Equivariant Riemann-Roch theorems for curves over perfect fields

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Abstract

We prove an equivariant Riemann-Roch formula for divisors on algebraic curves over perfect fields. By reduction to the known case of curves over algebraically closed fields, we first show a preliminary formula with coefficients in $\mathbb{Q}$. We then prove and shed some further light on a divisibility result that yields a formula with integral coefficients. Moreover, we give variants of the main theorem for equivariant locally free sheaves of higher rank.

Introduction

Let X be a smooth, projective, geometrically irreducible curve over a perfect field k and let G be a finite subgroup of the automorphism group $\mathrm{Aut}(X/k)$. For any locally free G-sheaf $\mathcal{E}$ on X, we are interested in computing the equivariant Euler characteristic

$$\chi(G, X, \mathcal{E}) := [H^0(X, \mathcal{E})] - [H^1(X, \mathcal{E})] \in K_0(G, k),$$

considered as an element of the Grothendieck group $K_0(G, k)$ of finitely generated modules over the group ring $k[G]$. The main example of a locally free G-sheaf we have in mind is the sheaf $\mathcal{L}(D)$ associated with a G-equivariant divisor $D = \sum_{P\in X} n_P P$ (that is $n_{\sigma(P)} = n_P$ for all $\sigma \in G$ and all $P \in X$). If two $k[G]$-modules are in the same class in $K_0(G, k)$, they are not necessarily isomorphic when the characteristic of k divides the order of G. In order to be able to determine the actual $k[G]$-isomorphism class of $H^0(X, \mathcal{E})$ or $H^1(X, \mathcal{E})$, we are therefore also interested in deriving conditions for $\chi(G, X, \mathcal{E})$ to lie in the Grothendieck group $K_0(k[G])$ of finitely generated *projective* $k[G]$-modules and in computing $\chi(G, X, \mathcal{E})$ within $K_0(k[G])$.

The equivariant Riemann-Roch problem goes back to Chevalley and Weil [CW], who described the G-structure of the space of global holomorphic differentials on a compact Riemann surface. Ellingsrud and Lønsted [EL] found a formula for the equivariant Euler characteristic of an arbitrary G-sheaf on a curve over an algebraically closed field of characteristic zero. Nakajima [Na] and Kani [Ka] independently generalized this to curves over arbitrary algebraically closed fields, under the assumption that the canonical morphism $X \to X/G$ be tamely ramified. These results have been revisited by Borne [Bo], who also found a formula that computes the difference between the equivariant Euler characteristics of two G-sheaves in the case of a wildly ramified cover $X \to X/G$. In the same setting, formulae for the equivariant Euler characteristic of a single G-sheaf have been found by the second author ([Kö1], [Kö2]). Using these formulae, new proofs for the reults of Ellingsrud-Lønsted, Nakajima and Kani have been given [Kö1].

In this paper, we concentrate on the case where the underlying field k is perfect. Our main theorem, Theorem 3.4, is an equivariant Riemann-Roch formula in $K_0(k[G])$ when the canonical morphism $X \to X/G$ is weakly ramified and $\mathcal{E} = \mathcal{L}(D)$ for some equivariant divisor D. By reduction to the known case of curves over algebraically closed fields, we first show a preliminary formula with coefficients in $\mathbb{Q}$. The divisibility result needed to obtain a formula with integral coefficients is then proved in two ways: Firstly, by applying the preliminary formula to suitably chosen equivariant divisors; and secondly, in two situations, by a local argument. The following paragraphs describe the content of each section in more detail.

It is well-known that a finitely generated $k[G]$-module M is projective if and only if $M \otimes_k \bar{k}$ is a projective $\bar{k}[G]$-module. In Section 2 we give a variant of this fact for classes in $K_0(G, k)$ rather than for $k[G]$-modules M (Corollary 2.2). This variant is much harder to prove and is an essential tool for the proof of our main result in Section 3.

The first results in Section 3 give both a sufficient condition and a necessary condition under which the equivariant Euler characteristic $\chi(G, X, \mathcal{E})$ lies in the image of the Cartan homomorphism $c : K_0(G, k) \to K_0(k[G])$. More precisely, when $\mathcal{E} = \mathcal{L}(D)$ for some equivariant divisor $D = \sum_{P\in X} n_P P$, this holds if the canonical projection $\pi : X \to X/G$ is weakly ramified and $n_P + 1$ is divisible by the wild part e_P^w of the ramification index e_P for all $P \in X$. When π is weakly ramified we furthermore derive from the corresponding result in [Kö2] the existence of the so-called ramification module $N_{G,X}$, a certain projective $k[G]$-module which embodies a global relation between the (local) representations

$\mathfrak{m}_P/\mathfrak{m}_P^2$ of the inertia group I_P for $P \in X$. If moreover D is an equivariant divisor as above, our main result, Theorem 3.4, expresses $\chi(G, X, \mathcal{L}(D))$ as an integral linear combination in $K_0(k[G])$ of the classes of $N_{G,X}$, the regular representation $k[G]$ and the projective $k[G]$-modules $\mathrm{Ind}_{G_P}^G(W_{P,d})$ (for $P \in X$ and $d \geq 0$) where the projective $k[G_P]$-module $W_{P,d}$ is defined by the following isomorphism of $k[G_P]$-modules:

$$\mathrm{Ind}_{I_P}^{G_P}(\mathrm{Cov}((\mathfrak{m}_P/\mathfrak{m}_P^2)^{\otimes(-d)})) \cong \bigoplus^{f_P} W_{P,d};$$

here Cov means taking the $k[I_P]$-projective cover and f_P denotes the residual degree.

Finding an equivariant Riemann-Roch formula without denominators amounts to showing that $W_{P,d}$ exists, i.e. that the left-hand side of the above is "divisible by f_P". To do this, we use our prototype formula *with* denominators, formula (4), and apply it to certain equivariant divisors D. If π is tamely ramified, we furthermore consider two situations where we can give a local proof of the divisibility result, yielding a more concrete description of $W_{P,d}$, see Proposition 3.5.

In Section 4, we give some variants of the main result that hold under slightly different assumptions. In particular, these variants hold for locally free G-sheaves that do not necessarily come from a divisor.

1 Preliminaries

The purpose of this section is to fix some notations used throughout this paper and to state some folklore results used later.

Throughout this section, let X be a scheme of finite type over a field k, and let $\bar{k}$ be an algebraic closure of k. For any (closed) point $P \in X$, let $k(P) := \mathcal{O}_{X,P}/\mathfrak{m}_P$ denote the residue field at P. Throughout this paper, let $\bar{X}$ denote the geometric fibre $X \times_k \bar{k}$, which is a scheme of finite type over $\bar{k}$, and let p denote the canonical projection $\bar{X} \to X$. Recall that p is a closed, flat morphism which is in general not of finite type. We will see later that in dimension 1, p is "unramified" in the sense that if $Q \in \bar{X}$ and $P = p(Q)$, then a local parameter at P is also a local parameter at Q. By Galois theory and Hilbert's Nullstellensatz, we have for every $P \in \bar{X}$:

$$\#p^{-1}(P) = \#\,\mathrm{Hom}_k(k(P), \bar{k}) \leq [k(P) : k] < \infty,$$

and equality holds if $k(P)/k$ is separable.

Let now G be a finite subgroup of $\mathrm{Aut}(X/k)$. Since the homomorphism

$$\mathrm{Aut}(X/k) \to \mathrm{Aut}(\bar{X}/\bar{k}), \sigma \mapsto \sigma \times \mathrm{id}$$

is injective, which is easy to check, we may view G as a subgroup of $\mathrm{Aut}(\bar{X}/\bar{k})$. Since the elements of G act on the topological space of X as homeomorphisms, G also acts on $|X|$, the set of closed points in X. Analogously, G acts on the set $|\bar{X}|$ of closed points in $\bar{X}$.

Definition 1.1. A *locally free G-sheaf* (of rank r) on X is a locally free $\mathcal{O}_X$-module $\mathcal{E}$ (of rank r) together with an isomorphism of $\mathcal{O}_X$-modules $v_\sigma : \sigma^*\mathcal{E} \to \mathcal{E}$ for every $\sigma \in G$, such that for all $\sigma, \tau \in G$, the following diagram commutes:

$$\begin{array}{ccc} \sigma^*\mathcal{E} & \xrightarrow{v_\sigma} & \mathcal{E} \\ \uparrow{\scriptstyle \sigma^* v_\tau} & \nearrow{\scriptstyle v_{\tau\sigma}} & \\ \sigma^*(\tau^*\mathcal{E}) = (\tau\sigma)^*\mathcal{E} & & \end{array}$$

If $\mathcal{E}$ is a locally free G-sheaf of finite rank, then the cohomology groups $H^i(X,\mathcal{E})$ ($i \in \mathbb{N}_0$) are k-representations of G. If moreover X is proper over k, then the $H^i(X,\mathcal{E})$ are finite-dimensional and vanish for $i >> 0$ (see Theorem III.5.2 in [Ha]).
We denote the Grothendieck group of all finitely generated $k[G]$-modules (i.e. finite-dimensional k-representations of G) by $K_0(G,k)$, as opposed to the notation $R_k(G)$ used by Serre in [Se2].

Definition 1.2. If X is proper over k, and $\mathcal{E}$ is a locally free G-sheaf of finite rank, then

$$\chi(G,X,\mathcal{E}) := \sum_i (-1)^i [H^i(X,\mathcal{E})] \in K_0(G,k)$$

is called the *equivariant Euler characteristic* of $\mathcal{E}$ on X.

For $P \in |X|$ or $P \in |\bar{X}|$, the *decomposition group* G_P and the *inertia group* I_P are defined as follows:

$$G_P := \{\sigma \in G | \sigma(P) = P\};$$
$$I_P := \{\sigma \in G_P | \bar{\sigma} = \mathrm{id}_{k(P)}\} = \ker(G_P \to \mathrm{Aut}(k(P)/k)).$$

Here $\bar{\sigma}$ denotes the endomorphism that σ induces on $k(P)$. Note that for all $Q \in |\bar{X}|$, we have $G_Q = I_Q$ and $G_Q = I_P$, where $P := p(Q) \in |X|$.
In the following lemma, we will assume for the first time that the field k is *perfect.*

Lemma 1.3. *Assume that k is perfect. Let $\mathcal{F}$ be a coherent sheaf on X, and let $\bar{\mathcal{F}} := p^*\mathcal{F}$. Let P be a point in X, and let $\mathcal{F}(P) := \mathcal{F}_P \otimes_{\mathcal{O}_{X,P}} k(P)$ be the fibre of $\mathcal{F}$ at P. Then the canonical homomorphism*

$$\mathcal{F}(P) \otimes_k \bar{k} \mapsto \bigoplus_{Q \in p^{-1}(P)} \bar{\mathcal{F}}(Q)$$

is an isomorphism. In particular, the canonical homomorphism

$$k(P) \otimes_k \bar{k} \to \bigoplus_{Q \in p^{-1}(P)} k(Q)$$

is an isomorphism.

Proof. It follows from Galois theory that for any separable finite field extension k'/k, the homomorphism

$$k' \otimes_k \bar{k} \to \bigoplus_{\mathrm{Hom}_k(k',\bar{k})} \bar{k}$$

defined by

$$y \otimes z \mapsto (\varphi(y) \cdot z)_{\varphi \in \mathrm{Hom}_k(k',\bar{k})}$$

is an isomorphism. Since k is perfect, by putting $k' = k(P)$ this implies the second part of the lemma, i.e. the special case where $\mathcal{F} = \mathcal{O}_X$.
Since the lemma is a local statement on X, we may assume that X is affine. The general case then follows from the special case together with the definitions and basic properties of coherent sheaves and fibred products. □

Proposition 1.4. *Assume that k is perfect. Let $\Omega_{X/k}$ be the sheaf of relative differentials of X over k. Then for every point $P \in |X|$, the canonical map*

$$\mathfrak{m}_P/\mathfrak{m}_P^2 \to \Omega_{X/k}(P)$$

is an isomorphism.

Proof. Let $\Omega_{k(P)/k}$ denote the module of relative differential forms of $k(P)$ over k. Using some basic properties of differentials and of the cotangent space in an affine setting, it follows from Corollary 6.5 in [Ku] that we have an exact sequence

$$0 \to \mathfrak{m}_P/\mathfrak{m}_P^2 \to \Omega_{X/k}(P) \to \Omega_{k(P)/k} \to 0.$$

By Corollary 5.3 in [Ku], $\Omega_{k(P)/k}$ is trivial, so the map $\mathfrak{m}_P/\mathfrak{m}_P^2 \to \Omega_{X/k}(P)$ is an isomorphism. □

Note that both Corollary 6.5 and Corollary 5.3 in [Ku] require $k(P)/k$ to be separable.

Both Lemma 1.3 and Proposition 1.4 can be turned into equivariant statements in the following sense. If we require $\mathcal{F}$ to be a locally free G-sheaf, then for every point $P \in |X|$, we obtain an action of the inertia group I_P on the fibre $\mathcal{F}(P)$ by $k(P)$-automorphisms. The action of I_P on the fibre $\Omega_X(P)$ of the canonical sheaf corresponds to the action on the cotangent space $\mathfrak{m}_P/\mathfrak{m}_P^2$ via the isomorphism from Proposition 1.4.

By letting I_P act trivially on $\bar{k}$, we can extend the action of I_P on $\mathcal{F}(P)$ to an action on the tensor product $\mathcal{F}(P) \otimes_k \bar{k}$. On the other hand, since $I_Q = I_P$ for any point $Q \in p^{-1}(P)$, I_P acts on the fibre $\mathcal{G}(Q)$ of any locally free G-sheaf $\mathcal{G}$ on $\bar{X}$ for any point $Q \in p^{-1}(P)$. In particular, this holds if $\mathcal{G} = p^*\mathcal{F}$ for a locally free G-sheaf $\mathcal{F}$ on X. With respect to these group actions, the isomorphism from Lemma 1.3 is an isomorphism of $\bar{k}[I_P]$-modules.

We also have an action of the decomposition group G_P on any fibre $\mathcal{F}(P)$, but G_P only acts on the fibre via k-automorphisms, whereas I_P acts via $k(P)$-automorphisms. G_P does act $k(P)$-*semilinearly* on the fibre, that is, for any $\sigma \in G_P, a \in k(P)$ and $x, y \in \mathcal{F}(P)$ we have $\sigma.(ax + y) = (\bar{\sigma}.a)(\sigma.x) + \sigma.y$, where $\bar{\sigma}$ denotes the automorphism of $k(P)/k$ induced by σ.

Let now X be a smooth, projective curve over a perfect field k. Assume further that X is geometrically irreducible, i.e. that the geometric fibre $\bar{X} = X \times_k \bar{k}$ is irreducible. Then the curve X itself is irreducible.

The following lemma shows that although the canonical morphism $p : \bar{X} \to X$ is usually not of finite type, it can be thought of as an "unramified" morphism in the common sense, a fact that will be used frequently throughout this paper.

Lemma 1.5. *Let $Q \in |\bar{X}|$ be a closed point, and let $P := p(Q)$. Then every local parameter at P is also a local parameter at Q.*

Proof. Let t_P be a local parameter at P. Then t_P must be an element of $\mathfrak{m}_P \setminus \mathfrak{m}_P^2$, so (the equivalence class of) t_P is a generator of the one-dimensional vector space $\mathfrak{m}_P/\mathfrak{m}_P^2$ over $k(P)$. Hence, $t_P \otimes 1$ is a generator of the rank-1 module $\mathfrak{m}_P/\mathfrak{m}_P^2 \otimes_k \bar{k}$ over $k(P) \otimes_k \bar{k}$. By Lemma 1.3 and Proposition 1.4, we have a canonical isomorphism

$$\mathfrak{m}_P/\mathfrak{m}_P^2 \otimes_k \bar{k} \to \bigoplus_{Q \in p^{-1}(P)} \mathfrak{m}_Q/\mathfrak{m}_Q^2$$

which we can view as an isomorphism of modules over $k(P) \otimes_k \bar{k} \cong \bigoplus_{Q \in p^{-1}(P)} k(Q)$. Since this isomorphism must map $t_P \otimes 1$ to a generator of the right-hand side over $\bigoplus_{Q \in p^{-1}(P)} k(Q)$, the image of $t_P \otimes 1$ in each component $\mathfrak{m}_Q/\mathfrak{m}_Q^2$ must be a generator of $\mathfrak{m}_Q/\mathfrak{m}_Q^2$, i.e. the image of t_P under each induced homomorphism $p_Q : \mathcal{O}_{X,P} \to \mathcal{O}_{\bar{X},Q}$ must be a local parameter at Q. $\square$

Let now G be a finite subgroup of $\mathrm{Aut}(X/k)$. It is a well-known result that the quotient scheme $Y := X/G$ is also a smooth projective curve, with function field $K(Y) = K(X)^G$. The canonical projection $X \to Y$ will be called π. Let $P \in X$ be a closed point, $R := \pi(P) \in Y$. Let v_p be the unique normed valuation of the function field $K(X)$ associated to P, and let v_R be the unique normed valuation of $K(Y)$ associated to R. Then v_P is equivalent to a valuation extending v_R. For $s \geq -1$, we define the *s-th ramification group $G_{P,s}$ at P* to be the s-th ramification group of the extension of local fields $K(X)_{v_P}/K(Y)_{v_R}$. In particular, we have $G_{P,-1} = G_P$ and $G_{P,0} = I_P$.
The canonical projection $\pi : X \to Y$ is called *unramified (tamely ramified, weakly ramified)* if $G_{P,s}$ is trivial for $s \geq 0$ ($s \geq 1, s \geq 2$) and for all $P \in X$. We denote the ramification index of π at the place P by e_P, its wild part by e_P^w and its tame part by e_P^t. In other words, $e_P = v_P(t_{\pi(P)}) = |G_{P,0}|$, $e_P^w = |G_{P,1}|$ and $e_P^t = |G_{P,0}/G_{P,1}|$.
If $Q \in |\bar{X}|$ is a closed point, $P := p(Q) \in |X|$, then for every $s \geq 0$, we have $G_{Q,s} = G_{P,s}$ (by Proposition 5 in Chapter IV in [Se1] and Lemma 1.5). In particular, we have $e_P = e_Q$, $e_P^w = e_Q^w$ and $e_P^t = e_Q^t$.

2 A Cartesian diagram of Grothendieck groups

A $k[G]$-module M is projective if and only if $M \otimes_k \bar{k}$ is a projective $\bar{k}[G]$-module. In this section, we will now show variants of this well-known fact for classes in $K_0(G,k)$ rather than $k[G]$-modules.
Let $K_0(k[G])$ denote the Grothendieck group of finitely generated projective $k[G]$-modules. This is a free group generated by the isomorphism classes of indecomposable projective $k[G]$-modules. The Cartan homomorphisms $c : K_0(k[G]) \to K_0(G,k)$ and $\bar{c} : K_0(\bar{k}[G]) \to K_0(G,\bar{k})$ are injective ([Se2], 16.1, Corollary 1 of Theorem 35), so $K_0(k[G])$ may be viewed as a subgroup of $K_0(G,k)$. The homomorphism

$$\beta : K_0(G,k) \to K_0(G,\bar{k})$$

defined by tensoring with $\bar{k}$ over k restricts to a homomorphism

$$\alpha : K_0(k[G]) \to K_0(\bar{k}[G]).$$

By Proposition (16.22) in [CR], both homomorphisms β, α are split injections.

Proposition 2.1. *The following diagram with injective arrows is Cartesian, i.e. it commutes and viewing the injections as inclusions, we have $K_0(\bar{k}[G]) \cap K_0(G,k) = K_0(k[G])$.*

$$\begin{array}{ccc} K_0(k[G]) & \xrightarrow{\alpha} & K_0(\bar{k}[G]) \\ {\scriptstyle c}\downarrow & & \downarrow{\scriptstyle \bar{c}} \\ K_0(G,k) & \xrightarrow[\beta]{} & K_0(G,\bar{k}) \end{array}$$

Proof. The commutativity is obvious. Now consider the extended diagram (with exact rows)

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K_0(k[G]) & \xrightarrow{\alpha} & K_0(\bar{k}[G]) & \longrightarrow & M & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle c} & & \downarrow{\scriptstyle \bar{c}} & & \downarrow{\scriptstyle f} & & \\ 0 & \longrightarrow & K_0(G,k) & \xrightarrow{\beta} & K_0(G,\bar{k}) & \longrightarrow & N & \longrightarrow & 0 \end{array}$$

where $M = \operatorname{cok}\alpha$, $N = \operatorname{cok}\beta$, and f is the homomorphism $M \to N$ induced by $\bar{c}$. By the Snake Lemma, there is an exact sequence of abelian groups

$$0 \to \ker c \to \ker \bar{c} \to \ker f \to \operatorname{cok} c,$$

the first two modules being trivial since c and $\bar{c}$ are injective. Since α is a split injection, $M = \operatorname{cok}\alpha$ is free over $\mathbb{Z}$, and therefore $\ker f$ must also be free over $\mathbb{Z}$. On the other hand, by Theorem (21.22) in [CR], we have $|G| \cdot \operatorname{cok} c = 0$, so $\operatorname{cok} c$ is a torsion module. Using the exactness of the sequence above, this implies $\ker f = 0$. Now an easy diagram chase completes the proof. □

Proposition 2.1 says that given a class $\mathcal{C}$ in $K_0(G,k)$, $\mathcal{C}$ lies in the image of c if and only if $\beta(\mathcal{C})$ lies in the image of $\bar{c}$. The following corollary appears to be only slightly different from this, yet some additional tools will be required for its proof.

Corollary 2.2. *Let $\mathcal{C}$ be a class in $K_0(G,k)$. Then $\mathcal{C}$ is the class of a projective $k[G]$-module if and only if $\beta(\mathcal{C})$ is the class of a projective $\bar{k}[G]$-module.*

Before proving Corollary 2.2, we will need a few preliminary results on $k[G]$-modules. Recall that a $k[G]$-module is called *simple* if it is nonzero and has no proper $k[G]$-submodules, and *indecomposable* if it is nonzero and is not a direct sum of proper $k[G]$-submodules.

Proposition 2.3. *(a) For every simple $k[G]$-module M, the $\bar{k}[G]$-module $M \otimes_k \bar{k}$ is semisimple.*

(b) Let $\{P_1, \ldots, P_s\}$ be a set of representatives of the isomorphism classes of indecomposable projective $k[G]$-modules, and let

$$P_i \otimes_k \bar{k} = \bigoplus_{j=1}^{r_i} \bar{Q}_{ij}, \quad \bar{Q}_{ij} \text{ indecomposable projective } \bar{k}[G]\text{-modules.}$$

Then every indecomposable $\bar{k}[G]$-module is isomorphic to some $\bar{Q}_{ij}$. Further $\bar{Q}_{ij} \cong \bar{Q}_{i'j'}$ implies that $i = i'$, i.e. there is no overlap between the sets of indecomposable $\bar{k}[G]$-modules which come from different indecomposable $k[G]$-modules.

Proof. This proposition is a variation of Theorem 7.9 in [CR]. In [CR], the algebraic closure $\bar{k}$ is replaced by a finite algebraic extension E of k, and part (b) is stated for *simple* modules rather than for indecomposable projective modules. Using only elementary algebraic methods, it can be shown that there is a finite algebraic extension E/k such that every simple $\bar{k}[G]$-module can be realized as a simple $E[G]$-module, i.e. every simple $\bar{k}[G]$-module M can be written as $M = N \otimes_E \bar{k}$ for some simple $E[G]$-module N. This suffices to derive part (a) from the result in [CR]. Furthermore, it is well-known that mapping every projective $k[G]$-module P to the $k[G]$-module $P/\operatorname{rad} P$ gives a 1-1 correspondence between the isomorphism classes of indecomposable projective $k[G]$-modules and the isomorphism classes of simple $k[G]$-modules, whose inverse is given by taking $k[G]$-projective covers. We can thus deduce our proposition from the result in [CR], using that projective covers are additive (by Corollary 6.25 (ii) in [CR]) and commute with tensor products (by Corollary 6.25 (i) in [CR]). □

Proof of Corollary 2.2. The "only if" direction is obvious. For the "if" direction, we note first of all that if $\mathcal{C}$ is a class in $K_0(G,k)$ and $\beta(\mathcal{C})$ is the class of a projective $\bar{k}[G]$-module, then Proposition 2.1 yields that $\mathcal{C}$ can be viewed as a class in $K_0(k[G])$. Hence it suffices to show the "if" direction for classes $\mathcal{C} \in K_0(k[G])$, replacing the homomorphism β by its restriction α.

Let $\{P_1, \ldots, P_s\}$ be a set of representatives of the isomorphism classes of indecomposable $k[G]$-modules. Every $\mathcal{C} \in K_0(k[G])$ can now be written as a $\mathbb{Z}$-linear combination of the classes $[P_i]$, and all coefficients of this linear combination are nonnegative if and only if $\mathcal{C}$ is the class of a projective module. Using Proposition 2.3, one now easily shows that if $\alpha(\mathcal{C})$ is the class of a projective module in $K_0(\bar{k}[G])$, then $\mathcal{C}$ is the class of a projective module in $K_0(k[G])$, which proves the assertion. □

3 The equivariant Euler characteristic in terms of projective $k[G]$-modules

By a theorem of Nakajima, the equivariant Euler characteristic of any locally free G-sheaf on X lies in the image of the Cartan homomorphism $c : K_0(k[G]) \to K_0(G,k)$, provided that the canonical projection $\pi : X \to Y = X/G$ is *tamely ramified.* In this section, we will also consider the more general case where π is *weakly ramified.* We give both a necessary condition and a sufficient condition for the equivariant Euler characteristic to lie in the image of c, provided that the G-sheaf in question has rank 1 (comes from a divisor). Under this condition, we state an equivariant Riemann-Roch formula in the Grothendieck group of projective $k[G]$-modules.

We make the same assumptions and use the same notations as in section 1. In particular p denotes the projection $\bar{X} = X \times_k \bar{k} \to X$. Additionally, let $\bar{\pi}$ denote the canonical projection $\bar{X} \to \bar{Y} := \bar{X}/G = Y \otimes_k \bar{k}$, and let $\tilde{p}$ denote the projection $\bar{Y} \to Y$. We have the following commutative diagram:

$$\begin{array}{ccc} \bar{X} & \xrightarrow{\;p\;} & X \\ {\scriptstyle\bar{\pi}}\big\downarrow & & \big\downarrow{\scriptstyle\pi} \\ \bar{Y} & \xrightarrow{\;\tilde{p}\;} & Y \end{array}$$

Theorem 3.1. *If π is tamely ramified and $\mathcal{E}$ is a locally free G-sheaf on X, then the equivariant Euler characteristic $\chi(G,X,\mathcal{E})$ lies in the image of the Cartan homomorphism $c : K_0(k[G]) \to K_0(G,k)$.*

Proof. Follows directly from Theorem 1 in [Na]. □

Theorem 3.2. *Let $D = \sum_{P \in |X|} n_P P$ be a G-equivariant divisor on X.*

(a) If π is weakly ramified and $n_P \equiv -1 \mod e_P^w$ for all $P \in X$, then the equivariant Euler characteristic $\chi(G,X,\mathcal{L}(D))$ lies in the image of the Cartan homomorphism $c : K_0(k[G]) \to K_0(G,k)$. If moreover one of the cohomology groups $H^i(X,\mathcal{L}(D))$, $i = 0,1$, vanishes, then the other one is a projective $k[G]$-module.

(b) *Let* $\deg D > 2g_X - 2$. *If the* $k[G]$*-module* $H^0(X, \mathcal{L}(D))$ *is projective, then* π *is weakly ramified and* $n_P \equiv -1 \mod e_P^w$ *for all* $P \in |X|$.

Proof. If k is algebraically closed, the theorem coincides with Theorem 2.1 in [Kö2].
In the general case, if π is weakly ramified and D satisfies the congruence condition "$n_P \equiv -1 \mod e_P^w$ for all P", then $\bar{\pi} : \bar{X} \to \bar{Y}$ is weakly ramified, and by Lemma 1.5, the divisor p^*D on $\bar{X}$ also satisfies the congruence condition. By the special case, $\chi(G, X, \mathcal{L}(p^*D)$ then lies in the image of $\bar{c}$. Hence by Proposition 2.1, $\chi(G, X, \mathcal{L}(D)$ lies in the image of c. Here we have used that $H^i(X, \mathcal{L}(D)) \otimes_k \bar{k} = H^i(\bar{X}, \mathcal{L}(p^*D))$ for every i (cf. Proposition III.9.3 in [Ha]). This also implies the rest of part (a).
For part (b), let $\deg D > 2g_X - 2$. and let $H^0(X, \mathcal{L}(D))$ be projective. Then $\deg p^*D > 2g_{\bar{X}} - 2$ and $H^0(\bar{X}, \mathcal{L}(D))$ is projective. Thus $\bar{\pi} : \bar{X} \to \bar{Y}$ is weakly ramified and the congruence condition holds. But then π is weakly ramified also, and the congruence condition holds for D, again by Lemma 1.5. □

The following theorem generalizes Theorem 4.3 in [Kö2] and will be used in the formulation of the (main) Theorem 3.4. We refer the reader to page 1101 of the paper [Kö2] for an account of the nature, significance and history of the "ramification module" $N_{G,X}$ and for simplifications of formulae (1) and (2) when π is tamely ramified.

Theorem 3.3. *Let* π *be weakly ramified. Then there is a projective* $k[G]$*-module* $N_{G,X}$ *such that*

$$\bigoplus^n N_{G,X} \cong \bigoplus_{P \in X} \bigoplus_{d=1}^{e_P^t - 1} \bigoplus^{e_P^w \cdot d} \mathrm{Ind}_{I_P}^G(\mathrm{Cov}((\mathfrak{m}_P/\mathfrak{m}_P^2)^{\otimes d})), \tag{1}$$

where Cov *denotes the* $k[I_P]$*-projective cover. The class of* $N_{G,X}$ *in* $K_0(G, k)$ *is given by*

$$[N_{G,X}] = (1 - g_Y)[k[G]] - \chi(G, X, \mathcal{L}(E)) \tag{2}$$

where E *denotes the* G*-equivariant divisor* $E := \sum_{P \in X}(e_P^w - 1) \cdot P$.

Proof. Theorem 4.3 in [Kö2] yields that there is a projective $\bar{k}[G]$-module $N_{G,\bar{X}}$ such that

$$\bigoplus^n N_{G,\bar{X}} \cong \bigoplus_{Q \in \bar{X}} \bigoplus_{d=1}^{e_Q^t - 1} \bigoplus^{e_Q^w \cdot d} \mathrm{Ind}_{G_Q}^G(\mathrm{Cov}((\mathfrak{m}_Q/\mathfrak{m}_Q^2)^{\otimes d})),$$

and that the class of $N_{G,\bar{X}}$ is given by

$$[N_{G,\bar{X}}] = (1 - g_{\bar{Y}})[\bar{k}[G]] - \chi(G, X, \mathcal{L}(\bar{E}))$$

where $\bar{E} := \sum_{Q \in \bar{X}}(e_Q^w - 1) \cdot Q = p^*E$. Thus $[N_{G,\bar{X}}] = \beta(\mathcal{C})$ where

$$\mathcal{C} := (1 - g_Y)[k[G]] - \chi(G, X, \mathcal{L}(E)) \in K_0(G, k).$$

By Corollary 2.2, $\mathcal{C}$ is the class of some projective $k[G]$-module, say $N_{G,X}$. Using Lemma 1.3 and the injectivity of β, one easily shows that $N_{G,X}$ satisfies Formula (1). □

For every point $P \in X$, let f_P denote the residual degree $[k(P) : k(\pi(P))]$.

Theorem 3.4 (Equivariant Riemann-Roch formula). *Let π be weakly ramified.*

(a) Let $P \in |X|$ be a closed point. For every $d \in \{0, \ldots, e_P^t - 1\}$, there is a unique projective $k[G_P]$-module $W_{P,d}$ such that

$$\mathrm{Ind}_{I_P}^{G_P}(\mathrm{Cov}((\mathfrak{m}_P/\mathfrak{m}_P^2)^{\otimes(-d)})) \cong \bigoplus^{f_P} W_{P,d}$$

as $k[G_P]$-modules.

(b) Let $D = \sum_{P \in X} n_P \cdot P$ be a divisor on X with $n_P \equiv -1 \mod e_P^w$ for all $P \in X$. For any $P \in X$, we write

$$n_P = (e_P^w - 1) + (l_P + m_P e_P^t) e_P^w$$

with $l_P \in \{0, \ldots, e_P^t - 1\}$ and $m_P \in \mathbb{Z}$. Furthermore, for any $R \in Y$, fix a point $\tilde{R} \in \pi^{-1}(R)$. Then we have in $K_0(k[G])_{\mathbb{Q}}$:

$$\begin{aligned} &\chi(G, X, \mathcal{L}(D)) \\ &\quad = -[N_{G,X}] + \sum_{R \in Y} \sum_{d=1}^{l_{\tilde{R}}} [\mathrm{Ind}_{G_P}^G(W_{P,d})] + \left(1 - g_Y + \sum_{R \in Y} [k(R):k] m_{\tilde{R}}\right) [k[G]]. \quad (3) \end{aligned}$$

Proof. We first show that under the preconditions of (b), the following holds in the Grothendieck group with rational coefficients $K_0(k[G])_{\mathbb{Q}}$:

$$\begin{aligned} \chi(G, X, \mathcal{L}(D)) = -[N_{G,X}] + \sum_{R \in Y} \frac{1}{f_{\tilde{R}}} \sum_{d=1}^{l_{\tilde{R}}} [\mathrm{Ind}_{I_{\tilde{R}}}^G(\mathrm{Cov}((\mathfrak{m}_{\tilde{R}}/\mathfrak{m}_{\tilde{R}}^2)^{\otimes(-d)}))] \\ + \left(1 - g_Y + \sum_{R \in Y} [k(R):k] m_{\tilde{R}}\right) [k[G]] \quad (4) \end{aligned}$$

With suitably chosen divisors D, Formula (4) will then be used to show part (a). Formula (4) and part (a) obviously imply part (b).

For curves over algebraically closed fields, we have $f_P = 1$ for all P, so Formula (4) coincides with Theorem 4.5 in [Kö2].

The injective homomorphism $\beta : K_0(G, k) \to K_0(G, \bar{k})$ maps $\chi(G, X, \mathcal{E})$ to $\chi(G, \bar{X}, p^*\mathcal{E})$, and by Theorem 3.2, both of these Euler characteristics lie in the image of the respective Cartan homomorphisms. Hence it suffices to show that β maps every summand of the right-hand side of formula (3) (applied to X, D) to the corresponding summand of the right-hand side applied to $\bar{X}, p^*D$.

From the proof of Theorem 3.3, we see that $\beta([N_{G,X}]) = [N_{G,\bar{X}}]$.

By Lemma 1.5, we have $l_Q = l_P$ and $m_Q = m_P$ whenever $Q \in p^{-1}(P)$. Furthermore, the number of preimages of a point $R \in Y$ under $\pi : X \to Y$ is $\frac{n}{e_{\tilde{R}} f_{\tilde{R}}}$. For any $S \in |\bar{Y}|$, fix a point $\tilde{S} \in \bar{\pi}^{-1}(S)$. Using Lemma 1.3, we see that

$$\beta\left(\sum_{R\in Y}\frac{1}{f_{\tilde{R}}}\sum_{d=1}^{l_{\tilde{R}}}[\mathrm{Ind}_{I_{\tilde{R}}}^{G}(\mathrm{Cov}((\mathfrak{m}_{\tilde{R}}/\mathfrak{m}_{\tilde{R}}^2)^{\otimes(-d)}))]\right)$$
$$= \sum_{Q\in\bar{X}}\frac{e_Q}{n}\sum_{d=1}^{l_Q}[\mathrm{Ind}_{G_Q}^{G}(\mathrm{Cov}((\mathfrak{m}_Q/\mathfrak{m}_Q^2)^{\otimes(-d)}))]$$
$$= \sum_{S\in\bar{Y}}\sum_{d=1}^{l_{\tilde{S}}}[\mathrm{Ind}_{G_{\tilde{S}}}^{G}(\mathrm{Cov}((\mathfrak{m}_{\tilde{S}}/\mathfrak{m}_{\tilde{S}}^2)^{\otimes(-d)}))]$$

Moreover, we have

$$\beta\left(\left(1-g_Y+\sum_{R\in Y}[k(R):k]m_{\tilde{R}}\right)\ [k[G]]\right) = \left(1-g_{\bar{Y}}+\sum_{S\in\bar{Y}}m_{\tilde{S}}\right)[\bar{k}[G]],$$

which completes the proof of Formula (4).

We now prove part (a). Let $P \in X$ be a closed point. For $d = 0$, the statement is obvious because $(\mathfrak{m}_P/\mathfrak{m}_P^2)^0$ is the trivial one-dimensional $k(P)$-representation of I_P, so it decomposes into f_P copies of the trivial one-dimensional $k(R)$-representation of I_P, where $R := \pi(P)$. Hence we only need to do the inductive step from d to $d+1$, for $d \in \{0,\dots,e_P^t-2\}$.

If π is unramified at P, then $e_P^t = 1$, so there is no $d \in \{0,\dots,e_P^t-2\}$. Hence we may assume that π is ramified at P. Set $H := G_P$, the decomposition group at P, and let π' denote the projection $X \to X/H =: Y'$. For every closed point $Q \in |X|$ and for every $s \geq -1$, let $H_{Q,s}$ be the s-th ramification group at Q with respect to that cover, as introduced in Section 1. Then we have $H_{Q,s} = G_P \cap G_{Q,s}$ for every $s \geq -1$ and every $Q \in |X|$. In particular, if π is weakly ramified, then so is π'. For $Q = P$, we get $H_{P,s} = G_{P,s}$ for all $s \geq -1$; in particular, the ramification indices and residual degrees of π and π' at P are equal.

Let now $D := \sum_{Q\in|X|} n_Q \cdot Q$ be the H-equivariant divisor with coefficients

$$n_Q = \begin{cases} (d+2)e_Q^w - 1 & \text{if} \quad Q = P \\ e_Q^w - 1 & \text{otherwise} \end{cases}$$

Then formula (4) applied to H, X, D gives

$$\chi(H,X,\mathcal{L}(D)) = -[N_{H,X}] + \frac{1}{f_P}\sum_{n=1}^{d}[\mathrm{Ind}_{I_P}^{H}(\mathrm{Cov}((\mathfrak{m}_P/\mathfrak{m}_P^2)^{\otimes(-n)}))]$$
$$+\frac{1}{f_P}[\mathrm{Ind}_{I_P}^{H}(\mathrm{Cov}((\mathfrak{m}_P/\mathfrak{m}_P^2)^{\otimes(-(d+1))}))] + (1-g_{Y'})[k[H]] \quad (5)$$

in $K_0(k[H])_{\mathbb{Q}}$. By the induction hypothesis, the sum from $n = 1$ to d in this formula is divisible by f_P in $K_0(k[H])$; hence the remaining fractional term $\frac{1}{f_P}[\mathrm{Ind}_{I_P}^{H}(\mathrm{Cov}((\mathfrak{m}_P/\mathfrak{m}_P^2)^{\otimes(-(d+1))}))]$ must lie in $K_0(k[H])$. In other words, when writing $\mathrm{Ind}_{I_P}^{H}(\mathrm{Cov}((\mathfrak{m}_P/\mathfrak{m}_P^2)^{\otimes(-(d+1))}))$ as a direct sum of indecomposable projective $k[H]$-modules, every summand occurs with a multiplicity divisible by f_P. This proves the assertion. □

In the proof of Theorem 3.4(a), we have used a preliminary version of the equivariant Riemann-Roch formula to show the divisibility of $\mathrm{Ind}_{I_P}^{G_P}(\mathrm{Cov}((\mathfrak{m}_P/\mathfrak{m}_P^2)^{\otimes(-d)}))$ by f_P, i.e. we have used a global argument to prove a local statement. This tells us very litte about the structure of the summands $W_{P,d}$, which leads to the question whether one could find a "local" proof for the divisibility. In two different situations, the following proposition provides such a proof, yielding a concrete description of $W_{P,d}$.

Proposition 3.5. *Assume that π is tamely ramified, let $P \in |X|$ and $d \in \{1, \dots, e_P^t - 1\}$.*

(a) If $\mathrm{Gal}(k(P)/k(\pi(P)))$ is abelian, then we have $W_{P,d} \cong (\mathfrak{m}_P/\mathfrak{m}_P^2)^{\otimes(-d)}$ as $k[G_P]$-modules.

(b) If I_P is central in G_P, then $W_{P,d}$ is of the form $W_{P,d} = \mathrm{Ind}_{I_P}^{G}(\chi_d)$ for some $k[I_P]$-module χ_d. If moreover $G_P \cong I_P \times G_P/I_P$, then $W_{P,d} \cong (\mathfrak{m}_P/\mathfrak{m}_P^2)^{\otimes(-d)}$ as $k[G_P]$-modules.

Note that since every Galois extension of a finite field is cyclic, the first part of this proposition gives a "local" proof of Theorem 3.4(a) for the important case where π is tamely ramified and the underlying field k is finite.

Proposition 3.5 can be deduced from the following purely algebraic result. Note that, in this result, we don't use the notations introduced earlier in this paper; when Proposition 3.6 is being applied to prove Proposition 3.5, the fields k and l become the fields $k(\pi(P))$ and $k(P)$, respectively, the group G becomes G_P and V becomes $(\mathfrak{m}_P/\mathfrak{m}_P^2)^{\otimes(-d)}$ which is viewed only as a representation of I_P (and not of G_P) in Theorem 4.6(a).

Proposition 3.6. *Let l/k be a finite Galois extension of fields. Let G be a finite group, and let I be a cyclic normal subgroup of G, such that $G/I \cong \mathrm{Gal}(l/k)$, i.e. we have a short exact sequence*

$$1 \to I \to G \to \mathrm{Gal}(l/k) \to 1.$$

Let V be a one-dimensional vector space over l such that G acts semilinearly on V, that is, for any $g \in G, \lambda \in l, v, w \in V$, we have $g.(\lambda v + w) = \bar{g}(\lambda)(g.v) + g.w$, where $\bar{g}$ denotes the image of g in $\mathrm{Gal}(l/k)$.

(a) If $\mathrm{Gal}(l/k)$ is abelian, then we have $\mathrm{Ind}_I^G \mathrm{Res}_I^G(V) \cong \bigoplus^{(G:I)} V$ as $k[G]$-modules.

(b) If I is central in G, then there is a (non-trivial) one-dimensional k-representation χ of I such that $\mathrm{Res}_I^G(V) \cong \bigoplus^{(G:I)} \chi$ as $k[I]$-modules.

If moreover $G = I \times \mathrm{Gal}(l/k)$, then we have $\mathrm{Ind}_I^G \chi \cong V$ and $\mathrm{Ind}_I^G \mathrm{Res}_I^G(V) \cong \bigoplus^{(G:I)} V$ as $k[G]$-modules.

Proof. (a) We have (isomorphisms of $k[G]$-modules):

$$\begin{aligned}
&\mathrm{Ind}_I^G \mathrm{Res}_I^G(V) \\
&\quad \cong V \otimes_k \mathrm{Ind}_I^G(k) && \text{by Corollary 10.20 in [CR]} \\
&\quad \cong V \otimes_k k[G/I] && \text{(cf. §10A in [CR])} \\
&\quad \cong V \otimes_k k[\mathrm{Gal}(l/k)] && \text{as } \mathrm{Gal}(l/k) \cong G/I \\
&\quad \cong V \otimes_k l \\
&\quad \cong \bigoplus_{\sigma \in \mathrm{Gal}(l/k)} V.
\end{aligned}$$

The last two isomorphisms can be derived as follows. By the normal basis theorem, there is an element $x_0 \in l$ such that $\{g(x_0) | g \in \mathrm{Gal}(l/k)\}$ is a basis of l over k. The resulting isomorphism

$$\begin{aligned} k[\mathrm{Gal}(l/k)] &\to l \quad \text{given by} \\ [g] &\mapsto g(x_0) \quad \text{for every } g \in \mathrm{Gal}(l/k). \end{aligned}$$

is obviously $k[G]$-linear. This is the second last isomorphism. For the last one, we define

$$\begin{aligned} \varphi : l \otimes_k V &\to \bigoplus_{\sigma \in \mathrm{Gal}(l/k)} V \quad \text{by} \\ a \otimes v &\mapsto (\sigma(a) \cdot v)_{\sigma \in \mathrm{Gal}(l/k)} \quad \text{for every } a \in l, v \in V. \end{aligned}$$

φ is an isomorphism of vector spaces over k, by the Galois Descent Lemma. If $\mathrm{Gal}(l/k)$ is commutative, then φ is also compatible with the G-action on both sides: Let $a \in l$, $v \in V$, $g \in G$, then we have

$$\begin{aligned} \varphi(g.(a \otimes v)) = \varphi(\bar{g}(a) \otimes g.v) = ((\sigma\bar{g})(a) \cdot g.v)_{\sigma \in \mathrm{Gal}(l/k)} &= ((\bar{g}\sigma)(a) \cdot g.v)_{\sigma \in \mathrm{Gal}(l/k)} \\ &= g.((\sigma(a) \cdot v)_{\sigma \in \mathrm{Gal}(l/k)}) = g.\varphi(a \otimes v). \end{aligned}$$

(b) Since I is cyclic, it acts by multiplication with e-th roots of unity, where e divides $|I|$. If I is central in G, then it follows that the e-th roots of unity are contained in k. For if h is a generator of I and $h.v = \zeta_e \cdot v$ for all $v \in V$, ζ_e an e-th root of unity, then we have for all $g \in G$ and all $v \in V$:

$$\bar{g}(\zeta_e)(g.v) = g.(\zeta_e v) = (gh).v = (hg).v = \zeta_e(g.v).$$

Hence for everg $\bar{g} \in \mathrm{Gal}(l/k)$, we have $\bar{g}(\zeta_e) = \zeta_e$, which means that ζ_e lies in k. Let now $\{x_1, \ldots, x_f\}$ be a k-basis of V, where $f = (G : I)$. Then we have $V = kx_0 \oplus \ldots \oplus kx_f$ not only as vector spaces over k, but also as $k[I]$-modules, since

$$Ix_i = \{\zeta_e^j x_i | j = 0, \ldots, e-1\} \subseteq kx_i$$

for every basis vector x_i. Furthermore, the summands kx_i are isomorphic as $k[I]$-modules because I acts on each of them by multiplication with the same roots of unity in k. Setting for example $kx_1 =: \chi$, we can write

$$\mathrm{Res}_I^G(V) \cong \bigoplus^f \chi$$

as requested.

Assume now that $G = I \times \mathrm{Gal}(l/k)$. Then by the Galois Descent Lemma, we have

$$V \cong l \otimes_k V^{\mathrm{Gal}(l/k)}$$

as $k[G]$-modules, where I acts trivially on l and $\mathrm{Gal}(l/k)$ acts trivially on $V^{\mathrm{Gal}(l/k)}$. This is isomorphic to $l \otimes_k \chi$, where χ is regarded as a $k[G]$-module via the projection $G = I \times \mathrm{Gal}(l/k) \to I$. By the normal basis theorem, we have

$$l \otimes_k \chi \cong \mathrm{Ind}_I^G(k) \otimes \chi = \mathrm{Ind}_I^G(\chi),$$

so $V \cong \mathrm{Ind}_I^G(\chi)$ as requested. Together with what we have shown before, this implies the last identity of the proposition:

$$\mathrm{Ind}_I^G \mathrm{Res}_I^G(V) = \mathrm{Ind}_I^G(\bigoplus^f \chi) = \bigoplus^f V.$$

□

4 Some variants of the main theorem

Throughout the previous section, we have concentrated on the case where $\pi : X \to Y$ is weakly ramified and where the locally free G-sheaf we are considering comes from an equivariant divisor. If π is tamely ramified, we have the following variant of Theorem 3.4 for locally free G-sheaves that need not come from a divisor. It generalizes Corollary 1.4(b) in [Kö1].

Theorem 4.1. *Let $\pi : X \to Y$ be tamely ramified. Let $\mathcal{E}$ be a locally free G-sheaf of rank r on X. For every closed point $P \in |X|$ and for $i = 1, \ldots, r$, let the integers $l_{P,i} \in \{0, \ldots, e_P - 1\}$ be defined by the following isomorphism of $k(P)[I_P]$-modules:*

$$\mathcal{E}(P) \cong \bigoplus_{i=1}^{r} \left(\mathfrak{m}_P/\mathfrak{m}_P^2 \right)^{\otimes l_{P,i}} .$$

For every $R \in |Y|$, let $\tilde{R} \in |X|$ and $W_{\tilde{R},d}$ be defined as in Theorem 3.4. Furthermore, let $N_{G,X}$ be the ramification module from Theorem 3.3. Then we have in $K_0(k[G])$:

$$\chi(G, X, \mathcal{E}) \equiv -r[N_{G,X}] + \sum_{R\in Y} \sum_{i=1}^{r} \sum_{d=1}^{l_{\tilde{R},i}} [\mathrm{Ind}_{G_{\tilde{R}}}^{G}(W_{\tilde{R},d})] \mod \mathbb{Z}[G].$$

Moreover, one can show an equivariant Riemann-Roch formula for *arbitrarily ramified covers* $\pi : X \to Y$. Recall that in Theorem 3.2, we have shown that in virtually all cases where the Euler characteristic lies in the image of the Cartan homomorphism, the cover π is weakly ramified. So in the general case, one cannot possibly find a formula in the Grothendieck group $K_0(k[G])$ of projective $k[G]$-modules. However, in the Grothendieck group $K_0(G, k)$ of all $k[G]$-modules, we have the following result, which generalizes Theorem 3.1 in [Kö2].

Theorem 4.2. *Let $\mathcal{E}$ be a locally free G-sheaf. Then we have in $K_0(G, k)$:*

$$n\,\chi(G, X, \mathcal{E}) = C_{G,X,\mathcal{E}}\,[k[G]] - \sum_{P\in|X|} e_P^w \sum_{d=0}^{e_P^t-1} d\,[\mathrm{Ind}_{I_P}^{G} \left(\mathcal{E}(P) \otimes_{k(P)} (\mathfrak{m}_P/\mathfrak{m}_P^2)^{\otimes d} \right)],$$

where

$$C_{G,X,\mathcal{E}} = r(1 - g_X) + \deg \mathcal{E} + \frac{r}{2} \sum_{P\in|X|} [k(P) : k](e_P^t - 1).$$

We omit the proofs of Theorem 4.1 and Theorem 4.2 due to their similarity with the proof of Theorem 3.4.

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