kW/m^2, water.

The oscillation is caused by resonance within the medium. With respect to wave propagation the slab behaves like a violin string. The heat production has a maximum if the wavelength λ_2 of the medium equals $2L/n$ for n = 1,2,3... The steady-state temperature follows from

$$D - 2h\,(T - T_0) = 0 \tag{3}$$

In order to have runaway three intersecting points of heat production and heat loss are necessary. This will be the case if the cosine has a maximum around 80 °C, which results in a thickness L of 1.6 cm for $n = 2$. This is the smallest thickness for which runaway might occur (Fig. 3).

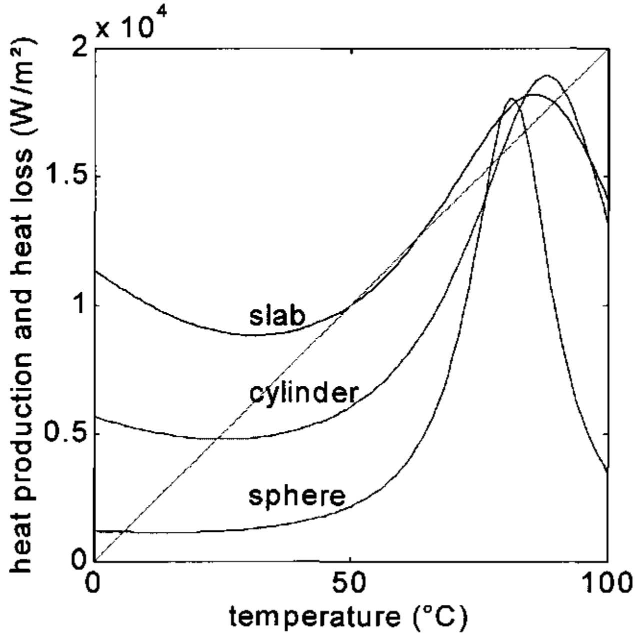

Figure 3. The absorbed microwave power oscillates as a function of temperature. This oscillation, combined with the heat loss (the straight line), is responsible for thermal runaway. Slab L=1.6 cm, P=40 kW/m^2; cylinder L=1.2 cm, P=9 kW/m^2; sphere L=1.4 cm, P=150 kW/m^2; h=100 W/m^2, ambient temperature is 0 °C; the medium is water.

A plot of the steady state temperature versus the microwave power for $L = 1.6$ cm shows the familiar S-shape (Fig. 4).

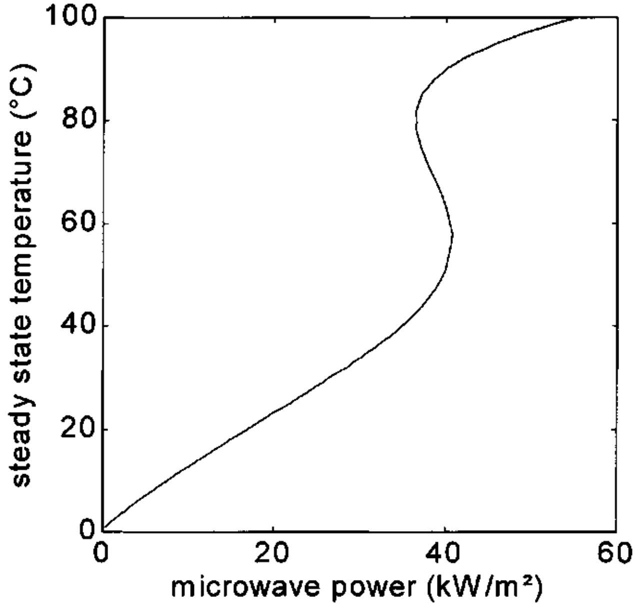

Figure 4. The S-shaped curve of a slab (water, L=1.6 cm). Increasing the microwave power from 40 to 45 kW/m^2 results in a temperature jump of about 40 °C.

5.3. THE ISOTHERMAL CYLINDER

A cylinder is irradiated by a plane harmonic wave from one side in the same way as the slab. The electric field vector of the incident field is parallel to the central symmetry axis of the cylinder (Fig. 5).

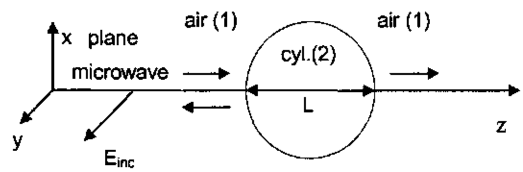

Figure 5. A cylinder, being irradiated from one side, in an echo-free cavity.

The equation of the heat balance differs a little from Eq.(1), because the surface is larger.

$$\frac{1}{2} L \rho C_p \frac{dT}{dt} = \frac{2D}{\pi L} - 2h\,(T-T_0) \tag{4}$$

In this case L is the diameter of the cylinder.

The heat production D is directly proportional to the absolute square of the electric field E_2 within the cylinder. In polar coordinates r and ϕ the field is[6]:

$$E_2 = A \sum_{n=-\infty}^{n=+\infty} c_n\, J_n(k_2 r)\, e^{in\phi} \tag{5}$$

where A is the amplitude of the incident electric field, J_n is the Bessel function of the first kind of order n, and k_2 is the complex wavenumber ($k_2 = \alpha_2 + i\beta_2$). The coefficients c_n determine the character of the field. They can be found by applying the boundary conditions at the surface of the cylinder.

$$c_n = \frac{k_1[J_n'(k_1R)H_n(k_1R) - J_n(k_1R)H_n'(k_1R)](i)^n}{k_2J_n'(k_2R)H_n(k_1R) - k_1J_n(k_2R)H_n'(k_1R)} \tag{6}$$

here H_n are Hankel functions of first kind of order n, R is the radius of the cylinder, and $k_1 = \alpha_1$. To achieve the total absorbed energy D of the isothermal cylinder the absolute square of the electric field has to be integrated over the cross section of the cylinder. This can be done analytically, but the result is a very complicated meaningless equation. A much better result is achieved by using the asymptotic expansion of the Bessel function.

$$J_n(x) \approx \sqrt{\frac{2}{\pi x}} \cos\left(x - \frac{n\pi}{2} - \frac{\pi}{4}\right) \tag{7}$$

where x is large.

Replacing Bessel functions by cosines and integrating over the cross section yields

$$D \sim [1 - 2|R_{12}|^2 e^{-2\beta_2 L} \cos(2\alpha_2 L)] \tag{8}$$

Except for the sign this equation is identical to Eq.(2) of the slab. A maximum heat production is expected for $L = \lambda_2(1+2n)/4$, where $n = 0,1,2,\ldots$. This means that the diameter L should equal 1.2 cm in order to have a maximum at 80 °C (Fig. 3) for $n=1$ (water). The consequence is thermal runaway (Fig. 6).

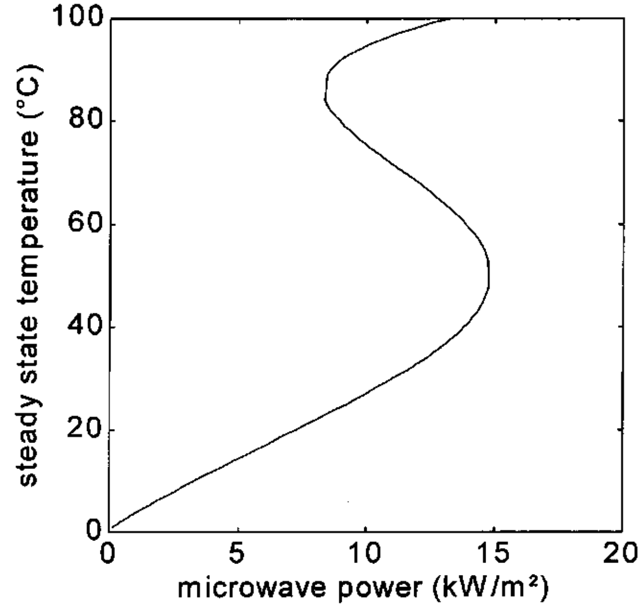

Figure 6. The S-shaped curve of a cylinder (water, L=1.2 cm). Increasing the microwave power from 14 to 15 kW/m^2 results in a temperature jump of about 50 °C.

The same kind of S shape has been calculated for ceramic cylinders in the small Biot number limit[7]. Numerical calculations demonstrate that, in the first approximation, the heat production is temperature independent. The runaway process for water is caused solely by the temperature dependence of the phase constant.

The phenomenon of thermal runaway has also been studied in the case of an isothermal cylinder irradiated from one side by a plane harmonic wave with the electric field vector lying in the plane of incidence (perpendicular to the vector drawn in figure 5). This leads to the same behavior as described above. Especially equation (8), the most important one, was exactly the same.

5.4. THE ISOTHERMAL SPHERE

A sphere is irradiated from one side by a plane harmonic wave in free space. In this case Fourier's law reads

$$\frac{1}{3} L \, \rho \, C_p \, \frac{dT}{dt} = \frac{2PD}{\pi L^2} - 2h \, (T - T_0) \tag{9}$$

where L is the diameter of the sphere. The mathematical description of the electromagnetic field is based on the so-called Mie theory[8]. In terms of spherical coordinates r, ϕ and θ the spherical components of the electric field within the sphere are

$$E_r = A \sum_{n=1}^{\infty} i^{n-1} \frac{2n+1}{k_2 r} \sqrt{\frac{\pi}{2k_2 r}} \, b_n \, J_{n+\frac{1}{2}}(k_2 r) \, \frac{dP_n(\cos\theta)}{d(\cos\theta)} \cos\phi \, \sin\theta \tag{10}$$

$$E_\theta = A \sum_{n=1}^{\infty} i^{n} \frac{2n+1}{n(n+1)} \sqrt{\frac{\pi}{2k_2 r}} \, [a_n \, J_{n+\frac{1}{2}}(k_2 r) \, \frac{dP_n(\cos\theta)}{d(\cos\theta)}$$

$$- \, ib_n \, \{ \frac{n+1}{k_2 r} J_{n+\frac{1}{2}}(k_2 r) - J_{n+\frac{3}{2}}(k_2 r) \} \, \frac{\partial P_n^1(\cos\theta)}{\partial\theta} \,] \cos\phi \tag{11}$$

$$E_\phi = A \sum_{n=1}^{\infty} i^{n} \frac{2n+1}{n(n+1)} \sqrt{\frac{\pi}{2k_2 r}} \, [-a_n \, J_{n+\frac{1}{2}}(k_2 r) \, \frac{\partial P_n^1(\cos\theta)}{\partial\theta}$$

$$+ ib_n \left\{ \frac{n+1}{k_2 r} J_{n+\frac{1}{2}}(k_2 r) - J_{n+\frac{3}{2}}(k_2 r) \right\} \frac{dP_n(\cos\theta)}{d(\cos\theta)} \Big] \sin\phi \tag{12}$$

where P_n are the Legendre polynomials and P^1 correspond to the associated Legendre functions of the first kind. The coefficients a_n and b_n can be found by applying the electromagnetic boundary conditions at the surface of the sphere. For a_n, respectively b_n, one yields

$$a_n = \frac{2i\sqrt{k_1 k_2}/\pi R}{\frac{(k_2 - k_1)(n+1)}{R} J_{n+\frac{1}{2}} H_{n+\frac{1}{2}} - k_2^2 J_{n+\frac{3}{2}} H_{n+\frac{1}{2}} + k_1^2 J_{n+\frac{1}{2}} H_{n+\frac{3}{2}}} \tag{13}$$

$$b_n = \frac{2i\sqrt{k_1 k_2}/\pi R}{\frac{(k_2 - k_1)(1-n)}{R} J_{n+\frac{1}{2}} H_{n+\frac{1}{2}} - k_1 k_2 J_{n+\frac{3}{2}} H_{n+\frac{1}{2}} + k_1 k_2 J_{n+\frac{1}{2}} H_{n+\frac{3}{2}}} \tag{14}$$

The Bessel functions have the argument $k_2 R$, while the Hankel functions always depend on $k_1 R$. Here R is the radius of the sphere. The total amount of absorbed heat D of the isothermal sphere is proportional to the integral over the volume of the sphere of the absolute square of the total electric field. The last one equals the sum of the absolute squares of the spherical field components. This (very elaborate) integration is analytically possible, but it also results in a meaningless equation. The problem is that the equation does not converge in such a way that the first order terms of the Bessel and Hankel functions are the most dominant ones. The most dominant term is determined by the diameter of the sphere. However, numerical analysis (in the case of water) shows that a simplification of the exact solution is possible. For relatively small L ($L < 3$ cm) the oscillating behavior of D is mainly described by $|a_1|^2$. If the diameter lies somewhere between 3 and 6 cm then Eq.(8) creates a perfect fit. The minimum diameter for which runaway occurs is 1.4 cm (Fig. 3). The high and sharp peak causes a large jump of the steady-state temperature

(Fig. 7). On the other hand this first order ($n = 1$) runaway effect is very sensitive to small changes of the diameter. Increasing or decreasing the diameter of 1.4 cm by 0.1 cm is enough to destroy the runaway phenomenon. Numerical analysis proves that also in the case of the sphere the absorbed heat D is temperature independent in the first approximation, resulting in the same kind of graph as in figure 2. Studies of a ceramic sphere inside a rectangular cavity[9] also show how the absorbed microwave power oscillates as a function of the radius.

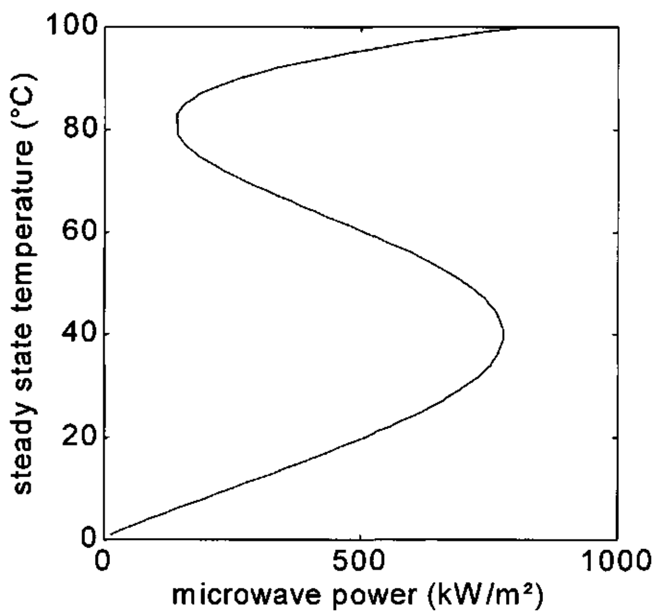

Figure 7. The S-shaped curve of a sphere (water, L=1.4 cm). Increasing the microwave power from 78 to 80 kW/m^2 results in a temperature jump of 60 °C.

5.5. DISCUSSION AND CONCLUSION

The isothermal slab, cylinder, and sphere, irradiated from one side by a plane harmonic microwave, behave in the same way in relation to thermal runaway. In the first approximation the dissipated microwave power is independent of temperature. The phenomenon of thermal runaway is basically caused by the temperature dependence of the phase constant of the irradiated medium in combination with heat loss. For watery objects the absorbed energy D is reflected by a simple approximation:

$$D \sim 1 \pm 2\,|R_{12}|^2\, e^{-2\beta_2 L} \cos(2\alpha_2 L) \tag{15}$$

where the + sign refers to the slab. L is the thickness of the slab or the diameter of the cylinder or sphere. In all three cases the oscillation is caused by resonance within the objects. This suggests that the geometry of an irradiated isothermal object is irrelevant in relation to thermal runaway. If there is a possibility for resonance of the electromagnetic waves within the object, thermal runaway will occur.

The calculations demonstrate that the characteristic dimension must be at least equal to a complete wavelength (slab) or to 3/4 of a wavelength (cylinder and sphere) at a relatively high temperature, in order to have runaway. At that specific temperature the heat production has a maximum. This maximum has to precede a minimum at low temperatures. This might already be the fact if a quarter of a period of oscillation is present.

Thus the conclusion is that any isothermal object with characteristic dimension L, irradiated from one side by microwaves, will never be overtreated or damaged by the phenomenon of thermal runaway if L is smaller than $\pi/4\Delta\alpha_2$, where $\Delta\alpha_2$ is the difference between the maximum and the minimum value of the phase constant α_2. For water this results in a dimension of about 1 cm.

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Chapter 6

Thermal profiles and thermal runaway in microwave heated slabs

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Journal of Applied Physics, **85** (7), 3774-3779, 1 April 1999

(Received 1 December 1998; accepted for publication 29 December 1998)

ABSTRACT

The microwave heating of slabs of water bound with a gel is modeled and analyzed without any restriction to the Biot number regime. Despite the fact that the temperature distribution over the slab is not uniform at all, the phenomenon of thermal runaway is basically caused by resonance of the electromagnetic waves within the object, combined with heat loss. A plot of the steady-state temperature at any position within the slab, versus the microwave power, is an S-shaped or a multi S-shaped curve. With respect to thermal runaway there is a strong similarity between isothermal and nonisothermal slabs. Using the average temperature of the nonisothermal slab, regardless of its Biot number, yields a reasonable approximation to describe the runaway. This is caused by the specific characteristic of the dielectric loss factor of water, which decreases with increasing temperature. This results in an almost constant absorption of energy over the whole slab without disturbing the wave character of the absorption. It turned out that this smoothing of the absorbed power plays a dominant role in the calculations of the temperature profiles. Any calculation where the temperature dependence of the permittivity is omitted, will not only pass the phenomenon of thermal runaway but its temperature profiles will differ substantially from the ones where the temperature dependence has been taken into account.

6.1. INTRODUCTION

The application of microwave heating (frequency range 2450 MHz) is seriously hampered by two problems, both having their roots in the basic physics of the heating process. The first difficulty is the uneven spatial absorption of energy within the irradiated object. The second difficulty is the catastrophic phenomenon of thermal runaway in which a slight change of microwave power causes the temperature of the object to increase rapidly. The aim of this investigation is to find the physical origin of the runaway process and hope that it leads to a general rule in preventing runaway. Temperature profiles of the irradiated slab are necessary in order to achieve this. These temperature profiles give a good insight in the uneven spatial absorption of energy. Both problems, the uneven spatial absorption and the thermal runaway, are related to each order. The process of thermal runaway is a non-linear problem and can be explained by taking the temperature dependence of the permittivity ε into account. This means that the temperature profiles, as shown in this article, will also reflect this temperature dependence. Many studies of microwave heated slabs of foodstuffs (see f.i. Dolande *et al*[1].) have been performed, but the majority ignores the temperature dependence of the permittivity arguing that this is a second order phenomenon. As will be shown in this study, the process of thermal runaway is caused by resonance within the irradiated medium due to the temperature dependence of ε. Resonance is always a very strong phenomenon and it must not be regarded as a second order effect. It has a major impact on the absorption of energy, and by this, as will be shown on the temperature profiles. The study of thermal runaway in microwave heated objects usually starts by formulating the equation of the absorbed power D.

$$D = \frac{1}{2}\omega\varepsilon''|E|^2 \tag{1}$$

where ω is the angular frequency, ε'' is the imaginary part of the permittivity, and E is the electric field within the object. Instead of ε'' one can also formulate Eq.(1) with the dielectric loss factor, which equals ε'' divided by ε_0, the permittivity of vacuum. Equation (1) is a very powerful equation because it is a general equation and does not depend on the geometry of the irradiated object. All problems involved in the geometry of the object are expressed by the field E. The absorbed power is always directly proportional to ε'' or the dielectric loss factor. The microwave sintering of ceramics especially is seriously hampered by the phenomenon of thermal runaway. The dielectric loss factor of ceramics strongly increases with increasing temperature. Looking at Eq.(1) the explanation of thermal runaway in ceramics is almost evident. While the temperature increases, the absorbed power will increase, resulting in a stronger increase of the temperature, which then results in more absorbed power, etc. This "hand-waving" argument can often be heard. Kriegsmann[2] formulated a plausible explanation of thermal runaway in ceramics in the small Biot number regime. Strangely enough the hand-waving argument does not play a major role in his explanation. Kriegsmann did not mention it but, in fact, he describes the phenomenon of resonance in an almost isothermal slab of ceramics. That the phenomenon of resonance within an isothermal slab results in thermal runaway became clear by the study of demineralized water (Stuerga *et al.*[3]). The dielectric loss factor of water hardly depends on temperature as compared to ceramics. It decreases a little bit with increasing temperature. It is obvious that in the case of water the hand-waving argument is not appropriate in explaining runaway. All of this does not mean that the argument is complete nonsense. Probably the hand-waving argument, combined with resonance, plays an important role in the description of runaway of ceramics in the large Biot regime. Resonance as the origin of thermal runaway in isothermal objects has been studied in earlier articles[4,5]. The aim of this study is to investigate the phenomenon of thermal runaway in nonisothermal slabs of foodstuffs without any limitations with respect to the Biot number.

Water is a major constituent of many foodstuffs and its Biot number can be quite large ($Bi \approx 14$). This is why water (bound with a gel) has been used to illustrate the theory. Because the dielectric loss factor of ceramics strongly increases with increasing temperature, which is completely opposite to water, the results of this investigation in the large Biot number regime cannot be used to explain the behavior of ceramics. From time to time some remarks will be made about the impact of certain parts of the theory in relation to ceramics. An analysis of this study probably makes it clear why it is suggested that the hand-waving argument in combination with resonance plays an important role in explaining thermal runaway in ceramics in the case of large Biot numbers.

Attention has to be paid to the notation. In contravention of the usual notation[6] the symbol α will be used as the phase constant and β as the attenuation constant. Both are real numbers. This notation is easier to read and it fits in better with the theory of wave propagation as described in classic books. The runaway phenomenon, as described in this article, is a result of the application of the wave propagation theory.

6.2. THEORY

Consider a layer of material specimen, irradiated from one side by microwave radiation with a frequency of 2450 MHz. The wave is a plane, harmonic one and impinges normally upon the material. In order to explain the principles of thermal runaway, the simplest possible system was conceived. The slab was located in free space, so no other waves than the incident one would be involved. Solving Maxwell's equations (see f.i. Stratton[7] or Ayappa *et al.*[8]) yields the classical wave equation in one dimension

$$\frac{d^2 E}{d x^2} + k^2(T)E = 0 \tag{2}$$

where the electric field E is a function of position x and temperature T at that position. The $k(T)$ stands for the temperature dependent complex wavenumber, which is connected to the permittivity ε of the medium by the following equation:

$$k^2 = \omega^2\mu_0(\varepsilon + \frac{i}{\omega}\frac{\partial\varepsilon}{\partial t} + \frac{i\sigma}{\omega}) \tag{3}$$

This equation takes into account that the permeability of the irradiated object almost equals μ_0, being the permeability in vacuum. Many materials treated by microwaves fulfill this requirement. The symbol σ is the ohmic electric conductivity, caused by the free charges of the object (free electrons, ions, etc.). The presence of free charges is not without consequences. The electric field causes surface charges and surface currents, which have their impact on the electromagnetic boundary conditions. The analytical description of a mixture of an isolator (dielectric) and conductor is nearly impossible without any numerical assumption of the ohmic conductivity. For this reason the ohmic conductivity is neglected, as is usually done in this kind of research, at least with respect to the electromagnetic boundary conditions. Therefore, this article deals with pure dielectrics. With this assumption the electromagnetic boundary conditions read

$$E_1 = E_2 \; ; \quad \frac{\mathrm{d}E_1}{\mathrm{d}x} = \frac{\mathrm{d}E_2}{\mathrm{d}x} \tag{4}$$

at the boundaries $x=0$ and $x=L$ (L is the thickness of the slab). The subscript 1 refers to vacuum or air and the subscript 2 is related to the irradiated medium.

The second term of Eq.(3) is very interesting because it can be written as the product of $\mathrm{d}\varepsilon/\mathrm{d}T$ times $\mathrm{d}T/\mathrm{d}t$. In the case of thermal runaway the temperature increases rapidly and this could result in a significant contribution of the second term during the temperature jump. On the other hand, this term is small compared to the first term ε of Eq.(3), because it is divided by ω ($2\pi \times 2450 \times 10^6$ s^{-1}). Including the second term in Eq.(1) of the absorbed power D results in

$$D = \frac{1}{2}\omega\left(\varepsilon'' + \frac{1}{\omega}\frac{\mathrm{d}\varepsilon'}{\mathrm{d}T}\frac{\mathrm{d}T}{\mathrm{d}t}\right)|E|^2 \tag{5}$$

where ε is written as the difference of a real and an imaginary part, according to $\varepsilon=\varepsilon'-i\varepsilon''$. For foodstuffs and also for ceramics, the temperature dependence of ε' is so small, that Eq.(5) might be replaced by the familiar Eq.(1) without any loss of generality. The second term has no consequences for the absorbed power. The interesting impact of the second term is found in the expression of the real wavenumber, or phase constant α. Omitting the small terms it yields

$$\alpha = \omega\sqrt{\frac{\mu_0}{2}\left(\varepsilon'' + \varepsilon' - \frac{1}{\omega}\frac{\mathrm{d}\varepsilon''}{\mathrm{d}T}\frac{\mathrm{d}T}{\mathrm{d}t}\right)} \tag{6}$$

For water ε'' decreases with increasing temperature, resulting in a decreasing wavenumber or increasing wavelength. At certain temperature the wavelength "fits" within the slab, causing resonance. Exactly at that moment the temperature will rise strongly and the time dependent term in Eq.(6) will become significant in an interesting way. It will resist the decrease of α, keeping the system in the resonant mode. This effect has been investigated with the aid of computer simulations applied on water. The conclusion is that compared to ε'', the time dependent term is just too small to have any influence. Perhaps in the case of ceramics, where the temperature dependence of the permittivity is much larger, there will be some effect.

To calculate the temperature within the slab as a function of position and time the three electromagnetic equations (1), (2) and (4) have to be combined with Fourier's law.

$$\rho C_p \frac{\partial T}{\partial t} = K \frac{\partial^2 T}{\partial x^2} + D \tag{7}$$

where ρ is the density, C_p is the thermal capacity, and K is the thermal conductivity. Although these parameters depend upon the temperature T, they are assumed constant for the following reason. The phenomenon of runaway is caused by resonance due to the temperature dependence of the wavelength within the slab, as will be shown later in this article. The temperature dependence of the three parameters mentioned above has no influence on the appearance of the resonance. The only consequence which can be found, is a small shift of the average temperature for which resonance occurs. The heat balance in Eq.(7) has to be accomplished by its boundary conditions. For small temperature differences between the surface of the slab and the ambient read:

$$K \frac{\partial T}{\partial x} = h(T - T_a) \qquad , x = 0 \tag{8}$$

$$K \frac{\partial T}{\partial x} = -h(T - T_a) \qquad , x = L \tag{9}$$

where h is some effective heat transfer coefficient related to the convective and radiative heat loss. T_a is the ambient temperature and also the initial temperature of the slab at $t=0$.

6.3. STEADY STATE SOLUTIONS

The heating process evolves as follows: while the temperature T increases, the electrical properties k and ε'' of the slab will change. This, in turn influences the absorbed power D, resulting in a new temperature. A complete analytical solution of this nonlinear problem is nearly impossible. Only a partially analytical solution can be formulated. In order to formulate this solution the slab has to be divided into a large number of small subslabs. The heating process starts with heating of an isothermal slab at $t=0$. The analytical description and the solutions of this process is well known. Thus at $t=\Delta t$ the temperature at each position x within the slab is known, where Δt is small. Assuming each subslab n is isothermal it is possible to calculate the electric field E_n in each subslab n with these data.

$$E_n(x,T) = A_n e^{ik_n(T)x} + B_n e^{-ik_n(T)x} \tag{10}$$

This is the solution of Eq. (2). The integration constants A_n and B_n depend on the temperature of subslab n and the temperature of its neighbours, the subslabs $n+1$ and $n-1$. With the aid of the electromagnetic boundary conditions it is possible to write (A_n,B_n) as a function of (A_{n+1},B_{n+1}) or as a function of (A_{n-1},B_{n-1}). This process of replacing the integration constants by its neighbours goes on until the surfaces of the slab has been reached. At $x=0$ and $x=L$ the boundary conditions are numerically known. This means that the value of A_n and B_n for each subslab n is known. Substituting field (10) in the expression of the absorbed power, generates new temperatures for each sub-slab at $t=2\Delta t$, and so on. This is how the computer program has been set up.

This computer program has been applied to demineralized water, bounded with a gel. See Kaatze[9] for the dielectric properties. Other data which has been used are: $C_p=4186$ J/kgK, $K=0.7$ W/mK, $\rho=1000$ kg/m^3, and $h=100$ W/Km^2. The thickness of the slab must be chosen. It is the only degree of freedom. In the case of an isothermal slab the thickness L plays a very important role. There is a kind of standing wave within the slab if $L=n\ \pi/\alpha$ ($n=1,2,3...$). However, not every standing wave causes thermal runaway. Thermal runaway is possible, but only if the temperature at which the standing wave occurs, has been preceded by a lower temperature at which the waves more or less cancel each other. These conditions, combined with an approximated formula for α as a function ($\alpha=470-0.9T$; T in °C) of temperature results in thicknesses where the phenomenon of thermal runaway should be possible. The minimum thickness for runaway in an isothermal slab is

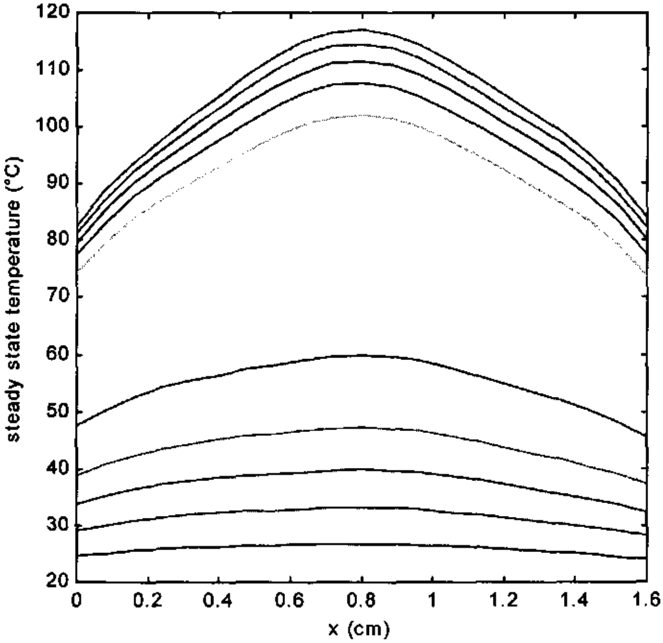

Figure 1. Steady-State temperature profiles in a 1.6 cm slab of water. The microwave power increases from 4 to 40 kW/m^2 in steps of 4 kW/m^2. Notice the large gap, illustrating the phenomenon of thermal runaway.

about 1.6 cm ($n=2$). In the case of $L=4$ cm, the effect of runaway is very pronounced ($n=5$). The behavior of a thick slab, $L=10$ cm , where two temperature jumps ($n=13,14$) are expected, is also interesting.

For these reasons nonisothermal slabs with a thickness of 1.6, 4.0, and 10.0 cm have been investigated. The steady-state temperature profiles as a function of the microwave power have been plotted in Fig.1, 2 and 3.

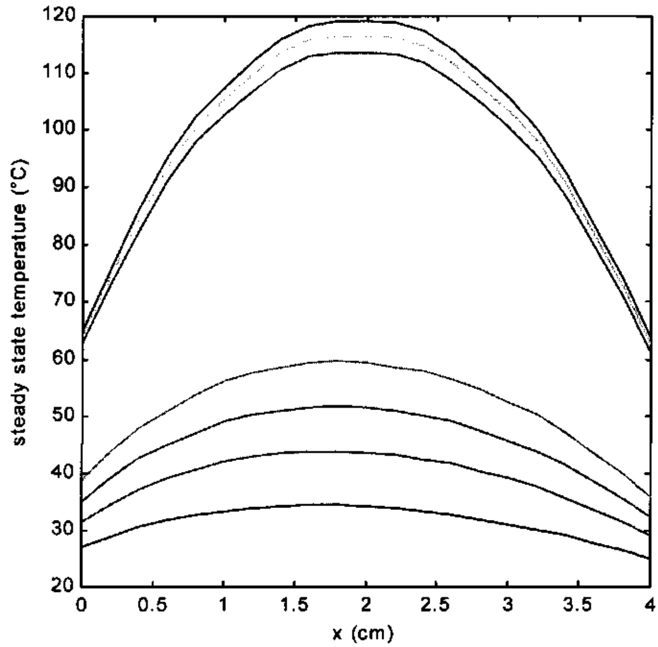

Figure 2. Steady-state temperature profiles in a 4 cm slab of water. The microwave power increases from 4 to 28 kW/m^2 in steps of 4 kW/m^2. Notice the large gap, illustrating the phenomenon of thermal runaway.

The jumps in temperature can be seen clearly and the conclusion is obvious. The behavior of the nonisothermal slab with regard to thermal runaway is the same as the behavior of the isothermal slab. The Biot number does not play a role. Even in the case of the relatively thick slab of 10 cm ($Bi = hL/K = 14.3$) the origin of thermal runaway is resonance once again. A plot of the steady-state temperature at every position x within the slab versus the microwave power will show an S-shaped curve or a multi S-shaped curve. The behavior of the complete slab can be described with the average temperature. To have a standing wave the only thing which counts is the way the wave fits within the slab. It makes no difference if the wavelength is large in the middle and small at the sides of the slab due to the temperature differences within the slab.

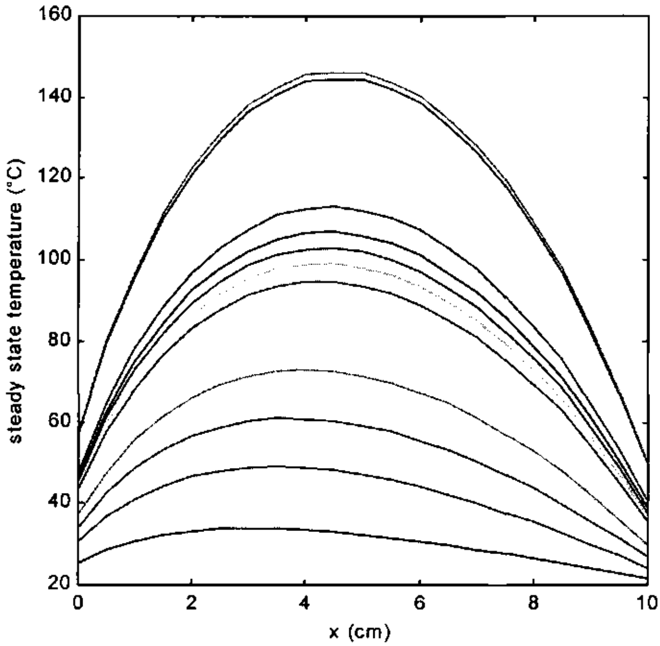

Figure 3. Steady-state temperature profiles in a 10 cm slab of water. The microwave power increases from 2 to 22 kW/m^2 in steps of 2 kW/m^2. Notice the two gaps, illustrating the phenomenon of thermal runaway.

If the wave fits then, there is resonance. This is one of the reasons why the slab, with respect to runaway, can be described with the average temperature. Replacing the local temperature T by the average temperature $\bar{T}$ in Eq.(7), and integrating over x yields

$$\rho C_p L \frac{d\bar{T}}{dt} = K\left(\frac{d\bar{T}}{dx}\right)_L - K\left(\frac{d\bar{T}}{dx}\right)_0 + D_{tot} \tag{11}$$

where D_{tot} is the total amount of absorbed power within the slab. Substitution of the thermal boundary conditions (8) and (9) results in

$$\rho C_p L \frac{\mathrm{d}\bar{T}}{\mathrm{d}t} = -2h(\bar{T} - T_a) + D_{tot} \tag{12}$$

This is the heat balance of an isothermal slab in free space.

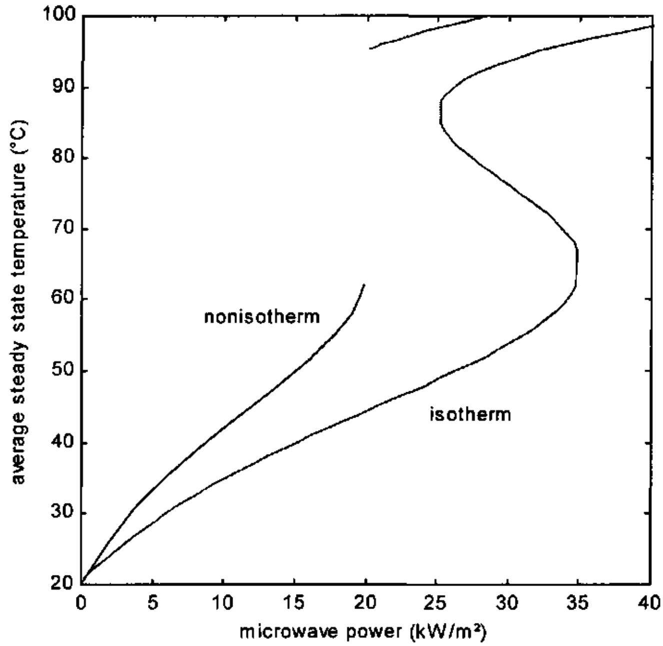

Figure 4. Steady-state response curves for an isothermal and nonisothermal slab of water (L= 4 cm). The graph of the nonisothermal slab is in fact an S-shaped curve.

Fig.4 shows the S-shaped curve of the approximated solution of Eq.(12), compared to the computational analysis of the correct Eq.(7) for L=4 cm. The correct curve shows the upper and lower branch of an S-shaped curve. If one starts the heating with the initial condition that the temperature of the slab equals the ambient temperature Eq.(7) has no physical or mathematical solution in the instable region. It does demonstrate the dramatic jump in temperature. The complete correct S-shaped curve will be found if one equals Eq.(7) to zero. The smaller the Biot number, the better the approximation, as has been proven by Kriegsmann, but the main conclusion, that the origin of thermal runaway in a nonisothermal slab (regardless of the value of the Biot number), is caused by resonance, still remains.

6.4. TIME DEPENDENT SOLUTIONS

That the phenomenon of thermal runaway can be described so successfully by regarding it as an isothermal slab with one temperature $\bar{T}$, while the real slab is not

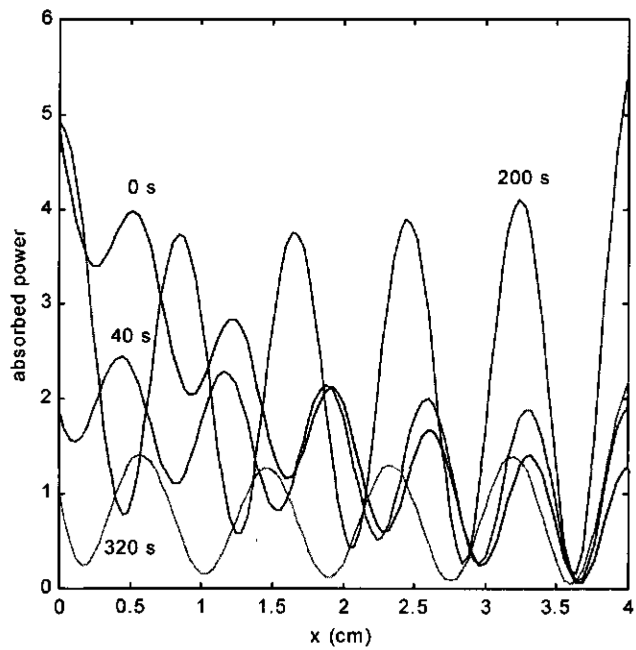

Figure 5. Evolution in time of the absorbed power (MW/m^2) in a 4 cm slab of water. The exponentially decreasing function (t=0 s) evolves to a constant oscillation (t=40 s and t=320 s). The ampitude of this oscillation increases strongly at t=200 s, because of the resonance. The microwave power is 200 kW/m^2.

isothermal at all, needs a more thorough explanation. The reason can be found in the specific character of water. The dielectric loss factor of water decreases by increasing temperature, while the penetration depth increases. These two factors are responsible for the fact that the absorbed power becomes constant in the average very soon. Only the small temperature dependence of the wavelength remains and this can change the absorbed power significantly because of the resonance. These phenomena have been illustrated in Fig.5. In the case of ceramics the dielectric loss factor increases and the penetration depth decreases by increasing temperature. The behavior of ceramics will be opposite to that of water. While water tends to smooth the electromagnetic energy over the whole slab, ceramics will concentrate the energy on the hot spot at the start of the heating process. Here it becomes clear why the hand-waving argument mentioned in the introduction, cannot be ignored and replaced by the argument of resonance only.

The steady-state solutions of the former chapter are interesting because they give a good insight in the origin of thermal runaway. On the other hand, in the industrial processing of foodstuffs, one does not usually wait until the steady state has been reached. It just takes too long; if one heats a 4 cm slab of water with a microwave power of 18 kW/m^2 under the circumstances as described in this article, it will take about 2.2 h before the steady state ($\bar{T}$ = 55 °C) has been reached. Using 200 kW/m^2 in stead of 18 kW/m^2 will result in an average temperature of 55 °C within 2 min. Fig.6 shows the temperature profiles as a function of the heating time. As could be expected, resonance is still present. At a certain moment the temperature rises strongly in a very short time. Suppose this jump in temperature is not wanted because it will overtreat the foodstuff, then the only way to prevent this kind of thermal runaway is by stopping the microwave heating at the right moment. Fig.7 shows the temperature profiles of water neglecting the temperature dependence of the dielectric constant, assuming that the value of the dielectric constant at 20 °C will result in a rather correct plot of the profiles. As can be seen, this is not the case. Not only the jump in temperature is missing, but the smoothing of the absorbed power by increasing temperature has also not been described. This demonstrates that the method of neglecting the temperature dependence of the dielectric constant yields a

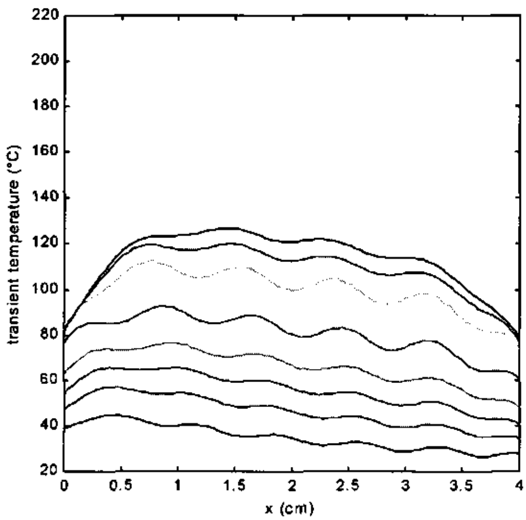

Figure 6. Transient temperature profiles in a 4 cm slab of water. The time increases from 0 to 320 s in steps of 40 s. Notice the gap between t=200 and t=240 s. The microwave power is 200 kW/m^2.

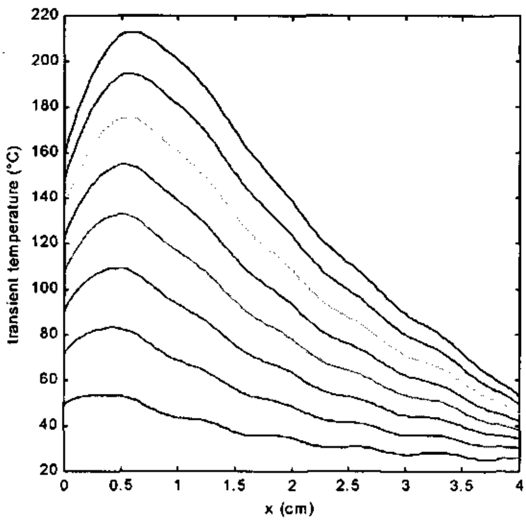

Figure 7. Same as Fig.6, except the temperature dependence of the permittivity has been omitted.

very poor approximation. The investigation of the uneven spatial absorption of the electromagnetic energy within the irradiated object (one of the main problems in the application of microwave heating) by omitting the temperature dependence of the permittivity has little value, at least with respect to watery objects.

6.5. CONCLUSIONS

The behavior of the nonisothermal slab in relation to thermal runaway is analogous to the behavior of the isothermal slab. The Biot number does not play a role. Even in the case of relatively large Biot numbers, the physical origin of thermal runaway is still the phenomenon of resonance. A plot of the steady-state temperature at any position within the slab, versus the microwave power, will be an S-shaped or a multi S-shaped curve.

It is possible to approximately describe the complete nonisothermal slab regarding thermal runaway by its average temperature similar to the description of the isothermal slab. This means that a nonisothermal slab with thickness L, irradiated from one side by microwaves, will never be overtreated or damaged by the phenomenon of thermal runaway if L is smaller than $\pi/4\Delta\alpha$, where $\Delta\alpha$ is the difference between the maximum and the minimum value of the phase constant α within the temperature interval of the heating process. For water this results in a dimension L of about 1 cm. The main reason for the similarity between isothermal and nonisothermal slabs can be found in the specific character of water, where the permittivity decreases by increasing temperature. The absorbed power within the slab will be small at hot spots and large at cold spots. The result is an almost constant absorption of energy over the whole slab. Only the wave character remains, causing runaway. This smoothing of the absorbed power during the heating process is very dominant. The calculations where the temperature dependence of the dielectric constant has been omitted, yield temperature profiles which substantially differ from the profiles where the temperature dependence has been taken into account.

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Chapter 7

Thermal profiles and thermal runaway in microwave heated ceramic slabs

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ABSTRACT

The microwave heating of ceramic slabs is modeled and analyzed with the aim of finding the physical origin of the phenomenon of thermal runaway. It is demonstrated that the strong increase of the dielectric loss factor with increasing temperature is not immediately responsible for the runaway process. The argument should be that it is the strong increase of the attenuation constant, while the phase factor remains constant in the relevant temperature interval. The phenomenon of thermal runaway is basically caused by the behavior of the attenuation constant, combined with heat loss. A plot of the steady-state temperature at any position within the slab, versus the microwave power, is an S-shaped curve. With respect to thermal runaway there is a strong similarity between isothermal and nonisothermal slabs. Using the average temperature of the nonisothermal slab, regardless of its Biot number, yields a reasonable approximation to describe the runaway. This explains the success of the isothermal approximation in explaining runaway. The isothermal approximation is also very useful in predicting the position within the slab, where the runaway will start and the slab will finally melt. In contrast to the microwave heating of foodstuffs the phenomenon of resonance hardly influences the thermal runaway process in ceramics.

7.1. INTRODUCTION

The application of microwave heating (frequency range 2450 MHz) is seriously hampered by two problems, both originating from the basic physics of the heating process. The first difficulty is the uneven spatial absorption of energy within the irradiated object. The second difficulty is the catastrophic phenomenon of thermal runaway in which a slight change of microwave power causes the temperature of the object to increase rapidly. The aim of this investigation is to find the physical origin of the runaway process and hope that it leads to a general rule in preventing runaway. Temperature profiles of the irradiated slab are necessary in order to achieve this. These temperature profiles give a good insight in the uneven spatial absorption of energy. Both problems, the uneven spatial absorption and the thermal runaway, are related to each other. The process of thermal runaway is a non-linear problem and can be explained by taking the temperature dependence of the permittivity ε into account. This means that the temperature profiles, as will be shown in this article, will also reflect this temperature dependence.

The study of thermal runaway in microwave heated objects usually starts by formulating the equation of the absorbed power D.

$$D = \frac{1}{2}\omega\varepsilon''|E|^2 = \frac{1}{2}\omega\varepsilon_0\kappa''|E|^2 \tag{1}$$

where ω is the angular frequency, ε'' is the imaginary part of the permittivity, and E is the electric field within the object. Instead of ε'' one can also formulate Eq.(1) with the dielectric loss factor κ'', which equals ε'' divided by ε_0, the permittivity of vacuum. Eq.(1) is a very powerful equation because it is a general equation and does not depend on the geometry of the irradiated object. All problems involved in the geometry of the object are expressed by the field E. The absorbed power is always directly proportional to ε'' or the dielectric loss factor. The microwave sintering of ceramics especially is seriously hampered by the phenomenon of thermal runaway. The dielectric loss factor of ceramics strongly increases with increasing temperature. Looking at Eq.(1) the explanation of thermal runaway in ceramics looks almost evident. As the temperature increases, the absorbed power increases, resulting in a stronger increase of the temperature, which then results in more absorbed power, etc. This hand-waving argument is often heard. Kriegsmann[1] formulated a plausible explanation of thermal runaway in ceramics in the small Biot number regime, but it does not become clear what the role of the hand-waving argument is in his mathematical analysis.

Not only the microwave sintering of ceramics, but also the microwave heating of foodstuffs (f.i. water) are hampered by the phenomenon of thermal runaway. The dielectric loss factor of water hardly depends on temperature as compared to ceramics. It decreases a little bit with increasing temperature and it is obvious, that in the case of water, the hand-waving argument is not appropriate in explaining runaway. The physical cause of runaway in water is the phenomenon of resonance within the slab, as became clear by the studies of Stuerga *et al.*[2] (isothermal slab) and Vriezinga[3] (nonisothermal slab).

Resonance is always a very strong phenomenon. It is expected that it will effect the microwave heating of ceramics. On the other hand the strong increase of the dielectric loss factor in Eq.(1) cannot be ignored.

Attention has to be paid to the notation. In contravention of the usual notation[4] the symbol α will be used as the phase constant and β as the attenuation constant. Both are real numbers. This notation is easier to read and it fits in better with the theory of wave propagation as described in classic books. The runaway phenomenon, as described in this article, is a result of the application of the wave propagation theory.

7.2. THEORY

Consider a layer of material specimen, irradiated from one side by microwave radiation with a frequency of 2450 MHz. The wave is a plane, harmonic one and impinges normally upon the material. In order to explain the principles of thermal runaway, the simplest possible system was conceived. The slab was located in free space so no other waves than the incident one would be involved. Solving Maxwell's equations (see f.i. Stratton[5] or Ayappa *et al.*[6]) yields the classical wave equation in one dimension:

$$\frac{d^2E}{dx^2} + k^2(T)E = 0 \tag{2}$$

where the electric field E is a function of position x and temperature T at that position. The $k(T)$ stands for the temperature dependent complex wavenumber, which is connected to the permittivity ε of the medium by the following equation.

$$k^2 = \omega^2 \mu_0 \varepsilon \tag{3}$$

This equation takes into account that the permeability of the irradiated object almost equals μ_0, being the permeability in a vacuum. Many materials treated by microwaves fulfil this requirement. Formally Eq.(3) should include the term $d(i\varepsilon)/d(\omega t)$, but this term is very small compared to ε itself and can be neglected. The equation also reflects the assumption that the irradiated medium is a pure dielectric (no free charges). With this condition the electromagnetic boundary conditions are:

$$E_1 = E_2 \quad ; \quad \frac{dE_1}{dx} = \frac{dE_2}{dx} \tag{4}$$

at the boundaries $x=0$ and $x=L$ (L is the thickness of the slab). The subscript 1 refers to vacuum or air and the subscript 2 is related to the irradiated medium. To calculate the temperature within the slab as a function of position and time the three electromagnetic equations (1), (2) and (4) have to be combined with Fourier's law.

$$\rho C_p \frac{\partial T}{\partial t} = K \frac{\partial^2 T}{\partial x^2} + D \tag{5}$$

where ρ is the density, C_p is the thermal capacity, and K is the thermal conductivity. Although these parameters depend upon the temperature T, they are assumed constant. They have no real impact on the phenomenon of runaway. The heat balance equation (5) has to be accomplished by its boundary conditions.

$$K\frac{\partial T}{\partial x} = h(T-T_a) + em.\sigma(T^4-T_a^4) \quad , x = 0 \tag{6}$$

$$K\frac{\partial T}{\partial x} = -h(T-T_a) - em.\sigma(T^4-T_a^4) \quad , x = L \tag{7}$$

where h is the heat transfer coefficient related to the convective heat loss. The radiative heat loss is expressed by the second part of the equations with *em* the thermal emissivity of the ceramic surface, and σ the Stefan-Boltzmann constant. T_a is the ambient temperature and also the initial temperature of the slab at $t=0$.

7.3. THE ISOTHERMAL SLAB

The heating process evolves as follows: while the temperature T increases, the electrical properties k and ε'' of the slab will change. This, in turn influences the absorbed power D, resulting in a new temperature for each position in the slab. At the start of the heating process the slab is isothermal. In that case it is possible to describe the heating process in formulas. At low slab temperatures we have:

$$D = P\frac{\omega}{c}\kappa_2''|T_{12}|^2 \quad \times$$
$$\frac{e^{-2\beta_2 x} - 2|R_{12}|e^{-2\beta_2 L}\cos(2\alpha_2(L-x)+\delta_{12}) + |R_{12}|^2 e^{-2\beta_2 L} e^{-2\beta_2(L-x)}}{1-2|R_{12}|^2 e^{-2\beta_2 L}\cos(2\alpha_2 L+2\delta_{12}) + |R_{12}|^4 e^{-4\beta_2 L}} \tag{8}$$

where P is the microwave power. The symbols T_{12} and R_{12} refer to the complex transmission and reflection coefficient, defined as follows

$$T_{12} = \frac{2k_1}{k_1+k_2} \quad ; \quad R_{12} = |R_{12}|e^{\delta_{12}} = \frac{k_1-k_2}{k_1+k_2} \tag{9}$$

where $k_1=\alpha_1$ is the wavenumber in air and $k_2=\alpha_2+i\beta_2$ the complex wavenumber in the isothermal medium. Integrated over the slab yields the total amount of absorbed power

$$D_{tot} = P\frac{\omega}{c}\kappa_2^{\prime\prime}|T_{12}|^2\frac{1}{2\beta_2} \qquad \times$$

$$\frac{(1-e^{-2\beta_2 L})(1+|R_{12}|^2 e^{-2\beta_2 L}) - \frac{4\beta_2}{\alpha_2}|R_{12}|e^{-2\beta_2 L}\sin(\alpha_2 L)\cos(\alpha_2 L+\delta_{12})}{1-2|R_{12}|^2 e^{-2\beta_2 L}\cos(2\alpha_2 L+2\delta_{12})+|R_{12}|^4 e^{-4\beta_2 L}} \tag{10}$$

Substituting the parameters of f.i. alumina at low temperatures and neglecting the small terms, reduces Eq.(10) to

$$D_{tot} = P\frac{\omega}{c}\kappa_2^{\prime\prime}|T_{12}|^2\frac{1}{2\beta_2}(1-e^{-2\beta_2 L}) \tag{11}$$

This is a very good approximation. It means that the heat absorption can be described by just one penetrating wave (Lambert-Beer approximation) at low temperatures. The ceramic slab is almost transparent to microwaves at the start of the heating process. The L-independent factor of Eq.(11) is very interesting. It can be rewritten into

$$P\frac{\omega}{c}\kappa_2^{\prime\prime}|T_{12}|^2\frac{1}{2\beta_2} = \frac{2P}{1+\frac{\alpha_2}{\alpha_1}+\frac{\alpha_1}{\alpha_2}(\frac{1+\kappa_2^{\prime}}{2})} \tag{12}$$

where κ_2' is the real part of the dielectric constant, according to $\kappa_2=\kappa_2'-i\kappa_2''$.

Because κ_2' hardly depends on temperature the total amount of absorbed power becomes a function of the phase factor α_2. This function has a maximum at $\alpha_2=\alpha_1\sqrt{((1+\kappa_2')/2)}$. In the case of alumina, the top proceeds at about 104 m^{-1}. If temperature increases from 27 °C to 3500 °C the phase factor will increase from 140 m^{-1} to 400 m^{-1}, resulting in a decreasing(!) amount of absorbed energy. On the temperature route, from ambient temperature (27 °C) to the melting point of alumina

(2050 °C) the phase constant remains even constant. The dielectric loss factor κ" of equation (1) is replaced by the α_2-dependent function (12). This function is almost a constant for the relevant temperature region, just as in the case of water. This means that the hand-waving argument, as stated in the introduction, is incorrect. The idea is that this is not only valid for isothermal objects, but also applicable to nonisothermal objects. How the temperature distribution within the slab looks, is not important; the expression of the total amount of absorbed power will always be

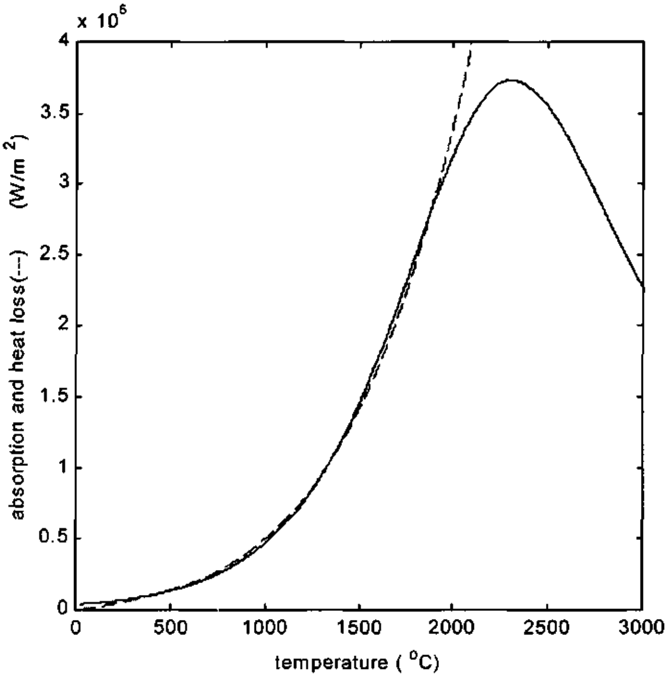

Figure 1. The curve of the absorbed heat (solid line) and the heat loss (dashed line) of a 3 cm isothermal slab of alumina almost coincide. The microwave power is 5500 kW/m^2.

proceeded by the dielectric loss factor times the square of a transmission coefficient divided by some effective attenuation factor. To explain the origin of thermal runaway we have to study the behavior of the electromagnetic field within the irradiated object. In the case of water this study lead to the phenomenon of resonance being basically responsible for runaway effects.

The isothermal formulation of the heat balance for ceramics is

$$\rho C_p L \frac{\mathrm{d}T}{\mathrm{d}t} = -2h(T-T_a) - 2em.\sigma(T^4-T_a^4) + D_{tot} \tag{13}$$

where D_{tot} refers to Eq.(10). Fig. 1 shows a plot of the heat loss and absorbed power versus temperature for a 3 cm slab of alumina. There are three intersection points, of which the first and the last one correspond to stable steady-state temperatures. The intersection point in the middle is an unstable point, so there will be runaway. A plot of the steady-state temperature as function of the microwave power (the solution of Eq.(13) with $\mathrm{d}T/\mathrm{d}t = 0$) shows the familiar S-shaped curve (Fig.2).

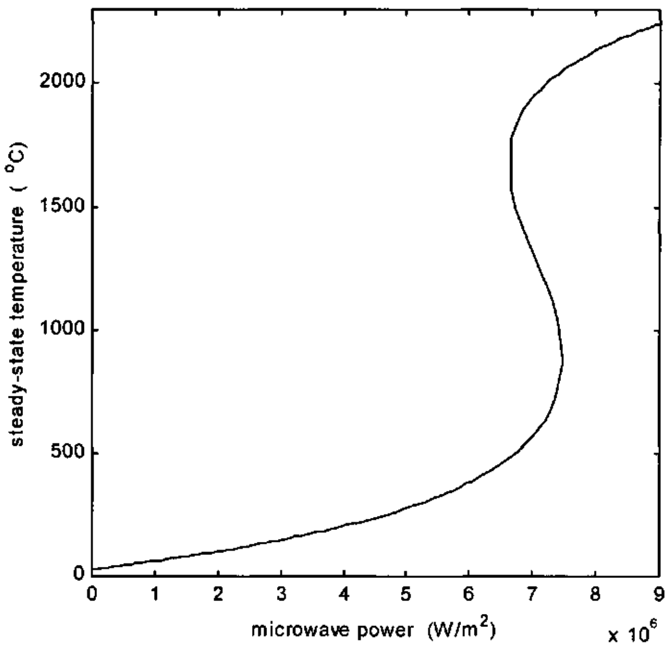

Figure 2. Typical *S*-shaped response curve of a 3 cm slab of alumina, illustrating the phenomenon of thermal runaway.

Looking at figure 1 one sees something very remarkable: The curve of the absorption and the curve of the heat loss almost coincide. This means that the system is very sensitive. In the calculations presented in this article, the ambient temperature is kept constant at 27 °C. Suppose that the experiment is performed in an oven, instead of free space, the wall temperature of the oven (the ambient temperature) will increase during the heating process, resulting in a different curve of the heat loss and because of that there will be another kind of runaway (Jackson *et al.*[7]).

There is another remarkable thing about Fig.*1*: At high temperatures the heat absorption decreases. This is caused by the function (12), illustrating the fact that the hand-waving argument is not appropriate to explain the phenomenon of runaway.

If the dielectric loss factor increases, then the phase factor and the attenuation constant will increase, according to

$$\alpha_2 = \alpha_1 \sqrt{\frac{\kappa_2'\left(\sqrt{1 + (\kappa_2''/\kappa_2')^2} + 1\right)}{2}} \tag{14}$$

$$\beta_2 = \alpha_1 \sqrt{\frac{\kappa_2'\left(\sqrt{1 + (\kappa_2''/\kappa_2')^2} - 1\right)}{2}} \tag{15}$$

As has been mentioned before, the phase factor remains constant in the relevant instable temperature region. The strong increase of the absorption is mainly caused by the strong increase of the attenuation constant β_2. This can be understood by looking at Eq.(11). The total amount of absorbed power is the constant (12) times the strongly increasing factor $1\text{-}\exp(-2\beta_2 L)$. At the start of the heating process the slab is almost transparent to microwaves. The absorption of energy is small and the temperature will increase slowly. Suddenly, at certain temperature, the penetration depth ($1/2\beta_2$) will decrease, meaning that more energy will be absorbed, resulting in a higher temperature, causing a smaller penetration depth, meaning more absorbed energy, etc. This is the cause of the phenomenon of thermal runaway in isothermal ceramic slabs. This looks like the hand-waving argument. One might say that the hand-waving argument is still correct, but no-one can come to such a conclusion by looking at the dielectric loss factor only. One has to translate the real and imaginary parts of the dielectric constant to the wave characteristics α and β for a more complete understanding. For insight one has to look inside the material.

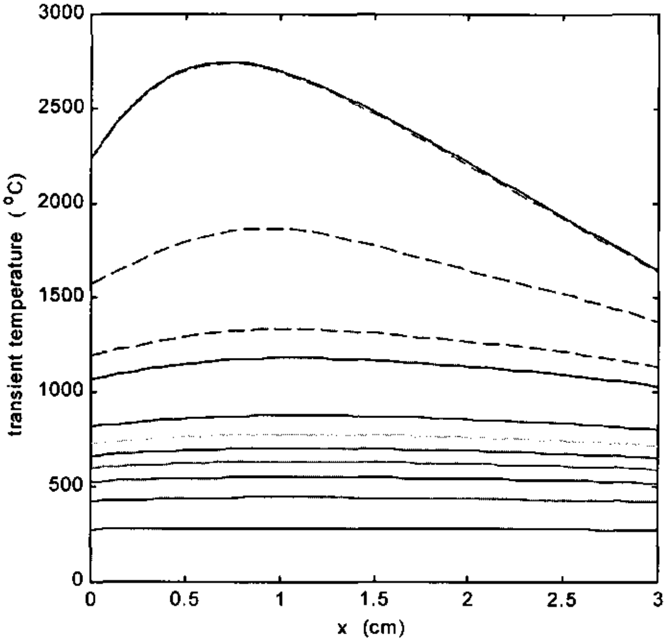

Figure 3. Transient temperature profiles in a 3 cm slab of alumina. The time increases from 0 s to 7200 s in steps of 800 s. Notice the gap between t = 6400 s and t = 7200 s. The dashed lines at 6500, 6600, and 6700 s are the temperature profiles during the runaway. The microwave power is 7000 kW/m^2.

7.4. THE NONISOTHERMAL SLAB

A complete analytical solution of this non-linear problem is almost impossible, but there is a way of escaping from this problem by looking at the Biot number. In this article this number will be defined as

$$Bi = \frac{(h + em.\sigma 8 T_m^3)L}{K} \tag{16}$$

where T_m is the average of the ambient and the surface temperature of the slab in Kelvin. For instance: the Biot number increases from 0.1 at 300 K to 0.2 at 2300 K for a 2 cm slab of alumina. This is relatively small, but because of the internal heat generation, it does not mean that a small slab of alumina might be regarded as isothermal. On the other hand the heating process will start irradiating an isothermal object. It is expected that at low temperatures the isothermal ($K=\infty$, $Bi=0$) approximation is appropriate. This will be confirmed if we look at the temperature

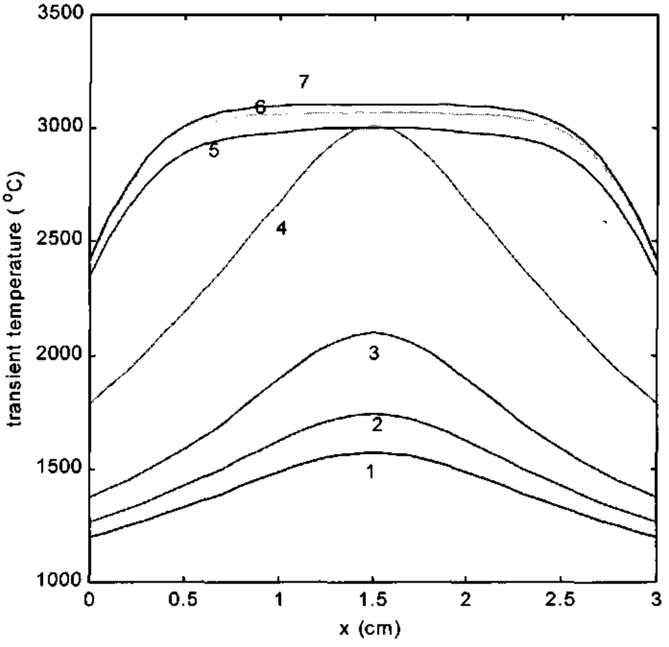

Figure 4. Temperature profiles in a 3 cm slab of alumina, irradiated from two sides. The runaway process starts at the center. One time step is 100 s. The microwave power is 8000 kW/m^2.

profiles of a nonisothermal slab (Fig. 3). Up to 1000 °C the dielectric constant of alumina[8] $\epsilon_0\omega\kappa''=0.002\ \exp(0.78(T-T_a)/T_a)$ is almost temperature independent, resulting in isothermal temperature profiles. The runaway process starts above the 1000 °C. The jump is very fast, about 20 times faster than the time needed to reach the 1000 °C. The figure also shows that initially the temperature increases the most at the position where Eq.(8) has a maximum. This is position x where the cosines equal one. With the aid of the isothermal formula of the absorbed power it is possible to predict the position within the slab where the runaway will start. This is also the position where the slab will start melting. At the end of the runaway process the $\exp(-2\beta_2 x)$ becomes very dominant, resulting in the heating of the skin of the slab.

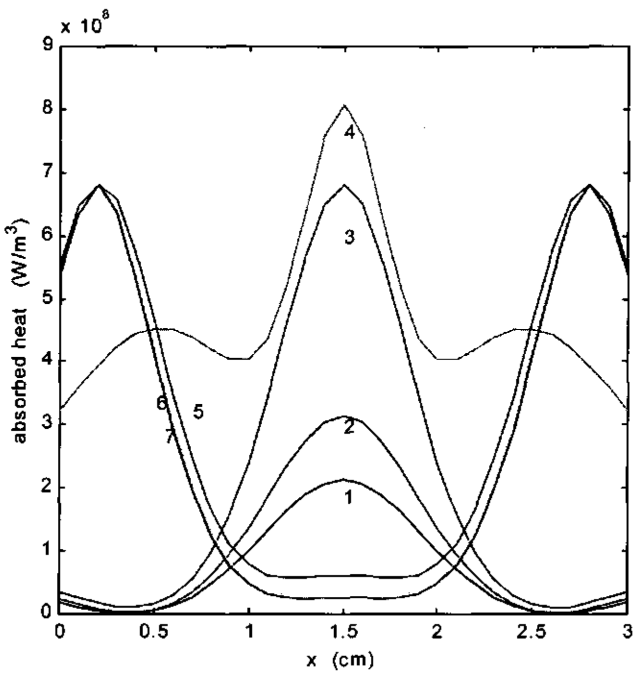

Figure 5. Evolution in time of the absorbed power in a 3 cm slab of alumina, irradiated from two sides. At first the center is heated, at the end the skin of the slab. This yields the temperature profiles of Fig.4

These phenomena are better illustrated by considering a slab irradiated from two sides. In this case the absorbed power is proportional to

$$D \sim e^{-2\beta_2 x} + e^{-2\beta_2 (L-x)} + 2e^{-\beta_2 L}\cos(\alpha_2 L - 2\alpha_2 x) \tag{17}$$

Fig.(4) shows the temperature profiles during the runaway process, illustrating that the slab will start melting in the center, according to Eq.(17). The profiles of the absorbed power emphasize the skin effect (Fig.5). This numerical analysis suggests that the temperature profile $T(x,\Delta t)$ during the runaway process can be found by assuming $K=0$ (so $Bi=\infty$!) in Eq.(5). This yields

$$T(x,\Delta t) \approx T_0 + \frac{D(x,T_0)\Delta t}{\rho C_p}\left(1 + \left(\frac{\partial D}{\partial T}\right)_{T_0} \frac{\Delta t}{\rho C_p}\right) \tag{18}$$

where Δt is the increase of time, starting at the isothermal solution at T_0, so $D(x,T_0)$ equals Eq.(8). The isothermal expression of the absorbed power can be used to approximately predict the temperature profiles during the jump, though this does not mean that the slab is isothermal. In spite of this fact the isothermal approach is still very useful, because it gives a very good insight in the process of thermal runaway by expressing it in the S-shaped curve of the steady-state temperature versus the microwave power. A plot of the steady-state temperature versus the microwave power at any position within the nonisothermal slab will also result in an S-shaped curve.

7.5. RESONANCE

For the isothermal slab the phenomenon of resonance is described by the denominator of Eq.(10) . In the case of microwave heated water the α_2 decreases with increasing temperature, resulting in a small denominator at certain temperature, which causes the runaway. The α_2 of alumina is a constant in the unstable temperature region. Resonance is not the cause of thermal runaway here, but the resonance does have an influence on the shape of the S-curve of Fig. (2), because the heat loss curve and the absorption curve almost coincide. Changing the ambient temperature will alter the curve of the heat loss; changing the denominator of Eq.(10) will alter the curve of the absorbed power. The folding of the S-shaped curve depends on $\mathrm{d}P/\mathrm{d}T$ in the relevant temperature region. One has a real folded S if $\mathrm{d}P/\mathrm{d}T$ is negative, which will happen if

$$\frac{1}{D_{tot}}\ \frac{\mathrm{d}D_{tot}}{\mathrm{d}T} > \frac{1}{V}\ \frac{\mathrm{d}V}{\mathrm{d}T} \tag{19}$$

where V is the heat loss of Eq.(13). If D_{tot} is large this demand is violated and the S will be unfolded.

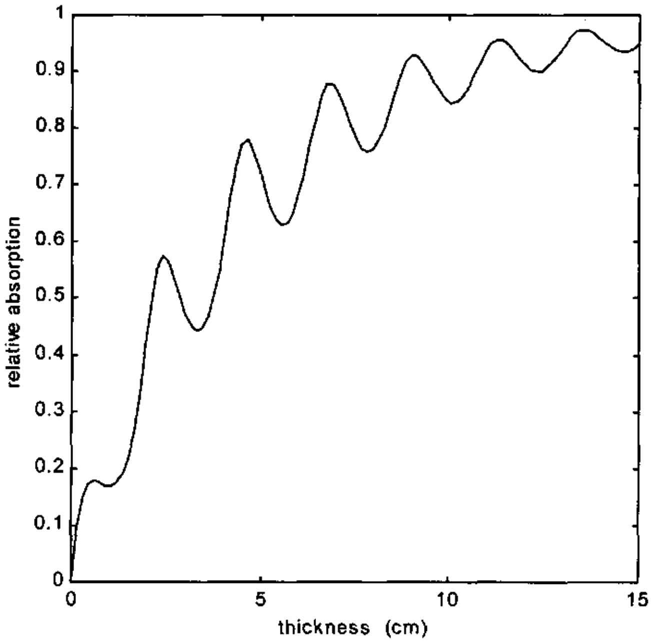

Figure 6. The relative absorbed heat in an isothermal slab of alumina versus the thickness of the slab.The oscillation is caused by the phenomenon of resonance.

Fig.6 shows D_{tot}/P as a function of the thickness L of the isothermal slab. No thermal runaway, as defined in the introduction, will occur at very thick slabs (> 9 cm) and in the neighborhood of 4.6 and 6.7 cm. The phenomenon of resonance prevents the runaway, unless the slab is thin (< 3 cm). In the case of water, resonance causes the runaway, in the case of isothermal ceramics it prevents the runaway. One has an unfolded S for L=4.6 cm (Fig. 7) and one has a folded S for L=5.6 cm (Fig. 8),

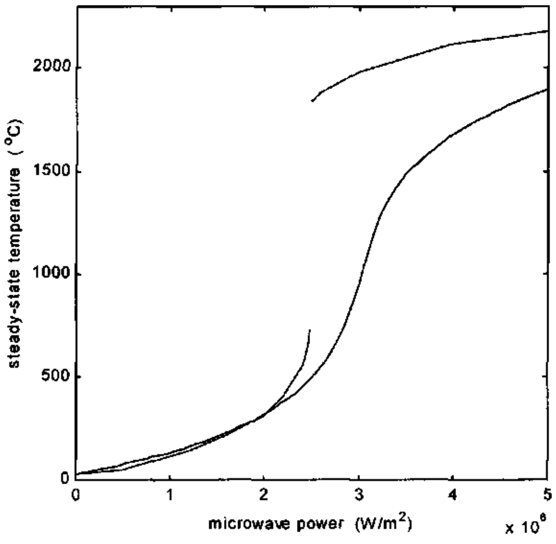

Figure 7. An unfolded *S*-shaped curve of the isothermal slab. The unfolding is caused by resonance. The response curve of the non-isothermal slab is still a folded *S*. Only the upper and lower branch of the folded *S* have been plotted. The thickness is 4.6 cm.

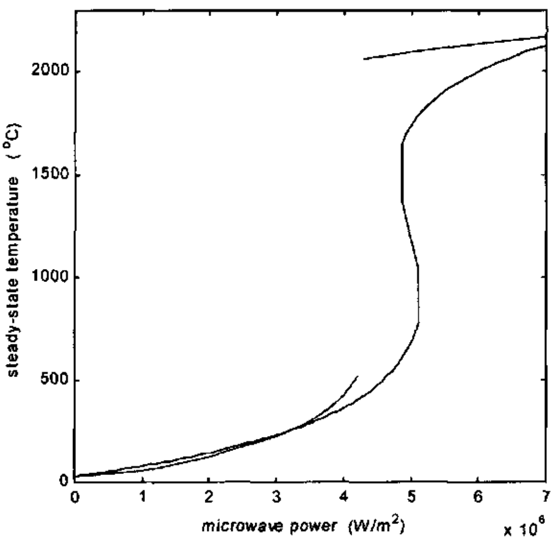

Figure 8. Same as Fig.7, except the thickness is 5.6 cm.

illustrating the oscillating character of the S-shaped curve. Unfortunately this is only correct in the isothermal approximation. The figures also show the upper and lower branch of the S-shaped curve of the nonisothermal slab. There will be no unfolding of the S, unless the slab is very thick. The isothermal formulation has its limitations.

7.6. CONCLUSIONS

The absorbed power in any microwave heated object is always proportional to the dielectric loss factor times the square of the absolute value of the electric field. This study has demonstrated that one cannot explain the phenomenon of thermal runaway in ceramics by looking at the dielectric loss factor only. One has to calculate the electric field within the irradiated object to understand the physical origin of the runaway process. These calculations show that the attenuation constant strongly increases with increasing temperature. At low temperatures a ceramic slab is almost transparent to microwaves (small absorption of energy), while at high temperatures the irradiated skin of the slab absorbs most of the energy. The behavior of the attenuation constant might be regarded as the origin of thermal runaway in ceramics. This also means that one has almost no possibilities to avoid runaway with respect to the thickness of the slab. Only the sintering of very thick slabs will not be hampered by the runaway process.

Numerical calculations show that the steady-state temperature versus the microwave power at any position within a nonisothermal slab, is an S-shaped curve. This explains why the isothermal approximation of the heating process is useful in understanding the runaway process, because it too generates an S-shaped curve, illustrating the instability of the system. The isothermal approach has another advantage. It is possible to predict where the runaway process will start by looking at the analytical formula of the adsorbed power. On the other hand one should not rely too much on the values of the steady-state temperatures generated by the isothermal approximation. This approximation suggests that an unfolded S is possible in case of resonance, but in reality (the nonisothermal slab) that will not happen.

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Chapter 8

Thermal runaway in microwave heated slabs of foodstuffs and ceramics: A comparison

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Proceedings of the International Conference on Microwave and High Frequency Heating, B1.4, 453-456, Valencia, Spain (1999)

ABSTRACT

The microwave heating of slabs is modelled and analysed with the aim to find the physical origin of the phenomenon of thermal runaway. This study demonstrates that one cannot explain the phenomenon of thermal runaway by looking at the dielectric loss factor. The process of thermal runaway can only be understood by translating the real and imaginary parts of the dielectric constant into the phase and attenuation constant. In the case of foodstuffs the phenomenon of resonance, combined with the heat loss, might be regarded as the origin of thermal runaway. The resonance is caused by the small decrease of the phase constant with increasing temperature. In the case of ceramics the phase constant is almost temperature independent in the relevant temperature region, but the attenuation constant increases strongly with increasing temperature. This, combined with the heat loss, causes the runaway process here. Also the impact of resonance within a ceramic slab is studied.

8.1. INTRODUCTION

One of the difficulties associated with the application of microwave heating (frequency range 2450 MHz) is the catastrophic phenomenon of thermal runaway in which a slight change of microwave power causes the temperature to increase rapidly. The microwave sintering of ceramics is seriously hampered by this phenomenon, but also foodstuffs might be over treated because of runaway effects. The aim of this investigation is to find the physical origin of the runaway process for both kind of material specimen and hope that it leads to a general rule in preventing runaway.

The study of thermal runaway in microwave heated objects usually starts by formulating the equation of the absorbed power *D*.

$$D = \frac{1}{2}\omega\varepsilon_0\kappa^{//}|E|^2 \qquad (1)$$

where ω is the angular frequency, ϵ_0 is the permittivity of vacuum, κ'' is the dielectric loss factor, and *E* is the electric field within the object. Eq. 1 is a general equation. All problems involved in the geometry of the object and environment are expressed by the field *E*. The absorbed power is always directly proportional to the dielectric loss factor. The dielectric loss factor of ceramics strongly increases with increasing temperature. Looking at Eq. 1 the explanation of thermal runaway in ceramics looks almost evident. While the temperature increases, the absorbed power will increase, resulting in a stronger increase of the temperature, which then results in more absorbed power, etc. This hand-waving argument can often be heard. Kriegsmann [1] formulated a plausible explanation of thermal runaway in ceramics in the small Biot number regime, but it does not become clear what the role is of the hand-waving argument in his analysis. The hand-waving argument is not appropriate in explaining the thermal runaway process in foodstuffs. The dielectric loss factor of f.i. water hardly depends on temperature as compared to ceramics. It decreases a little bit with increasing temperature. The physical cause of runaway in water is the phenomenon of resonance, as became clear by the studies of Stuerga *et al.* [2] (isothermal slab) and Vriezinga [3] (nonisothermal slab). So we have two explanations of runaway: The hand-waving argument for ceramics and the resonance argument for foodstuffs. The questions are: What is the correctness of the hand-waving argument and what is the impact of resonance in ceramic slabs.

8.2. THE HAND-WAVING ARGUMENT

Consider a pure dielectric (no free charges), irradiated from one side. The wave is a plane, harmonic one and impinges normally upon the surface of the slab. The slab is located in free space. The permeability of the slab equals the permeability in vacuum. Applying the wave propagation theory (see e.g. Stratton [4] or Ayappa [5]) yields the formula for the total amount of absorbed power D_{tot}, assuming the slab is isothermal.

$$D_{tot} = P\frac{\omega}{c}\kappa''|Tr|^2\frac{1}{2\beta} \quad \times$$

$$\frac{(1-e^{-2\beta L})(1+|R|^2e^{-2\beta L})-\frac{4\beta}{\alpha}|R|e^{-2\beta L}\sin(\alpha L)\cos(\alpha L+\delta)}{1-2|R|^2e^{-2\beta L}\cos(2\alpha L+2\delta)+|R|^4e^{-4\beta L}} \tag{2}$$

where P is the microwave power, c is the velocity of light, L is the thickness of the slab, α is the phase and β the attenuation constant, Tr is the transmission and $R = |R|\exp(i\delta)$ the reflection coefficient of air and the irradiated medium. This equation equals Eq.1, integrated over the slab, and because of this the total amount of absorbed power is proportional to the dielectric loss factor too.

It looks like that the hand-waving argument can still be used, but one has to consider the entire equation for a more complete understanding. For this reason the L-independent factor of Eq. 2 is transformed.

$$P\frac{\omega}{c}\kappa''|Tr|^2\frac{1}{2\beta} = \frac{2P}{1+\frac{\alpha}{\alpha_1}+\frac{\alpha_1}{\alpha}(\frac{1+\kappa'}{2})} \tag{3}$$

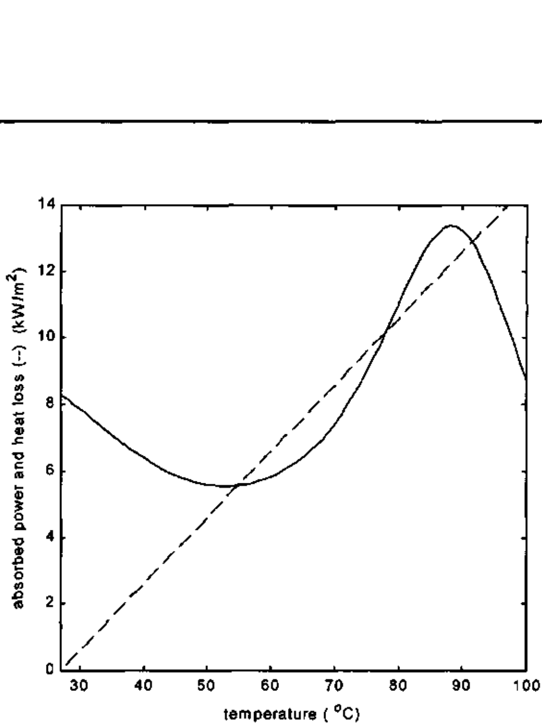

Figure 1.. Plot of the absorbed heat (solid line) and the heat loss (dashed line) of a 4 cm isothermal slab of water. P=25kW/m^2, h=100 W/m^{2}K, em=1, T_a=27 °C.

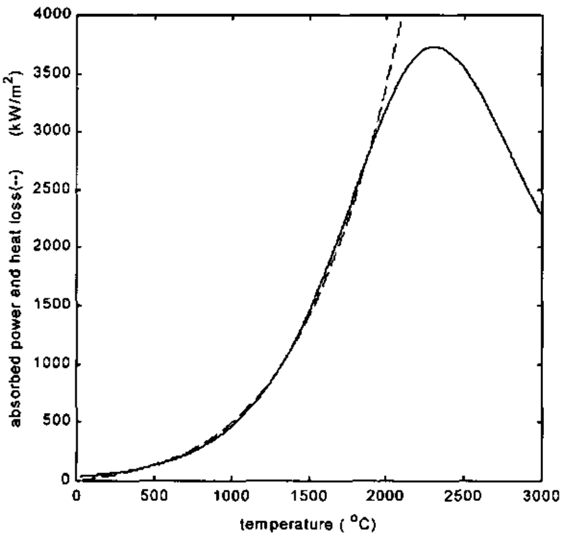

Figure 2. Plot of the absorbed heat (solid line) and the heat loss (dashed line) of a 3 cm isothermal slab of Alumina. P=5000 kW/m^2, h=100 W/m^{2}K, em=1, T_a=27 °C.

where α_1 is the phase constant in air (51.3 m^{-1}), and κ' is the real part of the dielectric constant, according to $\kappa=\kappa'-i\kappa''$. Foodstuffs and ceramics have in common that κ' is almost temperature independent. The phase constant of water (which will be used as an example of foodstuffs) is also more or less temperature independent. Surprisingly this is also the fact for alumina (an example of a ceramic) in the relevant temperature region between the initial slab temperature (27 °C) and the melting point (2050 °C). This means that the L-independent factor is almost a constant with increasing temperature, as well as for water (foodstuffs) and for alumina (ceramics). The temperature dependent behaviour of the dielectric loss factor in Eq. 1 is not directly reflected in the absorbed power. The hand-waving argument, as formulated in the introduction, is not correct. The idea is that this is not only valid for isothermal objects, but also applicable to nonisothermal (real) objects. How the temperature distribution within the slab looks like, is not important; the expression of the total amount of absorbed power will always be proceeded by the dielectric loss factor times the square of a transmission coefficient divided by some effective attenuation factor.

The physical origin of thermal runaway must be found in the second part of Eq. 2. The character of the wave within the slab determines the kind of runaway. Neglecting small terms the second part of Eq. 2 reduces to:

for water $$D_{tot} \sim \frac{1}{1-2|R|^2e^{-2\beta L}\cos(2\alpha L)} \tag{4}$$

for alumina $$D_{tot} \sim (1-e^{-2\beta L}) \tag{5}$$

In the case of water the small decrease of the phase constant with increasing temperature causes the absorbed power to increase (resonance). The origin of thermal runaway for ceramics is completely different. The strong increase of the attenuation constant with increasing temperature causes the absorbed power to increase strongly. For a complete description of the runaway phenomenon the absorbed power has to be combined with the heat loss. The heat loss V at the two surfaces of the isothermal slab is

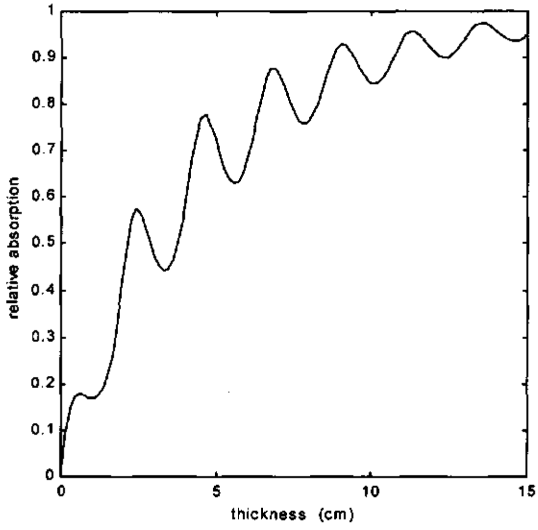

Figure 3. Isothermal alumina slab. The absorbed power oscillates due to resonance.

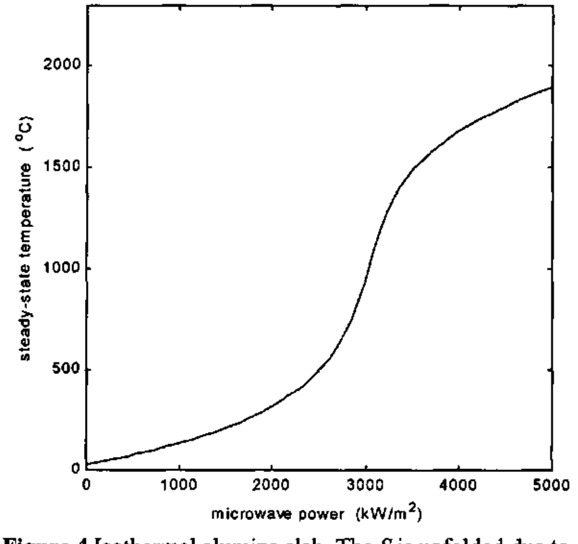

Figure 4.Isothermal alumina slab. The *S* is unfolded due to resonance at the thickness of 4.6 cm.

$$V = 2h(T-T_a) - 2em.\sigma(T^4-T_a^4) \tag{6}$$

where h is the heat transfer coefficient, em is the emissivity, σ is the Stefan-Boltzmann constant, and T is the isothermal slab temperature. T_a is the ambient temperature and also the initial temperature of the slab at $t=0$. Fig.(1) and (2) show plots of the heat loss and absorbed power versus temperature. There are three intersection points. The point in the middle is an unstable point, so there will be runaway. A plot of the steady-state temperature versus the microwave power results in the familiar S-shaped curve. It is important to notice that the mechanism (the physical origin and the combination with heat loss), which leads to the phenomenon of runaway as being developed for isothermal slabs, is also applicable to real nonisothermal slabs. This has been demonstrated in the case of water [3], but is it is also correct in the case of alumina.

8.3. RESONANCE

Resonance is always a very strong phenomenon. It is expected that it will effect the microwave heating of ceramics. As argued above resonance is not the cause of runaway here, but it has influence on the folding of the S-shaped curve (steady-state temperature versus microwave power), because the curves of the absorbed power and heat loss almost coincide (Fig. 2). Changing the denominator of Eq. 2 by changing the thickness of the slab will alter the curve of the absorbed power. The folding of the S-shaped curve depends on dP/dT in the relevant temperature region. One has a real folded S if dP/dT is negative, which will be the fact if

$$\frac{1}{D_{tot}} \frac{\mathrm{d}D_{tot}}{\mathrm{d}T} > \frac{1}{V} \frac{\mathrm{d}V}{\mathrm{d}T} \tag{7}$$

In the case of resonance D_{tot} is large, and the S will be unfolded. If the waves within the slab cancel each other more or less D_{tot} will be relatively small, and we have a folded S. Fig. 3 shows the L-dependent factor of Eq. 2 as a function of the thickness

L of the isothermal alumina slab. The folding of the S oscillates. No thermal runaway, as defined in the introduction will occur at very thick slabs ($>$ 9 cm) and in the neighbourhood of 4.6 cm (Fig.4) and 6.7 cm. The phenomenon of resonance prevents the runaway, unless the slab is too thin ($<$ 3cm). Unfortunately this does not work in reality. The S-shaped curve of the real nonisothermal slab of alumina will not be unfolded enough to prevent runaway.

8.4. CONCLUSIONS

The physical origin of thermal runaway in microwave heated slabs can be found by analysing the isothermal heating process. It is demonstrated that one cannot explain the phenomenon of thermal runaway by looking at the dielectric loss factor. The hand-waving argument explaining runaway in ceramics is false. The runaway in ceramics is caused by the strongly increasing attenuation constant with increasing temperature. In the case of foodstuffs the relatively small decrease of the phase constant with increasing temperature causes the runaway (resonance). The phenomenon of resonance might prevent the runaway when heating isothermal ceramic slabs.

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Chapter 9

Conclusions

the concept of bistability for an isothermal slab of water

This thesis starts with the analysis of a microwave heated isothermal slab of water, taking into account the temperature dependence of the dielectric constant. The analysis shows that bistability is not appropriately described by the temperature jump for a system for which the microwave power is directly proportional to time (Stuerga *et al.*[1]). It is shown that the behavior of the experimental curve of temperature versus time can be explained from the different time constants present in the nonlinear system. It belongs to a class of phenomena known as relaxation oscillations.

the analysis of thermal runaway in isothermal samples of water

An isothermal slab, cylinder and sphere of water, irradiated from one side, have been investigated with respect to thermal runaway. It has been demonstrated numerically as well as theoretically, that in the first approximation the absorbed power in every object (slab, cylinder, sphere) is independent of the temperature within the object. The direct influence of the dielectric loss factor is canceled by the attenuation constant. The runaway phenomenon can only be understood by looking at the behavior of the waves within the sample. The small decrease of the phase constant with increasing temperature results in oscillating absorption, because of the interference of the waves. Thermal runaway is possible if the resonance peak of the absorption is preceded by a de-interference of the waves during the heating process. Combined with the heat loss this produces an S-shaped response curve steady-state temperature versus microwave power. This explains the thermal runaway in which a slight change of microwave power causes the temperature to increase rapidly. It has been demonstrated that this is correct for the isothermal slab, cylinder and sphere.

The geometry of the irradiated sample has no influence on the presence of runaway. If resonance is possible, runaway will be there. The better understanding of the physical origin of the runaway process, provides rules in relation to the thickness of the object to avoid runaway. Relatively large objects will not be overheated by runaway, because the amplitude of the oscillating absorption as a function of the temperature is too small to create (with the heat loss) an S-shaped response curve. Also very small objects will not be damaged by runaway, because one needs at least one quarter of a wavelength within the object for a standing wave. Stated more precisely: Any isothermal object with characteristic dimension L, irradiated from one side by microwaves, will never be overheated or damaged by the phenomenon of thermal runaway if L is smaller than $\pi/(4\Delta\alpha)$, where $\Delta\alpha$ is the difference between the maximum and the minimum value of the phase constant α. For water this results in a dimension of about 1 cm.

the analysis of thermal runaway in nonisothermal samples of water

The study of a nonisothermal slab of water shows that there is a strong similarity between isothermal and nonisothermal slabs. A plot of the steady-state temperature at any position within the slab, versus the microwave power, will be an S-shaped or multi S-shaped curve. The main reason for this similarity can be found in the specific character of water, where the permittivity decreases by increasing temperature. The absorbed power within the slab will be small at hot spots and large at cold spots. The result is an almost constant absorption of the energy over the whole slab. Only the wave character remains, causing runaway. This smoothing of the absorbed power is very dominant. Any calculations where the temperature dependence of the permittivity is omitted, will not only pass the phenomenon of thermal runaway but its temperature profiles will differ substantially from the ones where the temperature dependence has been taken into account. The nonisothermal cylinder and sphere have not been studied in extension, because the principles of thermal runaway, developed for the nonisothermal slab, can easily be applied to objects with other geometry. No other effects in relation to thermal runaway are expected because of the geometry.

general conclusions about thermal runaway of foodstuffs in free space

Water stands for an example of foodstuffs. The foodstuffs, where the dielectric constant as a function of temperature is similar to that of water, will behave in the same way as water. Generalizing and summarizing the principles developed above: Thermal runaway of watery foodstuffs in free space is caused by the small decrease of the phase constant with increasing temperature (resonance). Very small and very large objects will not be overheated by the runaway effect. Analyzing an isothermal slab is sufficient to develop the principles of runaway. Omitting the temperature dependence of the permittivity yields poor results.

the analysis of thermal runaway in a nonisothermal slab of alumina

In addition to the analysis of the isothermal microwave heated ceramic slab by Kriegsmann[2] the real nonisothermal slab has been investigated. It has been demonstrated that the slab is almost isothermal up to the start of the runaway. During the runaway the slab is not isothermal. Good approximations of the temperature profiles for the unstable temperature region have been performed by neglecting the internal thermal conduction. This illustrates that the small Biot number has no physical meaning in this process with internal heat generation. A plot of the steady-state temperature at any position within the slab, versus the microwave power, is an S-shaped curve. This explains the success of the isothermal approximation in explaining runaway by the introduction of the S-shaped response curve. On the other hand one should not rely too much on the values of the steady-state temperatures generated by the isothermal approximation. Just as for foodstuffs the direct influence of the dielectric loss factor in the absorption is canceled by the attenuation constant. Also here the physical origin of thermal runaway must be found inside the slab. Calculations show that the attenuation constant strongly increases with increasing temperature. This behavior, combined with the heat loss, results in the S-shaped response curve, and therefore it could be regarded as the physical origin of runaway in ceramics. This means that one has almost no possibilities of avoiding runaway with respect to the thickness of the slab. Only the sintering of very thick slabs will not be hampered by the runaway process. The plots of the absorbed power and the heat loss versus the isothermal slab temperature almost coincide. The folding of the S-shaped curve will be influenced by a change of the heat loss or the absorption. It has been shown that for resonance within an isothermal slab the *S* will be unfolded. In reality (the nonisothermal slab) the *S* is not unfolded.

general conclusions about thermal runaway of ceramics in free space

Generalizing the developed principles: Thermal runaway of ceramics (with an dielectric loss factor exponentially increasing with increasing temperature) in free space is caused by the strong increase of the attenuation constant (skin effect). Only very thick objects will not be damaged by the runaway effect. Analyzing an isothermal slab is sufficient to develop the principles of runaway. The folding of the *S* is very sensitive to changes in the heat loss or absorption.

thermal runaway of ceramic samples in resonant cavities

The above-mentioned principles have been developed by the analysis of microwave heated samples in free space with a fixed ambient temperature. In real experiments the sample is always in a microwave oven or resonant cavity. The question is: Does there exist any meaningful correspondence between the free space model and the runaway phenomena, as has been observed in oven or cavity? This is especially a problem with respect to ceramics, where it actually has been reported that a small increase of the microwave power led to a rapid increase of the temperature, resulting in a melted sample (Brodwin *et al.*[3], Johnson[4]).

The main objections against the free space model are the following. In a cavity there are no plane waves, moreover, the power input to the cavity (the sum of the power absorbed in the sample and the power absorbed in the cavity walls), is the only objectively meaningful power to use in examining whether an S-shaped response curve of a characteristic sample temperature versus power could explain thermal runaway (Tian[5]). In later work, Tian *et al.*[6] found exact solutions of the coupled electromagnetic and thermal equations for an alumina sample that has a loss factor that increases rapidly with increasing temperature at high temperatures. They interpreted their calculated results as indicating occurrence of thermal runaway for some conditions. They did not plot steady-state temperature versus input power, so the criterion for an S-shaped heating curve was not examined. Jackson *et al.*[7] calculated exact simultaneous solutions of electromagnetic and thermal equations for a cylindrical TM_{010} mode and alumina samples having loss factors that at high temperatures increased rapidly with increasing temperature. Response curves for steady-state temperature versus input power to the cavity were calculated and plotted. No S-shaped response curve was found in those calculations. In later work, extensive parameter studies were conducted for heating curves on exact calculations (Jackson *et al.*[8]). Again no evidence for S-shaped response curves and thermal runaway was found in those studies. On the other hand Kriegsmann[9] has demonstrated that the microwave heating of a sample of alumina in a TE_{103} waveguide applicator could result in an S-shaped response curve. The *S* will be unfolded in case of a tuned cavity.

In this thesis it has been demonstrated that the folding or even the existence of an S-shaped response curve depends in a highly sensitive way on changes in heat loss and absorption. For resonance within an isothermal slab in free space the *S* is unfolded. It is remarkable how this corresponds with the unfolded *S* of Kriegsmann in a tuned cavity (i.e. resonance in an empty cavity). In a cavity one has resonance effects, an additional absorption in the walls, and another ambient temperature, which will certainly influence the heating response curve. Besides these obvious items one should pay attention to every detail which could have some impact on the response curve (e.g., the temperature dependence of the thermal conductivity), because the system is so sensitive. This could explain the difference between the results of the studies by Tian and Jackson (mainly based on the numerical analysis of realistic models) and the rather simplified, analytical approach by Kriegsmann.

thermal runaway of foodstuffs in ovens

The problem whether the concept of the S-shaped response curve is a correct explanation of the observed runaway phenomena, is not the subject of discussion in the microwave heating of foodstuffs. It has not been reported that a slight increase of the microwave power causes the temperature of the foodstuff sample to increase rapidly, except for the few experiments (Stuerga[10]) which have been performed to create experimental evidence for the correctness of the model with the S-shaped response curve. One can easily understand why this kind of runaway has not been reported. Under normal circumstances the temperature of the oven wall is higher than the 0 or 20 ^{0}C , which have been used as ambient temperatures in this thesis. This means that the steady-state temperatures, belonging to the lower branch of the S-shaped curve, will only be found at very small input powers. These small powers are normally not used in the food industry. The temperature immediately jumps to a steady-state temperature of the upper branch. This kind of runaway, where the transient temperature increases rapidly at a certain moment, have been reported (e.g., in potatoes, Dolanda[11]). It shows the influence of the phenomenon of resonance, as described in this thesis. An interesting application of this type of runaway can be found in the thawing of ice, where not only the temperature, but also the thickness of a layer of water will increase in time (Lee[12]).

follow-up research

The next step of the investigation as described in this thesis, could be the analysis of thermal runaway phenomena of ceramic and food samples in ovens and cavities.

final conclusion

The physical-mathematical modeling of the thermal runaway effect provides insight in the physical origin and the formulation of the conditions necessary for runaway. Based on this better understanding rules to prevent runaway have been formulated.

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Summary

The microwave heating of foodstuffs and ceramics can be seriously hampered by the phenomenon of thermal runaway, which means that the temperature of the irradiated sample suddenly increases rapidly, resulting in a temperature well above the desired one. The subject of this thesis is the analysis of the thermal runaway phenomena, which can be described with an S-shaped response curve of a characteristic sample steady-state temperature versus microwave power. This causes a jump of the steady-state temperature from the lower to the upper branch of the S by a slight increase of the microwave power. It has been reported in the microwave sintering of ceramics that a small increase of the applied power led to a rapid increase of the temperature. The description with the S-shaped response curve provides a plausible explanation of the observed phenomenon, but there are a number of studies of microwave heated ceramic samples, for which the response curve does not have an S-shape.

The aim of this thesis is to have a better understanding of thermal runaway and bistability, enabling the formulation of rules to avoid runaway. The main part of this thesis consists of the physical-mathematical modeling of microwave heated slabs, cylinders, and spheres. The models demonstrate that the response curve of water (as an example of foodstuffs) can be S-shaped and that this is caused by the phenomenon of resonance within the sample. The temperature dependency of the phase constant, causing resonance, can be regarded as the physical origin of thermal runaway in foodstuffs, even if there is no S-shaped response curve. This means that very small objects will never be overheated by thermal runaway, because one needs at least a characteristic dimension equal to a quarter of a wavelength for a standing wave. Also very large objects will not be overheated, because the increase of the absorbed power (due to resonance) diminishes with increasing thickness of the sample.

The study of the nonisothermal slab of water shows that the influence of the temperature dependency of the permittivity cannot be regarded as a second order phenomenon. Any calculations omitting the temperature dependency, will not only pass the phenomenon of thermal runaway but its temperature profiles will differ substantially from the correct ones. This study also analyzes the response curve at any location within the slab.

The mathematical aspects of the theory, describing a microwave heated isothermal slab of water with a time dependent microwave power, have been analyzed. In particular this has been done to formulate a theoretical base for the correct interpretation of an experimentally established curve reported in literature. It was shown that the plot of transient temperature versus time can be interpreted as a relaxation oscillation.

Besides foodstuffs, also ceramics are investigated. The dielectric loss factor of watery foodstuffs decreases with increasing temperature, in contrast to the loss factor of many ceramic material specimen, which increases strongly with increasing temperature (e.g. alumina). The absorbed power is proportional to the loss factor times the square of the amplitude of the electric field. Therefore one could easily think that the strong increase of the loss factor is the cause of runaway in ceramics. However, the study of a microwave heated slab of alumina demonstrates that the direct influence of the loss factor is always canceled by the attenuation constant. One has to consider the waves within the slab to explain the runaway. It turned out that the strongly increasing attenuation constant with increasing temperature should be regarded as the physical origin of thermal runaway of ceramics, even if there is no S-shaped response curve. The answer on the question whether or not an S-shaped curve exists, depends on the balance between the absorbed heat and the heat loss. This investigation shows that the balance is very delicate. A small change in absorption or heat loss has a strong impact on the shape of the response curve. This could explain the contradiction in the results of the studies of thermal runaway in ceramics, as mentioned in the beginning of this summary.

Samenvatting

De microgolf verwarming van levensmiddelen en keramiek kan ernstig gehinderd worden door het verschijnsel "thermal runaway". Hiermee wordt bedoeld dat de temperatuur van het te verwarmen object plotseling sterk toeneemt tot een waarde ver boven de gewenste temperatuur. Het onderwerp van dit proefschrift is de analyse van het verschijnsel thermal runaway, dat beschreven kan worden met een S-vormige respons curve van de stationaire temperatuur uitgezet tegen het microgolf vermogen. Dit betekent dat de stationaire temperatuur een sprong maakt van de lage naar de hoge tak van de S bij een kleine toename van het microgolf vermogen in het relevante gebied. Bij microgolf sintering van keramische materialen is waargenomen dat een kleine toename van het toegepaste vermogen leidde tot een snelle stijging van de temperatuur. De beschrijving met de S-vormige respons curve kan een verklaring inhouden van het waargenomen fenomeen, maar er bestaan ook een aantal studies van microgolf verwarmde keramische objecten, waarvoor de respons curve geen S-vorm heeft.

Het doel van dit proefschrift is het ontwikkelen van een beter begrip van thermal runaway en bistabiliteit, in de hoop dat dit leidt tot regels, waarmee runaway vermeden kan worden. Het hoofdbestanddeel van deze thesis bestaat uit de fysisch-mathematische modelering van microgolf verwarmde vlakke platen, cilinders en bollen. De modellen laten zien dat de respons curve van water (een belangrijk bestanddeel van levensmiddelen) S-vormig kan zijn en dat dit veroorzaakt wordt door resonantie in het object. De temperatuur afhankelijkheid van het golfgetal, verantwoordelijk voor de resonantie, kan worden aangemerkt als de fysische oorsprong van runaway in levensmiddelen, zelfs als er geen S-vormige respons curve is. Dit betekent dat zeer kleine objecten nooit een overbehandeling zullen ondergaan ten gevolge van de runaway, omdat er minstens een karakteristieke afmeting ter grootte van een kwart golflengte nodig is voor een staande golf. Ook relatief grote objecten zullen niet oververhit raken, omdat de stijging van het geabsorbeerd vermogen (t.g.v. de resonantie) minder is, naar mate het object groter is.

De studie van de niet isotherme vlakke plaat water toont aan dat de invloed van de temperatuur afhankelijkheid van de permittiviteit niet als een tweede orde effect beschouwd mag worden. Iedere berekening, waar de temperatuur afhankelijkheid genegeerd wordt, zal niet alleen het verschijnsel thermal runaway missen, maar tevens temperatuur profielen genereren, die substantieel afwijken van de correcte profielen. Deze studie omvat ook een onderzoek naar de aanwezigheid van een S-vormige respons curve voor iedere positie in de plaat.

De wiskundige aspecten van het gedrag van een isotherme vlakke plaat, die verwarmd wordt met een tijdsafhankelijk microgolf vermogen, worden beschreven. Dit is vooral gedaan om een theoretische grondslag te formuleren voor de correcte interpretatie van een experimenteel verkregen kromme. Aangetoond wordt dat de grafiek van de temperatuur, uitgezet tegen de tijd, als een relaxatie oscillatie kan worden geïnterpreteerd.

Naast levensmiddelen zijn ook keramische materialen onderzocht. De diëlektrische verliesfactor van waterachtige levensmiddelen neemt af met toenemende temperatuur, in tegenstelling tot de verliesfactor van vele keramische materialen, die sterk toeneemt met toenemende temperatuur (bijv. alumina). Het geabsorbeerd vermogen is evenredig met de verliesfactor maal het kwadraat van de amplitude van het elektrisch veld. Daarom is de verleiding groot om de sterke toename van de verliesfactor als oorzaak van thermal runaway in keramiek aan te merken. Echter, het onderzoek aan een met microgolven verwarmde vlakke plaat alumina toont aan dat de directe invloed van de verliesfactor altijd teniet gedaan wordt door de verzwakkingsfactor. Om runaway te verklaren moet men het gedrag van de golven in het object bestuderen. Het blijkt dat de sterke toename van de verzwakkingsfactor bij toenemende temperatuur beschouwd moet worden als de fysische oorzaak van runaway in keramiek, zelfs bij de afwezigheid van een S-vormige respons curve. Het antwoord op vraag of er wel of geen S-vormige kromme bestaat, is afhankelijk van de balans tussen geabsorbeerde warmte en warmteverlies. Dit onderzoek toont aan dat die balans zeer gevoelig is. Een kleine verandering in absorptie of verlies heeft grote gevolgen voor de vorm van de respons curve. Dit kan een verklaring zijn voor de tegenstrijdige onderzoeksresultaten op het terrein van thermal runaway in keramiek, waaraan in het begin van deze samenvatting gerefereerd wordt.

Curriculum Vitae

Op 10 januari 1950 werd in Velsen-Noord Cornelis Adrianus Vriezinga geboren. Hij is de auteur van dit proefschrift. Na het doorlopen van de lagere school (de Willem Hovy school te Beverwijk) volgde Kees Vriezinga voortgezet onderwijs aan de Prins Bernard HBS te IJmuiden, waar in juni 1968 het HBS-B diploma behaald werd. In dat zelfde jaar begon hij zijn studie in de Technische Natuurkunde aan de Technische Hogeschool te Delft. Daar bestudeerde hij bij voorkeur onderwerpen uit de theoretische fysica. Zijn experimentele werk verrichtte hij in de vakgroep Molecuul Analyse, waar hij zich bezig hield met ver-infra rood spectroscopie. In januari 1976 werd de studie afgesloten met het behalen van het ingenieursdiploma, waarbij tevens eerstegraads onderwijsbevoegdheid voor de vakken wis- en natuurkunde werd verkregen.

In oktober 1976 werd hij aangesteld als wetenschappelijk ambtenaar bij de vakgroep Natuur- en Weerkunde van de Landbouwhogeschool, hoofzakelijk belast met onderwijs in de natuurkunde. Dit werk bestond voor een groot deel uit het opzetten en geven van nieuwe of vernieuwde onderwijselementen. In de resterende schaarse tijd verdiepte hij zich in conceptuele problemen van de fysica, niet alleen uit interesse, maar ook om een stevig fundament onder de abstracte fysica te leggen, ten behoeve van het onderwijs.

Van oktober 1986 tot april 1988 werd hij gedetacheerd op de Universitas Brawijaja te Malang, Indonesië, in het kader van ontwikkelingssamenwerking, met als taak het ontwikkelen van onderwijs en onderzoek in de fysica.

In 1990, wanneer de onderwijsdruk in vergelijking met de jaren 80 is afgenomen, werd een start gemaakt met het onderzoek naar de microgolf verwarming van levensmiddelen, resulterend in dit proefschrift.