Parameterization of Rainfall Microstructure for Radar Meteorology and Hydrology

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Published before in December 1999 as doctoral thesis

Rapport 109

Sub-department Water Resources
Nieuwe Kanaal 11, 6709 PA Wageningen
The Netherlands
Internet: www.dow.wau.nl/whh

ISSN 0926-230X

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Parameterization of Rainfall Microstructure for Radar Meteorology and Hydrology

PROEFSCHRIFT
ter verkrijging van de graad van doctor
op gezag van de rector magnificus
van Wageningen Universiteit,
dr. C. M. Karssen,
in het openbaar te verdedigen
op dinsdag 21 december 1999
des namiddags te vier uur in de Aula.

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Technische Universiteit Delft, International Research Centre
for Telecommunications-transmission and Radar

Co-promotor: ir. J. N. M. Stricker
universitair hoofddocent bij de leerstoelgroep hydrologie
en kwantitatief waterbeheer

Financial support for the research in this thesis has been provided by the Commission of the European Communities under the EPOCH and ENVIRONMENT programmes (Contracts No. EPOC-CT90-0026, EV5V-CT92-0182 and ENV4-CT96-5030).

Opponenten:

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Druk: Universal Press, Veenendaal
Foto omslag: Quirin van Os / KNMI

CIP-DATA KONINKLIJKE BIBLIOTHEEK, DEN HAAG

Uijlenhoet, Remko

Parameterization of rainfall microstructure for radar meteorology and hydrology / Remko Uijlenhoet.
Doctoral thesis Wageningen University. – With ref. – With summary in Dutch.
ISBN 90-5808-156-7
Subject headings: hydrology / meteorology.

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Abstract

Uijlenhoet, R., 1999. *Parameterization of rainfall microstructure for radar meteorology and hydrology.* Doctoral thesis, Wageningen University, The Netherlands. 279 pp.

A comprehensive general framework for the description and analysis of the microstructure of rainfall is presented. The microstructure of rainfall is parameterized in terms of the raindrop size distribution, which determines both the macroscopic physical properties of rainfall relevant for radar meteorology and hydrology and their relationships.

To demonstrate that the definitions of rainfall related variables naturally lead to power law relationships, a rainfall parameterization based on the exponential raindrop size distribution is presented. The importance of the distinction between the properties of raindrops present in a volume of air and those of raindrops arriving at a surface is emphasized.

A general formulation for the raindrop size distribution as a scaling law is derived, based on the ubiquitous power law relationships between rainfall related variables. The scaling law formulation is independent of any a priori assumption regarding the functional form of the raindrop size distribution and unifies all previously published parameterizations. It allows a separation of the effects of changes in the shape of the raindrop size distribution from those of changes in the rain rate. The values of the scaling exponents indicate whether it is the raindrop concentration or the characteristic raindrop size which controls the variability of the raindrop size distribution. The gap between the scaling law and traditional parameterizations is bridged by providing explicit expressions for the scaling law for all analytical distributions proposed in the literature.

The scaling law formulation is verified experimentally using mean raindrop size distributions for various climatic settings (based on two classical parameterizations) and raw raindrop size distributions from The Netherlands. For the mean distributions the scaling procedure yields excellent results, for the raw distributions a residual amount of scatter about the mean curves remains, indicating that rain rate alone cannot explain all observed variability.

As an example of the application of the scaling law, a new method for establishing power law radar reflectivity–rain rate relationships is derived. The method is applied to the mentioned mean and raw distributions. The large inter-event variability of the coefficients indicates that climatological radar reflectivity–rain rate relationships will be of little practical use.

The Poisson homogeneity hypothesis, a fundamental assumption in radar meteorology, is tested on an extraordinary stationary time series of raindrop size distributions. The arrival rate fluctuations of the raindrops which contribute most to rain rate and radar reflectivity are found to behave according to Poisson statistics.

Finally, perspectives for future research are presented.

Additional index words: raindrop size distribution, scaling law, remote sensing, radar meteorology, hydrology.

Voorwoord

Hoewel universitaire bestuurders en politici daar nog wel eens anders over willen denken, valt wetenschappelijk onderzoek in het algemeen lastig te plannen. In dat verband is het illustratief de titel van een proefschrift eens met de titel van het oorspronkelijke onderzoeksvoorstel te vergelijken. In mijn geval luidde dat laatste: 'Toepassing van verschillende typen weerradar voor het schatten van neerslag over een verstedelijkt gebied ten behoeve van het waterbeheer'. Het zal duidelijk zijn dat dit proefschrift een enigszins afwijkend onderwerp behandelt. De aandacht is verschoven van de toepassing naar aspecten van de achterliggende theorie. Hetgeen overigens niet wegneemt dat de motivatie voor dit werk altijd de hydrologische toepassing is gebleven. Dat is ook de reden geweest waarom de EU dit onderzoek het afgelopen decennium heeft willen financieren. Uit experimenten uitgevoerd gedurende de eerste jaren van het project bleek echter dat we niet in staat waren om bevredigende verklaringen te vinden voor de discrepanties tussen wat weerradars waarnemen en wat de traditionele regenmeters meten. Daarom is de nadruk van het onderzoek in de loop der jaren meer komen te liggen op het beter begrijpen van de manier waarop de verschillende instrumenten regen meten en daarmee samenhangend het beter begrijpen van de structuur van regen zelf. Dit proefschrift behandelt enkele recente ontwikkelingen op dit laatste gebied. Met deze nieuwe inzichten kunnen we nu terug naar de radarmetingen om een poging te wagen de radarhydrologie vlot te trekken.

De eerste die ik op deze plaats moet bedanken is mijn directe begeleider, Han Stricker. Hij heeft het onderzoek naar de hydrologische toepassing van weerradar in Wageningen eind jaren tachtig een krachtige impuls gegeven door samenwerking met de radardeskundigen van de Technische Universiteit Delft te zoeken. De regen die voor telecommunicatieonderzoekers 'ruis' is, is voor hydrologen immers 'signaal'. Deze gezamenlijke interesse heeft de basis gelegd voor mijn promotieonderzoek en dit proefschrift is hiervan het uiteindelijke resultaat. Han heeft de inhoudelijke keuzes die ik de afgelopen jaren heb gemaakt altijd gesteund, ook toen het accent van het onderzoek verschoof van toegepast naar meer theoretisch. Hij omschreef dit proces eens als 'van de regen in de drup'. Aannemende dat hij dit in de letterlijke zin van het woord bedoelde, kan de essentie van dit proefschrift inderdaad niet kernachtiger worden omschreven.

Herman Russchenberg is vanaf het begin actief bij het onderzoek betrokken geweest. Mede door zijn toedoen is de 'signaal/ruis verhouding' van het Delftse radaronderzoek de laatste jaren aanzienlijk toegenomen. Regen en wolken zijn van alleen storende factoren een zelfstandig object van studie geworden aan het IRCTR. Dat is de samenwerking met Wageningse en andere onderzoekers vanzelfsprekend alleen maar ten goede gekomen. De interesse van Herman voor mijn werk gedurende de afsluitende fase van het onderzoek heb ik als zeer motiverend ervaren.

Mijn promotoren, Prof. Feddes en Prof. Ligthart, bedank ik voor het in mij gestelde vertrouwen. Hoewel ik vrees beiden meer dan eens tot wanhoop gedreven te hebben, zowel door de duur van het onderzoek op zich als door de stortvloed aan leeswerk in de laatste fase, hoop ik dat dit proefschrift toch het begin kan vormen van een hernieuwde Wagenings-Delftse samenwerking op het gebied van de radar remote sensing van neerslag.

Een speciaal woord van dank komt toe aan Herman Wessels. Niet alleen stelde hij ten behoeve van dit onderzoek de unieke druppelgroottemetingen beschikbaar die hij eind jaren zestig samen met zijn collega's van het KNMI verrichtte (en die in Hoofdstukken 5 en 6 geanalyseerd worden), hij las ook grote delen van het manuscript van dit proefschrift kritisch door. De discussies die we naar aanleiding daarvan hadden waren voor mij zeer leerzaam. Wat betreft de toekomstige samenwerking met het KNMI hoop ik dat ook de gegevens van de nieuwe dopplerradars hun weg zullen vinden naar het hydrologisch onderzoek.

I have spent the last two and a half years working as a research fellow at the Laboratoire d'étude des Transferts en Hydrologie et Environnement in Grenoble, France. First and foremost, I need to thank Dominique Creutin for providing me with the perfect environment to carry out my research and to develop myself as an independent researcher. His guidance, both in matters of science and project management, has been extremely helpful. Secondly, I thank Guy Delrieu, with whom I shared an office. The many discussions we had have been very stimulating and have helped me to set my research priorities for the future. I also acknowledge the support of Michel Vauclin, director of LTHE, for having welcomed me as a researcher in 'his' laboratory. My colleagues, both of the Equipe Hydrométéorologie and of the other research groups, have made me feel at home during my stay at LTHE. Merci à tous!

A particular acknowledgment goes to my Catalan colleagues and friends Pep Porrà and Daniel Sempere Torres. The many discussions we had have truly shaped my mind. My shorter and longer visits to Barcelona over the past couple of years have been unforgettable.

The optical disdrometer data analyzed in Chapter 7 of this thesis have been collected as part of the NERC Special Topic HYREX. They have been kindly provided to me by Bob Moore of the Institute of Hydrology in Wallingford, United Kingdom.

Tenslotte bedank ik Marjan en Sander voor de afleiding die ze me (soms ongewild) bezorgd hebben. In ieder geval weet ik na de voltooiing van dit proefschrift nu tenminste ook wat 'een zware bevalling' precies inhoudt.

Remko Uijlenhoet
Wageningen, november 1999

Contents

List of symbols

Symbol	Meaning	Dimension	Unit
a	63% quantile of $F_W(D)$	L	mm
c	Prefactor of power law $v(D)$ relationship	$\mathrm{L}^{1-\gamma}\,\mathrm{T}^{-1}$	$\mathrm{m\,s^{-1}\,mm^{-\gamma}}$
c_ω	Prefactor of power law ω–D relationship	–	–
c_{Ω_m}	Proportionality factor in definition of Ω_m	–	–
$\mathrm{erf}(\cdot)$	Error function	–	–
$f_{\underline{D}_{\mathrm{A,V}}}(D)$	Probability density function of raindrop diameters at a surface or in a volume	L^{-1}	mm^{-1}
$f_{D_{\mathrm{C}}}(\cdot)$	Dimensionless raindrop size distribution function	–	–
$f_R(D)$	Rain rate density function	L^{-1}	mm^{-1}
$f_W(D)$	Liquid rainwater density function	L^{-1}	mm^{-1}
$f_{\underline{\omega}_{\mathrm{A,V}}}(\omega)$	Probability density function of raindrop properties at a surface or in a volume	–	–
$f_{\Omega_{\mathrm{A,V}}}(D)$, $f_{\Omega_m}(D)$	Density function of rainfall flux or state variables	L^{-1}	mm^{-1}
$g(x)$, $g'(x)$	General raindrop size distribution function, R as reference variable	$\mathrm{L}^{-(4+\alpha)}\,\mathrm{T}^{\alpha}$	$\mathrm{mm^{-1}\,m^{-3}}\times$ $(\mathrm{mm\,h^{-1}})^{-\alpha}$
$g_\Psi(x)$, $g'_\Psi(x)$	General raindrop size distribution function, Ψ as reference variable	–	–
$h(x)$	General rain rate density function	$\mathrm{L}^{-(1-\beta)}\,\mathrm{T}^{-\beta}$	$\mathrm{mm^{-1}}\times$ $(\mathrm{mm\,h^{-1}})^{\beta}$
$h_{\Omega_m}(x)$	General moment density function	–	–
k	Specific microwave attenuation coefficient	L^{-1}	$\mathrm{dB\,km^{-1}}$
m	Raindrop mass	M	mg
	Order of moment	–	–
n	Number of raindrops	–	–
p	Exponent of power law a–R relationship	–	–
r	Range	L	km
	Exponent of power law W–R relationship	–	–
r^2	Empirical coefficient of determination	–	–
s	Proportionality factor	–	–
t	Time	T	s, min
v	Raindrop terminal fall speed	$\mathrm{L\,T^{-1}}$	$\mathrm{m\,s^{-1}}$

Symbol	Meaning	Dimension	Unit
$w_\Phi(D)$	Weighting function	–	–
x	Scaled raindrop diameter	–	–
A	Geometrical cross-section	L^2	cm^2
	Prefactor of power law a–R relationship	$L^{1-p}\,T^p$	$mm\,(mm\,h^{-1})^{-p}$
C	Radar constant	$M\,L\,T^{-3}$	$W\,km^2\times$ $(mm^6\,m^{-3})^{-1}$
	Prefactor of power law W–R relationship	$M\,L^{-r}\,T^r$	$mg\,(mm\,h^{-1})^{-r}$
C_R	Prefactor of power law R–Z relationship	$L^{1-3\gamma_R}\,T^{-1}$	$mm\,h^{-1}\times$ $(mm^6\,m^{-3})^{-\gamma_R}$
C_X	Prefactor of power law X–Ψ relationship	–	–
C_Z	Prefactor of power law Z–R relationship	$L^{3-\gamma_Z}\,T^{\gamma_Z}$	$mm^6\,m^{-3}\times$ $(mm\,h^{-1})^{-\gamma_Z}$
CK	Coefficient of kurtosis	–	–
Cov	Covariance operator	–	–
CS	Coefficient of skewness	–	–
CV	Coefficient of variation	–	–
D	Equivalent spherical raindrop diameter	L	mm
D_C	Characteristic diameter	L	mm
D_0	Median-volume diameter	L	mm
D_m	Volume-weighted mean diameter	L	mm
$\overline{D}_m$	Weighted mean diameter	L	mm
D_p	Predominant raindrop diameter	L	mm
D_V	Mean-volume diameter	L	mm
E	Expectation operator	–	–
E	Raindrop kinetic energy	$M\,L^2\,T^{-2}$	J
$F_R(D)$	Rain rate distribution function	–	–
$F_W(D)$	Liquid rainwater distribution function	–	–
$F_{\Omega_m}(D)$	Moment distribution function	–	–
$H_{\Omega_m}(x)$	General moment distribution function	–	–
K	$(\epsilon-1)\,/\,(\epsilon+2)$	–	–
M	Raindrop momentum	$M\,L\,T^{-1}$	$kg\,m\,s^{-1}$
$N_A(D)$	Raindrop size distribution at a surface	$L^{-3}\,T^{-1}$	$mm^{-1}\,m^{-2}\,s^{-1}$
$N(D)$, $N_V(D)$	Raindrop size distribution in a volume	L^{-4}	$mm^{-1}\,m^{-3}$
N_0	Parameter of gamma raindrop size distribution	$L^{-(4+\mu)}$	$mm^{-(1+\mu)}\,m^{-3}$
P	Rainfall pressure	$M\,L^{-1}\,T^{-2}$	Pa
$\overline{P}_r$	Mean power received by radar from targets at range r	$M\,L^2\,T^{-3}$	W
Q_b	Radar backscattering cross-section	L^2	cm^2
Q_t	Microwave extinction cross-section	L^2	cm^2
$Q_{t,o}$	Optical extinction cross-section	L^2	cm^2
R	Rain rate	$L\,T^{-1}$	$mm\,h^{-1}$
S	Optical extinction coefficient	L^{-1}	km^{-1}

Symbol	Meaning	Dimension	Unit
S_e	Self-consistency of exponent	–	–
S_p	Self-consistency of prefactor	–	–
U	Kinetic energy flux density	$\mathrm{M\,T^{-3}}$	$\mathrm{W\,m^{-2}}$
V	Raindrop volume	$\mathrm{L^3}$	$\mathrm{mm^3}$
Var	Variance operator	–	–
W	Liquid rainwater content	$\mathrm{M\,L^{-3}}$	$\mathrm{mg\,m^{-3}}$
Z	Radar reflectivity factor	$\mathrm{L^3}$	$\mathrm{mm^6\,m^{-3}}$
α, β, β'	Scaling exponents, R as reference variable	–	–
$\alpha_\Psi, \beta_\Psi, \beta'_\Psi$	Scaling exponents, Ψ as reference variable	–	–
γ	Exponent of power law $v(D)$ relationship	–	–
γ_R	Exponent of power law R–Z relationship	–	–
γ_X	Exponent of power law X–Ψ relationship	–	–
γ_Z	Exponent of power law Z–R relationship	–	–
γ_ω	Exponent of power law ω–D relationship	–	–
δ	Proportionality factor	–	–
ϵ	Complex dielectric constant of water	–	–
κ	Parameter of $g(x)$ and $h(x)$	–	–
λ	Radar wavelength	L	cm
	Prefactor of power law Λ–R relationship	$\mathrm{L^{-(1-\beta)}T^{-\beta}}$	$\mathrm{mm^{-1}\times}$ $\mathrm{(mm\,h^{-1})^{\beta}}$
μ	Parameter of gamma and lognormal raindrop size distributions	–	–
μ_x	Mean of x	–	–
$\mu'_{x,r}$	Moment of order r of x	–	–
ν	Parameter of generalized gamma raindrop size distribution	–	–
ρ_A	Mean raindrop arrival rate	$\mathrm{L^{-2}\,T^{-1}}$	$\mathrm{m^{-2}\,s^{-1}}$
ρ_V	Mean raindrop concentration	$\mathrm{L^{-3}}$	$\mathrm{m^{-3}}$
ρ_w	Density of water	$\mathrm{M\,L^{-3}}$	$\mathrm{kg\,m^{-3}}$
ρ^2	Theoretical coefficient of determination	–	–
σ	Parameter of lognormal raindrop size distribution	–	–
σ_x	Standard deviation of x	–	–
ω	Raindrop property	–	–
Λ	Parameter of gamma raindrop size distribution	$\mathrm{L^{-1}}$	$\mathrm{mm^{-1}}$
$\Gamma(\cdot)$	Gamma function	–	–
$\Gamma(\cdot,\cdot)$	Incomplete gamma function	–	–
Φ	Rainfall integral variable	–	–
Ψ	Reference variable	–	–
Ω, Ω_m	Rainfall related variable proportional to moment of raindrop size distribution	–	–
Ω_A	Rainfall flux variable	–	–
Ω_V	Rainfall state variable	–	–

Chapter 1

Introduction

1.1 Rationale

This thesis deals with *raindrops.* Anyone who has ever heard rain ticking on an umbrella or watched rain splashing in a pool knows very well that rain actually consists of individual raindrops which occur in varying numbers and have different positions, sizes and fall speeds. In other words, rainfall is a *discrete* process.

Nevertheless, hydrologists and to a lesser extent meteorologists have traditionally considered rainfall to be a *continuous* process. They have typically concentrated on the *average* properties of rainfall over sufficiently large volumes and time intervals. Knowledge of the exact positions, sizes and fall speeds of the individual raindrops is then no longer necessary. Consequently, the highly stochastic, discrete nature of rainfall at smaller spatial and temporal scales is treated only in a *statistical* sense. The true small scale variability of rainfall is represented by means of the statistical *distributions* of the numbers, positions, sizes and fall speeds of the raindrops within a reference volume or time interval. In other words, the *microstructure* of rainfall is not explicitly resolved at the larger scale, but only taken into account via a statistical model. Modeling small scale variability which is not resolved explicitly at a larger scale is called *parameterization.*

Hydrologists and meteorologists often make an additional simplification. They focus their attention entirely on only one aspect of the averaged rainfall process: the rain rate. This quantity is traditionally denoted as R and expressed in units of $\mathrm{mm\,h^{-1}}$. It represents the average mass flux density over a certain surface area and a certain time interval. Hence, it is a *macroscopic* property, related to the numbers, positions, sizes and fall speeds of the individual raindrops in a statistical manner[1].

In summary, in hydrology and meteorology the enormous complexity of the rainfall process tends to be reduced to a continuous *field* which describes the spatial and temporal variations of just one quantity: the rain rate. There are two main problems associated with this simplified representation of reality. The first is that R is just one

[1]In this sense, there is a clear analogy between rainfall and statistical mechanics. The rain rate R is related to the numbers, positions, sizes and speeds of the individual raindrops in much the same way as the macroscopic thermodynamic quantities (such as pressure) are related to the numbers, positions and speeds of the individual molecules (e.g. van Kampen, 1992).

of a whole range of possible quantities which could be used to characterize rainfall. It merely represents the average mass flux density at each point of the rainfall field[2]. Although this may be an appropriate quantity for many applications, it is not necessarily the most suitable for others. However, all macroscopic rainfall quantities and the relationships between them depend on the structure of rainfall on a *microscopic* scale. Therefore, if other quantities than R are needed for a particular application, or perhaps the relationships of those quantities with R, then a more detailed description of rainfall is required, one in which its discrete nature is taken into account, either explicitly or implicitly.

The second problem associated with the traditional description of rainfall as a continuous field is that at smaller and smaller spatial and temporal scales, the field approximation breaks down. The 'continuous' flux of water then becomes the highly intermittent, discrete process one observes in reality (Rodriguez-Iturbe et al., 1984; Fabry, 1996). To be able to account for this transition from a continuous to a discrete process, the continuum hypothesis has to be abandoned in favor of a more detailed description of the microstructure of rainfall.

The main reason why hydrologists have generally disregarded the microstructure of rainfall is because the spatial and temporal scales associated with it are thought to be insignificant as compared to the characteristic scales of typical hydrological processes such as rainfall-runoff transformations. There are two relatively recent developments which are stimulating the interest of the hydrological community in the microstructure of rainfall: (1) the increased use of weather radar for estimating the spatial and temporal distribution of rainfall (e.g. Collier, 1986a,b; Collier and Knowles, 1986; Krajewski, 1987; Creutin et al., 1988; Delrieu et al., 1988; Azimi-Zonooz et al., 1989; Seo et al., 1990a,b; Seo and Smith, 1991a,b; Smith, 1993b; Uijlenhoet et al., 1994, 1995, 1997, 1999a; Smith et al., 1996a,b; Andrieu et al., 1997; Creutin et al., 1997; Sempere Torres et al., 1999a); (2) the increased interest for processes at the land surface, such as rainfall interception by vegetation canopies (Calder, 1986; Dolman and Gregory, 1992; Eltahir and Bras, 1993; Hall and Calder, 1993; Calder, 1996a,b; Calder et al., 1996; Hall et al., 1996), soil detachment and erosion by raindrop impact (WMO, 1983; Rosewell, 1986; Sempere Torres et al., 1992; Sharma et al., 1993; Agassi et al., 1994), infiltration of rain water into the soil and surface runoff.

Knowledge of the microstructure of rainfall is indispensable for the hydrological application of weather radar because the relationship between measured radar reflectivity and surface rainfall (both macroscopic quantities) depends strongly on the microscopic structure of rainfall. For a proper understanding of the mentioned land surface processes knowledge of the microstructure of rainfall is required because these are in general highly nonlinear processes to which literally every single raindrop can make a significant contribution. Major hydroclimatological research programs where

[2]Classical examples of hydrometeorological models of the space and/or time structure of (continuous) rainfall fields are those due to Le Cam (1961), Waymire and Gupta (1981a,b,c), Smith and Karr (1983), Rodriguez-Iturbe et al. (1984), Waymire et al. (1984), Smith and Karr (1985), Rodriguez-Iturbe (1986), Rodriguez-Iturbe et al. (1986), Rodriguez-Iturbe and Eagleson (1987), Rodriguez-Iturbe et al. (1987), Smith (1987) and Rodriguez-Iturbe et al. (1988).

radar remote sensing of rainfall and land surface processes play a prominent role include NASA's Tropical Rainfall Measuring Mission (TRMM) (Simpson et al., 1988; Meneghini et al., 1999), the Global Energy and Water Cycle Experiment (GEWEX) (Chahine, 1992) and the Next Generation Weather Radar system (NEXRAD) (Hudlow et al., 1991). Two other disciplines where the microstructure of rainfall plays an important role are: (1) meteorology, e.g. radar meteorology (e.g. Doviak and Zrnić, 1993), cloud and precipitation physics (Rogers and Yau, 1996) and aerosol scavenging (e.g. Pruppacher and Klett, 1978); (2) telecommunications, e.g. the study of the distortion of radio signals on earth and space-based communication links (e.g. Crane, 1971; Olsen et al., 1978).

The subject of this thesis lies at the intersection of stochastic rainfall modeling and radar remote sensing of precipitation. At Wageningen Agricultural University exists a tradition of doctoral theses dealing with (stochastic) rainfall modeling (e.g. van Montfort, 1966; Buishand, 1977; Witter, 1984; de Lima, 1998). In a similar manner, several thesis research projects carried out at Delft University of Technology have dealt with radar remote sensing of precipitation (e.g. Klaassen, 1989; Russchenberg, 1992). This thesis in a sense bridges the gap between these traditions.

1.2 Rainfall microstructure

1.2.1 A static picture of rainfall

An example of a more detailed description of the microstructure of rainfall is provided by Fig. 1.1. Although it is merely a schematic representation of reality, it serves to show some of the features of the microstructure of rainfall which are relevant to this thesis. First of all, although the raindrops are distributed homogeneously in space *on the average*, their local concentration is not everywhere the same. For a volume of a given size, the numbers of raindrops it contains will therefore fluctuate in space and in time. On the average, 1 m^3 of air typically contains of the order of 10^3 raindrops. Closely related to the numbers of raindrops in a volume of air are the distances between them. Again, these will be subject to statistical fluctuations, but a typical mean distance would be of the order of 10 cm. A third and very prominent feature is that raindrops have different sizes. Their diameters range typically from 0.1 to 6 mm. Although Fig. 1.1 does not show this very clearly, in reality there are many more small raindrops than large ones. The majority of the raindrops encountered in nature are smaller than 3 mm (e.g. Rogers and Yau, 1996).

A fundamental property of rainfall in this respect is its so-called *raindrop size distribution* $N(D)$. In its traditional definition, the quantity $N(D)\mathrm{d}D$ represents the expected (mean) number of raindrops with diameters between D and $D+\mathrm{d}D$ present per unit volume of air. The dimensions of the function $N(D)$ are therefore L^{-4}, where L stands for length. With D expressed in mm and volume in m^3, the units of $N(D)$ become $\mathrm{mm}^{-1}\,\mathrm{m}^{-3}$. According to this definition, the notion of a raindrop size distribution is a mixture of two different concepts, namely that of the spatial distribution of raindrops in a volume of air (which governs the raindrop concentration)

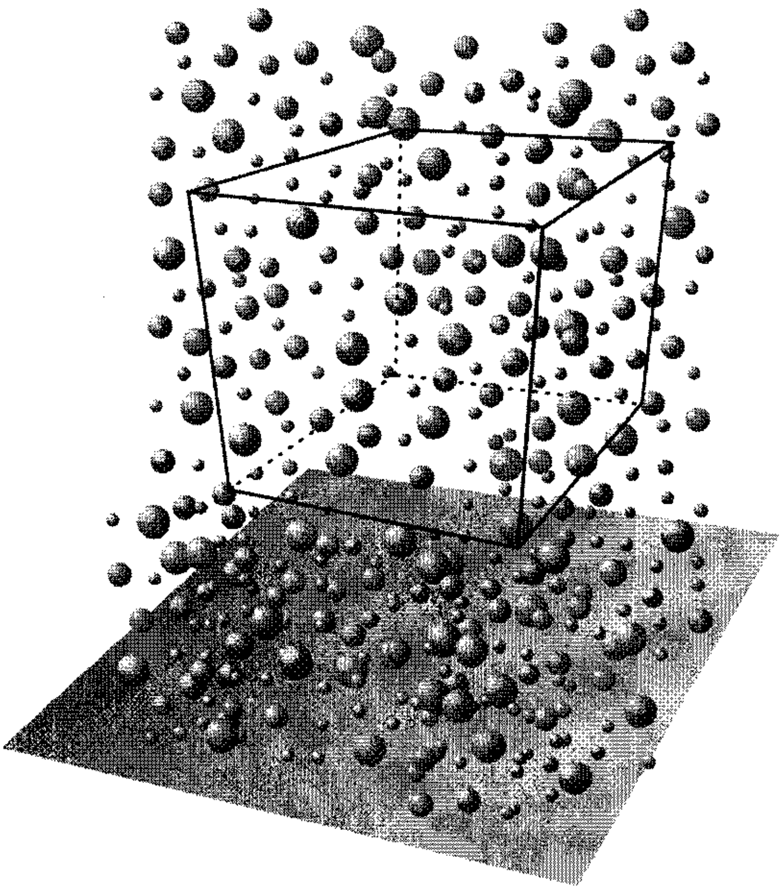

Figure 1.1: Schematic representation of the subject of this thesis: the spatial distribution of raindrops in a volume of air and the distribution of their sizes (Courtesy of J. M. Porrà).

and that of the probability distribution of their sizes. A fundamental but seldom explicitly mentioned hypothesis with regard to the existence of the function $N(D)$ is that it is independent of the size of the reference volume under consideration. This assumes a certain amount of spatial *homogeneity* and temporal *stationarity* of the rainfall process. See Porrà et al. (1998) for a review of the hypotheses on which $N(D)$ is based.

A comparison of the definition of the raindrop size distribution $N(D)$ with Fig. 1.1 shows that $N(D)$ is in fact a *parameterization* of the actual microstructure of rainfall within the reference volume. Its definition neglects the exact numbers, positions and sizes of the individual raindrops in the reference volume and merely provides an idea of the *average* conditions. The minimum spatial scale for which $N(D)$ can be considered an accurate representation of the *instantaneous* conditions is the scale for which the field approximation of rainfall breaks down. This *representative elementary volume* would roughly be a few tens of cubic meters[3].

[3]According to Orlanski's (1975) rational subdivision of scales for atmospheric processes, this corresponds to the micro-γ scale.

With regard to the shapes of raindrops, those in the figure are perfect spheres. This is a very good approximation to their true shapes. Only raindrops larger than 2 mm deviate significantly (i.e. more than 10%) from the perfect spherical shape. In contrast to common belief, these larger raindrops do not have 'teardrop' shapes, but more closely resemble oblate spheroids (Pruppacher and Pitter, 1971; Pruppacher and Klett, 1978; Beard and Chuang, 1987). Therefore, the raindrop diameter D actually represents an *equivalent spherical* raindrop diameter, i.e. the diameter of a sphere with the same volume as that of the raindrop under consideration. In this thesis, raindrops will be assumed perfect spheres. This has the additional advantage that the influence of wind and turbulence on the orientation of raindrops ('canting') (e.g. Brussaard, 1974; 1976) does not have to be considered.

1.2.2 A dynamic picture of rainfall

Fig. 1.1 provides a rather static picture of rainfall, in the sense that it suggests that the raindrops are not moving. However, nothing is less true. In still air, raindrops have terminal fall speeds which range from about 0.1 $\mathrm{m\,s^{-1}}$ for the smallest raindrops to more than 9 $\mathrm{m\,s^{-1}}$ for the largest raindrops. At altitudes well above sea level, the fall speeds tend to be somewhat higher (e.g. Foote and du Toit, 1969; Beard, 1976). However, in practical situations this effect of air density is likely to be small compared to the influence of wind (updrafts, downdrafts), turbulence and raindrop collisions.

Consider the flux of raindrops through part of the bottom of the reference volume indicated in Fig. 1.1. If the corresponding rain rates would be calculated on the basis of the volumes of the raindrops which pass that surface during subsequent time intervals of one second, then the resulting time series of rain rates might look like that provided by Fig. 1.2. This is actually a time series of rain rates with a temporal resolution of 1 s collected using a capacitor type raingauge with a surface area of 730 $\mathrm{cm^2}$ (Semplak and Turrin, 1969). For reference, a line corresponding to the 20 s moving average has been indicated in the figure. It will be clear that at least part of the fluctuations in the 1 s observations about the 20 s moving average must have been caused by purely random fluctuations in the numbers and sizes of the raindrops arriving at the raingauge. Note that there are rain rate differences from one second to the next of close to 100 $\mathrm{mm\,h^{-1}}$. The arrival of only one 6 mm raindrop at the raingauge during a 1 s time interval would already produce a mean rain rate of 5.6 $\mathrm{mm\,h^{-1}}$. Hence, the arrival of only a few large raindrops is able to cause the extreme rain rate differences observed at this time scale.

This is an example of a time scale for which the field approximation of rainfall breaks down. As a result, the observed rain rate fluctuations must be due 'both to statistical sampling errors and to real fine-scale physical variations which are not readily separable from the statistical ones' (Gertzman and Atlas, 1977). The terminology generally adopted for these two types of fluctuations is *sampling fluctuations* and *natural variability*, respectively. In this case, the 20 s moving average may be considered a first rough estimate of the natural variability for the considered time series and the deviations from this moving average consequently as an estimate of the

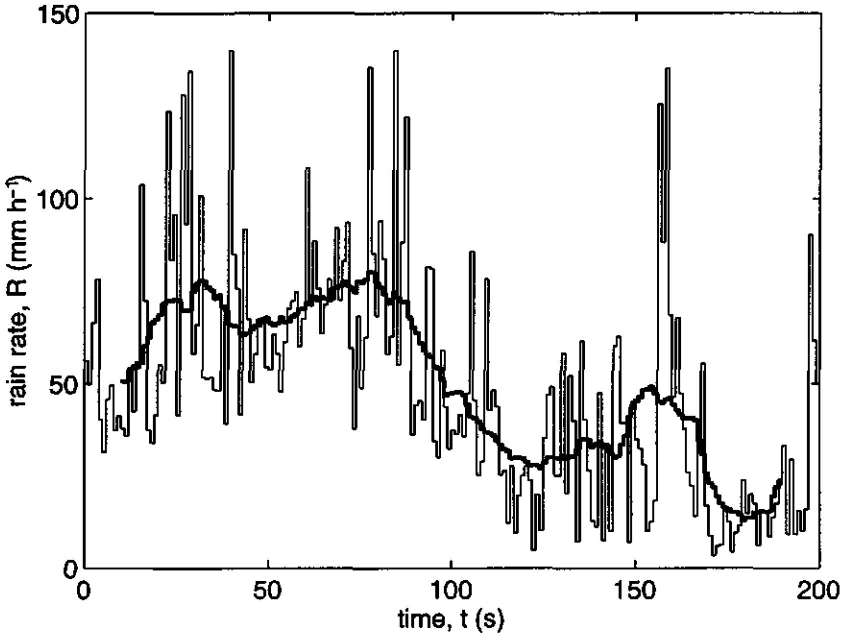

Figure 1.2: Thin line indicates 200 s time series of 1 s mean rain rates collected with a 730 cm^2 capacitor type raingauge at Bell Laboratories, New Jersey on July 21st, 1967 (Semplak and Turrin, 1969). Bold line indicates 20 s moving average.

sampling variability[4].

1.3 Radar meteorology and hydrology

1.3.1 The rainfall measurement problem

Accurate measurement and prediction of the spatial and temporal distribution of rainfall is a basic problem in hydrology because rainfall constitutes the main source of water for the terrestrial hydrological processes. As a result of the gradual development of radar technology over the past 50 years, ground-based weather radar is now finally becoming a tool for *quantitative* rainfall measurement. The advantages of ground-based weather radar over the traditionally used raingauge networks are: (1) they cover extended areas while measuring from a single point; (2) they allow rapid access for real-time hydrological applications; (3) their spatial and temporal resolution is generally high. Formerly, such results could only be achieved by very dense and therefore impractical raingauge networks. Potential areas of application of ground-based weather radar systems in operational hydrology include storm hazard assessment and flood forecasting, warning and control (Collier, 1989). The

[4]Results of applications of (multi)fractal analysis techniques to study the fluctuations in rain rate time series with comparable resolutions (e.g. Rodriguez-Iturbe et al., 1989; Rodriguez-Iturbe, 1991; Georgakakos et al., 1994) should therefore be interpreted with care.

current attention for the role of land surface hydrological processes in the climate system has stimulated research into the spatial and temporal variability of rainfall as well. A potential area of application of ground-based weather radar in this context is the validation and verification of sub-grid rainfall parameterizations for atmospheric mesoscale models and general circulation models (Collier, 1993).

A fundamental problem before radar derived rainfall amounts can be used for hydrological purposes is to make sure that they provide accurate and robust estimates of the spatially and temporally distributed rainfall amounts. The branch of hydrology dealing with this problem is now starting to be known as *radar hydrology*. The fundamental conversion associated with radar remote sensing of rainfall is that from the radar reflectivities measured aloft to rain rates at the ground. This so-called observer's problem is generally tackled in two main steps (e.g. Smith and Krajewski, 1993): (1) conversion of the reflectivity measured in the atmosphere to surface reflectivity; (2) conversion of surface reflectivity to rain rate. The exact manner in which these conversions are carried out will obviously affect the precision of the obtained radar rainfall estimates. Various aspects of the associated assumptions, errors and uncertainties are discussed among others by Battan (1973), Wilson and Brandes (1979), Sauvageot (1982), Doviak (1983), Zawadzki (1984), Clift (1985), Austin (1987), Joss and Waldvogel (1990), Jameson (1991), Andrieu et al. (1997) and Creutin et al. (1997).

1.3.2 The principle of weather radar

RADAR is the acronym for "RAdio Detection And Ranging". According to Battan (1973), radar can be defined as 'the art of detecting by means of radio echoes the presence of objects, determining their direction and range[5], recognizing their character and employing the data thus obtained'. The principle of radar remote sensing is based upon the transmission of a coded radio signal, the reception of a backscattered signal from the volume of interest and inferring the properties of the objects contained in that volume by comparing the transmitted and received signals. In the case of radar meteorology, the objects in the scattering volume are in principle hydrometeors (precipitation particles), although occasionally the ground surface may be detected as well. Hydrometeors can be raindrops, but snow flakes and ice crystals as well. The main interest in this thesis lies obviously in the raindrops.

The *weather radar equation* describes the relationship between the received power, the properties of the radar (transmitted power, wavelength/frequency, beamwidth, range resolution), the properties of the targets (sizes and composition) and the distance between the radar and the targets. The simplest form of the weather radar equation corresponds to the situation where a weather radar operating at a non-attenuated wavelength is observing a region which is homogeneously filled with raindrops. The weather radar equation then becomes (e.g. Battan, 1973)

$$\overline{P}_r = C\frac{|K|^2}{r^2}Z, \tag{1.1}$$

[5]range = distance.

where $\overline{P}_r$ (W) is the mean power received from raindrops at range r (km), C is the radar constant, $|K|^2$ (–) is a coefficient related to the dielectric constant of water (≈ 0.93) and Z ($\mathrm{mm^6\,m^{-3}}$) is the *radar reflectivity factor.* All radar properties are contained in C, all raindrop properties in $|K|^2$ and Z. In the Rayleigh limit (which holds for non-attenuated wavelengths) Z is defined as (e.g. Battan, 1973)

$$Z = \int_0^\infty D^6 N(D)\,\mathrm{d}D. \tag{1.2}$$

This definition shows that the radar reflectivity factor Z, notwithstanding its confusing name, is a *purely meteorological quantity* which is *independent of any radar property.* Z has been expressed here as an integral over the raindrop size distribution instead of a summation over all individual raindrops present in the sample volume at the moment of measurement. This is allowed since the radar sample volume is typically 1 $\mathrm{km^3}$, corresponding to an enormous number of raindrops (of the order of 10^{12}).

In the absence of wind (notably updrafts and downdrafts), turbulence and raindrop interaction the (stationary) rain rate R ($\mathrm{mm\,h^{-1}}$) can be defined in terms of the raindrop size distribution $N(D)$ ($\mathrm{mm^{-1}\,m^{-3}}$) according to

$$R = 6\pi \times 10^{-4} \int_0^\infty D^3 v(D)\, N(D)\,\mathrm{d}D, \tag{1.3}$$

where $v(D)$ represents the relationship between the raindrop terminal fall speed in still air v ($\mathrm{m\,s^{-1}}$) and the equivalent spherical raindrop diameter D (mm). Again, it is allowed to express the rain rate R as an integral because the number of raindrops in the sample volume is huge. A comparison of Eq. (1.3) with Eq. (1.2) demonstrates that it is the raindrop size distribution $N(D)$ (and to a lesser extent the $v(D)$ relationship as well) which ties Z and R together.

On the basis of measurements of raindrop size distributions (at the ground or in the air) and an assumption about the functional form of the $v(D)$ relationship, it is possible to derive so-called Z–R relationships (e.g. via regression analysis). Such relationships are generally found to follow power law relationships of the form

$$Z = C_Z R^{\gamma_Z}, \tag{1.4}$$

where C_Z and γ_Z are coefficients which may vary from one location to another and from season to the next, but which are independent of the rain rate R itself. Experience has learned that an appropriate average relationship in many situations is

$$Z = 200 R^{1.6} \tag{1.5}$$

(Marshall et al., 1955). In general, the coefficients C_Z and γ_Z will in some sense reflect the climatological character of a particular location or season, or more specifically the type of rainfall (e.g. stratiform, convective, orographic) for which they have been derived. If it would be possible to characterize such differences in rainfall regimes in terms of a limited number of parameters and associate with these parameters different Z–R relationships then ultimately this may lead to improved rainfall estimates using weather radar. Clearly, a parameterization of the microstructure of rainfall may provide a means of identifying such parameters.

1.4 Outline of this thesis

It has been demonstrated that it is the microstructure of rainfall and in particular the concept of the raindrop size distribution which ties all physical (i.e. mechanical and electromagnetic) properties of rainfall together. Moreover, it has been shown that it is the raindrop size distribution which, at least partly, determines the signature of rainfall once its field approximation is abandoned. Although many individual contributions have been made since "modern" scientific research in this domain started about a century ago (see Best, 1950b and references therein), a general framework for the treatment of raindrop size distributions and related rainfall properties has been lacking until now. It is the aim of this thesis to provide such a coherent framework for the description of the microstructure of rainfall.

The concrete objective of this thesis is to develop a parameterization of the microstructure of rainfall for applications in radar meteorology and hydrology. The term parameterization in this context means that the most important aspects of the microstructure of rainfall will be captured in a limited number of parameters, such as the raindrop concentration and characteristic raindrop sizes. The spatial and temporal variabilities of these parameters then determine those of any derived rainfall property and, moreover, determine the nature of the relationships between such properties. This may provide an improved understanding of the problems and opportunities associated with radar remote sensing of rainfall and rainfall-land surface interactions and may ultimately lead to improved estimates of the processes involved.

Chapter 2 is an introductory chapter. It re-introduces the concept of the raindrop size distribution and treats in detail the definitions of the rainfall microstructure and rainfall quantities which will be used extensively in the remainder of the thesis. As an example, it shows for the particular case of the classical *exponential* parameterization for the raindrop size distribution how the definitions of several hydrologically relevant rainfall quantities in terms of the raindrop size distribution naturally lead to the well known power law relationships between these quantities.

Chapter 3 forms the core of this thesis. It introduces a new parameterization for the raindrop size distribution and its properties. This parameterization is independent of any a priori assumption regarding the functional form of the raindrop size distribution. It contains the exponential parameterization presented in Chapter 2 as a special case. In this general framework the formulation for the raindrop size distribution takes the form of a *scaling law.* This scaling law formulation shows that it is possible to normalize experimental raindrop size distributions in such a way that the effects of changes in the *shape* of raindrop size distributions are separated from those of changes in the *rain rate* (or any other macroscopic quantity). Its parameters have a clear physical interpretation in terms of different types of rainfall.

Chapters 4 and 5 present two extensive experimental verifications of the scaling law formulation of the raindrop size distribution introduced in Chapter 3. In Chapter 4 the scaling law formulation is experimentally verified on the basis of *mean* raindrop size distributions collected in various climatic settings all over the world. In Chapter 5 it is experimentally verified using measurements of *raw* raindrop size distributions carried out at the Royal Netherlands Meteorological Institute in De Bilt, The Netherlands.

In Chapter 6 a new method for deriving radar reflectivity–rain rate relationships is presented, based on the scaling law formulation of Chapter 3. The method is applied to the experimental datasets presented in Chapters 4 and 5. Both the climatological variability and the inter-event variability of the coefficients of Z–R relationships is investigated and is related to different rainfall regimes. For the raw raindrop size distributions collected in The Netherlands, the new method is compared to the classical regression approach.

Both the theory of radar measurement of rainfall and the concept of the raindrop size distribution treated extensively in Chapters 2–6 are implicitly based on the Poisson homogeneity hypothesis. However, due to the strong natural variability of the rainfall process on many scales, this hypothesis is very difficult to verify experimentally. Chapter 7 presents such an experimental verification, based on an extraordinary stationary time series of raindrop size distributions.

Finally, in Chapter 8 the summary and conclusions of this thesis are presented.

Chapter 2

A consistent rainfall parameterization based on the exponential raindrop size distribution[1]

2.1 Introduction

2.1.1 Background

The existence of power law relationships between various rainfall related variables is experimentally well established in different fields of scientific research. They are probably most abundant in the field of radar meteorology, where the relationship between the radar reflectivity factor Z ($\mathrm{mm^6\,m^{-3}}$) and rain rate R ($\mathrm{mm\,h^{-1}}$) is of fundamental importance to the conversion of measured radar reflectivity factors to surface rain rates. Battan (1973) quotes a list of 69 empirical power law Z–R relationships collected in different climatic settings in various parts of the world. Since then, dozens of other Z–R relationships have been proposed.

However, Z is not the only rainfall related variable which has been related to R via a power law. Battan (1973) also gives examples of various power law relationships between the liquid rainwater content W ($\mathrm{mg\,m^{-3}}$) and R and between a characteristic raindrop size, the so-called median-volume diameter D_0 (mm) and R. The use of a power law D_0–R relationship has originally been proposed by Laws and Parsons (1943). Best (1950b) presents Z–R, W–R and D_0–R relationships for climatologically different locations in various parts of the world. Alternative Z–R, W–R and D_0–R relationships have been presented by Sekhon and Srivastava (1971), among others. D_0–R relationships have recently received increased attention in the hydrologic literature because of their useful application in rainfall interception modelling (e.g. Brandt, 1989; Calder, 1996a,b; Calder et al., 1996; Hall et al., 1996).

[1]Adapted version of Uijlenhoet, R. and Stricker, J. N. M. (1999). A consistent rainfall parameterization based on the exponential raindrop size distribution. *J. Hydrol.*, 218:101–127.

Examples of yet other variables which have been related to rain rate via power laws are the optical extinction coefficient in rainfall S (km^{-1}) (Atlas, 1953), the specific microwave attenuation coefficient k in rainfall (dB km^{-1}) (e.g. Atlas and Ulbrich, 1974; Delrieu et al., 1991) and the kinetic energy flux density (or rainfall power) U (W m^{-2}) (Sempere Torres et al., 1992; Smith and De Veaux, 1992). Although this overview is far from complete, it serves to show that there exists a large body of experimental evidence for the existence of power law relationships between rainfall related variables. Many of these are of direct hydrological interest.

2.1.2 Objectives

The purpose of this chapter is to explain that the power law dependence between rainfall related variables is not a coincidence, but a direct consequence of the fact that rainfall is not a continuous process, as is often assumed, but a discrete process consisting of individual raindrops with different sizes and fall speeds. A fundamental property of rainfall in this respect is its raindrop size distribution. Therefore, this chapter will start with a review of the measurement and parameterization of raindrop size distributions (Section 2.2). Subsequently, a second form of the raindrop size distribution will be introduced, which is important for a proper understanding of rainfall in terms of raindrop processes (Section 2.3). Next, it will be made clear in what manner the various hydrologically relevant rainfall related variables are related to both this new and the traditional form of the raindrop size distribution (Sections 2.4–2.6). In Section 2.7 it will be explained how the coefficients of power law relationships between these rainfall related variables are determined by the parameters of both forms of the raindrop size distribution. Section 2.8 will finally present the summary and conclusions of this chapter.

Three groups of rainfall related variables will be considered, namely properties of individual raindrops (size, speed, volume, mass, momentum and kinetic energy; Section 2.4), rainfall integral variables (raindrop concentration, raindrop arrival rate, liquid rainwater content, rain rate, rainfall pressure, rainfall power and radar reflectivity factor; Section 2.5) and characteristic sizes (median-volume diameter, volume-weighted mean diameter and mean-volume diameter; Section 2.6). Six different sets of power law relationships between these rainfall related variables on the one hand and rain rate on the other will be presented (Section 2.7). These will be based on different assumptions regarding the rain rate dependence of the parameters of the raindrop size distribution. Special attention will be paid to the internal consistency of the different sets of power law relationships.

In 1948, Marshall and Palmer have published their by now classical article (a technical note of barely more than one page) in which they introduce the exponential raindrop size distribution. Over the past five decades, this has become the most widely cited article in the field of radar meteorology (Rogers, 1997). Although several alternative parameterizations for the raindrop size distribution have been proposed over the years, the exponential parameterization has been found to realistically describe *averaged* raindrop size distributions in many parts of the world (e.g. Joss and Gori, 1978; Ulbrich and Atlas, 1998). Hence, the exponential raindrop size distribution will serve as the reference parameterization in the derivations to be presented in this chapter[2].

2.2 The measurement and parameterization of raindrop size distributions: a review

2.2.1 Measurement

There exist two types of instruments to estimate the raindrop size distribution, namely *volume integrating devices* and *time integrating devices*. The former provides direct estimates of $N(D)$ via instantaneous measurements of the numbers and sizes of raindrops present in a particular sample volume. Examples of such devices are the raindrop camera (Cataneo and Stout, 1968; Jones, 1992) and the optical array probe (Knollenberg, 1970). Although they lack the ability to resolve individual raindrops, vertically pointing Doppler radars belong to this class as well (e.g. Sekhon and Srivastava, 1971; Atlas et al., 1973; Hauser and Amayenc, 1981; Russchenberg, 1993). Time integrating devices provide indirect estimates of $N(D)$ via measurements of the numbers and sizes of raindrops arriving at a surface (generally at ground level) during a particular sample interval. Such measurements can be converted to the numbers and sizes of raindrops in a volume of air (and hence can be used to estimate $N(D)$), provided the terminal fall speeds of the raindrops are known (see Section 2.3). Formerly, such measurements used to be carried out using either the *flour* method (e.g. Laws and Parsons, 1943) or the *filter paper* method[3] (e.g. Marshall et al., 1947; Wessels, 1967). Nowadays, electromechanical *disdrometers* (Joss and Waldvogel, 1967) and optical *spectrometers* (e.g. Bradley and Stow, 1974a; Wang et al., 1979; Donnadieu, 1980; Hauser et al., 1984; Illingworth and Stevens, 1987; Salles et al., 1998) are the most widely used instruments.

Note that the *measured* shapes of raindrop size distributions in general differ from their *actual* shapes due to instrumental effects associated for instance with uncertainties in raindrop sizing (e.g. Illingworth and Stevens, 1987), raindrop splashing (e.g. Salles et al., 1998), sampling fluctuations associated with limited sample areas, sample volumes and integration times (e.g. Cornford, 1967, 1968; Joss and Waldvogel, 1969; Gertzman and Atlas, 1978) and instrument exposure to wind and turbulence (e.g. Folland, 1988; Salles et al., 1998). An additional source of uncertainty may come, depending on the type of instrument used for measuring raindrop size distributions, from the conversion of raindrop flux density to concentration (e.g. Illingworth and Stevens, 1987; Smith, 1993).

[2]The exponential raindrop size distribution is used as an *example* in this introductory chapter. A more general approach, independent of any a priori assumption regarding the exact functional form of the raindrop size distribution, will be presented in Chapter 3.

[3]The principle of the flour method is explained in Chapter 4 (Section 4.3) (in discussing Laws and Parsons' (1943) raindrop size data), that of the filter paper method is explained in Chapter 5 (Section 5.2) (in discussing Wessels' (1972) raindrop size data).

2.2.2 Parameterization

The actual shape of the raindrop size distribution is determined by the relative magnitude of the competing microphysical processes which lead to growth (coalescence, condensation) or decay (breakup, evaporation) of the raindrops as they fall to the ground (e.g. Pruppacher and Klett, 1978; Rogers and Yau, 1996). Their spatial and temporal variability then reflects variations in the relative importance of these processes, which in turn may be related to differences in the underlying precipitation generating mechanisms (e.g. stratiform, convective).

After almost a century of raindrop size distribution measurements, with different types of instruments and in different climatic settings in various parts of the world (e.g. Best, 1950b and references therein), it is by now experimentally well established that on average the mentioned microphysical processes tend to produce roughly unimodal, positively skewed raindrop size distributions. More recently, laboratory experiments of raindrop interactions, analytical calculations on the basis of the governing integro-differential equation (the so-called *stochastic collection equation*) and computer simulations of the temporal evolution of raindrop size distributions in zero-dimensional (box) and one-dimensional (shaft) models have given this observation a sound theoretical basis (e.g. Brazier-Smith et al., 1972; Srivastava, 1978, 1982; List et al., 1987; List and McFarquhar, 1990; Levin et al., 1991; Hu and Srivastava, 1995).

This has lead researchers to suggest that, instead of using the entire array of numbers describing the raindrop concentrations in all available size intervals, temporally averaged raindrop size distributions can be conveniently parameterized using only a few (up to three) parameters. A typical set of such parameters then consists of the raindrop concentration and the mean (as a measure of location) and the variance (as a measure of dispersion) of the raindrop diameters. These parameters correspond to the zeroth, first and second moment of the raindrop size distribution, respectively.

The most widely used analytical forms for the parameterization of observed raindrop size distributions are the *exponential distribution* (e.g. Marshall and Palmer, 1948; Sekhon and Srivastava, 1971; Atlas et al., 1973; Atlas and Ulbrich, 1974; Waldvogel, 1974; Joss and Gori, 1978; Ulbrich and Atlas, 1978; Uijlenhoet and Stricker, 1999a), the *gamma distribution* (e.g. Ulbrich, 1983; Willis, 1984; Delrieu et al., 1991; Russchenberg, 1993; Ulbrich and Atlas, 1998), the *lognormal distribution* (e.g. Bradley and Stow, 1974b; Markowitz, 1976; Feingold and Levin, 1986; Smith, 1993; Smith and De Veaux, 1994; Sauvageot and Lacaux, 1995) and the *generalized gamma distribution* (which comprises the Weibull distribution as a special case) (Best, 1950b; Wessels, 1972).

Fig. 2.1 provides a graphical representation of some members of Marshall and Palmer's classical family of exponential raindrop size distributions. The fact that only the slope of the raindrop size distribution on a semi-logarithmic plot is a function of rain rate (the intercept remains constant), is a characteristic property of Marshall and Palmer's parameterization.

The parameters of the functional forms used for the description of the raindrop size distribution (such as raindrop concentration and mean and variance of the raindrop

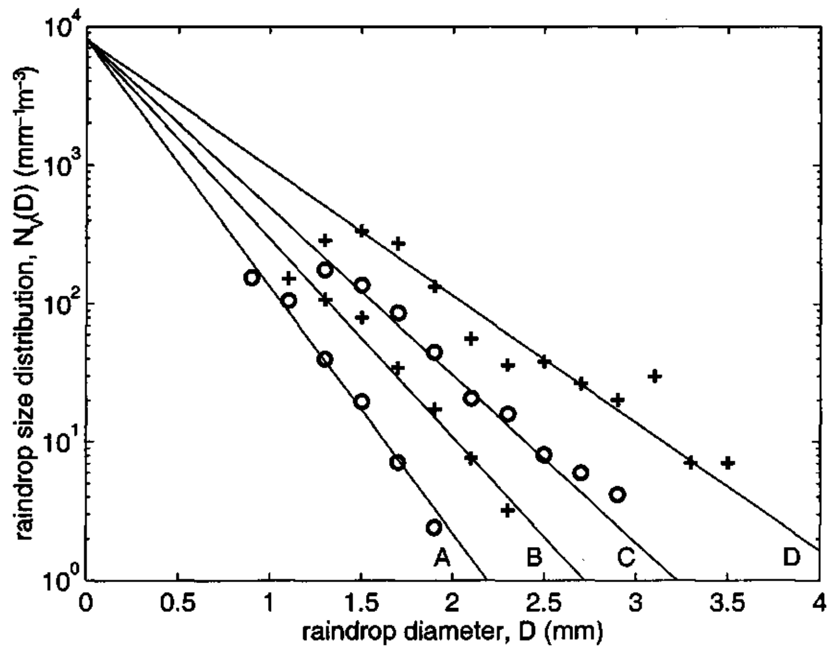

Figure 2.1: Experimental size distributions (circles, crosses) of raindrops present in a volume of air and fitted exponential parameterization $N_V(D) = N_0 \exp(-\Lambda D)$ mm^{-1} m^{-3} (solid lines) with $N_0 = 8.0 \times 10^3$ mm^{-1} m^{-3} and $\Lambda = 4.1R^{-0.21}$ mm^{-1} for different rain rates R (A: 1.0 mm h^{-1}; B: 2.8 mm h^{-1}; C: 6.3 mm h^{-1}; D: 23.0 mm h^{-1}) (after Marshall and Palmer, 1948).

diameter) are found not to fluctuate freely and independently in nature. Rather, some of them may be practically constant while others may depend more or less strongly on each other, the amount of dependence being a function of the type of rainfall (stratiform, convective) or the climatic setting (e.g. Smith and De Veaux, 1994). The result is that the *effective* number of free parameters of the raindrop size distribution is generally not two or three, but actually closer to one. If this were not the case, one rainfall related variable (e.g. Z) would never contain enough information about the raindrop size distribution to provide useful estimates of another (such as R). In other words, it is the fact that raindrop size distributions are often found to behave effectively as one-parameter distributions which renders power law relationships between rainfall related variables (such as Z–R relationships) of practical use. This point was made perfectly clear for the special case of the exponential raindrop size distribution by Ulbrich and Atlas (1978) by means of their so-called rain parameter diagram[4]. In a much more general fashion (i.e. independent of any

[4]Historically, in radar meteorology there is no clear distinction between "parameters" and "variables". This can be explained by the fact that the parameters of analytical parameterizations for the raindrop size distribution themselves are in general functions of rainfall related variables (such as R). The general framework to be presented in Chapter 3 considers this dependence explicitly and therefore renders the distinction between parameters and variables more clear.

assumption with regard to the exact functional form of the raindrop size distribution) this forms the basis of the general formulation for the raindrop size distribution in terms of a scaling law recently proposed and experimentally verified by Sempere Torres et al. (1994, 1998) and of its extensions to be presented in Chapter 3.

2.3 Two forms of the raindrop size distribution

2.3.1 Basic definitions

Recalling the definition given in Chapter 1, the quantity $N(D)\mathrm{d}D$ (m^{-3}) has traditionally been defined as the expected (mean) number of raindrops with diameters between D and $D+\mathrm{d}D$ (mm) present *per unit volume* of air. The units of $N(D)$ are therefore $\mathrm{mm}^{-1}\,\mathrm{m}^{-3}$. For a proper understanding of the various rainfall related variables (and their relationships) to be discussed in this thesis, however, it is important to recognize that there exists a second form of the raindrop size distribution. In this case, $N(D)\mathrm{d}D$ represents the expected number of raindrops with diameters between D and $D+\mathrm{d}D$ arriving at a surface *per unit area* and *per unit time.* The dimensions of this form of $N(D)$ are therefore $\mathrm{L}^{-3}\,\mathrm{T}^{-1}$ (where L stands for length and T for time). With D expressed in mm, area in m^2 and time in s, the units of this distribution $N(D)$ become $\mathrm{mm}^{-1}\,\mathrm{m}^{-2}\,\mathrm{s}^{-1}$. To distinguish between these two different forms of the raindrop size distribution, the former will be denoted by $N_\mathrm{V}(D)$ (the subscript V standing for volume) and the latter by $N_\mathrm{A}(D)$ (the subscript A standing for area), respectively. Using the terminology of Smith (1993), $N_\mathrm{V}(D)$ is the raindrop size distribution pertaining to the *sample volume process* and $N_\mathrm{A}(D)$ that pertaining to the *raindrop arrival process.*

If the effects of wind, turbulence and raindrop interaction are neglected, the relationship between $N_\mathrm{V}(D)$ and $N_\mathrm{A}(D)$ in stationary rainfall is (e.g. Austin, 1987; Hall and Calder, 1993)

$$\begin{cases} N_\mathrm{A}(D) = v(D)\,N_\mathrm{V}(D) \\ N_\mathrm{V}(D) = v(D)^{-1}\,N_\mathrm{A}(D) \end{cases}, \tag{2.1}$$

where $v(D)$ denotes the relationship between the terminal fall speed v ($\mathrm{m\,s}^{-1}$) of a raindrop in still air and its diameter D (mm). The fact that these two forms of the raindrop size distribution are fundamentally different was recognized decades ago, e.g. by Marshall et al. (1947) and Best (1950b), but has only recently been given a more formal basis (Smith, 1993).

Although it may seem merely a matter of taste which form of the raindrop size distribution should be considered as most fundamental, an argument in favor of $N_\mathrm{V}(D)$ is the fact that it only involves the static properties of the raindrop population (namely its concentration and size distribution). In addition to these static properties, $N_\mathrm{A}(D)$ involves the dynamic properties of the raindrop population as well (namely its velocity distribution). Hence, from the point of view of separating the dynamic from the static properties of the population, $N_\mathrm{A}(D)$ may be argued to be less fundamental. Both forms of the raindrop size distribution, however, implicitly assume that the rainfall process is stationary, at least over some minimum space and time scale. In any case, meteorologists and telecommunications researchers (being more concerned with the rainfall processes in the atmosphere) have generally preferred $N_V(D)$, whereas hydrologists (being more concerned with the rainfall fluxes at the earth's surface) may prefer $N_A(D)$. This is also reflected in the types of instruments these two communities have traditionally used to probe the rainfall process: weather radars versus rain gages. It should be noted, however, that since the study of raindrop size distributions has for the most part been the work of meteorologists, almost all publications on this subject deal with the parameterization of $N_V(D)$ instead of $N_A(D)$, even though $N_A(D)$ is the distribution which is actually measured in most cases. See the work of the hydrologists Laws and Parsons (1943) and Horton (1948) for notable exceptions.

2.3.2 The raindrop size distribution in a volume

Marshall and Palmer (1948) have proposed a simple *negative exponential* parameterization for the raindrop size distribution $N_V(D)$ as a fit to filter paper measurements of raindrop size spectra for rain rates between 1 and 23 $\mathrm{mm\,h^{-1}}$,

$$N_V(D) = N_0 \exp(-\Lambda D), \tag{2.2}$$

where N_0 ($\mathrm{mm^{-1}\,m^{-3}}$) is a shorthand notation for $N_V(0)$ and Λ ($\mathrm{mm^{-1}}$) is the slope of the $N_V(D)$-curve on a semi-logarithmic plot or equivalently, as will be demonstrated later (Eq. (2.8)), the inverse of the mean diameter of raindrops in a volume of air. They have found that N_0 is approximately constant for any rain rate,

$$N_0 = 8.0 \times 10^3, \tag{2.3}$$

and that Λ decreases with increasing rain rate R ($\mathrm{mm\,h^{-1}}$) according to the power law

$$\Lambda = 4.1 R^{-0.21}. \tag{2.4}$$

Fig. 2.1 compares this parameterization with the filter paper raindrop size measurements to which it was adjusted. Although these measurements correspond to rain rates not exceeding 23 $\mathrm{mm\,h^{-1}}$, the Marshall-Palmer parameterization has been found to remain a realistic representation of averaged raindrop size distributions for much higher rain rates (e.g. Hall and Calder, 1993). A more detailed description of the measurements is given by Marshall et al. (1947).

Note that since Marshall and Palmer's data are largely restricted to raindrop diameters in excess of 1 mm, their exponential fits (with the corresponding constant value for N_0) are extrapolations in the interval $D < 1$ mm. Indeed, it has been argued that exponential parameterizations for the raindrop size distribution tend to overestimate the numbers of small drops. This argument has been used to promote the use of gamma parameterizations (Ulbrich, 1983). However, gamma raindrop size distributions require an extra (third) parameter for their characterization. They will not be considered in this chapter. Moreover, as has been mentioned in Section 2.1, with sufficient temporal (or spatial) averaging, many raindrop size distributions tend to become approximately exponential (Joss and Gori, 1978; Ulbrich and Atlas, 1998).

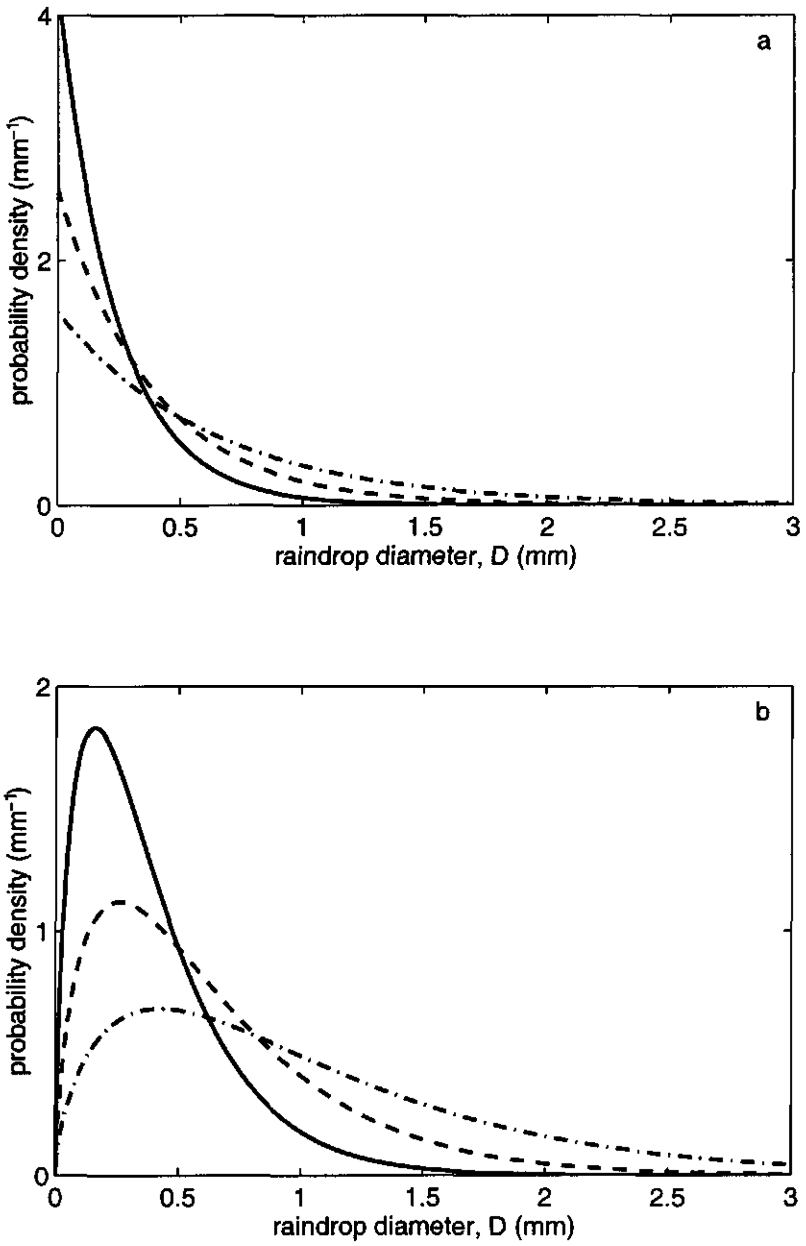

Figure 2.2: Probability density functions of equivalent spherical diameter D of raindrops (a) present per unit volume of air and (b) arriving at a surface per unit area and per unit time for different rain rates (solid line: $1\ \mathrm{mm\,h^{-1}}$; dashed line: $10\ \mathrm{mm\,h^{-1}}$; dash-dotted line: $100\ \mathrm{mm\,h^{-1}}$).

It should also be noted that the form of the raindrop size distribution which is actually estimated with filter paper measurements is $N_A(D)$ (a parameterization of this form of the raindrop size distribution will be derived later, see Eq. (2.11)). Indeed, Marshall et al. (1947) explain that they used the second equation in (2.1) to convert their measured $N_A(D)$ to the desired $N_V(D)$.

Since $N_V(D)$ can be interpreted as the distribution of the total raindrop concentration over all raindrop sizes, integration of $N_V(D)$ with respect to D yields an expression for the raindrop concentration ρ_V (m^{-3}),

$$\rho_V = \int_0^\infty N_V(D)\,dD = \frac{N_0}{\Lambda}. \tag{2.5}$$

Here, as in the sequel, the raindrop diameter integration limits are assumed to be 0 and ∞, respectively. In other words, the effects of truncation of the raindrop size distribution are not considered in this work. This is both common and convenient. Truncation of the raindrop size distribution can both be an instrumental artefact and a natural phenomenon. Ulbrich (1985) and Feingold and Levin (1986) discuss some of the errors involved in disregarding the effects of truncation for gamma and lognormal raindrop size distributions, respectively. Both studies conclude that the effects of truncation at the small diameter end of the raindrop size distribution will mostly affect its low order moments, whereas truncation at the large diameter end will mostly affect its high order moments, as would be expected. Ulbrich also demonstrates that the *exponents* of power law relationships between rainfall related variables will not be affected by truncation. Truncation effects are entirely contained in the *prefactors* of such relationships.

A probabilistic interpretation of $N_V(D)$ is that it is the product of the expected (mean) raindrop concentration ρ_V (m^{-3}) and the probability density function $f_{\underline{D}_V}(D)$ (mm^{-1}) of the *stochastic* diameter[5] $\underline{D}_V$ (mm) of raindrops in a volume of air, i.e.[6]

$$\begin{cases} N_V(D) = \rho_V f_{\underline{D}_V}(D) \\ f_{\underline{D}_V}(D) = \rho_V^{-1} N_V(D) \end{cases}. \tag{2.6}$$

This implies in the case of the exponential parameterization for $N_V(D)$ (Eq. (2.2))

$$f_{\underline{D}_V}(D) = \Lambda \exp(-\Lambda D); \quad \Lambda > 0; \quad D \geq 0, \tag{2.7}$$

which is the probability density function of an exponential distribution (Mood et al., 1974) with mean $\mu_{\underline{D}_V}$ (or expectation $E[\underline{D}_V]$, where $E[\cdot]$ is the expectation operator) and standard deviation $\sigma_{\underline{D}_V}$ (all in mm)

$$\mu_{\underline{D}_V} = E[\underline{D}_V] = \sigma_{\underline{D}_V} = \frac{1}{\Lambda}. \tag{2.8}$$

[5]Random variables are written as underlined quantities.

[6]This interpretation of $N_V(D)$ shows that the term *raindrop size distribution* is in fact ambiguous. Although it may be true that $N_V(D)$ represents the *distribution*, in an informal sense, of the raindrop concentration (i.e. numbers) over all raindrop sizes, statistically speaking it represents a probability *density* function, not a (cumulative) probability *distribution* function. Nevertheless, in accordance with the accepted usage, the term *raindrop size distribution* will be used here to denote $N_V(D)$ (or $N_A(D)$).

The median of this distribution, i.e. the diameter chosen such that the probability of a randomly selected diameter being smaller (or larger) is 1/2, is (in mm)

$$\text{median}_{\underline{D}_V} = \frac{\ln 2}{\Lambda}. \tag{2.9}$$

The mode of the exponential distribution, i.e. the raindrop diameter for which it attains its maximum probability density, is always equal to zero.

Fig. 2.2(a) shows examples of this probability density function for rain rates of 1, 10 and 100 mm h^{-1}. Note that for increasing rain rates, the proportion of large raindrops increases and consequently that of small drops decreases. The power law Λ–R relationship used in this and the subsequent figures is not exactly Eq. (2.4), but a slightly adapted (more consistent) relationship ($\Lambda = 4.23R^{-0.214}$) which will be derived later (Eq. (2.63)).

Substitution of Eqs. (2.3) and (2.4) into the expressions for ρ_V, $\mu_{\underline{D}_V}$, $\text{median}_{\underline{D}_V}$ and $\sigma_{\underline{D}_V}$ shows that for Marshall and Palmer's raindrop size parameterization both the raindrop concentration and the mean, median and standard deviation of the raindrop diameters in a volume increase with rain rate according to simple power laws.

2.3.3 The raindrop size distribution at a surface

To derive the raindrop size distribution $N_A(D)$ per unit area and per unit time corresponding to Marshall and Palmer's $N_V(D)$ parameterization (using the first equation in (2.1)), a particular $v(D)$ relationship needs to be assumed. Measurements of the terminal fall speeds of raindrops and water drops in still air have been reported by Laws (1941), Gunn and Kinzer (1949) and Foote and du Toit (1969), among others. Parameterizations of varying complexity have been proposed to mimic such measurements (e.g. Best, 1950a; Doherty, 1964; Atlas et al., 1973; Beard, 1976; Uplinger, 1981; Lhermitte, 1990; Rogers et al., 1993).

It will be demonstrated in Chapter 3 (Section 3.3.2) that in order to be able to derive a consistent set of power law relationships between rainfall related variables, the assumed relationship between raindrop terminal fall speed in still air v and raindrop diameter D should (at least effectively) be a power law as well, i.e.

$$v(D) = cD^{\gamma}. \tag{2.10}$$

Formulas of this type, with various values for the coefficients, have been in use for almost five decades now. Spilhaus (1948) proposes the coefficients $c = 4.49 \text{ m s}^{-1} \text{mm}^{-\gamma}$ and $\gamma = 0.5$. Sekhon and Srivastava (1971) cite the values $c = 3.352 \text{ m s}^{-1} \text{mm}^{-\gamma}$ and $\gamma = 0.8$, due to Liu and Orville (1968). An intermediate form is that proposed by Atlas and Ulbrich (1977). They demonstrate that Eq. (2.10) with $c = 3.778 \text{ m s}^{-1} \text{mm}^{-\gamma}$ and $\gamma = 0.67$ provides a close fit to the data of Gunn and Kinzer (1949) in the range $0.5 \leq D \leq 5.0$ mm (the diameter interval contributing most to rain rate). This has become the most widely used power law relationship nowadays and is the form which will be used throughout this thesis (Fig. 3.1(a), p. 68). Note that the effects of wind (updrafts, downdrafts), turbulence and raindrop interactions will cause a significant uncertainty in the coefficients of equations such as Eq. (2.10).

Substituting Eqs. (2.2) and (2.10) into the first equation in (2.1) leads to an expression for the raindrop size distribution per unit area and per unit time,

$$N_{\mathrm{A}}(D) = cN_0 D^{\gamma} \exp(-\Lambda D). \tag{2.11}$$

This must have been approximately the form of the raindrop size distribution measured by Marshall et al. (1947), providing the basis for the Marshall and Palmer (1948) parameterization.

Since $N_{\mathrm{A}}(D)$ can be interpreted as the distribution of the total raindrop arrival rate over all raindrop sizes, integration of $N_{\mathrm{A}}(D)$ with respect to D yields an expression for the raindrop arrival rate ρ_{A} ($\mathrm{m^{-2}\,s^{-1}}$),

$$\rho_{\mathrm{A}} = \int_0^{\infty} N_{\mathrm{A}}(D)\,\mathrm{d}D = cN_0 \frac{\Gamma(1+\gamma)}{\Lambda^{1+\gamma}}, \tag{2.12}$$

where $\Gamma(\cdot)$ denotes the (complete) gamma function ($\Gamma(1+\gamma) = 0.9033$ for $\gamma = 0.67$) (e.g. Abramowitz and Stegun, 1972).

The probabilistic interpretation of $N_{\mathrm{A}}(D)$ is that it is the product of the expected (mean) raindrop arrival rate ρ_{A} ($\mathrm{m^{-2}\,s^{-1}}$) and the probability density function $f_{\underline{D}_{\mathrm{A}}}(D)$ ($\mathrm{mm^{-1}}$) of the stochastic diameter $\underline{D}_{\mathrm{A}}$ (mm) of raindrops arriving at a surface per unit area and per unit time, i.e.

$$\begin{cases} N_{\mathrm{A}}(D) = \rho_{\mathrm{A}} f_{\underline{D}_{\mathrm{A}}}(D) \\ f_{\underline{D}_{\mathrm{A}}}(D) = \rho_{\mathrm{A}}^{-1} N_{\mathrm{A}}(D) \end{cases}. \tag{2.13}$$

This implies in the case of the exponential parameterization for $N_{\mathrm{V}}(D)$ (Eq. (2.2))

$$f_{\underline{D}_{\mathrm{A}}}(D) = \frac{\Lambda^{1+\gamma}}{\Gamma(1+\gamma)} D^{\gamma} \exp(-\Lambda D); \quad \gamma, \Lambda > 0; \quad D \geq 0, \tag{2.14}$$

which is the probability density function of a gamma distribution (Mood et al., 1974) with (all in mm) mean (or expectation)

$$\mu_{\underline{D}_{\mathrm{A}}} = \mathrm{E}[\underline{D}_{\mathrm{A}}] = \frac{1+\gamma}{\Lambda}, \tag{2.15}$$

median (in an approximation which was demonstrated by Ulbrich (1983) to be accurate to within 0.5% for raindrop diameter integration limits of 0 and ∞)

$$\mathrm{median}_{\underline{D}_{\mathrm{A}}} = \frac{0.67+\gamma}{\Lambda}, \tag{2.16}$$

mode

$$\mathrm{mode}_{\underline{D}_{\mathrm{A}}} = \frac{\gamma}{\Lambda} \tag{2.17}$$

and standard deviation

$$\sigma_{\underline{D}_{\mathrm{A}}} = \frac{(1+\gamma)^{1/2}}{\Lambda}. \tag{2.18}$$

The fact that the original exponential distribution for raindrop diameters in a volume of air changes to a non-exponential gamma distribution for diameters at a surface was noted by Smith (1993) as well. Clearly, Eq. (2.14) reduces to Eq. (2.7) for $\gamma = 0$.

Fig. 2.2(b) shows examples of this probability density function for rain rates of 1, 10 and 100 $\text{mm}\,\text{h}^{-1}$. The raindrop size distribution at a surface is clearly shifted towards larger raindrop diameters with respect to that in a volume (at the same rain rate): its mean, median and mode are all larger than that of the original exponential distribution. The relative dispersion of raindrop diameters (as measured by its coefficient of variation σ/μ, the relative root mean square deviation from the mean), however, is reduced (namely $(1+\gamma)^{-1/2} = 0.77$ for $\gamma = 0.67$ versus 1 for the original exponential distribution).

Substitution of Eqs. (2.3) and (2.4) into the expressions for ρ_{A}, $\mu_{\underline{D}_{\text{A}}}$, $\text{median}_{\underline{D}_{\text{A}}}$, $\text{mode}_{\underline{D}_{\text{A}}}$ and $\sigma_{\underline{D}_{\text{A}}}$ shows that for Marshall and Palmer's raindrop size parameterization the raindrop concentration and the mean, median, mode and standard deviation of the raindrop diameters at a surface increase with rain rate according to simple power laws[7].

2.4 Other properties of individual raindrops

2.4.1 Relationships with raindrop diameter

General observations

Table 2.1: Mechanical properties of individual raindrops ω written as power law relationships $\omega = c_\omega D^{\gamma_\omega}$ of the equivalent spherical raindrop diameter D (mm) ($\rho_{\text{w}} = 1000\ \text{kg}\,\text{m}^{-3}$ is the density of water).

Property ω	Symbol	Unit	Relation	c_ω	γ_ω
Diameter	D	mm	D	1	1
Fall speed	v	$\text{m}\,\text{s}^{-1}$	cD^γ	c	γ
Volume	V	mm^3	$(\pi/6)\,D^3$	$\pi/6$	3
Mass	m	mg	$10^{-3}\rho_{\text{w}}V$	$10^{-3}\,(\pi/6)\,\rho_{\text{w}}$	3
Momentum	M	$\text{kg}\,\text{m}\,\text{s}^{-1}$	$10^{-6}mv$	$10^{-9}\,(\pi/6)\,\rho_{\text{w}}c$	$3+\gamma$
Kinetic energy	E	J	$10^{-6}mv^2/2$	$10^{-9}\,(\pi/12)\,\rho_{\text{w}}c^2$	$3+2\gamma$

Apart from their diameter D (mm), Table 2.1 lists various other hydrologically relevant (mechanical) properties of individual raindrops, namely their terminal fall speed v ($\text{m}\,\text{s}^{-1}$), volume V (mm^3), mass m (mg), momentum M ($\text{kg}\,\text{m}\,\text{s}^{-1}$) and kinetic energy E (J). This table shows that each of these properties ω can be written as a power

[7] Table 3.5 on p. 85 compares the statistical properties derived in this section and those to be derived in Section 2.6 for the exponential distribution with the corresponding properties of the gamma and lognormal raindrop size distributions (both in a volume of air and at a surface).

Table 2.2: Electromagnetic properties of individual raindrops ω written as power law relationships $\omega = c_\omega D^{\gamma_\omega}$ of the equivalent spherical raindrop diameter D (mm) (λ (cm) is the radar wavelength; $K = (\epsilon - 1)/(\epsilon + 2)$ (–), where ϵ (–) is the complex dielectric constant of water; Im(·) denotes 'imaginary part of' and $|\cdot|$ 'modulus of'; $|K|^2 \approx 0.93$ for water at microwave frequencies, largely independent of temperature and wavelength, whereas $\mathrm{Im}(-K)$ is strongly temperature and wavelength dependent (Gunn and East, 1954)).

Property ω	Symbol	Unit	c_ω	γ_ω
Geometrical cross-section	A	cm^2	$10^{-2}\,(\pi/4)$	2
Optical extinction cross-section	$Q_{t,o} = 2A$	cm^2	$10^{-2}\,(\pi/2)$	2
Microwave extinction cross-section	Q_t	cm^2	$10^{-3}\,(\pi^2/\lambda)\,\mathrm{Im}\,(-K)$	3
Radar backscattering cross-section	Q_b	cm^2	$10^{-6}\,(\pi^5/\lambda^4)\,\lvert K\rvert^2$	6

of the raindrop diameter D according to

$$\omega = c_\omega D^{\gamma_\omega}, \tag{2.19}$$

provided the relationship between the terminal fall speed of a raindrop in still air and its diameter obeys the power law Eq. (2.10). If, in accordance with Atlas and Ulbrich (1977), γ is assumed to be equal to 0.67, then v, V, m, M and E become proportional to the powers of orders 0.67, 3, 3, 3.67 and 4.34 of the raindrop diameter D. In other words, given Eq. (2.10), several powers of the raindrop diameter have direct physical interpretations in terms of raindrop properties.

For the sake of completeness, Table 2.2 lists some additional (electromagnetic) properties of individual raindrops which can be written as powers of the raindrop diameter. These so-called extinction and scattering cross-sections represent the areas which, when multiplied by the incident intensity, give the total absorbed or scattered power (Gunn and East, 1954). Although perhaps less directly relevant to hydrology, they are basic ingredients for studies regarding remote sensing of rainfall, both in the optical and in the microwave range of the electromagnetic spectrum.

In Section 2.3, it has been shown that the diameters of raindrops present in a volume of air and those of raindrops arriving at a surface are in fact random variables (denoted by $\underline{D}_V$ and $\underline{D}_A$, respectively) which follow particular (interrelated) probability distributions. Therefore, the derived raindrop properties presented in Table 2.1 (and in Table 2.2) become random variables as well and the method of derived distributions can be invoked to derive their probability density functions (Appendix A).

Some remarks concerning the electromagnetic properties

The optical extinction cross-section $Q_{t,o}$ has been defined in accordance with Atlas (1953). This definition is strictly only valid if (1) D/λ, the ratio of the raindrop diameter to the wavelength of the incident radiation, tends to infinity (i.e. in the high frequency limit) and (2) 'the observation is made at a very great distance, i.e. far beyond the zone where a shadow can be distinguished' (van de Hulst, 1981)[8].

The microwave extinction and backscattering cross-sections Q_t and Q_b have been defined in accordance with Gunn and East (1954). These definitions are strictly only valid in the other limiting case, i.e. as D/λ approaches zero, known as the Rayleigh (i.e. low frequency) limit (Rayleigh, 1892). As a matter of fact, the definition of Q_t as given in Table 2.2 only takes extinction of microwaves due to absorption into account and neglects scattering altogether. It may be shown that for the backscattering cross-section, the Rayleigh approximation gives excellent results at S-band ($\lambda \approx 10$ cm) and remains reasonably accurate at C- and X-band ($\lambda \approx 5$ and 3 cm, respectively). For the extinction cross-section however, the Rayleigh approximation is essentially useless in rainfall at the typical wavelengths employed by meteorological radars (although for cloud droplets it still works satisfactorily) (Gunn and East, 1954; Battan, 1973; Ulaby et al., 1981). Hence, the rigorous scattering theory for spheres of arbitrary size (Mie theory) should be invoked (Mie, 1908; van de Hulst, 1981), which implies a departure from the perfect power law behavior as given by Eq. (2.19).

Log-log plots of the specific attenuation coefficient k (dB km^{-1}) versus the rain rate R (mm h^{-1}) (see Section 2.5 for the definitions of these *rainfall integral variables*) based on empirical raindrop size distributions and Mie scattering computations demonstrate that the resulting statistical k–R relationships still exhibit quite closely the theoretical power law behavior (e.g. Crane, 1971). Atlas and Ulbrich (1974) argue that 'power law regression equations between microwave attenuation [k] and rainfall rate R [...] also imply an "effective" power law dependence of the attenuation cross-section Q_t on drop diameter D'. The coefficients of such *effective* power law relationships then become functions of the temperature, the wavelength and the underlying family of raindrop size distributions (e.g. Olsen et al., 1978). Jameson (1991) shows that even the dependence of the extinction and backscattering cross-sections of *non-spherical* raindrops on their equivalent spherical diameter may effectively be described by power laws.

All this serves to show that some important electromagnetic properties of individual raindrops also obey the power law behavior given by Eq. (2.19) (albeit sometimes only effectively) and that, as a consequence, they fit into the general framework presented in this chapter. Since the main concern here is with the more directly hydrologically relevant properties however, the main focus of this chapter will remain on those listed in Table 2.1.

[8]The intuitively controversial fact that $Q_{t,o}$ equals *twice* the geometrical cross-section A is known as the 'extinction paradox' (e.g. van de Hulst, 1981; Stephens, 1994). It is caused by the fact that the total extinction in the high frequency limit is not only due to the simple geometrical effect of blocking, but to more subtle effects associated with diffraction of light around the particle's edge as well. The extinction of light by raindrops is but one aspect of a very complex and beautiful phenomenon. In his famous book on light scattering by small particles, van de Hulst (1981) devotes an entire chapter to the optics of a raindrop.

2.4.2 Probability density functions in a volume

As was the case for the raindrop diameters themselves, one should once again distinguish between the properties of raindrops present in a volume of air and those of raindrops arriving at a surface. For the former, the method of derived distributions (Appendix A) gives the general relationship

$$f_{\underline{\omega}_V}(\omega) = \frac{1}{c_\omega \gamma_\omega} \left(\frac{\omega}{c_\omega}\right)^{1/\gamma_\omega - 1} f_{\underline{D}_V}\left[\left(\frac{\omega}{c_\omega}\right)^{1/\gamma_\omega}\right]; \quad c_\omega, \gamma_\omega > 0; \quad \omega \geq 0. \qquad (2.20)$$

Substitution of Eq. (2.7) implies

$$f_{\underline{\omega}_V}(\omega) = \frac{\Lambda}{c_\omega \gamma_\omega} \left(\frac{\omega}{c_\omega}\right)^{1/\gamma_\omega - 1} \exp\left[-\Lambda \left(\frac{\omega}{c_\omega}\right)^{1/\gamma_\omega}\right]; \quad c_\omega, \gamma_\omega, \Lambda > 0; \quad \omega \geq 0, \qquad (2.21)$$

which can be recognized as the probability density function of a Weibull distribution (Mood et al., 1974) with (all in the units indicated in Table 2.1) mean (or expectation)

$$\mu_{\underline{\omega}_V} = \mathrm{E}\left[\underline{\omega}_V\right] = \frac{c_\omega \Gamma(1+\gamma_\omega)}{\Lambda^{\gamma_\omega}}, \qquad (2.22)$$

median (directly from Eq. (2.9))

$$\mathrm{median}_{\underline{\omega}_V} = c_\omega \left(\frac{\ln 2}{\Lambda}\right)^{\gamma_\omega}, \qquad (2.23)$$

mode

$$\mathrm{mode}_{\underline{\omega}_V} = \begin{cases} c_\omega \left(\frac{1-\gamma_\omega}{\Lambda}\right)^{\gamma_\omega} & ; \quad 0 < \gamma_\omega \leq 1 \\ 0 & ; \quad \gamma_\omega > 1 \end{cases} \qquad (2.24)$$

and standard deviation

$$\sigma_{\underline{\omega}_V} = \frac{c_\omega \left[\Gamma(1+2\gamma_\omega) - \Gamma^2(1+\gamma_\omega)\right]^{1/2}}{\Lambda^{\gamma_\omega}}. \qquad (2.25)$$

Clearly, if $c_\omega = \gamma_\omega = 1$ then $\underline{\omega}_V = \underline{D}_V$ and Eq. (2.21) reduces to Eq. (2.7).

Fig. 2.3(a), (c) and (e) shows the probability density functions in a volume of air of three of the raindrop properties mentioned in Table 2.1, namely its terminal fall speed, volume and kinetic energy (for rain rates of 1, 10 and 100 $\mathrm{mm\,h^{-1}}$). Note that the probability density function of raindrop mass (in mg) would be numerically equal to that of raindrop volume (in mm^3) because the density of water is 1000 $\mathrm{kg\,m^{-3}}$, which equals 1 $\mathrm{mg\,mm^{-3}}$. For raindrop volume and kinetic energy (both $\gamma_\omega > 1$), the probability density functions have an extreme reverse J-shape (their probability densities tend to infinity for values approaching zero). The probability density function of raindrop terminal fall speed ($\gamma_\omega < 1$) has a classical unimodal, positively skewed shape. The corresponding (cumulative) probability distribution functions are shown in Fig. 2.4(a), (c) and (e). These functions confirm that large fractions of the probability mass of both raindrop volume and kinetic energy correspond to small values, particularly for low rain rates.

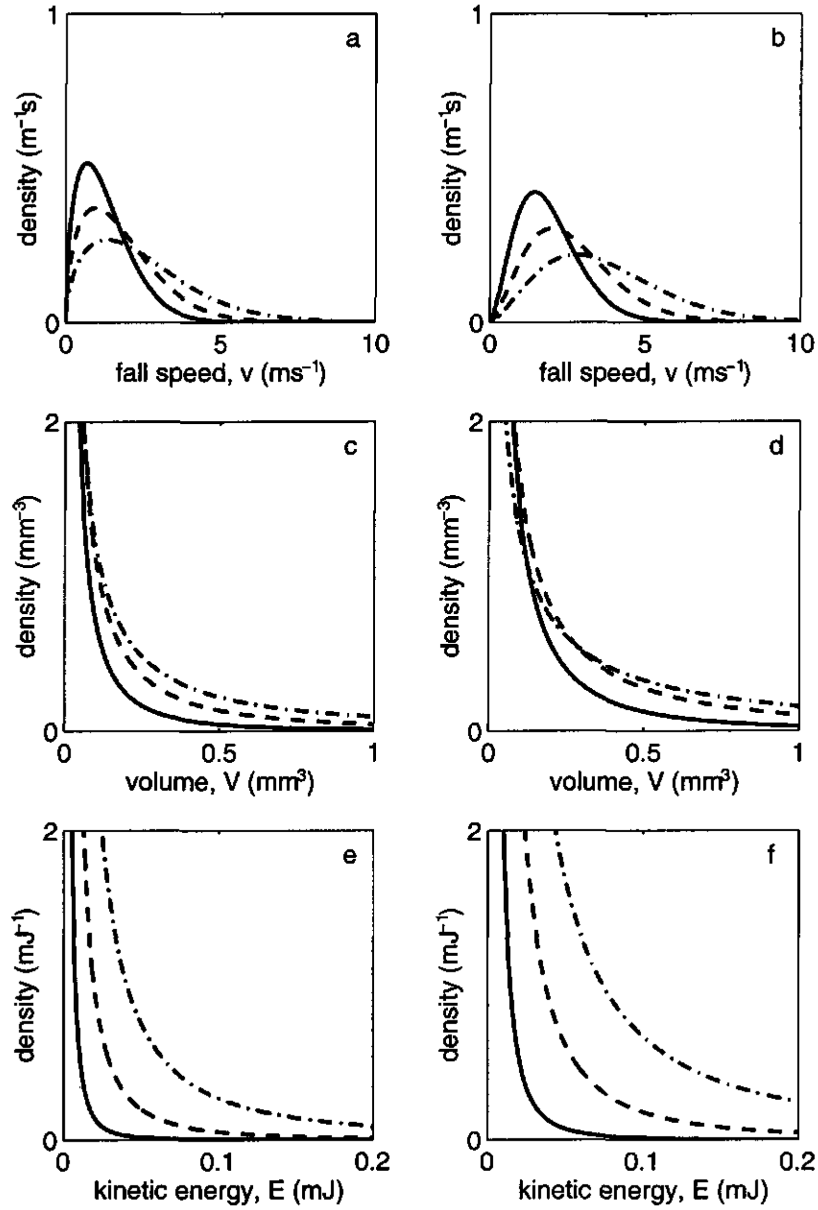

Figure 2.3: Probability density functions of raindrop terminal fall speed in still air v (a, b), raindrop volume V (c, d) and raindrop kinetic energy E (e, f) for different rain rates (solid line: 1 $\mathrm{mm\,h^{-1}}$; dashed line: 10 $\mathrm{mm\,h^{-1}}$; dash-dotted line: 100 $\mathrm{mm\,h^{-1}}$). (a), (c) and (e) Pertain to properties of raindrops present in a volume of air and (b), (d) and (f) to those of raindrops arriving at a surface.

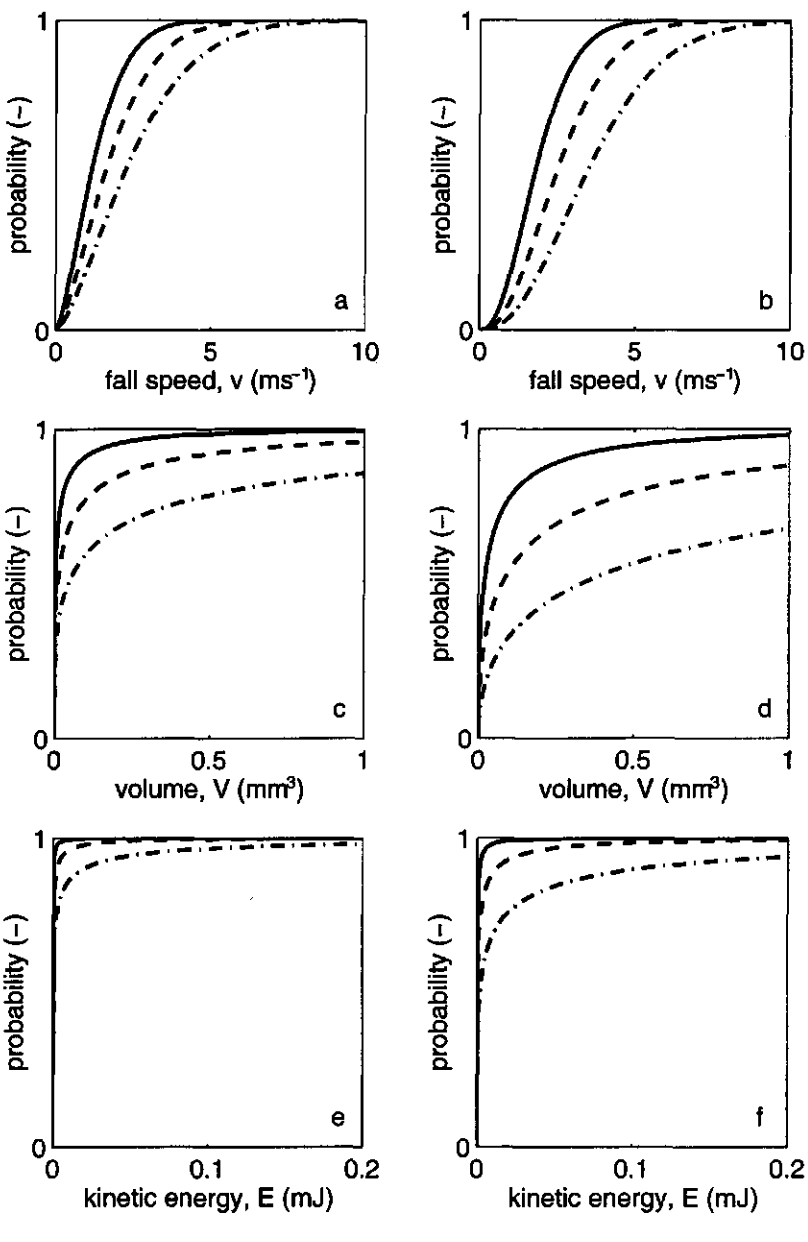

Figure 2.4: Probability distribution functions of raindrop terminal fall speed in still air v (a, b), raindrop volume V (c, d) and raindrop kinetic energy E (e, f) for different rain rates (solid line: 1 $\mathrm{mm\,h^{-1}}$; dashed line: 10 $\mathrm{mm\,h^{-1}}$; dash-dotted line: 100 $\mathrm{mm\,h^{-1}}$). (a), (c) and (e) Pertain to properties of raindrops present in a volume of air and (b), (d) and (f) to those of raindrops arriving at a surface.

2.4.3 Probability density functions at a surface

For the probability density functions of the properties of raindrops arriving at a surface, the method of derived distributions gives the general relationship

$$f_{\underline{\omega}_\mathrm{A}}(\omega) = \frac{1}{c_\omega \gamma_\omega}\left(\frac{\omega}{c_\omega}\right)^{1/\gamma_\omega - 1} f_{\underline{D}_\mathrm{A}}\left[\left(\frac{\omega}{c_\omega}\right)^{1/\gamma_\omega}\right]; \quad c_\omega, \gamma_\omega > 0; \quad \omega \geq 0. \tag{2.26}$$

Substitution of Eq. (2.14) implies

$$\begin{aligned} f_{\underline{\omega}_\mathrm{A}}(\omega) &= \frac{\Lambda^{1+\gamma}}{c_\omega \gamma_\omega \Gamma(1+\gamma)}\left(\frac{\omega}{c_\omega}\right)^{(1+\gamma)/\gamma_\omega - 1} \exp\left[-\Lambda\left(\frac{\omega}{c_\omega}\right)^{1/\gamma_\omega}\right]; \\ & \quad c_\omega, \gamma_\omega, \gamma, \Lambda > 0; \quad \omega \geq 0, \end{aligned} \tag{2.27}$$

which can be recognized as the probability density function of a so-called generalized gamma distribution (Stacy, 1962; Ashkar et al., 1988) with (all in the units indicated in Table 2.1) mean (or expectation)

$$\mu_{\underline{\omega}_\mathrm{A}} = \mathrm{E}\left[\underline{\omega}_\mathrm{A}\right] = \frac{c_\omega \Gamma(1+\gamma+\gamma_\omega)}{\Lambda^{\gamma_\omega}\Gamma(1+\gamma)}, \tag{2.28}$$

median (directly from Eq. (2.16))

$$\mathrm{median}_{\underline{\omega}_\mathrm{A}} = c_\omega \left(\frac{0.67+\gamma}{\Lambda}\right)^{\gamma_\omega}, \tag{2.29}$$

mode

$$\mathrm{mode}_{\underline{\omega}_\mathrm{A}} = \begin{cases} c_\omega \left(\frac{1+\gamma-\gamma_\omega}{\Lambda}\right)^{\gamma_\omega} & ; \quad 0 < \gamma_\omega \leq 1+\gamma \\ 0 & ; \quad \gamma_\omega > 1+\gamma \end{cases} \tag{2.30}$$

and standard deviation

$$\sigma_{\underline{\omega}_\mathrm{A}} = \frac{c_\omega}{\Lambda^{\gamma_\omega}}\left[\frac{\Gamma(1+\gamma+2\gamma_\omega)}{\Gamma(1+\gamma)} - \frac{\Gamma^2(1+\gamma+\gamma_\omega)}{\Gamma^2(1+\gamma)}\right]^{1/2}. \tag{2.31}$$

Again, if $c_\omega = \gamma_\omega = 1$ then $\underline{\omega}_\mathrm{A} = \underline{D}_\mathrm{A}$ and Eq. (2.27) reduces to Eq. (2.14). Moreover, if $\gamma = 0$ then Eq. (2.27) reduces to Eq. (2.21). For the special case of the raindrop terminal fall speed, it is possible to obtain general relations between the moments of the distributions of $\underline{v}_\mathrm{A}$ and $\underline{v}_\mathrm{V}$ without making assumptions regarding the functional forms of the raindrop size distribution and the $v(D)$ relationship (Appendix B).

Fig. 2.3(b), (d) and (f) shows the probability density functions and Fig. 2.4(b), (d) and (f) the corresponding (cumulative) distribution functions of the terminal fall speed, volume and kinetic energy of raindrops arriving at a surface (for rain rates of 1, 10 and 100 mm h^{-1}). Although the general shapes of the densities and distributions remain the same, the dominant raindrop properties at a surface are clearly shifted towards larger values with respect to those in a volume (at the same rain rate): their mean, median and mode are all increased. Their relative dispersions (as measured by their coefficients of variation σ/μ), however, are reduced.

2.5 Rainfall integral variables

2.5.1 Relationship with raindrop size distribution and raindrop properties

Using the raindrop properties ω presented in the previous section (Table 2.1), various hydrologically relevant rainfall integral variables Ω can be defined. These variables, as is suggested by their name, are characterized by the fact that they are integrals over the raindrop size distribution (either that in a volume or that at a surface). For each rainfall integral variable, a specific raindrop property is used as a weight in the integration, i.e.

$$\Omega = \int_0^\infty \omega(D)\, N(D)\, \mathrm{d}D. \tag{2.32}$$

It was demonstrated in Section 2.4 that if the terminal fall speed v of a raindrop is related to its diameter D via a power law, then any of the mentioned raindrop properties will be a power of this diameter as well. A consequence of this is that the corresponding rainfall integral variables Ω will become proportional to the *moments* of the raindrop size distribution, i.e.

$$\Omega = c_\omega \int_0^\infty D^{\gamma_\omega} N(D)\, \mathrm{d}D. \tag{2.33}$$

Two types of rainfall integral variables can be distinguished, namely *state variables* and *flux* (or *rate*) *variables*. The former describes the amount of a certain raindrop property (such as mass) present per unit volume of air (i.e. they are concentrations), the latter describes the amount of a certain raindrop property (such as volume, momentum or kinetic energy) arriving at a surface per unit area and per unit time (i.e. they are flux densities or rates). State variables are scalar quantities, i.e. they do not have directions, whereas flux variables are vector quantities, i.e. they have directions (namely vertically downward in the absence of wind and turbulence). A consequence of their definitions is that state variables are directly related to the size distribution $N_\mathrm{V}(D)$ of raindrops present in a volume of air and flux variables are directly related to the size distribution $N_\mathrm{A}(D)$ of raindrops arriving at a surface. If, in accordance with the notation used for these raindrop size distributions, state variables are denoted by Ω_V and flux variables by Ω_A, one can define

$$\begin{cases} \Omega_\mathrm{V} = c_\omega \int_0^\infty D^{\gamma_\omega} N_\mathrm{V}(D)\, \mathrm{d}D \\ \Omega_\mathrm{A} = c_\omega \int_0^\infty D^{\gamma_\omega} N_\mathrm{A}(D)\, \mathrm{d}D \end{cases}. \tag{2.34}$$

The two most fundamental state and flux variables have already been encountered in Section 2.3, namely the raindrop concentration ρ_V (Eq. (2.5)) and the raindrop arrival rate ρ_A (Eq. (2.12)). Tables 2.3 and 2.4 mention these and various other hydrologically relevant state and flux variables. If the definitions of the rainfall integral variables Ω presented in these tables are compared with the definitions of the corresponding raindrop properties ω given in Tables 2.1 and 2.2, the relationships between them (as defined by Eq. (2.34)) should become clear.

Using the conversion relationships between $N_V(D)$ and $N_A(D)$ (Eq. (2.1)) and the power law relationship between v and D (Eq. (2.10)), Ω_V can also be expressed in terms of $N_A(D)$ and Ω_A in terms of $N_V(D)$ according to

$$\begin{cases} \Omega_V = c_\omega c^{-1} \int_0^\infty D^{\gamma_\omega - \gamma} N_A(D)\,\mathrm{d}D \\ \Omega_A = c_\omega c \int_0^\infty D^{\gamma_\omega + \gamma} N_V(D)\,\mathrm{d}D \end{cases} . \tag{2.35}$$

If, in accordance with Atlas and Ulbrich (1977), γ is assumed to be equal to 0.67, then raindrop concentration ρ_V, raindrop arrival rate ρ_A, liquid rainwater content W, rain rate R, rainfall pressure P, rainfall power U and radar reflectivity factor Z become proportional to the moments of orders 0, 0.67, 3, 3.67, 4.34, 5.01 and 6 of the raindrop size distribution $N_V(D)$ in a volume. The corresponding moments of the raindrop size distribution $N_A(D)$ at a surface are −0.67, 0, 2.33, 3, 3.67, 4.34 and 5.33, respectively. In other words, given a plausible assumption regarding the dependence of raindrop terminal fall speed on diameter, several moments of the raindrop size distributions in a volume and at a surface have direct physical interpretations in terms of rainfall integral variables.

2.5.2 State variables

Since $N_V(D)$ can be regarded as the product of raindrop concentration ρ_V and the probability density function $f_{\underline{D}_V}(D)$ of the raindrop diameters in a volume (according to the first equation in (2.6)), the integral definition of the rainfall state variables (as given by the first equation in (2.34)) can be rewritten as

$$\Omega_V = \rho_V \mathrm{E}\left[\underline{\omega}_V\right], \tag{2.36}$$

where

$$\mathrm{E}\left[\underline{\omega}_V\right] = c_\omega \mathrm{E}\left[\underline{D}_V^{\gamma_\omega}\right] = c_\omega \int_0^\infty D^{\gamma_\omega} f_{\underline{D}_V}(D)\,\mathrm{d}D \tag{2.37}$$

is the expectation of the property $\underline{\omega}_V = c_\omega \underline{D}_V^{\gamma_\omega}$ of raindrops present in a volume of air. In other words, Ω_V can simply be interpreted as the product of the mean number of raindrops present per unit volume of air times the mean of a particular property of those raindrops (as demonstrated in Table 2.3 for raindrop concentration ρ_V, optical extinction coefficient S, liquid rainwater content W, specific attenuation coefficient k and radar reflectivity factor[9] Z).

[9]In general, the so-called *effective* or *equivalent* radar reflectivity factor Z_e is defined as (with Z_e in $\mathrm{mm^6\,m^{-3}}$, λ in cm, Q_b in $\mathrm{cm^2}$ and $N_V(D)\mathrm{d}D$ in $\mathrm{m^{-3}}$)

$$\begin{aligned} Z_e &= \frac{10^6 \lambda^4}{\pi^5 |K|^2} \int_0^\infty Q_b(D)\, N_V(D)\,\mathrm{d}D \\ &= \frac{10^6 \lambda^4}{\pi^5 |K|^2} \rho_V \mathrm{E}\left[\underline{Q}_b\right] \end{aligned}$$

(e.g. Battan, 1973). If the Rayleigh approximation for $Q_b(D)$ (Table 2.2) is substituted in this expression then Z_e reduces to Z, the radar reflectivity factor defined in Table 2.3.

Table 2.3: Rainfall state variables Ω_V written as moments $c_\omega \int_0^\infty D^{\gamma_\omega} N_V(D)\mathrm{d}D$ of the raindrop size distribution $N_V(D)$ ($\mathrm{mm^{-1}\,m^{-3}}$) in a volume or equivalently as moments $c_\omega c^{-1} \int_0^\infty D^{\gamma_\omega-\gamma} N_A(D)\mathrm{d}D$ of the raindrop size distribution $N_A(D)$ ($\mathrm{mm^{-1}\,m^{-2}\,s^{-1}}$) at a surface. S and k are both *one-way* extinction coefficients and k and Z are *Rayleigh* approximations ($\rho_w = 1000\ \mathrm{kg\,m^{-3}}$ is the density of water; c ($\mathrm{m\,s^{-1}\,mm^{-\gamma}}$) and γ (–) are the coefficients of the power law relationship $v(D) = cD^\gamma$; λ (cm) is the radar wavelength; $\mathrm{Im}(K)$ and $|K|^2$ are functions of the complex refractive index of water, see Table 2.2).

State variable Ω_V	Symbol	Unit	Relation	c_ω	γ_ω
Raindrop concentration	ρ_V	$\mathrm{m^{-3}}$	ρ_V	1	0
Optical extinction coefficient	S	$\mathrm{km^{-1}}$	$10^{-1}\rho_V \mathrm{E}\left[\underline{Q}_{t,o}\right]$	$10^{-3}(\pi/2)$	2
Liquid rainwater content	W	$\mathrm{mg\,m^{-3}}$	$\rho_V \mathrm{E}[\underline{m}_V]$	$10^{-3}(\pi/6)\rho_w$	3
Specific attenuation coefficient	k	$\mathrm{dB\,km^{-1}}$	$\frac{\rho_V \mathrm{E}[\underline{Q}_t]}{\ln 10}$	$10^{-3}\frac{\pi^2\,\mathrm{Im}(-K)}{\lambda \ln 10}$	3
Radar reflectivity factor	Z	$\mathrm{mm^6\,m^{-3}}$	$\rho_V \mathrm{E}\left[\underline{D}_V^6\right]$	1	6

The expression for Ω_V corresponding to the exponential raindrop size distribution $N_V(D)$ in a volume (Eq. (2.2)) can now be obtained by substituting Eq. (2.5) for ρ_V and Eq. (2.22) for $\mathrm{E}[\underline{\omega}_V]$ in Eq. (2.36). This yields

$$\Omega_V = \frac{c_\omega N_0 \Gamma(1+\gamma_\omega)}{\Lambda^{1+\gamma_\omega}}. \tag{2.38}$$

For various applications (notably the study of sampling fluctuations) it is of interest to have an idea of the relative contribution of each infinitesimal raindrop diameter interval $[D, D+\mathrm{d}D]$ to a rainfall state variable and to compare the results for different state variables. A normalized measure (in the sense that it has unit area) for this is the ratio $f_{\Omega_V}(D)$ ($\mathrm{mm^{-1}}$) of the integrand in the definition of Ω_V (the first equation in (2.34)) to Ω_V itself. Using Eqs. (2.2) and (2.38) this gives for the exponential raindrop size distribution $N_V(D)$ in a volume

$$f_{\Omega_V}(D) = \frac{\Lambda^{1+\gamma_\omega}}{\Gamma(1+\gamma_\omega)} D^{\gamma_\omega} \exp(-\Lambda D); \quad \gamma_\omega, \Lambda > 0; \quad D \geq 0, \tag{2.39}$$

which, as can also be seen when compared with Eq. (2.14), is equivalent to the probability density function of a gamma distribution (Mood et al., 1974) with (all in mm) mean

$$\mu_{\Omega_V} = \frac{1+\gamma_\omega}{\Lambda}, \tag{2.40}$$

median (using the Ulbrich, 1983 approximation)

$$\mathrm{median}_{\Omega_V} = \frac{0.67+\gamma_\omega}{\Lambda}, \tag{2.41}$$

mode

$$\mathrm{mode}_{\Omega_\mathrm{V}} = \frac{\gamma_\omega}{\Lambda} \tag{2.42}$$

and standard deviation

$$\sigma_{\Omega_\mathrm{V}} = \frac{(1+\gamma_\omega)^{1/2}}{\Lambda}. \tag{2.43}$$

Note that although the notation in Eqs. (2.39)–(2.43) is similar to that in Eqs. (2.14)–(2.18), the subscripts Ω_V are not underlined here. As such, these quantities do not represent the properties of a random variable[5] Ω_V.

If one compares Eq. (2.40) with Eq. (2.42), it follows that for exponential raindrop size distributions $N_\mathrm{V}(D)$, the *mean* of the distribution of the γ_ωth moment of $N_\mathrm{V}(D)$ over all raindrop sizes apparently corresponds to the *mode* of the distribution of the $(\gamma_\omega + 1)$th moment of $N_\mathrm{V}(D)$ over all raindrop sizes. This property was noted by Joss and Gori (1978) as well.

Since the functions $f_{\Omega_\mathrm{V}}(D)$ have unit area and are non-negative on the entire interval $D \geq 0$, they are by definition probability density functions. However, they do not in general represent probability density functions of any diameter with a physical interpretation in terms of raindrops. Rather, they describe the normalized (unit area) distribution of the γ_ωth moment of the raindrop size distribution $N_\mathrm{V}(D)$ in a volume over all raindrop sizes. In distribution theory, a branch of statistics, such functions are known as the *moment distributions* of the probability density functions $f_{\underline{D}_\mathrm{V}}(D)$ (Kendall and Stuart, 1977). Only if $\gamma_\omega = 0$, $f_{\Omega_\mathrm{V}}(D)$ reduces to the probability density function of a diameter which can be interpreted physically in terms of raindrops. It then namely represents the normalized distribution of the raindrop concentration over all raindrop sizes, which is by definition equal to the probability density function $f_{\underline{D}_\mathrm{V}}(D)$ of the diameters of the raindrops present in a volume of air (Eq. (2.7)).

Fig. 2.5(a), (c) and (f) shows the density functions and Fig. 2.6(a), (c) and (f) the corresponding (cumulative) distribution functions with respect to raindrop diameter of the raindrop concentration ρ_V, the liquid rainwater content W and the radar reflectivity factor Z (for rain rates of 1, 10 and 100 mm h^{-1}). Note that Fig. 2.5(a) equals Fig. 2.2(a). These figures clearly show that the higher order moments of the raindrop size distribution put more weight on the larger raindrop diameters than the lower order moments, as would be expected. This has important implications for the estimation of such high order moments (such as Z) from observed raindrop size distributions. Since the number of large raindrops in a sample will generally be much more sensitive to sampling fluctuations than the number of small raindrops (because there are fewer of them on the average), from a statistical point of view the estimation of high order moments will be much more uncertain than that of low order moments (Joss and Waldvogel, 1969; Gertzman and Atlas, 1977)). A general framework for the treatment of such sampling fluctuations will be presented in Chapter 7. From a practical point of view, however, the estimation of low order moments poses serious problems as well. This is due to their sensitivity to the instrumental effects associated with the measurement of small raindrops. These are close to the detection limit of many raindrop sampling devices and, moreover, they are more sensitive to the effects of wind, turbulence and splash than large raindrops (e.g. Salles et al., 1998).

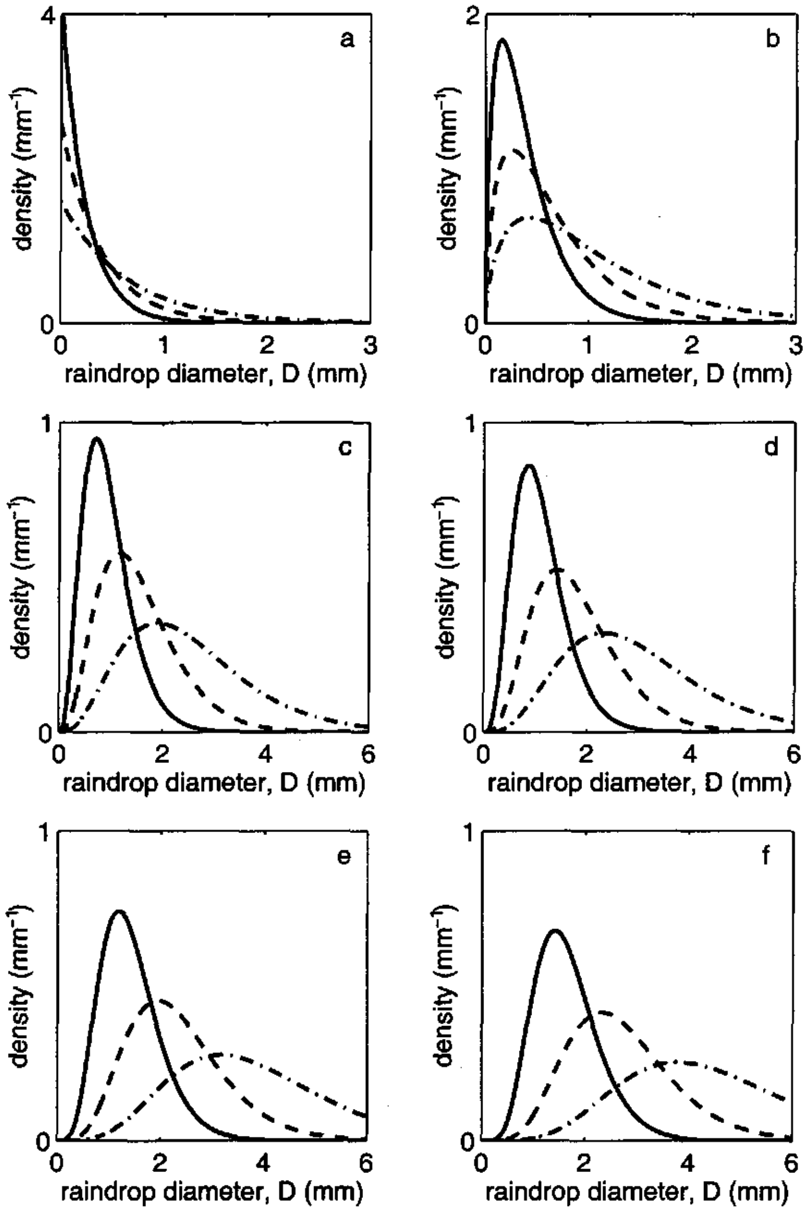

Figure 2.5: Density functions with respect to raindrop diameter D of raindrop concentration ρ_V (a), raindrop arrival rate ρ_A (b), liquid rainwater content W (c), rain rate R (d), rainfall power U (e) and radar reflectivity factor Z (f) for different rain rates (solid line: 1 $\mathrm{mm\,h^{-1}}$; dashed line: 10 $\mathrm{mm\,h^{-1}}$; dash-dotted line: 100 $\mathrm{mm\,h^{-1}}$).

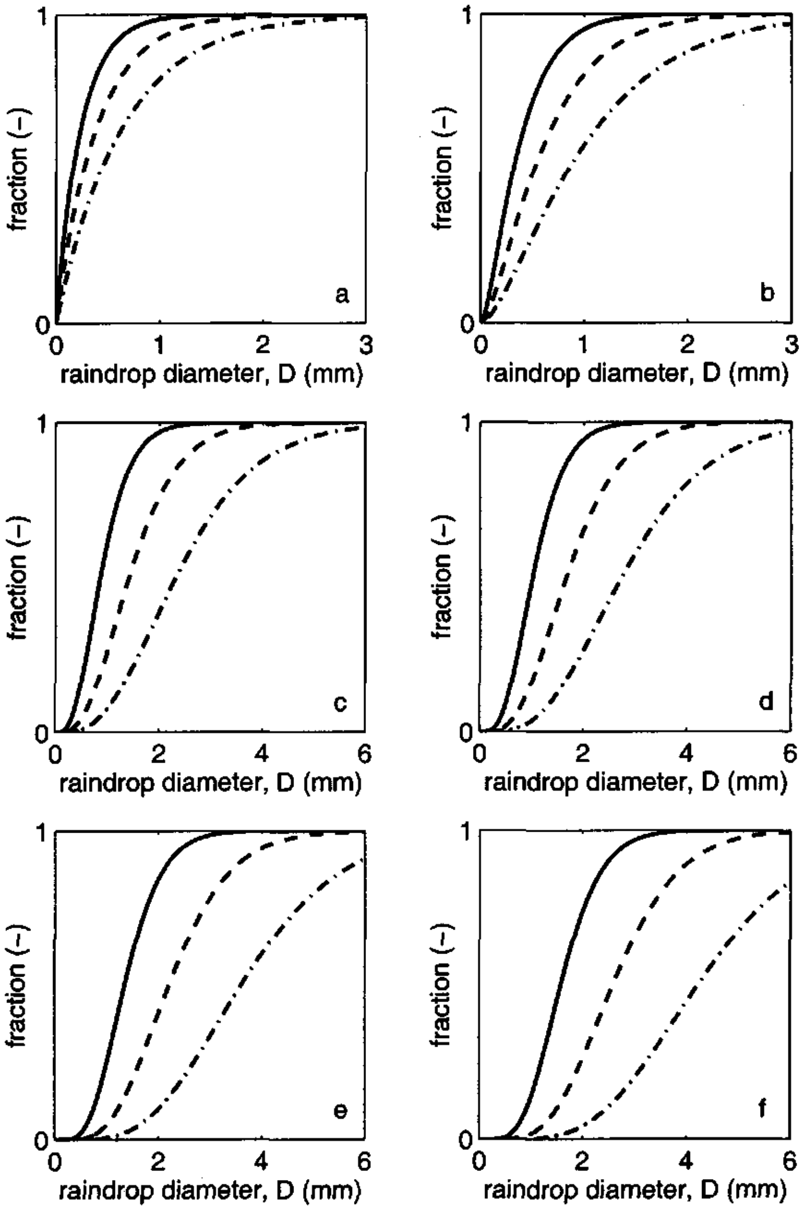

Figure 2.6: Distribution functions with respect to raindrop diameter D of raindrop concentration ρ_V (a), raindrop arrival rate ρ_A (b), liquid rainwater content W (c), rain rate R (d), rainfall power U (e) and radar reflectivity factor Z (f) for different rain rates (solid line: 1 $\mathrm{mm\,h^{-1}}$; dashed line: 10 $\mathrm{mm\,h^{-1}}$; dash-dotted line: 100 $\mathrm{mm\,h^{-1}}$).

Figs. 2.5(f) and 2.6(f) also reveal a fundamental physical limitation of the (non-truncated) exponential raindrop size parameterization $N_V(D)$. Namely, that according to this raindrop size distribution, a significant fraction (about 20%, see Fig. 2.6(f)) of the radar reflectivity factor Z can be attributed to raindrops with diameters larger than 6 mm at a rain rate R of 100 mm h^{-1}. Although this corresponds to only a very small fraction of the total raindrop concentration (see Fig. 2.6(a)), raindrops of this size are very rare in nature (Pruppacher and Klett, 1978) and therefore it seems questionable whether such raindrops would indeed contribute 20% to Z at $R = 100$ mm h^{-1} in reality. Hence, for combinations of such high rain rates and high order moments of the raindrop size distribution (Z is the 6th moment of $N_V(D)$), the non-truncated exponential distribution (and probably *any* non-truncated distribution) is not a suitable raindrop size parameterization.

2.5.3 Flux (rate) variables

Due to their similarity in definition (Eq. (2.34)), the treatment of the flux variables largely follows that of the state variables. In this case, $N_A(D)$ is the product of raindrop arrival rate ρ_A and the probability density function $f_{\underline{D}_A}(D)$ of the raindrop diameters at a surface (according to the first equation in (2.13)). This suggests that the integral definition of the rainfall flux variables (as given by the second equation in (2.34)) can be rewritten as

$$\Omega_A = \rho_A \mathrm{E}\left[\underline{\omega}_A\right], \tag{2.44}$$

where

$$\mathrm{E}\left[\underline{\omega}_A\right] = c_\omega \mathrm{E}\left[\underline{D}_A^{\gamma_\omega}\right] = c_\omega \int_0^\infty D^{\gamma_\omega} f_{\underline{D}_A}(D)\,\mathrm{d}D \tag{2.45}$$

is the expectation of the property $\underline{\omega}_A = c_\omega \underline{D}_A^{\gamma_\omega}$ of raindrops arriving at a surface. Therefore, Ω_A can be regarded as the product of the mean number of raindrops arriving at a surface per unit area and per unit time and the mean of a particular property of those raindrops. This is shown in Table 2.4 for raindrop arrival rate ρ_A, rain rate R, rainfall pressure P and rainfall power U.

Table 2.4: Rainfall flux variables Ω_A written as moments $c_\omega \int_0^\infty D^{\gamma_\omega} N_A(D)\mathrm{d}D$ of the raindrop size distribution $N_A(D)$ (mm^{-1} m^{-2} s^{-1}) at a surface or equivalently as moments $c_\omega c \int_0^\infty D^{\gamma_\omega+\gamma} N_V(D)\mathrm{d}D$ of the raindrop size distribution $N_V(D)$ (mm^{-1} m^{-3}) in a volume ($\rho_w = 1000$ kg m^{-3} is the density of water; c (m s^{-1} mm$^{-\gamma}$) and γ (–) are the coefficients of the power law relationship $v(D) = cD^\gamma$).

Flux variable Ω_A	Symbol	Unit	Relation	c_ω	γ_ω
Raindrop arrival rate	ρ_A	m^{-2} s^{-1}	ρ_A	1	0
Rain rate	R	mm h^{-1}	$3.6 \times 10^{-3} \times \rho_A \mathrm{E}[\underline{V}_A]$	$6\pi \times 10^{-4}$	3
Rainfall pressure	P	Pa	$\rho_A \mathrm{E}[\underline{M}_A]$	$10^{-9}\,(\pi/6)\,\rho_w c$	$3+\gamma$
Rainfall power	U	W m^{-2}	$\rho_A \mathrm{E}[\underline{E}_A]$	$10^{-9}\,(\pi/12)\,\rho_w c^2$	$3+2\gamma$

The expression for Ω_A corresponding to the exponential raindrop size distribution $N_V(D)$ in a volume can now be obtained by substituting Eq. (2.12) for ρ_A and Eq. (2.28) for $E[\underline{\omega}_A]$ in Eq. (2.44). This yields

$$\Omega_A = c_\omega c N_0 \frac{\Gamma(1+\gamma_\omega+\gamma)}{\Lambda^{1+\gamma_\omega+\gamma}}. \tag{2.46}$$

This result could have been obtained as well by noting that Ω_A is $c_\omega c$ times the moment of order $\gamma_\omega+\gamma$ of the raindrop size distribution $N_V(D)$ in a volume (according to the second equation in (2.35)). This then leads directly from Eq. (2.38) to Eq. (2.46).

The density function $f_{\Omega_A}(D)$ (mm^{-1}) of the flux variables Ω_A with respect to raindrop diameter is the same as that of the state variables Ω_V (Eq. (2.39)), with γ_ω replaced by $\gamma_\omega + \gamma$, i.e.

$$f_{\Omega_A}(D) = \frac{\Lambda^{1+\gamma_\omega+\gamma}}{\Gamma(1+\gamma_\omega+\gamma)} D^{\gamma_\omega+\gamma} \exp(-\Lambda D); \quad \gamma_\omega, \gamma, \Lambda > 0; \quad D \geq 0. \tag{2.47}$$

This is the probability density function of a gamma distribution (Mood et al., 1974) with (all in mm) mean

$$\mu_{\Omega_A} = \frac{1+\gamma_\omega+\gamma}{\Lambda}, \tag{2.48}$$

median (using the Ulbrich, 1983 approximation)

$$\text{median}_{\Omega_A} = \frac{0.67+\gamma_\omega+\gamma}{\Lambda}, \tag{2.49}$$

mode

$$\text{mode}_{\Omega_A} = \frac{\gamma_\omega+\gamma}{\Lambda} \tag{2.50}$$

and standard deviation

$$\sigma_{\Omega_A} = \frac{(1+\gamma_\omega+\gamma)^{1/2}}{\Lambda} \tag{2.51}$$

(note again that these quantities do not represent the properties of a random variable Ω_A). If $\gamma_\omega = 0$, $f_{\Omega_A}(D)$ represents the normalized distribution of the raindrop arrival rate over all raindrop sizes, which is by definition equal to the probability density function $f_{\underline{D}_A}(D)$ of the diameters of the raindrops arriving at a surface (Eq. (2.14)).

Fig. 2.5(b), (d) and (e) shows the density functions and Fig. 2.6(b), (d) and (e) the corresponding (cumulative) distribution functions with respect to raindrop diameter of the raindrop arrival rate ρ_A, the rain rate R and the rainfall power U (for rain rates of 1, 10 and 100 mm h^{-1}). Note that Fig. 2.5(b) equals Fig. 2.2(b). With respect to the raindrop size distributions in a volume $N_V(D)$ and at a surface $N_A(D)$, Figs. 2.5(a)–(f) and 2.6(a)–(f) correspond to sequences of moments of increasing orders (0, 0.67, 3, 3.67, 5.01, 6 and −0.67, 0, 2.33, 3, 4.34, 5.33, respectively). As mentioned before, in statistics the curves in Fig. 2.6(a)–(f) are known as the *moment distributions* of the probability density functions $f_{\underline{D}_V}(D)$ and $f_{\underline{D}_A}(D)$ (Kendall and Stuart, 1977).

2.6 Characteristic raindrop sizes

2.6.1 Relationship with raindrop size distribution and rainfall integral variables

For several reasons it may be useful to have an idea of characteristic raindrop sizes corresponding to raindrop size distributions. For instance, they may be used to characterize the shape and scale of experimental (Joss and Gori, 1978) or theoretical (Ulbrich, 1983; Ulbrich and Atlas, 1998) raindrop size distributions. Another application is in stochastic rainfall models, where it is not always feasible to take into account the entire raindrop size distribution and it is sometimes assumed that all raindrops have the same (characteristic) size (such that the raindrop size distribution is a Dirac δ function concentrated at that particular size). An example of such a model is the Calder (1996a,b) two layer stochastic model of rainfall interception, in which the so-called median-volume raindrop diameter (to be discussed later) is used as the characteristic raindrop size (see Uijlenhoet and Stricker, 1999b for a comment on this approach).

Depending on the application, there are different ways to define characteristic raindrop sizes. As a matter of fact, the parameters of the raindrop size distributions in a volume and at a surface given in Section 2.3 (Eqs. (2.8), (2.9) and (2.15)–(2.18)) and those of the density functions of the rainfall state and flux variables presented in the previous section (Eqs. (2.40)–(2.43) and (2.48)–(2.51)) are all characteristic sizes. Selecting a suitable candidate for a specific application, however, cannot be done objectively. Here, a well-established approach is followed by restricting the treatment to characteristic sizes which are related to the *third* moments of either $N_{\mathrm{V}}(D)$ or $N_{\mathrm{A}}(D)$. Not only do these moments correspond to rainfall integral variables of direct hydrological interest (namely liquid rainwater content and rain rate), they are also central in the range of moments (0–6) which have been discussed in the previous section.

2.6.2 Median-volume diameter

A first class of characteristic sizes treated here is that of the median diameters, defined as those diameters which divide the distributions of the rainfall integral variables over all raindrop sizes into two equal parts. These have already been encountered in the previous section as $\mathrm{median}_{\Omega_{\mathrm{V}}}$ (Eq. (2.41)) and $\mathrm{median}_{\Omega_{\mathrm{A}}}$ (Eq. (2.49)). Their general definition is

$$\int_0^{\mathrm{median}_\Omega} D^{\gamma_\omega} N(D)\,\mathrm{d}D = \int_{\mathrm{median}_\Omega}^{\infty} D^{\gamma_\omega} N(D)\,\mathrm{d}D, \tag{2.52}$$

where $N(D)$ can either be $N_{\mathrm{V}}(D)$ or $N_{\mathrm{A}}(D)$. Note that this equation could have been written just as well in terms of $f_{\underline{D}_{\mathrm{V}}}(D)$ and $f_{\underline{D}_{\mathrm{A}}}(D)$. The most widely used among the median_Ω is undoubtedly $\mathrm{median}_{\Omega_{\mathrm{V}}}$ for $\gamma_\omega = 3$ (Eq. (2.41)). This is the *median-volume diameter*, traditionally written as D_0, and defined as 'that drop diameter which divides the drop [size] distribution in such a fashion that half [the liquid] water content is contained in drops greater than D_0' (Battan, 1973). In other words, D_0

divides the liquid rainwater content W into two equal parts. Defined in this way, D_0 was introduced by Atlas (1953) as an alternative to Marshall and Palmer's Λ for the (scale) characterization of exponential raindrop size distributions. Ulbrich (1983) later generalized its range of applications to the gamma raindrop size distribution (which includes the exponential distribution as a special case).

A slightly different version of the median-volume diameter, however, was already proposed by Laws and Parsons (1943). They defined D_0 as 'the raindrop diameter dividing the drops of larger and smaller diameter into groups of equal volume'. This may seem to be the same definition as that given by Battan. However, Laws and Parsons were not concerned with the raindrop size distribution $N_{\mathrm{V}}(D)$ in a volume (as Atlas and Battan were), but with that at a surface, $N_{\mathrm{A}}(D)$. Therefore, Laws and Parsons' definition of D_0 does not divide the liquid rainwater content W into two equal parts, but the rain rate R. In other words, their D_0 corresponds to $\mathrm{median}_{\Omega_{\mathrm{A}}}$ for $\gamma_\omega = 3$ (Eq. (2.49)). This latter definition was also used recently by Calder (1996a,b) in his stochastic model of rainfall interception.

Using the framework developed in the previous sections, it is easy to distinguish between the two definitions of D_0. Battan's D_0 can be obtained from Eq. (2.41), whereas Laws and Parsons' D_0 can be obtained from Eq. (2.49) (both for $\gamma_\omega = 3$). If the former is denoted by $D_{0,\mathrm{V}}$ (since it is defined with respect to $N_{\mathrm{V}}(D)$) and the latter by $D_{0,\mathrm{A}}$ (since it is defined with respect to $N_{\mathrm{A}}(D)$), the definitions become (both in mm)

$$D_{0,\mathrm{V}} = \frac{3.67}{\Lambda} \tag{2.53}$$

and

$$D_{0,\mathrm{A}} = \frac{3.67 + \gamma}{\Lambda}. \tag{2.54}$$

2.6.3 Volume-weighted mean diameter

A second class of characteristic sizes that has found wide application is that of the weighted mean diameters. These are defined as the ratios of the $(\gamma_\omega + 1)$th moment to the γ_ωth moment of the raindrop size distribution (e.g. Joss and Gori, 1978), i.e.

$$\frac{\int_0^\infty D^{\gamma_\omega+1} N(D)\,\mathrm{d}D}{\int_0^\infty D^{\gamma_\omega} N(D)\,\mathrm{d}D}, \tag{2.55}$$

where $N(D)$ can either be $N_{\mathrm{V}}(D)$ or $N_{\mathrm{A}}(D)$ and again this equation could have been written just as well in terms of $f_{\underline{D}_{\mathrm{V}}}(D)$ or $f_{\underline{D}_{\mathrm{A}}}(D)$. Just as in the case of the median diameters, the weighted mean diameters have already been encountered in the previous section, namely as $\mu_{\Omega_{\mathrm{V}}}$ (Eq. (2.40)) and $\mu_{\Omega_{\mathrm{A}}}$ (Eq. (2.48)). They are the mean raindrop diameters, weighted with respect to the γ_ωth moment of the raindrop size distribution. Several values of γ_ω have been used in the literature, e.g. $\gamma_\omega = 1$ (Joss and Gori, 1978), $\gamma_\omega = 2$ (Cerro et al., 1997), $\gamma_\omega = 3$ (Ulbrich, 1983; Ulbrich and Atlas, 1998) and $\gamma_\omega = 5$ (Joss and Gori, 1978) (all with respect to $N_{\mathrm{V}}(D)$). Clearly, for $\gamma_\omega = 0$ the weighted mean raindrop diameters reduce to the ordinary mean diameters (Eqs. (2.8) and (2.15)).

Table 2.5: Definitions of the characteristic sizes of raindrops present in a volume of air (subscript V) and of those arriving at a surface (subscript A) and the resulting values in case of an exponential raindrop size distribution $N_V(D)$ (characteristic sizes in mm if Λ in mm^{-1}). γ (–) is the power of the relationship $v(D) = cD^\gamma$, $f_{\underline{D}_V}(D)$ and $f_{\underline{D}_A}(D)$ (both in mm^{-1}) are the probability density functions corresponding to the raindrop size distributions $N_V(D)$ (mm^{-1} m^{-3}) and $N_A(D)$ (mm^{-1} m^{-2} s^{-1}), respectively.

Characteristic diameter	Definition	Value
Median-volume	$\int_0^{D_{0,V}} D^3 N_V(D)\mathrm{d}D = \int_{D_{0,V}}^\infty D^3 N_V(D)\mathrm{d}D$	$\frac{3.67}{\Lambda}$
	$\int_0^{D_{0,A}} D^3 N_A(D)\mathrm{d}D = \int_{D_{0,A}}^\infty D^3 N_A(D)\mathrm{d}D$	$\frac{3.67+\gamma}{\Lambda}$
Volume-weighted mean	$D_{m,V} = \frac{\int_0^\infty D^4 N_V(D)\mathrm{d}D}{\int_0^\infty D^3 N_V(D)\mathrm{d}D}$	$\frac{4}{\Lambda}$
	$D_{m,A} = \frac{\int_0^\infty D^4 N_A(D)\mathrm{d}D}{\int_0^\infty D^3 N_A(D)\mathrm{d}D}$	$\frac{4+\gamma}{\Lambda}$
Mean-volume	$D_{V,V} = \left[\int_0^\infty D^3 f_{\underline{D}_V}(D)\,\mathrm{d}D\right]^{1/3}$	$\frac{6^{1/3}}{\Lambda}$
	$D_{V,A} = \left[\int_0^\infty D^3 f_{\underline{D}_A}(D)\,\mathrm{d}D\right]^{1/3}$	$\frac{[(3+\gamma)(2+\gamma)(1+\gamma)]^{1/3}}{\Lambda}$

For the same reasons as the median-volume diameter was selected above, the weighted mean diameter corresponding to $\gamma_\omega = 3$ seems to be the most suitable candidate here. This is the *volume-* (or mass-) *weighted mean diameter*, usually denoted as D_m. Again, a distinction will be made between the volume-weighted mean diameter corresponding to $N_V(D)$, which will be denoted by $D_{m,V}$, and that corresponding to $N_A(D)$, denoted by $D_{m,A}$. For the exponential raindrop size distribution $N_V(D)$ in a volume, the former can be obtained from Eq. (2.40) for $\gamma_\omega = 3$, yielding

$$D_{m,V} = \frac{4}{\Lambda}, \tag{2.56}$$

and the latter from Eq. (2.48) for $\gamma_\omega = 3$, yielding

$$D_{m,A} = \frac{4+\gamma}{\Lambda}. \tag{2.57}$$

These values are slightly larger than that for $D_{0,V}$ and $D_{0,A}$, respectively. The main advantage of the volume-weighted mean diameters over the median-volume diameters lies in the fact that the uncertainty associated with their estimation from measured raindrop size distributions is less (Joss and Gori, 1978; Ulbrich, 1983; Ulbrich and Atlas, 1998).

2.6.4 Mean-volume diameter

A third class of characteristic raindrop sizes is defined here not in terms of the rainfall integral variables, such as the median diameters and weighted mean diameters discussed above, but in terms of the rainfall properties presented in Section 2.4. Particularly for rainfall simulation purposes (e.g. Calder, 1996a,b), it may be useful to have an idea of the diameters corresponding to the *mean* raindrop fall speed, volume, mass, momentum or kinetic energy, both with respect to $N_V(D)$ and with respect to $N_A(D)$. In general, the characteristic diameters defined in this manner correspond to the γ_ωth root (the $(1/\gamma_\omega)$th power) of the γ_ωth moment of the raindrop size distribution, i.e.

$$\mathrm{E}\left[\underline{D}^{\gamma_\omega}\right]^{1/\gamma_\omega} = \left[\int_0^\infty D^{\gamma_\omega} f_{\underline{D}}(D)\,\mathrm{d}D\right]^{1/\gamma_\omega}, \tag{2.58}$$

where the expectation can either be with respect to $\underline{D}_V$ or with respect to $\underline{D}_A$. In other words, the integration can be over $f_{\underline{D}_V}(D)$ or over $f_{\underline{D}_A}(D)$.

For $\gamma_\omega = 3$, the *mean-volume diameter*, i.e. the raindrop diameter corresponding to the mean raindrop volume (or mass), is obtained. This characteristic size will be denoted as $D_{V,V}$ when defined with respect to $N_V(D)$ and as $D_{V,A}$ when defined with respect to $N_A(D)$. For exponential raindrop size distributions $N_V(D)$ in a volume, the former can be obtained from Eq. (2.22) for $c_\omega = 1$ and $\gamma_\omega = 3$, yielding

$$D_{V,V} = \frac{\Gamma^{1/3}(4)}{\Lambda} = \frac{6^{1/3}}{\Lambda}, \tag{2.59}$$

and the latter from Eq. (2.28) for $c_\omega = 1$ and $\gamma_\omega = 3$, yielding

$$D_{V,A} = \frac{\Gamma^{1/3}(4+\gamma)}{\Lambda\Gamma^{1/3}(1+\gamma)} = \frac{\left[(3+\gamma)\,(2+\gamma)\,(1+\gamma)\right]^{1/3}}{\Lambda}. \tag{2.60}$$

These mean-volume diameters are significantly smaller than both the median-volume diameters and the volume-weighted mean diameters. They have direct physical interpretations in that $(\pi/6)\,D_{V,V}^3$ is the mean volume of the raindrops present in a volume of air (hence $W \sim \rho_V D_{V,V}^3$) and $(\pi/6)\,D_{V,A}^3$ is the mean volume of the raindrops arriving at a surface (hence $R \sim \rho_A D_{V,A}^3$). Table 2.5 provides a summary of the hydrologically relevant characteristic raindrop diameters discussed in this section.

2.7 Resulting power law relationships

2.7.1 Self-consistency

In the preceding sections, three groups of rainfall related variables have been presented: 1) the parameters of the probability density functions of raindrop properties (Sections 2.3 and 2.4); 2) rainfall integral variables and the parameters of their density functions (Section 2.5); 3) various characteristic raindrop sizes (Section 2.6). These rainfall related variables have been expressed in terms of four different parameters: N_0 and Λ, the parameters of the exponential size distribution of raindrops in a volume of air (Eq. (2.2)), and c and γ, the parameters of the power law relationship between raindrop terminal fall speed and diameter (Eq. (2.10)). The latter two are constants, without any functional dependence on rainfall related variables. If the remaining two parameters, N_0 and Λ, were allowed to fluctuate freely and independently, then power law relationships between rainfall related variables would never be possible. This is because the existence of power law relationships implies the effective number of free parameters to be *one*. Therefore, to be able to derive such power laws, one of the parameters N_0 or Λ should either be constant, or both parameters should be related to each other via a power law themselves.

An important requirement of sets of power law relationships between rainfall related variables is that they should be *consistent*. Self-consistency generally implies that the number of degrees of freedom of a model (i.e. its number of free parameters) is constrained. In this case, it means that power law relationships between rainfall related variables should satisfy the definitions of these variables in terms of the parameters of the raindrop size distribution. For example, N_0–R and Λ–R relationships should not, when substituted in the defining expression for R (Eq. (2.61)), lead to contradictions (i.e. they should yield $R = R$). This so-called self-consistency requirement has been considered explicitly by Bennett et al. (1984), among others. The resulting constraints on the coefficients of power law relationships between rainfall related variables were treated recently in a much more general fashion by Sempere Torres et al. (1994), as part of their general formulation for the raindrop size distribution.

Imposing the self-consistency requirement has the advantage that all rainfall related variables only need to be expressed as functions of one common variable, usually referred to as the *reference variable*. The resulting set of power law relationships will automatically imply all other possible power law relationships. Since its introduction by Marshall and Palmer (1948), it has become common practice to use rain rate R as the reference variable (Eqs. (2.2)–(2.4)). This will be done here as well, since rain rate is the most widely measured rainfall related variable.

2.7.2 Consistency of N_0 and Λ with the $v(D)$ relationship

In Marshall and Palmer's parameterization, N_0 is the constant 8.0×10^3 mm^{-1} m^{-3} (Eq. (2.3)). Given values for c and γ, the only free parameter left is therefore Λ. This determines a particular Λ–R relationship and consequently an entire set of power law relationships between rainfall related variables. The Λ–R relationship proposed by Marshall and Palmer is $\Lambda = 4.1R^{-0.21}$ mm^{-1} (Eq. (2.4)). Does this power law relationship satisfy the self-consistency requirement? In other words, to what extent is it consistent with the definition of R in terms of N_0 and Λ, and for what values of the parameters c and γ?

Substituting $c_\omega = 6\pi \times 10^{-4}$ and $\gamma_\omega = 3$ (Table 2.4) into Eq. (2.46) yields for the definition of the rain rate R (mm h^{-1}) in terms of N_0 (mm^{-1} m^{-3}), Λ (mm^{-1}), c (m s^{-1} mm$^{-\gamma}$) and γ (–)

$$R = 6\pi \times 10^{-4} c N_0 \frac{\Gamma(4+\gamma)}{\Lambda^{4+\gamma}}. \tag{2.61}$$

With a constant N_0, this expression can be inverted to yield an expression for Λ explicitly in terms of R, N_0, c and γ,

$$\Lambda = \left[6\pi \times 10^{-4} c N_0 \Gamma(4+\gamma)\right]^{1/(4+\gamma)} R^{-1/(4+\gamma)}. \tag{2.62}$$

Apparently, with a constant N_0, the power of the Λ–R relationship is completely determined by the power γ of the $v(D)$ relationship. Substituting the Marshall and Palmer value for N_0 and the Atlas and Ulbrich (1977) values for c and γ ($c = 3.778$ $\mathrm{m\,s^{-1}\,mm^{-\gamma}}$ and $\gamma = 0.67$) yields

$$\Lambda = 4.23R^{-0.214}. \tag{2.63}$$

This Λ–R relationship differs only a little from that proposed by Marshall and Palmer, which is surprising given their entirely different methods of derivation. Eq. (2.63) is the result of an analytical derivation based on a theoretical parameterization for the raindrop size distribution, whereas Eq. (2.4) is the result of a sort of regression analysis based on experimentally determined raindrop size distributions.

Although the small difference between Eqs. (2.63) and (2.4) falls entirely within the limits of uncertainty normally associated with this type of relationship, it shows that the latter is not fully consistent with the Atlas and Ulbrich (1977) raindrop terminal fall speed parameterization (at least not for raindrop diameter integration limits of 0 and ∞). The coefficients of the power law $v(D)$ relationship which are consistent with the Marshall and Palmer raindrop size parameterization (Eqs. (2.2)–(2.4)) can be obtained by forcing the coefficients of the general Λ–R relationship (Eq. (2.62)) to be 4.1 and -0.21, respectively. Assuming $N_0 = 8.0 \times 10^3$ $\mathrm{mm^{-1}\,m^{-3}}$, this yields $c = 3.25$ $\mathrm{m\,s^{-1}\,mm^{-\gamma}}$ and $\gamma = 0.762$. These values for c and γ should be regarded as *effective* values, however, and should not be confused with values obtained from actual fits of Eq. (2.10) to measurements of raindrop terminal fall speeds (such as the values given by Atlas and Ulbrich (1977)).

2.7.3 Consistency of N_0 and Λ with the Z–R relationship

The most widely used power law relationship between rainfall related variables is that between Z, the radar reflectivity factor, and R, rain rate. Such Z–R relationships are of fundamental importance to the hydrological application of weather radar as they provide a way of converting measured radar reflectivity factors to surface rain rates. The most widely used Z–R relationship is (Marshall et al., 1955)[10]

$$Z = 200R^{1.6}, \tag{2.64}$$

with Z in $\mathrm{mm^6\,m^{-3}}$ and R in $\mathrm{mm\,h^{-1}}$. The popularity of this particular Z–R relationship merits a verification of its consistency with the definitions of Z and R in terms of N_0, Λ, c and γ.

Substituting the values $c_\omega = 1$ and $\gamma_\omega = 6$ (Table 2.3) into Eq. (2.38) yields for the definition of the radar reflectivity factor Z ($\mathrm{mm^6\,m^{-3}}$) in terms of N_0 ($\mathrm{mm^{-1}\,m^{-3}}$)

[10]This relationship is generally attributed to Marshall and Palmer (1948). However, in their 1948 article, Marshall and Palmer give $Z = 220R^{1.6}$, a revision of a relationship published the year before (Marshall et al., 1947). The famous $Z = 200R^{1.6}$ is presented only several years later (Marshall et al., 1955), as a 'slight revision' of the 1948 relationship. This revision has perhaps been inspired by the relationship $Z = 199R^{1.600}$, derived by Best (1950a) for the data of Marshall and Palmer (1948). The erroneous reference can be traced back to Battan (1973, Eq. (7.15a), p. 89).

and Λ (mm^{-1})

$$Z = \frac{N_0\Gamma(7)}{\Lambda^7} = \frac{720N_0}{\Lambda^7}. \tag{2.65}$$

For the Marshall-Palmer raindrop size distribution, with $N_0 = 8.0 \times 10^3$ mm^{-1} m^{-3} and $\Lambda = 4.1R^{-0.21}$ mm^{-1}, this yields $Z = 296R^{1.47}$, an expression reported by Marshall and Palmer as well. This is significantly different from Eq. (2.64), although both are based on the same data. Their methods of derivation are very different, however. $Z = 296R^{1.47}$ is the result of an analytical derivation based on a theoretical parameterization for the raindrop size distribution, whereas Eq. (2.64) is the result of a regression analysis based on experimentally determined raindrop size distributions. In any case, Eq. (2.64), although it is commonly known as the Marshall-Palmer Z–R relationship, is *not* consistent with the Marshall-Palmer raindrop size parameterization. It is not consistent with Atlas and Ulbrich's raindrop terminal fall speed parameterization, either. A Z–R relationship consistent with that parameterization can be obtained by substituting Eq. (2.63) into Eq. (2.65). This yields $Z = 237R^{1.50}$, a Z–R relationship consistent with $N_0 = 8.0 \times 10^3$ mm^{-1} m^{-3}, $c = 3.778$ m s^{-1} mm$^{-\gamma}$ and $\gamma = 0.67$.

With a constant N_0, Λ can be eliminated from Eq. (2.65) through the substitution of Eq. (2.62). This gives a general expression for the Z–R relationship in terms of N_0, c and γ,

$$Z = 720N_0 \left[6\pi \times 10^{-4} cN_0\Gamma(4+\gamma)\right]^{-7/(4+\gamma)} R^{7/(4+\gamma)}. \tag{2.66}$$

As was the case for the Λ–R relationship (Eq. (2.62)), with a constant N_0, the power of the Z–R relationship is completely determined by the power γ of the $v(D)$ relationship. The coefficients of the power law $v(D)$ relationship which are consistent with Eq. (2.64) can be obtained by forcing the coefficients of Eq. (2.66) to be 200 and 1.6, respectively. Assuming $N_0 = 8.0 \times 10^3$ mm^{-1} m^{-3}, this yields $c = 4.15$ m s^{-1} mm$^{-\gamma}$ and $\gamma = 0.375$. These values should again be regarded as *effective* values. The corresponding Λ–R relationship can be obtained by substituting these values for c and γ into Eq. (2.62). This yields $\Lambda = 4.34R^{-0.229}$.

2.7.4 Consistent sets of power law relationships

On the basis of the N_0–R, Λ–R, $v(D)$ and Z–R power law relationships $N_0 = 8.0 \times 10^3$ mm^{-1} m^{-3}, $\Lambda = 4.1R^{-0.21}$ mm^{-1} (Marshall and Palmer, 1948), $v(D) = 3.778D^{0.67}$ (Atlas and Ulbrich, 1977) and $Z = 200R^{1.6}$ (Marshall et al., 1955), a total of six different consistent sets of power law relationships between rainfall related variables can be constructed. This is because, as has been shown above, each combination of two power law relationships out of these four implies the other two. Out of the four variables N_0, Λ, v and Z, six different combinations of two variables can be selected. Each of these pairs corresponds to a different (consistent) set of power law relationships.

Table 2.6 gives the basic N_0–R, Λ–R, $v(D)$ and Z–R power law relationships for these six sets. The first three of them have already been encountered. For the last

Table 2.6: Basic power law relationships with rain rate R (mm h^{-1}) or equivalent spherical raindrop diameter D (mm) for six different consistent sets of power law relationships between rainfall related variables. Each set corresponds to a particular pair selected from the four rainfall related variables N_0 (mm^{-1} m^{-3}), Λ (mm^{-1}), v (m s^{-1}) and Z (mm^6 m^{-3}).

Set	$N_0 \times 10^{-3}$	Λ	v	Z
N_0, Λ	8.00	$4.10R^{-0.210}$	$3.25D^{0.762}$	$296R^{1.47}$
N_0, v	8.00	$4.23R^{-0.214}$	$3.78D^{0.670}$	$237R^{1.50}$
N_0, Z	8.00	$4.34R^{-0.229}$	$4.15D^{0.375}$	$200R^{1.60}$
Λ, v	$6.91R^{0.019}$	$4.10R^{-0.210}$	$3.78D^{0.670}$	$255R^{1.49}$
Λ, Z	$5.41R^{0.130}$	$4.10R^{-0.210}$	$4.71D^{0.143}$	$200R^{1.60}$
v, Z	$11.3R^{-0.203}$	$4.55R^{-0.258}$	$3.78D^{0.670}$	$200R^{1.60}$

three sets, it is necessary to drop the Marshall and Palmer (1948) assumption of a constant N_0. In those cases, N_0 becomes a power of the rain rate, too. Although not widely used, the possibility of such power law N_0–R relationships has been suggested already a long time ago (e.g. Sekhon and Srivastava, 1971). In principle, it is possible to construct even more consistent sets of power law relationships by selecting yet other pairs of rainfall related variables. The methodology for obtaining these sets, however, is similar to that used for obtaining the current six. The treatment here is restricted to these six as they contain the most widely used power law relationships between rainfall related variables.

Appendix C provides power law relationships with rain rate of the most important rainfall related variables presented in the previous sections for the six different consistent sets. It is difficult to make general statements about differences in quality between these six sets. The reliability of a particular set depends on the plausibility of the corresponding raindrop size parameterization. Perhaps the sets which are consistent with the raindrop terminal fall speed parameterization of Atlas and Ulbrich (1977) should be given a slight preference, as they seem to be most physically realistic. Fig. 2.7 compares the raindrop size parameterizations for these three sets with the Marshall and Palmer (1948) data. Although the differences between the coefficients of the power law relationships for these sets seem to be significant, they all provide reasonable fits to the data.

2.8 Summary and conclusions

There exists an impressive body of experimental evidence confirming the existence of power law relationships between various rainfall related variables. Many of these variables (such as rain rate, radar reflectivity factor and kinetic energy flux density) have a direct relevance for hydrology and related disciplines (hydrometeorology, rainfall interception by vegetation canopies, soil erosion, infiltration). There is one fundamental property of rainfall which ties all these variables together, namely *the raindrop size distribution.* In this introductory chapter, the classical exponential raindrop size dis-

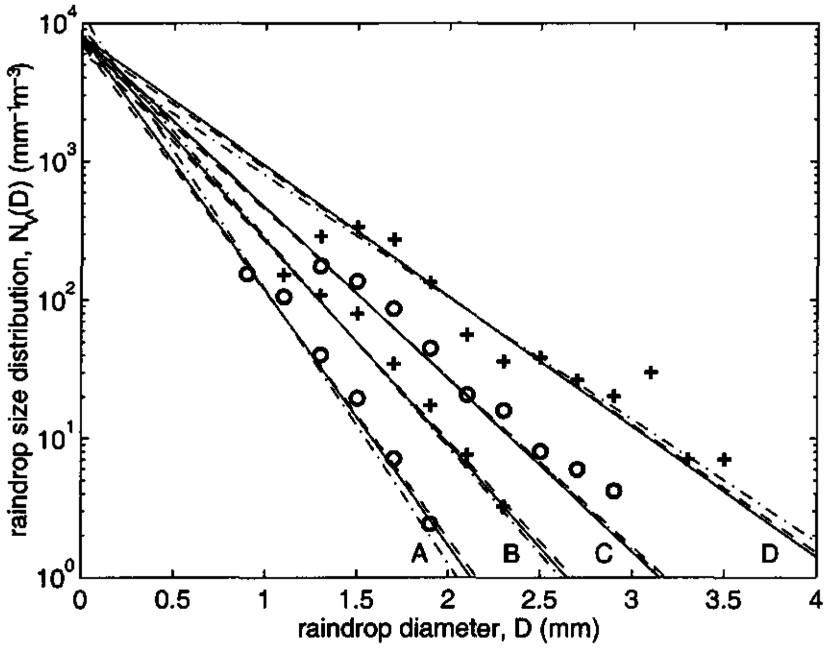

Figure 2.7: Experimental size distributions (circles, crosses) of raindrops present in a volume of air (after Marshall and Palmer, 1948) and fitted exponential parameterizations $N_V(D) = N_0 \exp(-\Lambda D)$ mm^{-1} m^{-3} with $N_0 = 8.00 \times 10^3$ mm^{-1} m^{-3} and $\Lambda = 4.23R^{-0.214}$ mm^{-1} (solid lines), $N_0 = 6.91 \times 10^3 R^{0.019}$ mm^{-1} m^{-3} and $\Lambda = 4.10R^{-0.210}$ mm^{-1} (dashed lines) and $N_0 = 1.13 \times 10^4 R^{-0.203}$ mm^{-1} m^{-3} and $\Lambda = 4.55R^{-0.258}$ mm^{-1} (dashed-dotted lines) for different rain rates R (A: 1.0 mm h^{-1}; B: 2.8 mm h^{-1}; C: 6.3 mm h^{-1}; D: 23.0 mm h^{-1}).

tribution has been used as an example of a family of raindrop size distributions. First of all, it has been explained that there exist two fundamentally different forms of the raindrop size distribution, namely that *per unit volume of air* and that *per unit surface area and per unit time.*

Subsequently, it has been shown how various hydrologically relevant rainfall variables are related to both these forms of the raindrop size distribution. Three groups of rainfall related variables have been considered, namely *properties of individual raindrops* (size, speed, volume, mass, momentum and kinetic energy), *rainfall integral variables* (raindrop concentration, raindrop arrival rate, liquid rainwater content, rain rate, rainfall pressure, rainfall power and radar reflectivity factor) and *characteristic raindrop sizes* (median-volume diameter, volume-weighted mean diameter and mean-volume diameter). In the treatment of these variables, the importance of the distinction between the properties of raindrops present in a volume of air and those of raindrops arriving at a surface has been emphasized. For the rainfall integral variables, this has lead to a distinction between *state variables*, representing concentrations, and *flux (or rate) variables*, representing flux densities.

Finally, it has been demonstrated how the coefficients of power law relationships between such rainfall variables are determined by the parameters of both forms of the raindrop size distribution, i.e. by the parameters N_0 and Λ of the exponential raindrop size distribution and the coefficients c and γ of the power law relationship between raindrop terminal fall speed and equivalent spherical diameter. Six different consistent sets of power law relationships between the rainfall related variables and rain rate have been derived, based on different assumptions regarding the rain rate dependence of N_0 and Λ. Special attention has been paid to the *internal consistency* of the different sets of power law relationships.

Chapter 3

A general framework for the analysis of raindrop size distributions and their properties[1]

3.1 Introduction

3.1.1 Background

It has been shown in Chapter 2 (Section 2.2) that it is an experimentally well established fact that the microstructure of rainfall exhibits two interrelated characteristics: (1) raindrop size distributions can often be conveniently parameterized using only a few (in general dependent) parameters, typically the raindrop concentration and the mean and the variance of the raindrop diameters; (2) rainfall related variables (i.e. variables defined in terms of the raindrop size distribution) are on average related to each other via power laws.

Despite the plethora of power law relationships which have been proposed over the years, it is an empirical fact that log-log plots of rainfall related variables against each other generally exhibit significant scatter around some mean relationship. This suggests that any power law relationship between two such rainfall related variables is in fact merely a *statistical* relationship, representing the regression (i.e. conditional mean) of one variable with respect to another (Haddad and Rosenfeld, 1997). However, it has been common practice for over 50 years now to regard power law relationships as being *deterministic* relationships, i.e. without any uncertainty attached to them. If one assumes that this is indeed the case, then the immediate implication is that all raindrop size distributions must effectively behave as functions with only *one* free (rainfall related) parameter. It is then the spatial and temporal variability of this single parameter which governs the corresponding variabilities of the shape of the raindrop size distribution. Moreover, the functional form of the distribution must then be such as to yield power law relationships between all possible rainfall

[1]Adapted version of Uijlenhoet, R., Creutin, J.-D., and Stricker, J. N. M. (1999). Physical interpretation of a scaling law for the raindrop size distribution. *Q. J. R. Meteorol. Soc.* (submitted).

variables which can be calculated from it. Or, in the words of Ulbrich and Atlas (1978): 'The assumption of a specific dependence of one rainfall parameter on another automatically implies all the possible relationships between all other pairs of rainfall parameters'.

The result of all this is that of the three previously mentioned typical raindrop size distribution parameters (raindrop concentration and mean and variance of the raindrop diameters) either two must be constant, or one must be constant (typically the variance) with the other two related to each other via a power law. For the special case of exponential raindrop size distributions, for which the variance is uniquely related to the mean (their coefficient of variation by definition equals one), this point has been made perfectly clear by Ulbrich and Atlas (1978) by means of their so-called rain parameter diagram. Sempere Torres et al. (1994, 1998) have recently recognized that this has indeed been the basic premise of *all* parameterizations for the raindrop size distribution which have been proposed over the years, regardless of their exact functional form. They present and experimentally verify a general formulation for the raindrop size distribution in terms of a scaling law. Their formulation is independent of any a priori assumption regarding the functional form of the raindrop size distribution and is consistent with the ubiquitous power law relationships between rainfall related variables. In the scaling law formulation, the role of the remaining free parameter of the raindrop size distribution is played by what is called a *reference variable*. This variable therefore reflects the spatial and temporal variability of the raindrop size distribution. The reference variable can in principle be any rainfall related variable, although it is typically the rain rate.

3.1.2 Objectives

The main objective of this chapter is to present a comprehensive general framework for the treatment of raindrop size distributions and their properties. The ultimate aim of this framework is to facilitate the extraction of physically relevant information from (typically large amounts of) empirical raindrop size distributions. A careful interpretation of such information may ultimately lead to an improved understanding of the physical processes which shape the spatial and temporal variability of rainfall on both weather and climate scales. As a result, the proposed framework may find application in meteorological, hydrological and telecommunications research. The scaling law formulation for the raindrop size distribution introduced by Sempere Torres et al. (1994, 1998) will serve as the starting point for the developments of this chapter.

The extension and generalization of the scaling law formulation proposed in this chapter covers the following aspects: (1) a new method for the identification of the scaling exponents is presented, using the weighted mean raindrop diameters rather than the moments of the raindrop size distribution; (2) a physical interpretation of the scaling exponents is proposed; (3) the physical interpretation of the general raindrop size distribution function $g(x)$ is clarified; (4) a new function $h(x)$ is introduced, the general rain rate density function, which in contrast to $g(x)$ has the advantage of behaving as a probability density function; (5) an objective method for the identification of the parameters of the general functions $g(x)$ and $h(x)$ is presented, employing the moments of the empirical rain rate density function; (6) explicit expressions for the general functions $g(x)$ and $h(x)$ and associated quantities are provided for all analytical forms of the raindrop size distribution which have been proposed in the literature over the years (exponential, gamma, generalized gamma, Best and lognormal), thus bridging the gap between the scaling law formulation and the traditional analytical parameterizations.

In Section 3.2 the empirical and theoretical basis of the general formulation for the raindrop size distribution as a scaling law will first be thoroughly reviewed and clarified where necessary. Subsequently, several aspects of the scaling law formulation which may evoke discussion will be put in perspective (Section 3.3). Then, in an effort to bridge the gap with the traditional analytical parameterizations for the raindrop size distribution and at the same time provide a more coherent way for the identification of its parameters, the scaling law formulation will be significantly extended and generalized (Section 3.4). To demonstrate the power of the proposed general framework as a convenient summary of previously published parameterizations, two examples of such parameterizations will be discussed in the light of the new developments (Section 3.5). Section 3.6 will finally present the summary and conclusions of this chapter.

3.2 Empirical and theoretical basis of the scaling law formulation

3.2.1 Derivation of the scaling law

In its traditional definition, the concept of a raindrop size distribution is in fact a mixture of two different notions, namely that of the spatial distribution of raindrops in the air (governing the raindrop concentration) and that of the probability distribution of raindrop sizes in the air. Hence

$$N_V(D) = \rho_V f_{\underline{D}_V}(D), \tag{3.1}$$

where $N_V(D)$ [L^{-4}] is the raindrop size distribution in a volume of air (defined such that $N_V(D)\mathrm{d}D$ represents the mean number of raindrops with equivalent spherical diameters between D and $D+\mathrm{d}D$ [L] present per unit volume of air [L^3]), ρ_V [L^{-3}] the expected (mean) raindrop concentration (which can be obtained from $N_V(D)$ via integration with respect to D) and $f_{\underline{D}_V}(D)$ [L^{-1}] the probability density function of raindrop diameters in the air (the total integral of which, by definition, equals one).

There are several rather fundamental but seldom explicitly mentioned hypotheses (see Porrà et al. (1998) for a notable exception) which form the basis of the concept of the raindrop size distribution as defined by Eq. (3.1): (1) that the numbers of raindrops in the representative elementary volume to which $N_V(D)$ pertains are independent of their sizes; (2) that ρ_V, or indeed any other rainfall state variable (i.e. rainfall related variable expressed as a concentration, e.g. W or Z), does not depend on the size of the sample volume (implying that raindrops are uniformly distributed in space); (3) that it is not necessary to know the (multivariate) statistical properties (e.g. size, speed and position) of the entire raindrop population in the sample volume exactly (including the interdrop dependencies), but sufficient to have an idea of the *average* raindrop size properties, as expressed by the (univariate) probability density function of raindrop sizes $f_{\underline{D}_V}(D)$. An additional hypothesis for the application of Eq. (3.1) in radar meteorology is that the moments of $f_{\underline{D}_V}(D)$, at least up to order six, exist and are finite. These hypotheses imply that the concept of the raindrop size distribution, at least in its traditional interpretation, is incompatible with the recently proposed (multi-)fractal descriptions of rainfall, which allow for non-uniform spatial distributions of raindrops and so-called fat-tailed (power law) probability distributions of rain rate, with the associated divergence of moments (e.g. Lovejoy and Schertzer, 1990)[2].

A crucial step in the derivation of the scaling law is to recognize that the probability density function $f_{\underline{D}_V}(D)$ can be rendered dimensionless using its first moment (i.e. the mean raindrop diameter), or in general any other rainfall related variable obtained from $f_{\underline{D}_V}(D)$ with dimensions [L] (Sempere Torres et al., 1998). Such variables represent in fact characteristic raindrop sizes, the existence of which has implicitly been assumed in all parameterizations for the raindrop size distribution that have been proposed over the years (Porrà et al., 1998). If D_C [L] denotes such a characteristic diameter, then Eq. (3.1) can be rewritten in terms of the physical parameters ρ_V and D_C as

$$N_V(D) = \frac{\rho_V}{D_C} f_{D_C}\left(\frac{D}{D_C}\right), \tag{3.2}$$

where $f_{D_C}(\cdot)$ is now a dimensionless function. This is a general result, based purely on dimensional considerations and valid for any parameterization for the raindrop size distribution[3].

In addition to the two *scale* parameters that have been identified (ρ_V and D_C), $N_V(D)$ (hence $f_{D_C}(\cdot)$) may depend on one or more dimensionless *shape* parameters. An example of the latter is the coefficient of variation of the raindrop diameters, i.e. the ratio of their standard deviation to their mean. However, it has been argued in Section 1 that for Eq. (3.2) to give rise to unique power law relationships between all pairs of rainfall related variables, (1) it must effectively behave as a function with only one free parameter and (2) its functional form must be such as to yield power law relationships. Although this has not been proved rigorously, two necessary (and sufficient) conditions for this to hold seem to be (1) that any dimensionless shape

[2]These (multi-)fractal descriptions of rainfall are also known as *scaling* theories, a term which should not be confused with the use of the word 'scaling' in the present context, i.e. with regard to the general description of raindrop size distributions. The same holds for the use of the term *power laws*, in (multi-)fractal contexts generally pertaining to the shape of the tails of probability distributions, here merely to indicate the type of functional dependence between two rainfall related variables (an exception is the microscopic cloud model of Provata and Nicolis (1994), which indeed gives rise to power law (cloud) drop size distributions).

[3]In collision-coalescence theory, this similarity transformation for raindrop size distributions is known as the so-called *self-preserving distribution* (Pruppacher and Klett, 1978, p. 402). It represents an asymptotic solution to the stochastic collection equation, i.e. the integro-differential equation governing the temporal evolution of raindrop size distributions.

parameter on which $N_V(D)$ and $f_{D_C}(\cdot)$ may depend is a constant (i.e. is independent of any rainfall related variable) and (2) that the two remaining parameters (ρ_V and D_C) depend in a power law fashion on some reference variable Ψ (and hence on each other), i.e.

$$\rho_V = C_{\rho_V}\Psi^{\gamma_{\rho_V}} \text{ and } D_C = C_{D_C}\Psi^{\gamma_{D_C}}. \tag{3.3}$$

Substitution of these two power law relationships in Eq. (3.2) finally yields

$$N_V(D,\Psi) = \Psi^{\alpha_\Psi} g_\Psi\left(\Psi^{-\beta_\Psi} D\right), \tag{3.4}$$

where

$$\alpha_\Psi = \gamma_{\rho_V} - \gamma_{D_C} \tag{3.5}$$

and

$$\beta_\Psi = \gamma_{D_C} \tag{3.6}$$

are (dimensionless) scaling exponents. The prefactors C_{ρ_V} and C_{D_C} (whose dimensions depend on those of ρ_V, D_C and Ψ and on the values of the exponents γ_{ρ_V} and γ_{D_C}) have been assimilated in the so-called *general raindrop size distribution function* $g_\Psi(x)$, where x is a scaled raindrop diameter defined as

$$x = \Psi^{-\beta_\Psi} D. \tag{3.7}$$

It follows that $g_\Psi(x)$, as opposed to $f_{D_C}(\cdot)$, is no longer a dimensionless function. As a matter of fact, its dimensions will depend on those of Ψ and $N_V(D,\Psi)$ and on the values of the scaling exponents[4]. Also note that the values of α_Ψ and β_Ψ and the form and dimensions of $g_\Psi(x)$ depend on the *choice* of the reference variable Ψ, but do not bear any functional dependence on its *value*. Eq. (3.4) represents the scaling law for the raindrop size distribution, as introduced and experimentally verified by Sempere Torres et al. (1994, 1998). The hypotheses made in the course of its derivation are not specific to the scaling law, however, but have (although often implicitly) formed the basis of most previously proposed parameterizations for the raindrop size distribution.

3.2.2 Functional form of the scaling law

The notation $N_V(D,\Psi)$ for the raindrop size distribution (instead of $N_V(D)$), originally introduced by List et al. (1987) and List (1988), has been used to explicitly account for the dependence of the shape of the raindrop size distribution on Ψ. In fact, the latter represents the one remaining free parameter of the raindrop size distribution. Due to the basic assumption of deterministic dependence between all rainfall related variables, its role can in principle be played by any such variable. Since first proposed by Marshall and Palmer (1948), however, the rain rate R (mm h^{-1}) has found the widest application.

It will be shown in Section 3.2.3 that Eq. (3.4) is in fact the most general formulation for the raindrop size distribution which gives rise to the ubiquitous power law

[4]To indicate its dependence on the choice of Ψ, the notation $g_\Psi(x)$ for the general distribution function is prefered here to the notation $g(x)$ used by Sempere Torres et al. (1994). This also renders the notation for the general distribution function consistent with that for the scaling exponents.

relationships between rainfall related variables. Its functional form is that of a *scaling law*, well known from different applications in physics and chemistry. As noted by Sempere Torres et al. (1994), 'the peculiarity here is that the scaling variable Ψ is an integral function of $N_V(D, \Psi)$'. As a result, this 'peculiarity' will give rise to self-consistency constraints on the general raindrop size distribution function and the corresponding scaling exponents. These will be extensively discussed in Section 3.2.3. In any case, rather than representing any formal *theory* of the evolution of raindrop size distributions (in the sense of that advanced for instance by Srivastava, 1988), the scaling law description is used here in a heuristic manner, i.e. as a convenient summary describing all previously published parameterizations for the raindrop size distribution.

Two particularly interesting special cases of the scaling law may be obtained when Ψ is taken to be either a characteristic diameter D_C or the raindrop concentration ρ_V. In the first case, substituting $\Psi = D_C$ in Eq. (3.3) implies that $C_{D_C} = \gamma_{D_C} = 1$. This is what is called a *self-consistency constraint.* Substituting $\gamma_{D_C} = 1$ in Eqs. (3.5) and (3.6) yields for the scaling exponents

$$\alpha_{D_C} = \gamma_{\rho_V} - 1 \text{ and } \beta_{D_C} = 1. \tag{3.8}$$

Substituting these finally in Eq. (3.4) yields an alternative form of the scaling law, namely

$$N_V(D, D_C) = D_C^{\alpha_{D_C}} g_{D_C}\left(\frac{D}{D_C}\right), \tag{3.9}$$

where the only free scaling exponent is now $\alpha_{D_C} = \gamma_{\rho_V} - 1$. Recall that the general raindrop size distribution function $g_{D_C}(\cdot)$ is a non-dimensionless function. As noted by Sempere Torres et al. (1994), this particular form of the scaling law implies that $N_V(D, D_C)$ must be a homogeneous function, i.e. a function satisfying

$$N_V(\delta D, \delta D_C) = \delta^{\alpha_{D_C}} N_V(D, D_C), \tag{3.10}$$

which is a classical way of formulating a scaling law.

The second special case of the scaling law, not explicitly treated by Sempere Torres et al. (1994), can be obtained when the raindrop concentration ρ_V is used as reference variable Ψ. In this case, substitution of $\Psi = \rho_V$ in Eq. (3.3) implies that $C_{\rho_V} = \gamma_{\rho_V} = 1$, a new self-consistency constraint. Substituting $\gamma_{\rho_V} = 1$ in Eqs. (3.5) and (3.6) yields for the scaling exponents

$$\alpha_{\rho_V} = 1 - \gamma_{D_C} \text{ and } \beta_{\rho_V} = \gamma_{D_C}. \tag{3.11}$$

The corresponding form of the scaling law (Eq. (3.4)) now becomes

$$N_V(D, \rho_V) = \rho_V^{\alpha_{\rho_V}} g_{\rho_V}\left(\rho_V^{-\beta_{\rho_V}} D\right), \tag{3.12}$$

with apparently $\alpha_{\rho_V} + \beta_{\rho_V} = 1$. In other words, as was the case when D_C was used as reference variable, there is actually only *one* free scaling exponent. In Section 3.2.3 it will be demonstrated that D_C and ρ_V are really only two special cases of the reference variable Ψ. The fact that there is only one free scaling exponent is a general property of the scaling law, valid for any choice of the reference variable. It is a consequence of the imposed self-consistency, implied by Eq. (3.3).

3.2.3 Constraints on the general raindrop size distribution function and scaling exponents

As the raindrop size distribution $N_V(D, \Psi)$ depends on a reference variable Ψ which itself is a function of the raindrop size distribution, it seems obvious that the form of $g_\Psi(x)$ and the values of α_Ψ and β_Ψ cannot be chosen freely, but should somehow be constrained. In a particular but fairly general case the corresponding self-consistency constraint can be obtained explicitly.

Assume Φ to be a rainfall integral variable, i.e. a rainfall related variable defined as a weighted integral over the raindrop size distribution,

$$\Phi = \int_0^\infty w_\Phi(D)\, N_V(D, \Psi)\, dD, \tag{3.13}$$

where $w_\Phi(D)$ is some weighting function of the raindrop diameter specific to the particular rainfall integral variable Φ. A large class of rainfall related variables can be written in this manner. Substituting the scaling law (Eq. (3.4)) in the right-hand side of Eq. (3.13) yields

$$\Phi = \Psi^{\alpha_\Psi} \int_0^\infty w_\Phi(D)\, g_\Psi\left(\Psi^{-\beta_\Psi} D\right) dD. \tag{3.14}$$

A change of variable in Eq. (3.14) from D to $x = \Psi^{-\beta_\Psi} D$ gives

$$\Phi = \Psi^{\alpha_\Psi + \beta_\Psi} \int_0^\infty w_\Phi\left(\Psi^{\beta_\Psi} x\right) g_\Psi(x)\, dx. \tag{3.15}$$

This is a general expression for the relation between any rainfall integral variable Φ and any reference variable Ψ. Clearly, a power law relationship between Φ and Ψ starts to emerge. A perfect power law however, i.e. a power law with coefficients which do not functionally depend on the value of Ψ, can only be obtained if $w_\Phi(D)$ has a particular functional form, namely if it follows a power law itself (Sempere Torres et al., 1994).

Now consider the special case of Eq. (3.15) which can be obtained when the reference variable Ψ is assumed to be equal to the rainfall integral variable Φ. Then Eq. (3.15) takes the form

$$\Phi = \Phi^{\alpha_\Phi + \beta_\Phi} \int_0^\infty w_\Phi\left(\Phi^{\beta_\Phi} x\right) g_\Phi(x)\, dx. \tag{3.16}$$

Hence, self-consistency requires that

$$\Phi^{\alpha_\Phi + \beta_\Phi - 1} \int_0^\infty w_\Phi\left(\Phi^{\beta_\Phi} x\right) g_\Phi(x)\, dx = 1, \tag{3.17}$$

independent of the value of Φ. This is a general self-consistency constraint implied by the scaling law, which any rainfall integral variable Φ should obey. It puts constraints on the scaling exponents α_Φ and β_Φ, on the general raindrop size distribution function $g_\Phi(x)$ *and* on the form of the function $w_\Phi(D)$. Although this has not been proved rigorously, it seems clear that for Eq. (3.17) to hold independently of the value of Φ,

$w_\Phi(D)$ should necessarily follow a power law (or be a constant, i.e. follow a power law with exponent zero).

The simplest example of a rainfall integral variable Φ has already been encountered in Section 3.2.2, namely the raindrop concentration ρ_V. By definition $w_{\rho_V}(D) = 1$. Hence, the self-consistency requirement for ρ_V (Eq. (3.17)) leads to two constraints, one on the general raindrop size distribution function $g_{\rho_V}(x)$, namely $\int_0^\infty g_{\rho_V}(x)\mathrm{d}x = 1$, and another on the corresponding scaling exponents, namely $\alpha_{\rho_V} + \beta_{\rho_V} = 1$ (confirming the result obtained in Section 3.2.2). In the next three subsections, the self-consistency constraints for three important types of rainfall integral variables will be discussed.

Case 1: the reference variable is proportional to a moment of the raindrop size distribution

It has been shown in Chapter 2 (Section 2.5) that many (hydro)meteorologically relevant rainfall related variables are proportional to moments of the raindrop size distribution. This implies that such variables can be written as

$$\Omega_m = c_{\Omega_m} \int_0^\infty D^m N_V(D, \Psi)\, \mathrm{d}D, \tag{3.18}$$

where c_{Ω_m} is a proportionality constant which takes into account, among others, the necessary unit conversions and m is the order of the moment (which lies typically in the range 0–6, although it is not necessarily an integer). Note that this is a general expression, valid for any Ψ. It is a special case of Eq. (3.13), obtained when $w_{\Omega_m}(D) = c_{\Omega_m} D^m$. The particular reference variable of interest here will be specified only later.

Substituting the scaling law (Eq. (3.4)) in the right-hand side of Eq. (3.18) yields the power law

$$\Omega_m = C_{\Omega_m} \Psi^{\gamma_{\Omega_m}}, \tag{3.19}$$

with prefactor

$$C_{\Omega_m} = c_{\Omega_m} \int_0^\infty x^m g_\Psi(x)\, \mathrm{d}x \tag{3.20}$$

(where $x = \Psi^{-\beta_\Psi} D$ is the scaled raindrop diameter) and exponent

$$\gamma_{\Omega_m} = \alpha_\Psi + (m+1)\,\beta_\Psi. \tag{3.21}$$

Hence, the scaling law implies (1) a power law relationship between Ω_m and Ψ and (2) a linear relationship between the exponent of the power law and the order of the moment. The functional forms of these two relationships do not depend on that of $g_\Psi(x)$, only the prefactor C_{Ω_m} depends on it (Sempere Torres et al., 1994). This means that for given values of the scaling exponents α_Ψ and β_Ψ (i.e. for a given type of rainfall or a given climatic setting), the exponent γ_{Ω_m} of any power law Ω_m–Ψ relationship will be *independent* of the shape of the raindrop size distribution, only the prefactor C_{Ω_m} will depend on it. This implies for instance that possible effects of truncation of the raindrop size distribution must be entirely contained in the prefactor of such power law relationships, the exponent remaining unaffected (something which Ulbrich (1985) demonstrates for the particular case of gamma raindrop size distributions).

If Ψ is now assumed to be proportional to a moment of $N_V(D, \Psi)$ then the number of degrees of freedom in Eqs. (3.20) and (3.21) can be reduced with 2 if the self-consistency requirement that $C_{\Omega_m} = \gamma_{\Omega_m} = 1$ when $\Psi = \Omega_m$ is imposed. This yields the constraints

$$\int_0^\infty x^m g_{\Omega_m}(x)\,\mathrm{d}x = c_{\Omega_m}^{-1} \tag{3.22}$$

and

$$\alpha_{\Omega_m} = 1 - (m+1)\,\beta_{\Omega_m}. \tag{3.23}$$

Hence, the mth order moment of $g_{\Omega_m}(x)$ is fixed and, more importantly, there remains only *one* free scaling exponent (say β_{Ω_m}, for reasons which will become clear in Chapter 5, Section 5.4). The resulting form of the scaling law is

$$N_V(D, \Omega_m) = \Omega_m^{\alpha_{\Omega_m}} g_{\Omega_m}\left(\Omega_m^{-\beta_{\Omega_m}} D\right), \tag{3.24}$$

where $g_{\Omega_m}(x)$ has to satisfy Eq. (3.22) and α_{Ω_m} and β_{Ω_m} are related via Eq. (3.23). The particular functional form of the scaling law which is obtained when Ω_m is taken to be the raindrop concentration ρ_V (which is by definition Ω_0 with $c_{\Omega_m} = 1$) is given by Eq. (3.12). For $m = 0$, Eq. (3.22) implies $\int_0^\infty g_{\rho_V}(x)\mathrm{d}x = 1$ and Eq. (3.23) implies $\alpha_{\rho_V} + \beta_{\rho_V} = 1$, as has been demonstrated above.

Case 2: the reference variable is a weighted mean raindrop diameter

Another group of widely used rainfall related variables, important in the characterization of raindrop size distributions, are the weighted mean raindrop diameters, defined as

$$\overline{D}_m = \frac{\Omega_m}{\Omega_{m-1}} \tag{3.25}$$

with $c_m = c_{m-1}$ (see Chapter 2, Section 2.6). It follows directly from Eqs. (3.19)–(3.21) that the scaling law implies that $\overline{D}_m$ is related to Ψ via the power law

$$\overline{D}_m = C_{\overline{D}_m} \Psi^{\gamma_{\overline{D}_m}}, \tag{3.26}$$

with prefactor

$$C_{\overline{D}_m} = \frac{C_{\Omega_m}}{C_{\Omega_{m-1}}} = \frac{\int_0^\infty x^m g_\Psi(x)\,\mathrm{d}x}{\int_0^\infty x^{m-1} g_\Psi(x)\,\mathrm{d}x} \tag{3.27}$$

and exponent

$$\gamma_{\overline{D}_m} = \gamma_{\Omega_m} - \gamma_{\Omega_{m-1}} = \beta_\Psi. \tag{3.28}$$

Hence, the exponent is now a constant, independent of α_Ψ and m. The fact that all characteristic raindrop diameters are related to Ψ via a power law with exponent β_Ψ has been one of the basic assumptions in the derivation of the scaling law (see Eq. (3.6)) and should therefore not be surprising. The self-consistency requirement now yields the constraints

$$\int_0^\infty x^m g_{\overline{D}_m}(x)\,\mathrm{d}x = \int_0^\infty x^{m-1} g_{\overline{D}_m}(x)\,\mathrm{d}x \tag{3.29}$$

and

$$\beta_{\overline{D}_m} = 1. \tag{3.30}$$

The resulting form of the scaling law therefore is

$$N_{\mathrm{V}}\left(D, \overline{D}_m\right) = \overline{D}_m^{\alpha_{\overline{D}_m}} g_{\overline{D}_m}\left(\frac{D}{\overline{D}_m}\right), \tag{3.31}$$

where $g_{\overline{D}_m}(x)$ has to satisfy Eq. (3.29). Obviously, Eq. (3.31) has the same functional form as the scaling law for any other characteristic diameter D_{C} (Eqs. (3.9) and (3.10)).

Case 3: rain rate is the reference variable

Finally, consider the most widely used rainfall related variable, the rain rate R. For raindrop diameter integration limits of 0 and ∞, R ($\mathrm{mm\,h^{-1}}$) is defined as

$$R = 6\pi \times 10^{-4} \int_0^\infty D^3 v(D)\, N_{\mathrm{V}}(D, \Psi)\, \mathrm{d}D, \tag{3.32}$$

where $v(D)$ ($\mathrm{m\,s^{-1}}$) is the terminal fall speed (in still air) of raindrops with equivalent spherical diameters of D (mm) and $N_{\mathrm{V}}(D, \Psi)$ is the raindrop size distribution ($\mathrm{mm^{-1}\,m^{-3}}$). Substituting the scaling law for $N_{\mathrm{V}}(D, \Psi)$ (Eq. (3.4)) in the right-hand side of Eq. (3.32) yields

$$R = \left[6\pi \times 10^{-4} \int_0^\infty x^3 v\left(\Psi^{\beta_\Psi} x\right) g_\Psi(x)\, \mathrm{d}x\right] \Psi^{\alpha_\Psi + 4\beta_\Psi}. \tag{3.33}$$

Clearly, this only corresponds to a power law relationship between R and Ψ if R is proportional to a moment of the raindrop size distribution. This is only true if the function $v(D)$ is a power law as well. Substitution of Eq. (2.10) (p. 28) in Eq. (3.33) yields the power law

$$R = C_R \Psi^{\gamma_R}, \tag{3.34}$$

with prefactor

$$C_R = 6\pi \times 10^{-4} c \int_0^\infty x^{3+\gamma} g_\Psi(x)\, \mathrm{d}x \tag{3.35}$$

and exponent

$$\gamma_R = \alpha_\Psi + (4 + \gamma)\, \beta_\Psi. \tag{3.36}$$

In accordance with Eqs. (3.22) and (3.23), the self-consistency requirement for R (i.e. $\Omega_{3+\gamma}$ with $c_{\Omega_{3+\gamma}} = 6\pi \times 10^{-4} c$) leads to the constraints[5]

$$\int_0^\infty x^{3+\gamma} g(x)\, \mathrm{d}x = \frac{10^4}{6\pi c} \tag{3.37}$$

and

$$\alpha = 1 - (4 + \gamma)\, \beta. \tag{3.38}$$

[5]In case of R, the subscripts of $g(x)$, α and β are omitted for convenience in notation (Sempere Torres et al., 1994, 1998).

The resulting form of the scaling law becomes

$$N_V(D,R) = R^{\alpha} g\left(R^{-\beta} D\right), \tag{3.39}$$

where $g(x)$ has to satisfy Eq. (3.37) and α and β are related via Eq. (3.38). This form of the scaling law will serve as basis for the applications which will be presented in Section 3.5 and in Chapters 4–6. Table 3.1 summarizes the self-consistency constraints for the three types of rainfall integral variables discussed in this section.

Table 3.1: Summary of the implications of the scaling law formulation for three rainfall integral variables Φ: general moment Ω_m, weighted mean raindrop diameter $\overline{D}_m$ (mm) and rain rate R (mm h^{-1}) (D in mm, $N_V(D)$ in mm^{-1} m^{-3} and $v(D) = cD^{\gamma}$ in m s^{-1}). Note that R equals $\Omega_{3+\gamma}$ with $c_{\Omega_{3+\gamma}} = 6\pi \times 10^{-4} c$. Properties given for each variable: their definitions in terms of the raindrop size distribution $N_V(D,\Psi)$; the coefficients of their power law relationships $\Phi = C_\Phi \Psi^{\gamma_\Phi}$ with the reference variable Ψ; the constraints on their general raindrop size distribution functions g_Φ and the associated scaling exponents α_Φ and β_Φ; their self-consistent functional forms of the raindrop size distribution $N_V(D,\Phi)$.

Property	$\Phi = \Omega_m$	$\Phi = \overline{D}_m$	$\Phi = R$
definition	$c_{\Omega_m} \int_0^\infty D^m \times$ $N_V(D,\Psi)\,\mathrm{d}D$	$\frac{\int_0^\infty D^m N_V(D,\Psi)\mathrm{d}D}{\int_0^\infty D^{m-1} N_V(D,\Psi)\mathrm{d}D}$	$\frac{6\pi c}{10^4} \int_0^\infty D^{3+\gamma} \times$ $N_V(D,\Psi)\,\mathrm{d}D$
C_Φ	$c_{\Omega_m} \int_0^\infty x^m g_\Psi(x)\,\mathrm{d}x$	$\frac{\int_0^\infty x^m g_\Psi(x)\mathrm{d}x}{\int_0^\infty x^{m-1} g_\Psi(x)\mathrm{d}x}$	$\frac{6\pi c}{10^4} \int_0^\infty x^{3+\gamma} g_\Psi(x)\,\mathrm{d}x$
γ_Φ	$\alpha_\Psi + (m+1)\,\beta_\Psi$	β_Ψ	$\alpha_\Psi + (4+\gamma)\,\beta_\Psi$
$g_\Phi(x)$	$\int_0^\infty x^m g_{\Omega_m}(x)\,\mathrm{d}x =$ $c_{\Omega_m}^{-1}$	$\int_0^\infty x^m g_{\overline{D}_m}(x)\,\mathrm{d}x =$ $\int_0^\infty x^{m-1} g_{\overline{D}_m}(x)\,\mathrm{d}x$	$\int_0^\infty x^{3+\gamma} g(x)\,\mathrm{d}x =$ $\frac{10^4}{6\pi c}$
α_Φ, β_Φ	$\alpha_{\Omega_m} =$ $1 - (m+1)\,\beta_{\Omega_m}$	$\beta_{\overline{D}_m} = 1$	$\alpha = 1 - (4+\gamma)\,\beta$
$N_V(D,\Phi)$	$\Omega_m^{\alpha_{\Omega_m}} g_{\Omega_m}\left(\Omega_m^{-\beta_{\Omega_m}} D\right)$	$\overline{D}_m^{\alpha_{\overline{D}_m}} g_{\overline{D}_m}\left(\frac{D}{\overline{D}_m}\right)$	$R^{\alpha} g\left(R^{-\beta} D\right)$

Different aspects of these self-consistency requirements have been touched upon every now and again in the scientific literature on raindrop size distributions (e.g. Olsen et al., 1978). However, their importance has only been recognized completely since Bennett et al. (1984) have considered them explicitly for the case of the exponential raindrop size distribution (with rain rate as reference variable).

3.3 The scaling law formulation in perspective

3.3.1 Alternative approaches to scaling raindrop size distributions

A normalization procedure similar to the scaling law approach has already been proposed by Sekhon and Srivastava (1971) and has later been applied by Willis (1984) and Willis and Tattelman (1989). With regard to this procedure, Sempere Torres et al. (1994) remark that 'this methodology requires setting the shape of the general DSD [drop size distribution] before the normalization procedure, and even if the parameters are fitted in a more robust way the shape is a priori chosen and does not follow from the data'. However, things are in fact slightly more subtle. Although in deriving their normalization Sekhon and Srivastava did indeed impose an a priori functional form for the raindrop size distribution (namely the exponential), their procedure can be justified for *any* two-parameter form of the raindrop size distribution (although it must be admitted that from the way in which they derived their normalization procedure it is not clear whether they realized this). In fact, the normalization proposed by Sekhon and Srivastava is nothing but a particular case of Eq. (3.2), obtained when $N_V(D)\,D_C/\rho_V$ is plotted against D/D_C. Hence, Sekhon and Srivastava's 'universal' distribution is intimately related to the dimensionless function $f_{D_C}(\cdot)$ of Eq. (3.2)[6]. Their experimental data (22 raindrop size distributions derived from measurements with a vertically pointing Doppler radar) confirm that a negative exponential function indeed provides a satisfactory fit to their empirical 'universal' distribution. Willis (1984) and Willis and Tattelman (1989) show that this fit may sometimes be improved upon by using the more versatile gamma distribution[7].

In summary, the two main differences between Sekhon and Srivastava's approach and that discussed here are (1) that Sekhon and Srivastava require *two* parameters to scale their experimental raindrop size distributions (namely the median-volume diameter D_0 and the liquid rainwater content W), whereas according to the scaling law formulation this can be achieved using only *one* parameter (namely Ψ) and (2) that as a result their approach is not intrinsically consistent with the ubiquitous power law relationships between rainfall related variables, whereas the scaling law approach is.

3.3.2 Power law raindrop terminal fall speed – diameter relationships

It follows from Eq. (3.33) that any other functional form for $v(D)$ than a power law will be *inconsistent* with deterministic power law relationships between R and Ψ and hence with the scaling law formulation. Even if not stated explicitly, therefore, any power law relationship involving R (such as a Z–R relationship) must, at least *effectively*, be based on a power law $v(D)$ relationship (as argued by Sempere Torres et al. (1994))[8]. The motivation to employ such power law relationships in the present

[6]Instead of D_C/ρ_V Sekhon and Srivastava (1971) use $\rho_w D_0^4/W$ and instead of $1/D_C$ they use $1/D_0$, where ρ_w [M L^{-3}] is the density of water, D_0 [L] is the median-volume raindrop diameter and W [M L^{-3}] is the liquid rainwater content. However, in both cases the dimensions of the scaling factors are the same, namely [L^4] and [L^{-1}], respectively.

[7]Incidentally, Willis (1984) himself does not realize the generality of Sekhon and Srivastava's normalization either when he states that 'part of the scatter at small and large sizes is caused by the distribution not being strictly exponential, as incorrectly assumed in the [Sekhon and Srivastava (1971)] normalization'.

[8]In much the same way as power law relationships between the specific microwave attenuation coefficient k (dB km^{-1}) and rain rate R (mm h^{-1}) imply 'effective' power law relationships between

context is much the same as that of Ulbrich and Atlas (1998), who state that 'the assumption of a power-law dependence of raindrop fall speed on diameter is not intended to be an accurate representation of the actual fall speeds. Rather, it is used to show that the results found for the coefficients and exponents in the empirical fits are in agreement with that which is predicted by theory'.

Fig. 3.1(a) compares the Atlas and Ulbrich (1977) power law raindrop terminal fall speed parameterization (Eq. (2.10)) with the theoretically more accurate formula proposed by Best (1950a). Clearly, the former starts to deviate significantly from the latter for equivalent spherical raindrop diameters in excess of 3 mm. Since it is a monotonously increasing function of D, the power law is not able to cope with the effects of raindrop deformation, causing larger drops to attain a certain asymptotic fall speed (Pruppacher and Klett, 1978). However, as these drops contribute relatively little to rain rate, the differences in rain rate density $f_R(D)$ (i.e. the distribution of the rain rate over all raindrop diameters, normalized to unit area) for both fall speed parameterizations will be much less pronounced. Fig. 3.1(b) shows the $f_R(D)$-curves for rain rates of 1, 10 and 100 $\text{mm}\,\text{h}^{-1}$ corresponding to these parameterizations on the basis of Best's (1950b) parameterization for the raindrop size distribution (see Chapter 4). This figure clearly shows that a power law raindrop terminal fall speed parameterization yields realistic distributions of rain rate over drop size, notwithstanding its physical shortcomings[9]. In this case it happens to yield more accurate results as well. Whereas the areas under the curves in Fig. 3.1(b) should equal one for reasons of self-consistency, the area under the 100 $\text{mm}\,\text{h}^{-1}$ curve corresponding to Best's fall speed parameterization is only 0.89 (versus 0.98 for that corresponding to the power law fall speed parameterization).

3.3.3 A scaling law for the raindrop size distribution at a surface

Although the scaling law (Eq. (3.4)) has been derived on the basis of the raindrop size distribution per unit volume of air $N_\text{V}(D,\Psi)$ ($\text{mm}^{-1}\,\text{m}^{-3}$), it could have been derived just as well on the basis of that per unit area and per unit time $N_\text{A}(D,\Psi)$ ($\text{mm}^{-1}\,\text{m}^{-2}\,\text{s}^{-1}$). The two forms of the raindrop size distribution are related via (Eq. 2.1)

$$N_\text{A}(D,\Psi) = v(D)\,N_\text{V}(D,\Psi)\,, \tag{3.40}$$

the microwave extinction cross-section Q_t (cm^2) and raindrop diameter D (mm) (Atlas and Ulbrich, 1974; Olsen et al., 1978) (Chapter 2, Section 2.4).

[9]That it not to say, however, that a power law $v(D)$ relationship will *always* provide satisfactory results. For instance, the well-known effect that the width of the Doppler spectrum as measured by vertically pointing radars in rainfall tends to decrease at the highest rain rates (e.g. Russchenberg, 1993) can only be explained if the power law $v(D)$ relationship is abandoned in favor of a more realistic parameterization with an asymptotic terminal fall speed bahavior for large raindrops (such as those of Best (1950b) or Atlas et al. (1973)).

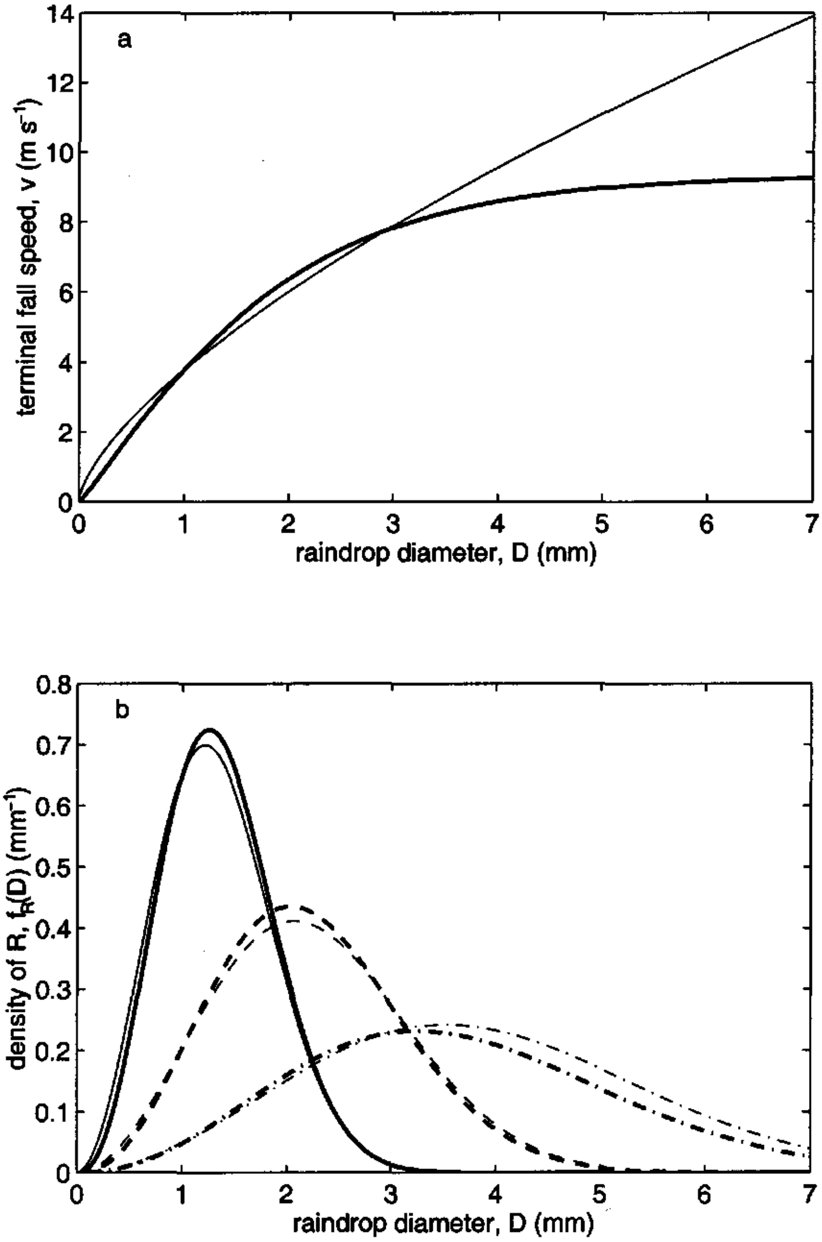

Figure 3.1: (a) Two parameterizations for the relationship between raindrop terminal fall speed (at mean sea level) v ($\mathrm{m\,s^{-1}}$) and equivalent spherical raindrop diameter D (mm): $v(D) = 9.32\left\{1 - \exp\left[-(D/1.77)^{1.147}\right]\right\}$ (Best, 1950a; bold line) and $v(D) = 3.778D^{0.67}$ (Atlas and Ulbrich, 1977; thin line). (b) Implied rain rate density functions (those corresponding to Best's terminal fall speeds in bold) on the basis of Best's (1950b) parameterization for the raindrop size distribution (solid lines: 1 $\mathrm{mm\,h^{-1}}$; dashed lines: 10 $\mathrm{mm\,h^{-1}}$; dash-dotted lines: 100 $\mathrm{mm\,h^{-1}}$).

where $v(D)$ is the raindrop terminal fall speed v ($\mathrm{m\,s^{-1}}$) – diameter D (mm) relationship. Substitution of Eqs. (2.10) and (3.4) yields

$$N_\mathrm{A}(D,\Psi) = \Psi^{\alpha_\Psi} c D^\gamma g_\Psi\Big(\Psi^{-\beta_\Psi} D\Big), \tag{3.41}$$

which can also be written as

$$N_\mathrm{A}(D,\Psi) = \Psi^{\alpha_\Psi+\gamma\beta_\Psi} c \left(\Psi^{-\beta_\Psi} D\right)^\gamma g_\Psi\Big(\Psi^{-\beta_\Psi} D\Big). \tag{3.42}$$

This shows that a scaling law for the raindrop size distribution at a surface can be defined as

$$N_\mathrm{A}(D,\Psi) = \Psi^{\alpha'_\Psi} g'_\Psi\Big(\Psi^{-\beta_\Psi} D\Big), \tag{3.43}$$

with

$$g'_\Psi(x) = c x^\gamma g_\Psi(x) \tag{3.44}$$

and

$$\alpha'_\Psi = \alpha_\Psi + \gamma\beta_\Psi \tag{3.45}$$

(where $x = \Psi^{-\beta_\Psi} D$ is a scaled raindrop diameter). Hence, the general appearance of the alternative scaling law is the same as that of the original (Eq. (3.4)), but one of the scaling exponents has changed and so has the general raindrop size distribution function. The scaling exponent β_Ψ remains the same because it is the exponent of the characteristic raindrop diameters, whose dimensions are not affected by the transformation.

In principle, the entire analysis presented in Section 3.2.3 could now be repeated on the basis of the new form of the scaling law (Eq. (3.43)). However, the attention will be restricted here to the rain rate R. The treatment for other choices of the reference variable will be largely analogous. The definition of R ($\mathrm{mm\,h^{-1}}$) in terms of $N_\mathrm{A}(D,\Psi)$ is

$$R = 6\pi \times 10^{-4} \int_0^\infty D^3 N_\mathrm{A}(D,\Psi)\,\mathrm{d}D. \tag{3.46}$$

Upon substitution of Eq. (3.43) this leads to the power law relationship

$$R = C_R \Psi^{\gamma_R}, \tag{3.47}$$

with prefactor

$$C_R = 6\pi \times 10^{-4} \int_0^\infty x^3 g'_\Psi(x)\,\mathrm{d}x \tag{3.48}$$

and exponent

$$\gamma_R = \alpha'_\Psi + 4\beta_\Psi. \tag{3.49}$$

Substitution of Eqs. (3.44) and (3.45) shows that this prefactor and exponent are equal to those given by Eqs. (3.35) and (3.36), as would be expected. The self-consistency requirement in case $\Psi = R$ now yields for the constraint on the general raindrop size distribution function

$$\int_0^\infty x^3 g'(x)\,\mathrm{d}x = \frac{10^4}{6\pi} \tag{3.50}$$

and for that on the scaling exponents

$$\alpha' = 1 - 4\beta, \tag{3.51}$$

which again are equivalent to the constraints given by Eqs. (3.37) and (3.38). The resulting form of the scaling law becomes

$$\begin{aligned} N_{\mathrm{A}}(D,R) &= R^{\alpha'} g'\left(R^{-\beta} D\right) \\ &= R^{1-4\beta} g'\left(R^{-\beta} D\right), \end{aligned} \tag{3.52}$$

where $g(x)$ has to satisfy Eq. (3.50).

From a purely theoretical point of view, Eq. (3.52) is a rather interesting form of the scaling law, because it abandons the hypothesis of a power law $v(D)$ relationship, necessary for the original form of the scaling law (Eq. (3.39)). As a matter of fact, it is independent of *any* assumption regarding the functional form of the $v(D)$ relationship. This is because, as has been demonstrated in Chapter 2 (Section 2.5), the rain rate R is a flux (or rate) variable[10], a type of rainfall related variable which is intimately related to $N_{\mathrm{A}}(D)$. Since what is actually measured with most ground-based instruments is $N_{\mathrm{A}}(D)$ and not $N_{\mathrm{V}}(D)$, Eq. (3.52) also has practical relevance.

3.4 Extension and application of the scaling law formulation

Up to this point, the scaling law formulation has remained merely a theoretical development, providing a convenient summary of the intimate relation between the ubiquitous power law relationships between rainfall related variables and the general shape of the raindrop size distribution. The ultimate aim of this formulation and its extensions to be presented in this section however, is to facilitate the extraction of physically relevant information from (typically large amounts of) empirical raindrop size distributions. That would imply a confrontation of the theory with actual data. A comprehensive application of the scaling law formulation to empirical raindrop size distributions may in total comprise six main steps:

1. the selection of an appropriate reference variable Ψ;
2. the estimation of the corresponding scaling exponents α_Ψ and β_Ψ (Section 3.4.1);
3. the identification of the empirical general raindrop size distribution function (Sections 3.4.2 and 3.4.3);
4. the adjustment of an analytical parameterization for $g_\Psi(x)$ to the empirical general raindrop size distribution function (Section 3.4.4);

[10] For the calculation of power law relationships between state variables, such as the radar reflectivity factor Z, and R on the basis of Eq. (3.52), the assumption of a power law $v(D)$ function is still needed.

5. the interpretation of the values of the scaling exponents and of the parameters of the analytical $g_\Psi(x)$ in terms of rainfall physics (Section 3.5);

6. further applications such as the derivation of Z–R relationships (Chapter 6).

The Steps 1–4 are largely methodological and form the core of this chapter. Steps 5 and 6, on the other hand, are important for application in radar meteorology and hydrology. They will receive ample attention in Chapters 4 and 6.

With regard to Step 1, there can be little discussion. For most meteorological and hydrological applications, it is appropriate to take the rain rate R as the reference variable. It is certainly the most widely measured rainfall integral variable and, moreover, it is proportional to a moment of the raindrop size distribution with an order (roughly 3.67) which lies central in the range 0–6 of common interest (see Chapter 2). Indeed, in their discussion of experimental evidence for the scaling law formulation, Sempere Torres et al. (1998) choose R to be the reference variable.

In the original presentation of their scaling law formulation (which has been re-derived here in a coherent manner in Section 3.2.1), the main focus of Sempere Torres et al. (1994) is on the Steps 2 and 3 mentioned above. They propose particular methodologies to estimate the scaling exponents and to identify the general distribution function. Hardly any attention is paid to Step 4. In their second article, Sempere Torres et al. (1998) show some preliminary results of adjustments of analytical expressions to empirical general raindrop size distribution functions, but the employed methods seem to be rather "ad hoc". It is the aim of this section to propose a framework which allows one to perform this in a more systematic manner, thus bridging the gap with the more traditional analytical parameterization for the raindrop size distribution.

In this section, after a discussion of the method proposed originally by Sempere Torres et al. (1994), a new method for the identification of the scaling exponents will be presented. Subsequently, the physical interpretation of the general raindrop size distribution function $g(x)$ will be clarified. Then, a new general function $h(x)$ will be introduced, which in contrast to $g(x)$ has the advantage of behaving as a probability density function. Finally, explicit expressions for the general functions $g(x)$ and $h(x)$ and various associated quantities will be provided for all analytical forms of the raindrop size distribution which have been proposed in the literature (exponential, gamma, generalized gamma, Best and lognormal).

3.4.1 Estimation of the scaling exponents

It has been demonstrated in Section 3.2.3 that the exponents γ_Φ of power law relationships between rainfall integral variables Φ and any reference variable Ψ are completely determined by the values of the scaling exponents α_Ψ and β_Ψ. More precisely, the values of γ_Φ depend in a linear fashion on those of α_Ψ and β_Ψ (Table 3.1). They are independent of the shape of the general raindrop size distribution function $g_\Psi(x)$. This implies that the estimation of the exponents γ_{Φ_1} and γ_{Φ_2} of empirical power law relationships between *two* rainfall integral variables Φ_1 and Φ_2 and a particular reference variable Ψ (via some regression procedure) should in principle be enough to determine unambiguously the values of the corresponding scaling exponents α_Ψ and β_Ψ.

One could go even one step further and *impose* the appropriate self-consistency constraint which theoretically ties the scaling exponents α_Ψ and β_Ψ together (Table 3.1). In that case, the exponent of only *one* empirical power law Φ–Ψ relationship would suffice to determine the value of the remaining free scaling exponent (say β_Ψ). For instance, assume that the rainfall integral variable at hand is proportional to the mth order moment of the raindrop size distribution (i.e. $\Phi = \Omega_m$) and that the rain rate plays the role of reference variable (i.e. $\Psi = R$). The scaling law formulation then implies that the exponent of the power law Ω_m–R relationship is related to the scaling exponents according to

$$\gamma_{\Omega_m} = \alpha + (m+1)\,\beta \tag{3.53}$$

(from Eq. (3.21)) and that at the same time the two scaling exponents are related to each other via $\alpha = 1 - (4+\gamma)\,\beta$ (Eq. (3.38)). Substitution of the latter in the former yields

$$\gamma_{\Omega_m} = 1 + [m - (3+\gamma)]\,\beta. \tag{3.54}$$

Hence, β can be in principle obtained from one single empirically determined γ_{Ω_m}, which subsequently implies α via the self-consistency constraint on the exponents[11]. A similar procedure could be devised for any other choice of the reference variable.

All this leads to the question as to *which* rainfall integral variable (in Eq. (3.54): which order of the moment) or which pair of variables should be used to estimate the scaling exponents. In theory, *any* choice would yield the same values of the scaling exponents. In reality, however, different variables will put their principal weight on different parts of the raindrop size distribution. For instance, low order moments tend to weight the small diameter end of the raindrop size distribution more heavily, whereas high order moments tend to put more weight on the tail of the distribution (see Chapter 2, Section 2.5). Since the statistical uncertainty associated with different parts of the raindrop size distribution (associated both with natural variability and with sampling fluctuations) is different, the choice of the variable or pair of variables will influence the results. In short, the estimated values of the scaling exponents will depend on the selected rainfall related variable or pair of variables[12].

Sempere Torres et al. (1994, 1998) have proposed and successfully applied an estimation procedure intended to overcome this subjectivity problem. Their claim is that the estimation of the scaling exponents will be more objective (and hence more robust) if, instead of only one or two variables, a whole range of rainfall related

[11] A particularly interesting special case of Eq. (3.53) is obtained for $m = 6$. Since Ω_6 equals the radar reflectivity factor Z, this implies that the exponent γ_Z of a power law Z–R relationship is related to β via $\gamma_Z = 1 + 2.33\beta$ (if $\gamma = 0.67$). This relationship will be used extensively in Chapter 6.

[12] An example of the mentioned effects of statistical uncertainty associated with rainfall related variables can be found in empirical power law relationships between such variables (e.g. Chapter 5, Section 5.4). In practice, these are never the perfect (i.e. deterministic) relationships which form the fundamental hypothesis of the scaling law.

variables, covering the entire raindrop size distribution, are employed. Of course, this leads to an overdetermined problem, having more than two estimators and only two parameters to estimate. Therefore, the problem must be solved in a regression framework. The idea is to establish empirical power law relationships between a whole series of raindrop size distribution moments Ω_m of different orders m, typically in the range 0–6, and some reference variable Ψ, typically the rain rate R (to which the analysis will be restricted here). The scaling law formulation predicts that the exponents γ_{Ω_m} of these power law relationships must increase linearly with the order of the moment m (Eq. (3.53)). More precisely, α must be the intercept and β the slope of a plot of γ_{Ω_m} versus $(m+1)$. However, due to uncertainties associated with the model, the data and the analysis procedures, in reality the estimated exponents will never lie exactly on this straight line, i.e. there will be some scatter about it. Hence, given a number (> 2) of empirically determined power law exponents, a simple linear regression of γ_{Ω_m} on $(m+1)$ will provide estimates of the scaling exponents α and β.

A first remark with regard to this procedure is that Sempere Torres et al. (1994, 1998) restrict their analyses to integer values of the moment order m. However, there is of course no reason why intermediate values of m could not be taken into account. This would only increase the objectivity and hence the robustness of the method. A second remark concerns the procedure itself. It will be clear that the proposed "method of moments" is but one of a plethora of possible methods to estimate α and β. Any type of rainfall related variable could in principle be used for that purpose. The moments of the raindrop size distribution are but one example. For instance, an alternative method would be to establish empirical power law relationships between the weighted mean raindrop diameters $\overline{D}_m$ (Section 3.2.3) of orders $m = 1$–6 and R. According to the scaling law formulation the exponents $\gamma_{\overline{D}_m}$ of these power laws should all equal β (Eq. (3.28)). A reasonable estimate of β could then be obtained by simply taking the arithmetic mean of the individual exponents. Self-consistency may subsequently be imposed to obtain an estimate of α (Eq. (3.38)). This method would combine the advantage of the method originally proposed by Sempere Torres et al. (1994), namely that a whole range of rainfall integral variables is employed in the estimation, with the advantage of the method which employs only one single exponent (Eq. 3.54), namely that self-consistency is guaranteed[13]. An additional (numerical) advantage over the original procedure would be that all weighted mean raindrop diameters of orders 1–6 are roughly of the same order of magnitude, whereas the moments of the same orders may cover several orders of magnitude. Experience has learned that the latter may cause numerical problems due to non-convergence in iterative nonlinear power law regression procedures. Both methods will be applied and compared on actual rainfall data in Chapters 4 and 5 (Fig. 4.10 on p. 128 and Fig. 5.10 on p. 157).

[13]The advantage of an estimation procedure which takes α as a free parameter is obviously that it provides a check as to the degree of self-consistency implied by the data themselves. Moreover, if desired the procedure originally proposed by Sempere Torres et al. (1994) could be adapted so as to guarantee self-consistency.

3.4.2 Identification and interpretation of the general raindrop size distribution function

It will be clear from the manner in which the scaling law for the raindrop size distribution (Eq. (3.4)) has been derived (Section 3.2.1) that the function $g_\Psi(x)$ will not be dimensionless. As a matter of fact, its dimensions depend on those of the reference variable Ψ and those of $N_V(D,\Psi)$ and also on the values of the corresponding scaling exponents. Although its dimensionality is a logical consequence of the functional form of the scaling law, it does not contribute to a clear physical *interpretation* of $g_\Psi(x)$.

The *identification* of $g_\Psi(x)$ from empirical raindrop size distributions, on the other hand, is rather straightforward. First, the scaling exponents have to be estimated. This has been discussed in the previous subsection. It follows from Eq. (3.4) that once α_Ψ and β_Ψ are known, the function $g_\Psi(x)$ can be identified from experimental data by scaling the available empirical raindrop size distributions ($\Psi^{-\alpha_\Psi} N_V(D,\Psi)$) and corresponding raindrop diameters ($\Psi^{-\beta_\Psi} D$) and plotting them against each other. If this is done on log-log paper, then it is seen that the actual scaling consists of vertical and horizontal displacements of the original raindrop size distributions proportional to $\log \Psi$ (where Ψ is the value of the reference variable for each individual distribution). The factor of proportionality for the vertical displacements (i.e. those of the raindrop concentrations) is $-\alpha_\Psi$ and that for the horizontal displacements (i.e. those of the diameters) $-\beta_\Psi$. In theory such a displacement procedure, aimed at matching as good as possible the individual distributions through vertical and horizontal displacements, could be employed as a graphical procedure for the estimation of the scaling exponents (Sempere Torres et al., 1994). However, in practice it would be quite a tedious procedure and moreover not very objective.

Incidentally, this manner of identification also provides a clear physical interpretation of the general raindrop size distribution function $g_\Psi(x)$. It simply represents the *equivalent* raindrop size distribution at $\Psi = 1$ (e.g. at $R = 1$ mm h^{-1} if the rain rate R would be the reference variable). Indeed, it follows from Eq. (3.4) that

$$g_\Psi\left(\Psi^{-\beta_\Psi} D\right) = \Psi^{-\alpha_\Psi} N_V(D,\Psi)\,, \tag{3.55}$$

which implies

$$g_\Psi(D) = N_V(D,1)\,. \tag{3.56}$$

According to this interpretation of $g_\Psi(x)$, each experimental raindrop size distribution in a sample can be scaled to an equivalent distribution for $\Psi = 1$ using the value of Ψ obtained from that distribution and an a priori knowledge of the scaling exponents α_Ψ and β_Ψ. This approach avoids the common but statistically less robust methodology of grouping measured raindrop size distributions into classes of Ψ (most often classes of rain rate) and then computing a mean distribution for each class (Laws and Parsons, 1943; Marshall and Palmer, 1948; Delrieu et al., 1991). Moreover, it allows to identify the empirical shape of $g_\Psi(x)$ using *all* available raindrop size distributions at once, without having to impose an a priori functional form.

3.4.3 A new class of general functions

After having estimated the scaling exponents α_Ψ and β_Ψ and used them to identify the empirical general raindrop size distribution function, one can in principle proceed to Step 4 of the application of the scaling law formulation (p. 70): the adjustment of an analytical parameterization for $g_\Psi(x)$ to the empirical general raindrop size distribution function. The problem with this adjustment is that not just any analytical function which visually (or according to a more objective numerical goodness-of-fit criterion) describes the empirical distribution satisfactorily, represents a valid parameterization for $g_\Psi(x)$. This is because the self-consistency requirement imposes a certain constraint on $g_\Psi(x)$, the specific form of which depends on the choice of the reference variable (see Table 3.1). Ideally, the procedure used to fit an analytical function to the empirical general raindrop size distribution function should intrinsically comprise this constraint. Sempere Torres et al. (1998) give some examples of negative exponential fits and one example of a gamma fit to empirical general raindrop size distribution functions from several locations in Europe. They use a nonlinear least-squares regression procedure which takes the self-consistency requirement automatically into account. In their normalization procedure (described in Section 3.3), Willis (1984) and Willis and Tattelman (1989) use similar curve-fitting procedures to fit gamma functions to their normalized empirical raindrop size distributions.

One could question however, whether such curve-fitting procedures will always yield the most satisfactory results. Classically, the procedure used to adjust analytical parameterizations to empirical raindrop size distributions is the *method of moments*, typically using moments of orders 3–6 (e.g. Ulbrich, 1983; Feingold and Levin, 1986; Tokay and Short, 1996; Ulbrich and Atlas, 1998). Although the employed methodologies are sometimes questionable from a statistical point of view, the motivation for using the method of moments is clear: the moments represent "remote measurables", i.e. quantities which may be determined using (optical or microwave) remote sensing techniques. Indeed, it has been demonstrated in Chapter 2 (Section 2.5) that various of such remote measurables are, under certain assumptions, proportional or closely proportional to moments of the raindrop size distribution.

As shown in Table 3.1, the moments of the general raindrop size distribution function $g_\Psi(x)$ have a clear physical interpretation as the prefactors C_{Ω_m} of power law relationships between the moments of the raindrop size distribution Ω_m and the reference variable Ψ. How can these be used in a classical statistical framework, where the sample moments are used to estimate the parameters of *probability density functions*? For that purpose, it will be necessary to devise a normalized version of $g_\Psi(x)$, i.e. a function with unit area. At the same time, it is desirable to have a function available which somehow automatically takes into account the self-consistency constraint on $g_\Psi(x)$, as explained above.

If the treatment is restricted to the common case where a moment Ω_m plays the role of reference variable Ψ, then the definition of a function with the two required properties follows directly from the definition of the self-consistency constraint on the general raindrop size distribution function $g_{\Omega_m}(x)$ (Eq. (3.22)). A function $h_{\Omega_m}(x)$

defined as the integrand of that self-consistency constraint, i.e. as

$$h_{\Omega_m}(x) = c_{\Omega_m} x^m g_{\Omega_m}(x)\,, \tag{3.57}$$

will namely by definition satisfy

$$\int_0^\infty h_{\Omega_m}(x)\,\mathrm{d}x = 1. \tag{3.58}$$

Hence, this is the desired function. What is its physical interpretation? If, in accordance with Chapter 2 (Section 2.5), the density function $f_{\Omega_m}(D)$ (mm^{-1}) of a moment Ω_m is defined as the ratio of the integrand in the definition of Ω_m (Eq. (3.18)) to Ω_m itself then

$$f_{\Omega_m}(D, \Omega_m) = c_{\Omega_m} \Omega_m^{-1} D^m N_\mathrm{V}(D, \Omega_m)\,, \tag{3.59}$$

with Ω_m explicitly included as reference variable. Substitution of the corresponding form of the scaling law (Eq. (3.24)) and of the constraint on the scaling exponents (Eq. (3.23)) yields

$$f_{\Omega_m}(D, \Omega_m) = c_{\Omega_m} \Omega_m^{-(m+1)\beta_{\Omega_m}} D^m g_{\Omega_m}\left(\Omega_m^{-\beta_{\Omega_m}} D\right). \tag{3.60}$$

Multiplying both sides with $\mathrm{d}D$ yields

$$f_{\Omega_m}(D, \Omega_m)\,\mathrm{d}D = c_{\Omega_m} \Omega_m^{-(m+1)\beta_{\Omega_m}} D^m g_{\Omega_m}\left(\Omega_m^{-\beta_{\Omega_m}} D\right)\mathrm{d}D, \tag{3.61}$$

which can also be written as

$$f_{\Omega_m}(D, \Omega_m)\,\mathrm{d}D = c_{\Omega_m} \left(\Omega_m^{-\beta_{\Omega_m}} D\right)^m g_{\Omega_m}\left(\Omega_m^{-\beta_{\Omega_m}} D\right)\mathrm{d}\left(\Omega_m^{-\beta_{\Omega_m}} D\right). \tag{3.62}$$

Now, a change of variable from D to $x = \Omega_m^{-\beta_{\Omega_m}} D$ gives

$$f_{\Omega_m}(D, \Omega_m)\,\mathrm{d}D = c_{\Omega_m} x^m g_{\Omega_m}(x)\,\mathrm{d}x, \tag{3.63}$$

which upon substitution of Eq. (3.57) becomes

$$f_{\Omega_m}(D, \Omega_m)\,\mathrm{d}D = h_{\Omega_m}(x)\,\mathrm{d}x. \tag{3.64}$$

This finally implies

$$\begin{aligned} f_{\Omega_m}(D, \Omega_m) &= h_{\Omega_m}(x(D))\frac{\mathrm{d}x(D)}{\mathrm{d}D} \\ &= \Omega_m^{-\beta_{\Omega_m}} h_{\Omega_m}\left(\Omega_m^{-\beta_{\Omega_m}} D\right). \end{aligned} \tag{3.65}$$

Hence, the function $h_{\Omega_m}(x)$ is simply a scaled version of the moment density function $f_{\Omega_m}(D, \Omega_m)$. Since it does not bear any functional dependence on the value of Ω_m (it merely depends on the choice of Ω_m), it plays the same role for $f_{\Omega_m}(D, \Omega_m)$ as $g_{\Omega_m}(x)$ plays for $N_\mathrm{V}(D, \Omega_m)$. Therefore, in analogy with the terminology proposed by Sempere Torres et al. (1994) for $g_{\Omega_m}(x)$, the function $h_{\Omega_m}(x)$ will from now on be called the *general moment density function*.

In the common case where Ω_m is the rain rate R, the equivalent of $h_{\Omega_m}(x)$ (Eq. (3.57)) is

$$h(x) = 6\pi \times 10^{-4} c x^{3+\gamma} g(x) , \qquad (3.66)$$

which will be called the *general rain rate density function.* Similarly, the equivalent of the moment density function (Eq. (3.65)) is

$$f_R(D, R) = R^{-\beta} h\Big(R^{-\beta} D\Big) , \qquad (3.67)$$

where $f_R(D, R)$ is now the rain rate density function. In accordance with the notation used for $g(x)$, α and β (Section 3.2.3), the subscript R of $h(x)$ is omitted for convenience in notation. The function $f_R(D, R)$ represents the normalized distribution of the rain rate R over all raindrop sizes. Since R is proportional to the $(3 + \gamma)$th moment of the raindrop size distribution, it lies central in the range of moments (0–6) of common interest in meteorological, hydrological and telecommunications research (Chapter 2, Section 2.5). Eq. (3.67) will therefore, together with the corresponding definition for $N_V(D, R)$ in terms of $g(x)$ (Eq. (3.39)), serve as basis for the applications to be presented in Section 3.5 and in Chapters 4 and 6.

The identification of $h_{\Omega_m}(x)$ from empirical moment density functions (which can be obtained directly from the empirical raindrop size distributions using Eq. (3.59)) can be performed in a similar manner as that of $g_{\Omega_m}(x)$, namely by means of scaling the available empirical moment densities $(\Omega_m^{\beta_{\Omega_m}} f_{\Omega_m}(D, \Omega_m))$ and corresponding raindrop diameters $(\Omega_m^{-\beta_{\Omega_m}} D)$ and plotting them against each other. Since the densities are multiplied by the same amount by which the diameters are divided, it is clear that the empirical $h_{\Omega_m}(x)$-curves will have the same area as the empirical $f_{\Omega_m}(D, \Omega_m)$-curves on which they are based (namely unity). If, in analogy with the general raindrop size distribution function (Section 3.4.2), the scaling is done on log-log paper, then it is seen that it actually consists of vertical and horizontal displacements of the original moment density functions proportional to $\log \Omega_m$, with proportionality factors for the vertical (i.e. those of the moment densities) and for the horizontal displacements (i.e. those of the diameters) of β_Ψ and $-\beta_\Psi$, respectively. Correspondingly, $h_{\Omega_m}(x)$ can be interpreted as the *equivalent* moment density function for $\Omega_m = 1$ (see Eq. (3.56)).

In summary, empirical general moment density functions $h_{\Omega_m}(x)$ will be normalized to unit area, as opposed to the corresponding empirical general raindrop size distribution functions $g_{\Omega_m}(x)$. This has the advantage that the analytical functions which may be adjusted to them can in principle be any of the theoretical continuous probability density functions found in statistical textbooks, provided their domain is limited to non-negative values (e.g. Kendall and Stuart, 1977). As these have by definition unit area, the self-consistency requirement will be guaranteed[14]. Moreover, the application of the method of moments to estimate the parameters of such theoretical probability density functions will be straightforward, as will be shown in Chapter 4.

[14]In practical applications, it is sometimes preferable to work with the (cumulative) distribution function instead of the corresponding density function (see Chapter 2, Section 2.5). Since the function $h_{\Omega_m}(x)$ behaves as a probability density function, such a transformation is particularly straightforward in this case. A *general (cumulative) moment distribution function* may be defined

3.4.4 Analytical parameterizations for the general functions

In order to be able to relate the described general framework for the analysis of raindrop size distributions and their properties to the traditional analytical parameterizations for the raindrop size distribution which have become commonplace over the past decades, it is necessary to propose explicit expressions for $g_{\Omega_m}(x)$, $h_{\Omega_m}(x)$ and the related functions $N_V(D, \Omega_m)$ and $f_{\Omega_m}(D, \Omega_m)$. This is done in the current section for the common case where Ω_m equals the rain rate R, although the expressions which are going to be presented are easily generalized to the case where Ω_m is proportional to *any* moment of the raindrop size distribution (using the expressions given in Table 3.1 and Section 3.4.3).

Tables 3.2–3.4 summarize the explicit expressions for $g(x)$, $h(x)$ and the related functions $N_V(D, R)$ and $f_R(D, R)$ corresponding to the primary candidates for the parameterization of the general raindrop size distribution function: the exponential, gamma, generalized gamma, Best and lognormal distributions[15]. Except for the rather special multi-modal distributions introduced for equilibrium rainfall conditions (Section 3.5.2), these five cases comprise *all* major analytical parameterizations which have been proposed to describe raindrop size distributions since the beginning of scientific research in this domain around the turn of the last century[16]. Besides explicit expressions for the mentioned quantities, these tables also provide equations for the moments and the associated (dimensionless) coefficients of variation, skewness and kurtosis (peakedness) of the general rain rate density function $h(x)$. These will be employed in Chapter 4, where the method of moments is used to adjust the analytical functions for $h(x)$ to empirical rain rate density functions.

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as

$$H_{\Omega_m}(x) = \int_0^x h_{\Omega_m}(\xi)\,\mathrm{d}\xi.$$

In accordance with the (cumulative) probability distribution functions known from statistical theory, $H_{\Omega_m}(x)$ will be a non-decreasing function of x, with limiting values $H_{\Omega_m}(0) = 0$ and $H_{\Omega_m}(\infty) = 1$. A transformation of variable from x to D shows that

$$\begin{aligned} H_{\Omega_m}\left(\Omega_m^{-\beta_{\Omega_m}} D\right) &= \int_0^D f_{\Omega_m}(D', \Omega_m)\,\mathrm{d}D' \\ &= F_{\Omega_m}(D, \Omega_m), \end{aligned}$$

where $F_{\Omega_m}(D, \Omega_m)$ is the (cumulative) moment distribution function of x. Hence, $H_{\Omega_m}(x)$ can be identified by plotting $F_{\Omega_m}(D, \Omega_m)$ versus the scaled raindrop diamters $\Omega_m^{-\beta_{\Omega_m}} D$.

[15]In Chapter 4 (Section 4.2) it will be demonstrated that the parameterization Best (1950a) has proposed for the cumulative distribution of the liquid rainwater content W over raindrop diameter D corresponds to a raindrop size distribution $N_V(D, R)$ which is a special case of the generalized gamma distribution.

[16]Power law (fat-tailed, hyperbolic) distributions have to the best of the author's knowledge never been proposed to parameterize raindrop size distributions. They have been used, however, for size distributions of other atmospheric particles, such as aerosol particles (i.e. the so-called Junge distribution) (e.g. Rogers and Yau, 1996) and cloud droplets (Provata and Nicolis, 1994).

Table 3.2: General definitions and specific expressions for the exponential case of the self-consistent forms of the general raindrop size distribution function $g(x)$ and the general rain rate density function $h(x)$ (where D in mm, $N_V(D,R)$ in $\mathrm{mm^{-1}\,m^{-3}}$, R in $\mathrm{mm\,h^{-1}}$ and $v(D) = cD^\gamma$ in $\mathrm{m\,s^{-1}}$), their properties (specifically the moments $\mu'_{x,r}$ of the scaled raindrop diameter $x = R^{-\beta}D$ with respect to $h(x)$ and the corresponding mean μ_x and coefficients of variation CV_x, skewness CS_x and kurtosis CK_x) and their relationships with the raindrop size distribution $N_V(D,R)$ and the rain rate density function $f_R(D,R)$.

Quantity	Definition	Exponential
$g(x)$	$N_V(x,1)$	$\kappa \exp(-\lambda x)$
domain	$0 \le x < \infty$	$\kappa, \lambda > 0$
κ	solution of $\frac{6\pi c}{10^4}\int_0^\infty x^{3+\gamma} g(x)\mathrm{d}x = 1$	$\frac{10^4}{6\pi c}\frac{\lambda^{4+\gamma}}{\Gamma(4+\gamma)}$
$h(x)$	$\frac{6\pi c}{10^4} x^{3+\gamma} g(x) = f_R(x,1)$	$\frac{\lambda^{4+\gamma}}{\Gamma(4+\gamma)} x^{3+\gamma} \exp(-\lambda x)$
	$\left(\int_0^\infty h(x)\mathrm{d}x = 1\right)$	(gamma)
$\mu'_{x,r}$	$\int_0^\infty x^r h(x)\mathrm{d}x$	$\frac{\Gamma(4+\gamma+r)}{\lambda^r \Gamma(4+\gamma)}$
μ_x	$\mu'_{x,1}$	$\frac{4+\gamma}{\lambda}$
CV_x	$\frac{\mu_{x,2}^{1/2}}{\mu_x} = \frac{\left(\mu'_{x,2}-\mu_x^2\right)^{1/2}}{\mu_x} = \left(\frac{\mu'_{x,2}}{\mu_x^2} - 1\right)^{1/2}$	$\frac{1}{(4+\gamma)^{1/2}}$
CS_x	$\frac{\mu_{x,3}}{\mu_{x,2}^{3/2}} = \frac{\mu'_{x,3}-3\mu'_{x,2}\mu_x+2\mu_x^3}{\left(\mu'_{x,2}-\mu_x^2\right)^{3/2}}$	$\frac{2}{(4+\gamma)^{1/2}} = 2\mathrm{CV}_x$
CK_x	$\frac{\mu_{x,4}}{\mu_{x,2}^2} - 3 = \frac{\mu'_{x,4}-4\mu'_{x,3}\mu_x+6\mu'_{x,2}\mu_x^2-3\mu_x^4}{\left(\mu'_{x,2}-\mu_x^2\right)^2} - 3$	$\frac{6}{4+\gamma} = 6\mathrm{CV}_x^2$
$N_V(D,R)$	$R^{1-(4+\gamma)\beta} g\left(R^{-\beta}D\right)$	$\kappa R^{1-(4+\gamma)\beta} \exp\left(-\lambda R^{-\beta}D\right)$
$f_R(D,R)$	$\frac{6\pi c}{10^4} R^{-1} D^{3+\gamma} N_V(D,R) =$	$\frac{\lambda^{4+\gamma}}{\Gamma(4+\gamma)} R^{-(4+\gamma)\beta} D^{3+\gamma} \times$
	$R^{-\beta} h\left(R^{-\beta}D\right)$	$\exp\left(-\lambda R^{-\beta}D\right)$

General properties of the analytical parameterizations

A few remarks concerning the expressions in Tables 3.2–3.4 are in place. First of all, it is seen that all expressions for the general raindrop size distribution function $g(x)$ contain a proportionality factor κ. These should not be confused with the free parameters of the functions $g(x)$, however. The self-consistency requirement when R is the reference variable (Eq. (3.37)) imposes a constraint on κ which ties it to the other parameters of $g(x)$. Expressions for κ in terms of these other parameters are provided in the tables for each of the five distributions types. It follows that the actual number of free parameters (i.e. the number of degrees of freedom) of the general functions $g(x)$ and $h(x)$ for the different distribution types is: one for the exponential distribution (λ); two for the gamma distribution (λ, μ); three for the generalized gamma distribution (λ, μ, ν); two for the Best distribution (λ, ν); two for

Table 3.3: Idem for the gamma and generalized gamma cases.

Quantity	Gamma	Generalized gamma
$g(x)$	$\kappa x^{\mu} \exp(-\lambda x)$	$\kappa x^{\mu} \exp(-\lambda x^{\nu})$
domain	$\kappa, \lambda > 0;\ \mu > -1$	$\kappa, \lambda, \nu > 0;\ \mu > -1$
κ	$\frac{10^4}{6\pi c} \frac{\lambda^{4+\gamma+\mu}}{\Gamma(4+\gamma+\mu)}$	$\frac{10^4}{6\pi c} \frac{\nu\lambda^{(4+\gamma+\mu)/\nu}}{\Gamma[(4+\gamma+\mu)/\nu]}$
$h(x)$	$\frac{\lambda^{4+\gamma+\mu}}{\Gamma(4+\gamma+\mu)} x^{3+\gamma+\mu} \exp(-\lambda x)$ (gamma)	$\frac{\nu\lambda^{(4+\gamma+\mu)/\nu}}{\Gamma[(4+\gamma+\mu)/\nu]} x^{3+\gamma+\mu} \exp(-\lambda x^{\nu})$ (generalized gamma)
$\mu'_{x,r}$	$\frac{\Gamma(4+\gamma+\mu+r)}{\lambda^r \Gamma(4+\gamma+\mu)}$	$\frac{\Gamma[(4+\gamma+\mu+r)/\nu]}{\lambda^{r/\nu}\Gamma[(4+\gamma+\mu)/\nu]}$
μ_x	$\frac{4+\gamma+\mu}{\lambda}$	$\frac{\Gamma[(5+\gamma+\mu)/\nu]}{\lambda^{1/\nu}\Gamma[(4+\gamma+\mu)/\nu]}$
CV_x	$\frac{1}{(4+\gamma+\mu)^{1/2}}$	$\left\{\frac{\Gamma[(6+\gamma+\mu)/\nu]\Gamma[(4+\gamma+\mu)/\nu]}{\Gamma^2[(5+\gamma+\mu)/\nu]} - 1\right\}^{1/2}$
CS_x	$\frac{2}{(4+\gamma+\mu)^{1/2}} = 2\mathrm{CV}_x$	$\frac{\mu'_{x,3}/\mu_x^3 - 3\mu'_{x,2}/\mu_x^2 + 2}{\mathrm{CV}_x^3}$
CK_x	$\frac{6}{4+\gamma+\mu} = 6\mathrm{CV}_x^2$	$\frac{\mu'_{x,4}/\mu_x^4 - 4\mu'_{x,3}/\mu_x^3 + 6\mu'_{x,2}/\mu_x^2 - 3}{\mathrm{CV}_x^4} - 3$
$N_V(D,R)$	$\kappa R^{1-(4+\gamma+\mu)\beta} D^{\mu} \exp\left(-\lambda R^{-\beta} D\right)$	$\kappa R^{1-(4+\gamma+\mu)\beta} D^{\mu} \exp\left(-\lambda R^{-\beta\nu} D^{\nu}\right)$
$f_R(D,R)$	$\frac{\lambda^{4+\gamma+\mu}}{\Gamma(4+\gamma+\mu)} R^{-(4+\gamma+\mu)\beta} D^{3+\gamma+\mu} \times \exp\left(-\lambda R^{-\beta} D\right)$	$\frac{\nu\lambda^{(4+\gamma+\mu)/\nu}}{\Gamma[(4+\gamma+\mu)/\nu]} R^{-(4+\gamma+\mu)\beta} D^{3+\gamma+\mu} \times \exp\left(-\lambda R^{-\beta\nu} D^{\nu}\right)$

the lognormal distribution (μ, σ)[17]. Note that the generalized gamma distribution contains the exponential (for $\mu = 0$ and $\nu = 1$), gamma (for $\nu = 1$) and Best (for $\mu = \nu - 4$) distributions as special cases[18]. The functions $N_V(D,R)$ and $f_R(D,R)$ have one free parameter more than the corresponding general functions, namely the scaling exponent β.

Self-consistency requires all these parameters to be functionally *independent* of the rain rate R. If not, the general functions would be dependent on R, thereby ceasing to be general and accordingly provoke a violation of the self-consistency. It will therefore be a challenge to investigate how these parameters are related to the type of rainfall or the climatic setting to which the empirical data pertain, to the type of measurement device used to collect the data and possibly, as suggested by Sempere Torres et al. (1998), to each other[19].

Another point of interest is that in the transformation from $g(x)$ to $h(x)$, the exponential distribution changes to a gamma distribution. The same obviously holds for the (related) transformation from $N_V(D,R)$ to $f_R(D,R)$ (Table 3.2). This effect

[17]The λs and the μs do not represent the same parameter for each of these distributions.

[18]The Best distribution is sometimes erroneously referred to as the Weibull distribution. However, the Weibull distribution is obtained from the generalized gamma distribution for $\mu = \nu - 1$ (e.g. Mood et al., 1974).

[19]Because the units of the parameter λ depend on the value of the scaling exponent β, however, there is a risk of *spurious correlation* (e.g. Haan, 1977) (Chapter 6).

Table 3.4: Idem for the Best and lognormal cases.

Quantity	Best	Lognormal
$g(x)$	$\kappa x^{\nu-4} \exp\left(-\lambda x^{\nu}\right)$	$\kappa x^{-1} \exp\left[-\frac{1}{2}\left(\frac{\ln x-\mu}{\sigma}\right)^{2}\right]$
domain	$\kappa, \lambda > 0;\ \nu > 3$	$-\infty < \mu < \infty;\ \sigma > 0$
κ	$\frac{10^4}{6\pi c} \frac{\nu\lambda^{1+\gamma/\nu}}{\Gamma(1+\gamma/\nu)}$	$\frac{10^4}{6\pi c} \frac{\exp\left[-(3+\gamma)\mu-\frac{1}{2}(3+\gamma)^2\sigma^2\right]}{\sqrt{2\pi}\sigma}$
$h(x)$	$\frac{\nu\lambda^{1+\gamma/\nu}}{\Gamma(1+\gamma/\nu)} x^{\gamma+\nu-1} \exp\left(-\lambda x^{\nu}\right)$	$\frac{1}{\sqrt{2\pi}\sigma x} \exp\left\{-\frac{1}{2}\left[\frac{\ln x-\left(\mu+(3+\gamma)\sigma^2\right)}{\sigma}\right]^{2}\right\}$
	(generalized gamma)	(lognormal)
$\mu'_{x,r}$	$\frac{\Gamma[1+(r+\gamma)/\nu]}{\lambda^{r/\nu}\Gamma(1+\gamma/\nu)}$	$\exp\left\{r\mu+\left[(3+\gamma)\,r+\frac{1}{2}r^2\right]\sigma^2\right\} =$
		$\mu_x^r\left(1+\mathrm{CV}_x^2\right)^{r(r-1)/2}$
μ_x	$\frac{\Gamma[1+(1+\gamma)/\nu]}{\lambda^{1/\nu}\Gamma(1+\gamma/\nu)}$	$\exp\left[\mu+\left(3\frac{1}{2}+\gamma\right)\sigma^2\right]$
CV_x	$\left\{\frac{\Gamma[1+(2+\gamma)/\nu]\Gamma[1+\gamma/\nu]}{\Gamma^2[1+(1+\gamma)/\nu]}-1\right\}^{1/2}$	$\left[\exp\left(\sigma^2\right)-1\right]^{1/2}$
CS_x	$\frac{\mu'_{x,3}/\mu_x^3-3\mu'_{x,2}/\mu_x^2+2}{\mathrm{CV}_x^3}$	$3\mathrm{CV}_x+\mathrm{CV}_x^3$
CK_x	$\frac{\mu'_{x,4}/\mu_x^4-4\mu'_{x,3}/\mu_x^3+6\mu'_{x,2}/\mu_x^2-3}{\mathrm{CV}_x^4}-3$	$16\mathrm{CV}_x^2+15\mathrm{CV}_x^4+6\mathrm{CV}_x^6+\mathrm{CV}_x^8$
$N_{\mathrm{V}}(D,R)$	$\kappa R^{1-(\gamma+\nu)\beta} D^{\nu-4}\times$	$\kappa R^{1-(3+\gamma)\beta} D^{-1}\times$
	$\exp\left(-\lambda R^{-\beta\nu} D^{\nu}\right)$	$\exp\left\{-\frac{1}{2}\left[\frac{\ln D-(\mu+\beta\ln R)}{\sigma}\right]^{2}\right\}$
$f_R(D,R)$	$\frac{\nu\lambda^{1+\gamma/\nu}}{\Gamma[1+\gamma/\nu]} R^{-(\gamma+\nu)\beta} D^{\gamma+\nu-1}\times$	$\frac{1}{\sqrt{2\pi}\sigma D}\times$
	$\exp\left(-\lambda R^{-\beta\nu} D^{\nu}\right)$	$\exp\left\{-\frac{1}{2}\left[\frac{\ln D-\left(\mu+(3+\gamma)\sigma^2+\beta\ln R\right)}{\sigma}\right]^{2}\right\}$

is comparable to that observed when transforming an exponential raindrop size distribution in a volume of air ($N_{\mathrm{V}}(D,R)$) to a gamma distribution for that at a surface ($N_{\mathrm{A}}(D,R)$) (as demonstrated in Chapter 2, Fig. 2.2, p. 26). The gamma, generalized gamma and lognormal distributions all retain their original general forms, albeit with changed parameters. It is noteworthy that in case of the lognormal distribution, a transformation from $g(x)$ to $h(x)$ does not affect the logarithmic standard deviation σ. Since σ uniquely determines the coefficient of variation of x (CV_x) (defined as the ratio of the standard deviation to the mean, see Table 3.4), this implies that in this particular case the *relative* dispersion of $g(x)$ equals that of $h(x)$. In other words, the transformation from $g(x)$ to $h(x)$ only induces a shift in the logarithmic mean of the distribution. The same is obviously true for the (related) transformation from $N_{\mathrm{V}}(D,R)$ to $f_R(D,R)$ (Table 3.4). The latter has been noted earlier by Feingold and Levin (1986).

Relations with the most important traditional analytical parameterizations

The exponential raindrop size distribution The exponential distribution has for a long time been the most widely used analytical parameterization for the raindrop size distribution. In its traditional form it is written as

$$N_V(D) = N_0 \exp(-\Lambda D); \quad N_0, \Lambda > 0; \quad D \geq 0, \tag{3.68}$$

where N_0 ($\mathrm{mm^{-1}\,m^{-3}}$) is the intercept of the distribution with the vertical axis ($D = 0$) and Λ ($\mathrm{mm^{-1}}$) is the slope of the distribution on a semi-logarithmic plot, or equivalently the inverse mean raindrop diameter. Chapter 2 (Section 2.3) gives some additional statistical properties of this distribution.

In the form of Eq. (3.68), the exponential raindrop size distribution has been introduced originally by Marshall and Palmer (1948), who considered the special case when N_0 is a fixed parameter (Section 3.5.1). It has been generalized, first by Sekhon and Srivastava (1971) and some years later by Waldvogel (1974). Both consider N_0 to be a variable parameter, the former explicitly related to the rain rate R via a power law relationship and the latter as a time-varying parameter exhibiting sudden variations ('jumps') during transitions from one mesoscale precipitation area to another.

Eq. (3.68) can be compared directly with the self-consistent functional form $N_V(D, R)$ of the exponential raindrop size distribution as implied by the scaling law formulation (Table 3.2). This shows that Eq. (3.68) will only be self-consistent if N_0 and Λ are power law functions of the rain rate R. More precisely, they should obey

$$\begin{aligned} N_0 &= \kappa R^{1-(4+\gamma)\beta} \\ &= \frac{10^4}{6\pi c} \frac{\lambda^{4+\gamma}}{\Gamma(4+\gamma)} R^{1-(4+\gamma)\beta} \end{aligned} \tag{3.69}$$

and

$$\Lambda = \lambda R^{-\beta}, \tag{3.70}$$

where D is expressed in mm, $N_V(D, R)$ in $\mathrm{mm^{-1}\,m^{-3}}$, R in $\mathrm{mm\,h^{-1}}$ and $v(D) = cD^{\gamma}$ in $\mathrm{m\,s^{-1}}$. These expressions allow a verification of the self-consistency of the N_0–R and Λ–R relationships reported in the literature for exponential raindrop size distributions. Although not mentioned explicitly, they have been used in the derivation of the consistent sets of power law relationships between rainfall related variables presented in Chapter 2 (Section 2.7).

The gamma raindrop size distribution Since the mid-70s many studies have indicated that, unless sufficiently averaged in space and/or time, the exponential raindrop size distribution overestimates both the concentrations of the very small and the very large raindrops. The magnitude of these effects is found to be more pronounced the shorter the time periods and the smaller the areas over which the averaging takes place. To account for these effects, Ulbrich (1983) has re-introduced the gamma distribution. The particular functional form he proposes is

$$N_V(D) = N_0 D^{\mu} \exp(-\Lambda D); \quad N_0, \Lambda > 0; \quad \mu > -1; \quad D \geq 0, \tag{3.71}$$

where N_0 (now expressed in the inconvenient units of $\mathrm{mm}^{-(1+\mu)}\,\mathrm{m}^{-3}$) is a concentration parameter, μ (–) is a shape parameter[20] and Λ (mm^{-1}) is a scale parameter. Some statistical properties of this distribution are given in Chapter 2 (Section 2.3). As a matter of fact, the raindrop size distribution at a surface $N_A(D)$ treated in that section, reduces to the $N_V(D)$ of Eq. (3.71) for $c = 0$ and $\gamma = \mu$. Note that the raindrop concentration ρ_V, the mean raindrop diameter $\mu_{\underline{D}_V}$ and the coefficient of variation of the diameters are only defined for $\mu > -1$. The approximative expression for $\mathrm{median}_{\underline{D}_V}$ is only valid if $\mu > -0.67$, although it will yield only accurate results for $\mu \geq 0$. The mode of $N_V(D)$ obviously becomes zero for $\mu \leq 0$. By the same token, the general requirement for the size distribution of raindrops arriving at a surface ($N_A(D)$) is that $\mu > -(1+\gamma)$. The approximation for $\mathrm{median}_{\underline{D}_A}$ is only valid for $\mu > -(0.67+\gamma)$ and the mode of $N_A(D)$ becomes zero for $\mu \leq -\gamma$.

In a similar manner as has been done for the exponential distribution, Eq. (3.71) can be compared with the self-consistent functional form $N_V(D, R)$ of the gamma raindrop size distribution as implied by the scaling law formulation (Table 3.3). The result is that Eq. (3.71) will again only be self-consistent if N_0 and Λ are power law functions of the rain rate R. In this case, they must obey

$$\begin{aligned} N_0 &= \kappa R^{1-(4+\gamma+\mu)\beta} \\ &= \frac{10^4}{6\pi c}\frac{\lambda^{4+\gamma+\mu}}{\Gamma(4+\gamma+\mu)} R^{1-(4+\gamma+\mu)\beta} \end{aligned} \tag{3.72}$$

and

$$\Lambda = \lambda R^{-\beta}. \tag{3.73}$$

These expressions will be used extensively in Chapter 6, where Z–R relationships implied by the scaling law formulation for the gamma raindrop size distribution will be investigated. They also allow a verification of the self-consistency of the N_0–R and Λ–R relationships reported in the literature for gamma raindrop size distributions (for given values of μ). Clearly, for $\mu = 0$ Eqs. (3.71) and (3.72) reduce to Eqs. (3.68) and (3.69).

The lognormal raindrop size distribution Feingold and Levin (1986) have revitalized the lognormal raindrop size distribution as an alternative to the gamma distribution. After the exponential and gamma distributions, it is the third important analytical parameterization for the raindrop size distribution. Its traditional form is (e.g. Mood et al., 1974)

$$\begin{aligned} N_V(D) &= \frac{\rho_V}{\sqrt{2\pi}\sigma_{\ln \underline{D}_V} D} \exp\left[-\frac{1}{2}\left(\frac{\ln D - \mu_{\ln \underline{D}_V}}{\sigma_{\ln \underline{D}_V}}\right)^2\right]; \\ & \rho_V, \sigma_{\ln \underline{D}_V} > 0; \quad -\infty < \mu_{\ln \underline{D}_V} < \infty; \quad D \geq 0, \end{aligned} \tag{3.74}$$

[20]According to Ulbrich (1983) 'μ can have any positive or negative value'. However, unless the gamma distribution is truncated at some minimum diameter $D_{\min}$ (Ulbrich takes $D_{\min} = 0$), the raindrop concentration ρ_V (m^{-3}) will diverge if $\mu \leq -1$. All non-negative moments of the gamma distribution exist only as long as $\mu > -1$ (e.g. Mood et al., 1974). Note that Ulbrich reports values for μ as low as -3.42 (see Chapter 6, Section 6.5).

where ρ_V (m^{-3}) is the raindrop concentration, $\mu_{\ln \underline{D}_V}$ (–) is a scale parameter (namely the mean of the log-transformed raindrop diameters in a volume of air) and $\sigma_{\ln \underline{D}_V}$ (–) is a shape parameter (the standard deviation of the log-transformed raindrop diameters in a volume of air).

By definition, if the random variable $\underline{D}$ follows a lognormal distribution, then the random variable $\ln \underline{D}$ follows a normal distribution. If the Central Limit Theorem states that the normal (Gaussian) distribution should be the asymptotic distribution of the *sum* of many (not necessarily independent) random variables (i.e. the limit of *additive* processes), the lognormal distribution should be the asymptotic distribution of the *product* of many of such variables (i.e. the limit of *multiplicative* processes). The formation of raindrop size distributions can be seen as the outcome of such a multiplicative stochastic process, involving interactions between raindrops through collisions and breakup. This type of (informal) reasoning provides some theoretical justification for the lognormal distribution as a suitable parameterization for raindrop size distributions (Feingold and Levin, 1986).

Eq. (3.74) can again be compared with the self-consistent functional form $N_V(D, R)$ of the lognormal raindrop size distribution as implied by the scaling law formulation (Table 3.4). This shows that Eq. (3.74) will only be self-consistent if $\sigma_{\ln \underline{D}_V}$ is a constant equal to σ, i.e.

$$\sigma_{\ln \underline{D}_V} = \sigma, \tag{3.75}$$

$\mu_{\ln \underline{D}_V}$ is linearly related to $\ln R$ according to

$$\mu_{\ln \underline{D}_V} = \mu + \beta \ln R \tag{3.76}$$

and ρ_V is a power law function of R according to

$$\begin{aligned} \rho_V &= \sqrt{2\pi}\sigma\kappa R^{1-(3+\gamma)\beta} \\ &= \frac{10^4}{6\pi c} \exp\left[-(3+\gamma)\mu - \frac{1}{2}(3+\gamma)\sigma^2\right] R^{1-(3+\gamma)\beta}. \end{aligned} \tag{3.77}$$

That Eq. (3.76) actually represents a power law as well can be seen as follows. By analogy with the normal distribution, it follows directly from Eq. (3.74) that $\mu_{\ln \underline{D}_V}$ represents the natural logarithm of the median of $N_V(D)$, i.e. the median raindrop diameter $\text{median}_{\underline{D}_V}$. This is the diameter which divides the distribution of the raindrop concentration ρ_V over all raindrop diameters into two equal parts[21]. Hence

$$\text{median}_{\underline{D}_V} = \exp\left(\mu_{\ln \underline{D}_V}\right) = \exp(\mu) R^{\beta}. \tag{3.78}$$

In other words, the scaling law formulation implies that median raindrop diameter should be a power law function of the rain rate R as well, with an exponent equal to the scaling exponent β. That any characteristic raindrop diameter is related to R via a power law with exponent β has of course been one of the basic hypotheses of the scaling law formulation (Section 3.2.1), so this result should not come as a surprise.

[21] The median raindrop diameter should not be confused with the median-*volume* raindrop diameter, which divides the distribution of the liquid rainwater content (or that of rain rate) over all raindrop diameters into two equal parts (Chapter 2, Section 2.6).

Incidentally, Eq. (3.78) by definition equals the *geometric mean* of the raindrop diameters (in a volume of air). That for the lognormal distribution the median equals the geometric mean is a consequence of the symmetry of the normal distribution, which has the median equal to the (arithmetic) mean.

Table 3.5: Summary of the statistical properties of the most widely used analytical forms for the raindrop size distribution per unit volume of air: the exponential, gamma and lognormal distributions. The top half of the table pertains to raindrops present in a volume of air (subscripts V), the bottom half to those arriving at a surface (subscripts A). The listed properties are: raindrop concentration (m^{-3}) / raindrop arrival rate ($\mathrm{m}^{-2}\,\mathrm{s}^{-1}$), mean diameter (mm), median diameter (mm), modal diameter (mm), coefficient of variation of the diameters (–), median-volume diameter (mm) and volume-weighted mean diameter (mm).

Property	Exponential	Gamma	Lognormal
ρ_{V}	$\frac{N_0}{\Lambda}$	$N_0\frac{\Gamma(1+\mu)}{\Lambda^{1+\mu}}$	$\frac{10^4 R}{6\pi c}\exp\left[-(3+\gamma)\,\mu_{\ln \underline{D}_{\mathrm{V}}} -\frac{1}{2}(3+\gamma)\,\sigma^2_{\ln \underline{D}_{\mathrm{V}}}\right]$
$\mu_{\underline{D}_{\mathrm{V}}}$	$\frac{1}{\Lambda}$	$\frac{1+\mu}{\Lambda}$	$\exp\left(\mu_{\ln \underline{D}_{\mathrm{V}}}+\frac{1}{2}\sigma^2_{\ln \underline{D}_{\mathrm{V}}}\right)$
$\mathrm{median}_{\underline{D}_{\mathrm{V}}}$	$\frac{\ln 2}{\Lambda}$	$\frac{0.67+\mu}{\Lambda}$	$\exp\left(\mu_{\ln \underline{D}_{\mathrm{V}}}\right)$
$\mathrm{mode}_{\underline{D}_{\mathrm{V}}}$	0	$\frac{\mu}{\Lambda}$	$\exp\left(\mu_{\ln \underline{D}_{\mathrm{V}}}-\sigma^2_{\ln \underline{D}_{\mathrm{V}}}\right)$
$\mathrm{CV}_{\underline{D}_{\mathrm{V}}}$	1	$\frac{1}{(1+\mu)^{1/2}}$	$\left[\exp\left(\sigma^2_{\ln \underline{D}_{\mathrm{V}}}\right)-1\right]^{1/2}$
$D_{0,\mathrm{V}}$	$\frac{3.67}{\Lambda}$	$\frac{3.67+\mu}{\Lambda}$	$\exp\left(\mu_{\ln \underline{D}_{\mathrm{V}}}+3\sigma^2_{\ln \underline{D}_{\mathrm{V}}}\right)$
$D_{m,\mathrm{V}}$	$\frac{4}{\Lambda}$	$\frac{4+\mu}{\Lambda}$	$\exp\left(\mu_{\ln \underline{D}_{\mathrm{V}}}+3\frac{1}{2}\sigma^2_{\ln \underline{D}_{\mathrm{V}}}\right)$
ρ_{A}	$cN_0\frac{\Gamma(1+\gamma)}{\Lambda^{1+\gamma}}$	$cN_0\frac{\Gamma(1+\gamma+\mu)}{\Lambda^{1+\gamma+\mu}}$	$c\rho_{\mathrm{V}}\exp\left(\gamma\mu_{\ln \underline{D}_{\mathrm{V}}}+\frac{1}{2}\gamma^2\sigma^2_{\ln \underline{D}_{\mathrm{V}}}\right)$
$\mu_{\underline{D}_{\mathrm{A}}}$	$\frac{1+\gamma}{\Lambda}$	$\frac{1+\gamma+\mu}{\Lambda}$	$\exp\left[\mu_{\ln \underline{D}_{\mathrm{V}}}+\left(\frac{1}{2}+\gamma\right)\sigma^2_{\ln \underline{D}_{\mathrm{V}}}\right]$
$\mathrm{median}_{\underline{D}_{\mathrm{A}}}$	$\frac{0.67+\gamma}{\Lambda}$	$\frac{0.67+\gamma+\mu}{\Lambda}$	$\exp\left(\mu_{\ln \underline{D}_{\mathrm{V}}}+\gamma\sigma^2_{\ln \underline{D}_{\mathrm{V}}}\right)$
$\mathrm{mode}_{\underline{D}_{\mathrm{A}}}$	$\frac{\gamma}{\Lambda}$	$\frac{\gamma+\mu}{\Lambda}$	$\exp\left[\mu_{\ln \underline{D}_{\mathrm{V}}}-(1-\gamma)\,\sigma^2_{\ln \underline{D}_{\mathrm{V}}}\right]$
$\mathrm{CV}_{\underline{D}_{\mathrm{A}}}$	$\frac{1}{(1+\gamma)^{1/2}}$	$\frac{1}{(1+\gamma+\mu)^{1/2}}$	$\left[\exp\left(\sigma^2_{\ln \underline{D}_{\mathrm{V}}}\right)-1\right]^{1/2}$
$D_{0,\mathrm{A}}$	$\frac{3.67+\gamma}{\Lambda}$	$\frac{3.67+\gamma+\mu}{\Lambda}$	$\exp\left[\mu_{\ln \underline{D}_{\mathrm{V}}}+(3+\gamma)\,\sigma^2_{\ln \underline{D}_{\mathrm{V}}}\right]$
$D_{m,\mathrm{A}}$	$\frac{4+\gamma}{\Lambda}$	$\frac{4+\gamma+\mu}{\Lambda}$	$\exp\left[\mu_{\ln \underline{D}_{\mathrm{V}}}+\left(3\frac{1}{2}+\gamma\right)\sigma^2_{\ln \underline{D}_{\mathrm{V}}}\right]$

In contrast to what was the case for the exponential and gamma raindrop size distributions, the statistical properties of the lognormal distribution have not been treated before in this thesis. Its most important properties will therefore be recalled here. The moments of the lognormal raindrop size distribution are given by (e.g. Mood et al., 1974)

$$\mathrm{E}\left[\underline{D}_{\mathrm{V}}^r\right] = \exp\left(r\mu_{\ln \underline{D}_{\mathrm{V}}}+\frac{1}{2}r^2\sigma^2_{\ln \underline{D}_{\mathrm{V}}}\right). \tag{3.79}$$

Hence, its mean $\mu_{\underline{D}_V}$ is

$$\mu_{\underline{D}_V} = \mathrm{E}\left[\underline{D}_V\right] = \exp\left(\mu_{\ln \underline{D}_V} + \frac{1}{2}\sigma^2_{\ln \underline{D}_V}\right) \tag{3.80}$$

and its coefficient of variation $\mathrm{CV}_{\underline{D}_V}$ follows

$$\mathrm{CV}_{\underline{D}_V} = \left[\exp\left(\sigma^2_{\ln \underline{D}_V}\right) - 1\right]^{1/2}. \tag{3.81}$$

Using Eqs. (3.80) and (3.81), an alternative general expression for the moments can be derived as

$$\mathrm{E}\left[\underline{D}_V^r\right] = \mu^r_{\underline{D}_V}\left(1 + \mathrm{CV}^2_{\underline{D}_V}\right)^{r(r-1)/2}. \tag{3.82}$$

This shows clearly the particular multiplicative form of the lognormal moments. An expression of this form has been used by Smith and De Veaux (1994) to model rain rates (which are proportional to the $(3+\gamma)$th moment of the raindrop size distribution in a volume of air). The median (or geometric mean) of the lognormal distribution has already been encountered (Eq. (3.78)). In terms of $\mu_{\underline{D}_V}$ and $\mathrm{CV}_{\underline{D}_V}$ it can be rewritten as

$$\begin{aligned}\mathrm{median}_{\underline{D}_V} &= \exp\left(\mu_{\ln \underline{D}_V}\right) \\ &= \mu_{\underline{D}_V}\left(1 + \mathrm{CV}^2_{\underline{D}_V}\right)^{-1/2}. \end{aligned} \tag{3.83}$$

A final parameter of interest here is the mode (peak) of the lognormal distribution, which can be found by setting the derivative of Eq. (3.74) with respect to D equal to zero. This yields

$$\begin{aligned}\mathrm{mode}_{\underline{D}_V} &= \exp\left(\mu_{\ln \underline{D}_V} - \sigma^2_{\ln \underline{D}_V}\right) \\ &= \mu_{\underline{D}_V}\left(1 + \mathrm{CV}^2_{\underline{D}_V}\right)^{-3/2}. \end{aligned} \tag{3.84}$$

These results show that for the lognormal raindrop size distribution (as for any unimodal, positively skewed distribution), the measures of location follow each other in reverse alphabetical order: mode $<$ median $<$ mean (Kendall and Stuart, 1977). Finally, note that self-consistent forms of Eqs. (3.79)–(3.84) can be obtained by substituting Eq. (3.75) for $\sigma_{\ln \underline{D}_V}$ and Eq. (3.76) for $\mu_{\ln \underline{D}_V}$.

This completes the treatment of the three most widely used analytical parameterizations for the raindrop size distribution in a volume of air: the exponential, the gamma and the lognormal distribution. Table 3.5 summarizes the most important statistical properties of these distributions in terms of their traditional parameters. Table 3.6 lists the self-consistency constraints on the concentration and scale parameters in case the rain rate R is the reference variable. Note that self-consistency requires the shape parameters of these distributions to be constants (i.e. for a given type of rainfall or climatic setting), independent of R. The corresponding results for the Best (1950b) distribution involve more elaborate manipulations and will be postponed to Chapter 4 (Table 4.1, p. 111). There, this much less widely used but

Table 3.6: Self-consistency requirements for the concentration and scale parameters of the traditional forms of the exponential, gamma and lognormal distributions in case the rain rate R (mm h^{-1}) is the reference variable (N_0 in mm$^{-(1+\mu)}$ m^{-3}; Λ in mm^{-1}; ρ_V in m^{-3}; $\exp(\mu_{\ln D_V})$ in mm).

Distribution	Concentration parameter	Scale parameter
Exponential	$N_0 = \frac{10^4}{6\pi c}\frac{\lambda^{4+\gamma}}{\Gamma(4+\gamma)}R^{1-(4+\gamma)\beta}$	$\Lambda = \lambda R^{-\beta}$
Gamma	$N_0 = \frac{10^4}{6\pi c}\frac{\lambda^{4+\gamma+\mu}}{\Gamma(4+\gamma+\mu)}R^{1-(4+\gamma+\mu)\beta}$	$\Lambda = \lambda R^{-\beta}$
Lognormal	$\rho_V = \frac{10^4}{6\pi c}\exp\left[-(3+\gamma)\mu - \frac{1}{2}(3+\gamma)\sigma^2\right]R^{1-(3+\gamma)\beta}$	$\exp\left(\mu_{\ln D_V}\right) = \exp(\mu)\, R^{\beta}$

nevertheless interesting distribution will be extensively revisited in the light of the framework developed in this chapter. Here, the treatment of the exponential, gamma and lognormal distributions merely serves to show the practical implications of the adopted systematic approach: the expressions given in Tables 3.2–3.4 allow to bridge the gap between the presented scaling law formulation for the raindrop size distribution and its extensions on the one hand and the traditional analytical parameterizations on the other.

3.5 Three special cases of the scaling law

Extensive experimental verifications of the scaling law formulation will be presented in Chapters 4 and 5. In this section, the scaling law will be confronted with some analytical parameterizations for the raindrop size distribution. Three special cases will be considered: (1) the Marshall-Palmer raindrop size distribution, corresponding to $\alpha = 0$ (Section 3.5.1); (2) the equilibrium raindrop size distribution, corresponding to $\beta = 0$ (Section 3.5.2); (3) the case where $\alpha + \beta = 0$ (Section 3.5.3).

3.5.1 The Marshall-Palmer raindrop size distribution: $\alpha = 0$

Formulation in terms of the scaling law and verification of the self-consistency

The most widely used analytical parameterization for the raindrop size distribution during the past five decades has without any doubt been that proposed by Marshall and Palmer (1948). It is an exponential distribution of the form of Eq. (3.68), with a constant value of N_0 equal to 8.0×10^3 mm^{-1} m^{-3} and Λ (mm^{-1}) related to R (mm h^{-1}) via the power law $4.1R^{-0.21}$ (Chapter 2, Eqs. (2.2)–(2.4)). The Marshall-Palmer distribution is generally believed to be a reasonably accurate representation of the average raindrop size distribution during stratiform rainfall (e.g. Joss and Waldvogel, 1969; Battan, 1973). Comparison with Eq. (3.39) and Table 3.2 shows that the Marshall-Palmer distribution can be recast in a form which is consistent with the scaling law, if the general raindrop size distribution function is taken to be

$$g(x) = 8.0 \times 10^3 \exp(-4.1x) \tag{3.85}$$

(i.e. $\kappa = 8.0 \times 10^3$ and $\lambda = 4.1$), with the scaling exponents $\alpha = 0$ and $\beta = 0.21$.

It has already been noted in Chapter 2 (Section 2.7) that the Marshall-Palmer distribution is not entirely consistent with the raindrop terminal fall speed – diameter relationship of Atlas and Ulbrich (1977), adopted as the reference relationship in this thesis. Only an adjustment of the Λ–R relationship proposed by Marshall and Palmer to $\Lambda = 4.23R^{-0.214}$ (Eq. (2.63)) would render their distribution self-consistent in the sense implied by Eqs. (3.37) and (3.38). This lack of self-consistency of the Marshall-Palmer distribution has been noted before, e.g. by Bennett et al. (1984) and Zawadzki and de Agostinho Antonio (1988).

To quantify the extent to which $g(x)$, α and β satisfy (or violate) the self-consistency constraints posed by Eqs. (3.37) and (3.38), two error coefficients are introduced. The first coefficient,

$$S_{\mathrm{p}} = 6\pi \times 10^{-4} c \int_0^\infty x^{3+\gamma} g(x)\, \mathrm{d}x, \tag{3.86}$$

quantifies the self-consistency of the prefactor, and the second,

$$S_{\mathrm{e}} = \alpha + (4 + \gamma)\,\beta, \tag{3.87}$$

that of the exponent. These coefficients are nothing but the prefactor (Eq. (3.35)) and the exponent (Eq. (3.36)) of a power law relationship between the rain rate R and a reference variable Ψ (Eq. (3.34)) in case $\Psi = R$. Hence, if a parameterization for the raindrop size distribution would be fully self-consistent then both S_{p} and S_{e} should equal one. Substituting Eq. (3.85) in Eq. (3.86), with $c = 3.778$ and $\gamma = 0.67$, yields $S_{\mathrm{p}} = 1.16$. Substitution of $\alpha = 0$ and $\beta = 0.21$ in Eq. (3.87) (again assuming $\gamma = 0.67$) gives $S_{\mathrm{e}} = 0.98$. In other words, at least for diameter integration limits of 0 and ∞, the Marshall-Palmer distribution is not self-consistent, particularly not with regard to the prefactor.

Interpretation of the scaling exponents and the general distribution function

Fig. 3.2(a) shows the location of the self-consistent form (i.e. with $\beta = 0.214$) of the Marshall-Palmer distribution in the parameter space spanned by the scaling exponents α and β. For reference, the theoretical self-consistency relationship between the scaling exponents, corresponding to Eq. (3.87) for $S_{\mathrm{e}} = 1$ (i.e. $\beta = \frac{1-\alpha}{4+\gamma}$), has been drawn for three different values of the exponent γ of a power law $v(D)$ relationship: 0.8, 0.67 and 0.5. It is seen that in this region of the parameter space, the self-consistency relationship between α and β is not very sensitive to such differences in γ. Fig. 3.2(a) is a generalization of a type of plot originally introduced by Sempere Torres et al. (1994).

What is the physical interpretation of the particular location of the Marshall-Palmer distribution on the self-consistency lines of Fig. 3.2? Returning to Section 3.2.1

(Eqs. (3.2)–(3.6)), it is seen that $\alpha = 0$ implies that $\gamma_{\rho_V} = \gamma_{D_C}$, where γ_{ρ_V} and γ_{D_C} are the exponents of power law relationships between the raindrop concentration ρ_V and the rain rate R and between any characteristic raindrop diameter D_C and R, respectively. In other words, the Marshall-Palmer distribution corresponds to the special case where ρ_V and D_C depend in exactly the same fashion on R, such that their ratio ρ_V/D_C becomes independent of R, implying that they must be proportional. For exponential raindrop size distributions ρ_V equals N_0/Λ (Table 3.5), which means that ρ_V/D_C is in general proportional to N_0. In the specific case where D_C is the mean raindrop diameter $1/\Lambda$, ρ_V/D_C exactly equals N_0. In the Marshall-Palmer parameterization N_0 is indeed a constant.

This becomes even more clear from Fig. 3.2(b), which shows the location of the Marshall-Palmer distribution on a plot of β versus $\alpha + \beta$. It follows from Eqs. (3.5) and (3.6) that this is equivalent to a plot of γ_{D_C} versus γ_{ρ_V}. The self-consistency relationship between β (γ_{D_C}) and $\alpha + \beta$ (γ_{ρ_V}) now becomes

$$\beta = \frac{1 - (\alpha + \beta)}{3 + \gamma}, \tag{3.88}$$

or

$$\gamma_{D_C} = \frac{1 - \gamma_{\rho_V}}{3 + \gamma}. \tag{3.89}$$

The particular location of a point on the self-consistency line (e.g. for the Marshall-Palmer distribution the intersection with $\gamma_{\rho_V} = \gamma_{D_C}$) now has a clear physical interpretation in terms of the exponents of power law relationships between D_C, ρ_V and R.

Fig. 3.3(a) and (b) show the general raindrop size distribution function $g(x)$ and the corresponding general rain rate density function $h(x)$ for the self-consistent form (i.e. with $\lambda = 4.23$) of the Marshall-Palmer raindrop size distribution. Recall that $g(x)$ and $h(x)$ can be interpreted as the equivalent raindrop size distribution $N_V(D, R)$ and the equivalent rain rate density function $f_R(D, R)$ for $R = 1\ \mathrm{mm\,h^{-1}}$, respectively. Clearly, the exponential functional form for $g(x)$ changes to a gamma form for $h(x)$ (Table 3.2), with a corresponding shift in the center of gravity towards larger (scaled) raindrop sizes. This effect has already been observed in Chapter 2 (Fig. 2.5(a) and (d), p. 41). There however, different curves for different rain rates had to be drawn. Here, the influence of the rain rate on the appearance of the curves has been filtered out completely through the applied scaling.

Since the Marshall-Palmer distribution has a scaling exponent α of zero, the vertical axis in Fig. 3.3(a) is effectively not scaled at all. It can therefore simply be interpreted as the value of the raindrop size distribution $N_V(D, R)$ itself. As a matter of fact, the intercept with the vertical axis (denoted as κ in the exponential parameterization for $g(x)$, Table 3.2) in this case equals N_0 ($8.0 \times 10^3\ \mathrm{mm^{-1}\,m^{-3}}$). Then it is seen that an increase in R for the Marshall-Palmer distribution merely corresponds to a blow-up of the horizontal (raindrop size) axis. On a semi-logarithmic plot of $N_V(D, R)$ versus D this change of scale would then correspond to a counter-clockwise rotation of the raindrop size distribution around the pivotal point $N_V(0, R) = N_0$ (as shown in Fig. 2.1, p. 23).

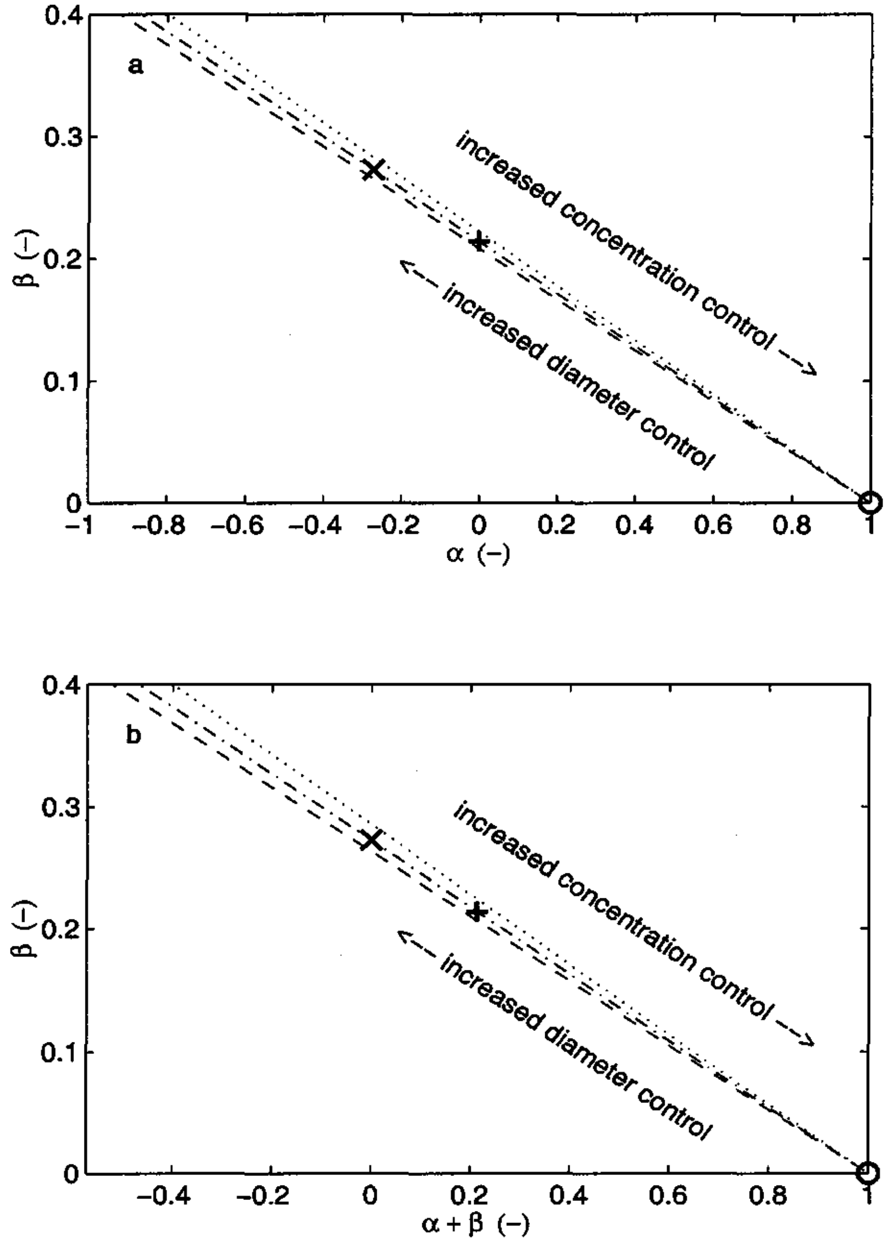


Figure 3.2: (a) Theoretical self-consistency relationship between the scaling exponents α (–) and β (–), $\beta = \frac{1-\alpha}{4+\gamma}$, for three different values of the exponent γ of a power law relationship between raindrop terminal fall speed and equivalent spherical raindrop diameter (dashed line: $\gamma = 0.8$; dash-dotted line: $\gamma = 0.67$; dotted line: $\gamma = 0.5$). The cross at the point with coordinates $(\alpha, \beta) = (-0.27, 0.27)$ corresponds to raindrop size controlled rainfall, the plus at the point with coordinates $(\alpha, \beta) = (0, 0.21)$ to Marshall and Palmer's (1948) exponential raindrop size distribution, the circle at the point with coordinates $(\alpha, \beta) = (1, 0)$ to equilibrium rainfall (raindrop concentration controlled) conditions. (b) Idem in the transformed parameter space spanned by the exponents $\alpha + \beta = \gamma_{\rho_V}$ and $\beta = \gamma_{D_C}$.

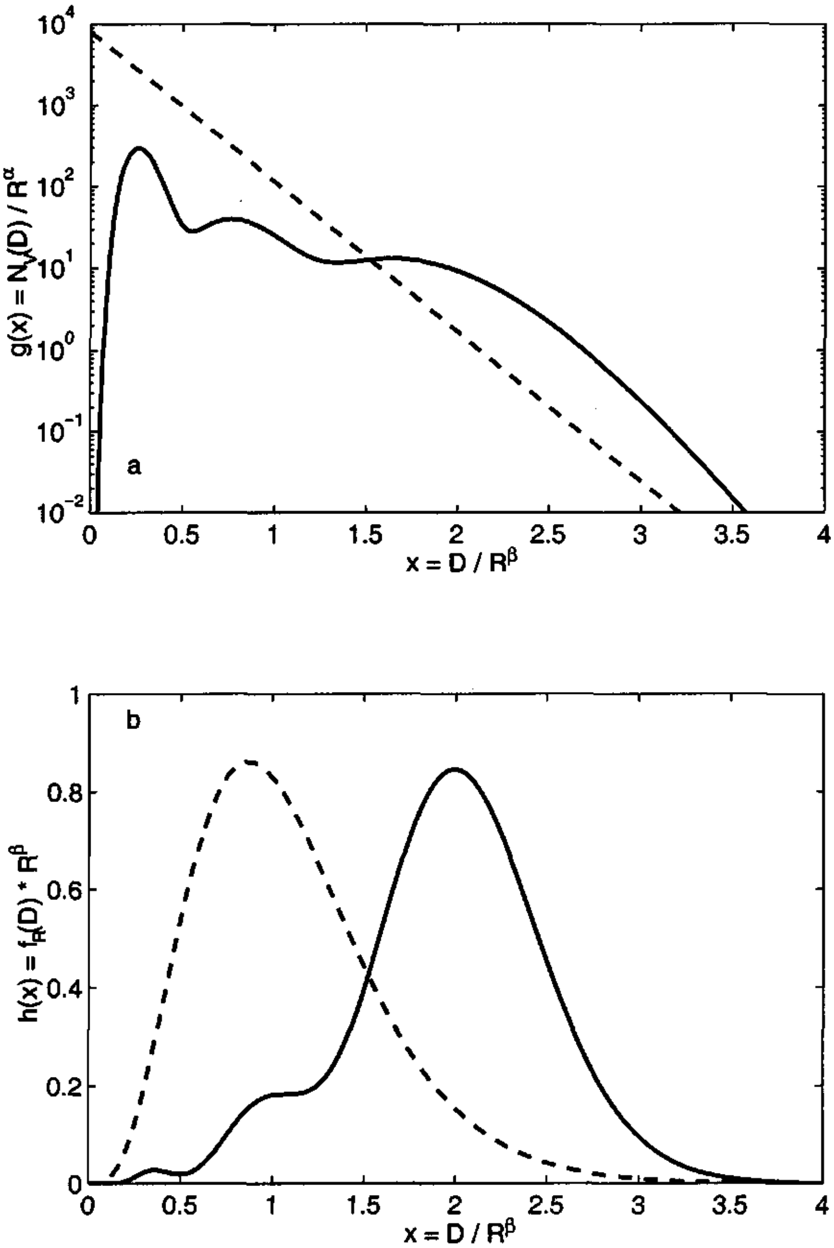


Figure 3.3: (a) General raindrop size distribution functions $g(x)$ corresponding to Marshall and Palmer's (1948) exponential raindrop size distribution (dashed line) and List's (1988) parameterization for the three-peak equilibrium distribution as the sum of three gamma distributions (solid line). (b) Corresponding general rain rate density functions $h(x)$.

3.5.2 The equilibrium raindrop size distribution: $\beta = 0$

Verification of the self-consistency and interpretation of the scaling exponents

As a result of compensating effects in the competing microphysical processes shaping raindrop size distributions, 'any raindrop distribution will develop with time into an equilibrium distribution regardless of the initial spectrum' (List, 1988). On the basis of the stationary form of the stochastic collection equation (the integro-differential equation governing the temporal evolution of the raindrop size distribution) List et al. (1987) show analytically that equilibrium raindrop size distributions are by definition the product of the rain rate R (or any other rainfall integral variable) and a generic shape function, i.e. that they are a family of curves defined by

$$N_{\mathrm{V}}(D,R) = Rg(D)\,. \tag{3.90}$$

Experimental evidence for such a proportionality in persistent tropical rain is reported by List et al. (1988) and Zawadzki and de Agostinho Antonio (1988).

Comparison with Eq. (3.39) shows immediately that Eq. (3.90) is in fact a limiting case of the scaling law, obtained for $\alpha = 1$ and $\beta = 0$. Because these values satisfy the constraint posed by Eq. (3.38), they form a self-consistent pair. Hence $S_{\mathrm{e}} = 1$ (Eq. (3.87)). Fig. 3.2(a) shows the location of the equilibrium distribution in the parameter space spanned by the scaling exponents α and β. Because $\beta = 0$ in equilibrium, Eq. 3.38 implies that $\alpha = 1$, independent of γ. That is why the three theoretical self-consistency lines indicated in the figure meet in this point.

Eqs. (3.2)–(3.6) (p. 58) show that $\alpha = 1$ and $\beta = 0$ imply $\gamma_{\rho_{\mathrm{V}}} = 1$ and $\gamma_{D_{\mathrm{C}}} = 0$. In other words, the raindrop concentration ρ_{V} must depend linearly on the rain rate R and at the same time any characteristic raindrop diameter must remain constant, unaffected by changes in R. Apparently, under equilibrium rainfall conditions, all spatial and temporal variability of the shape of the raindrop size distribution is controlled by variations in the raindrop concentration ρ_{V}. The probability density function of the raindrop diameters remains constant. For the specific case of the weighted mean raindrop diameters, this also follows from Eq. (3.28) ($\gamma_{\overline{D}_m} = \beta = 0$). Fig. 3.2(b) shows the location of the equilibrium distribution in the parameter space spanned by the exponents $\gamma_{\rho_{\mathrm{V}}}$ and γ_{D_C}. Again, the three theoretical self-consistency lines meet in the equilibrium point.

Substitution of $\alpha = 1$ and $\beta = 0$ in Eq. (3.21) leads to $\gamma_{\Omega_m} = 1$ (independent of m), implying that under equilibrium rainfall conditions *any* moment of the raindrop size distribution, not just ρ_{V}, must be proportional to R and hence to any other moment. This confirms one of List's (1988) main conclusions. In effect, this proportionality is even a more general property of equilibrium rainfall. Eq. (3.15) shows that when $\alpha = 1$ and $\beta = 0$ *any* rainfall integral variable, not necessarily a moment of the raindrop size distribution, will be proportional to the rain rate R and hence to any other rainfall integral variable.

Self-consistency and interpretation of the general distribution function

With regard to the functional form of $g(D)$, computer simulations of the temporal evolution of raindrop size distributions in both zero-dimensional (box) and one-dimensional (shaft) models have demonstrated that raindrop size distributions evolve with time to equilibrium distributions with three peaks, the so-called *three-peak equilibrium distributions* (3-PED) (e.g. List et al., 1987; List and McFarquhar, 1990; Hu and Srivastava, 1995). Various investigations have indeed provided evidence for such multiple peak behavior in empirical raindrop size distributions (e.g. Steiner and Waldvogel, 1987; List, 1988; Zawadzki and de Agostinho Antonio, 1988). It has recently become clear however, that all reported peaks have in fact been instrumental artifacts caused by the signal-processing electronics of the Joss-Waldvogel (1967) disdrometers used for the measurements (Sheppard, 1990; McFarquhar and List, 1993; Sauvageot and Lacaux, 1995). Nevertheless, McFarquhar and List (1993) conclude that 'it cannot be categorically stated that raindrop size distributions with multiple peaks do not exist'. Indeed, using another type of instrument, the (airborne) optical array probe, multi-modal distributions have been observed in tropical rain (Willis, 1984).

List (1988) present an approximation to the theoretically predicted equilibrium form of $g\,(D)$ (Eq. (3.90)) as the sum of three gamma distributions with the peak positions mode_x (mm), peak heights $g(\text{mode}_x)$ ($\text{mm}^{-1}\,\text{m}^{-3}\,(\text{mm}\,\text{h}^{-1})^{-1}$) and shape parameters μ (–) of the component distributions as indicated in Table 3.7. A comparison with the gamma parameterization for $g(x)$ proposed earlier (Table 3.3) shows that the peak position and peak height of each of the component distributions are related to the parameters κ, λ and μ via

$$\text{mode}_x = \frac{\mu}{\lambda} \tag{3.91}$$

and consequently

$$g(\text{mode}_x) = \kappa\left(\frac{\mu}{\lambda}\right)^{\mu}\exp\left(-\mu\right) = \kappa\left(\frac{\mu}{e\lambda}\right)^{\mu}, \tag{3.92}$$

where e denotes the base of the natural logarithm. From these expressions the parameters κ and λ for each of the three component distributions can be obtained from the given values of mode_x, $g(\text{mode}_x)$ and μ (Table 3.7). These define a general raindrop size distribution function $g(x)$ for equilibrium rainfall.

However, substitution of the resulting three-peak form for $g(x)$ in Eq. (3.86) shows that it does not satisfy the self-consistency constraint ($S_\text{p} = 0.91$). This may be associated with the fact that List (1988) does not use a power law relationship for $v(D)$, as implicitly assumed in Eq. (3.86), but the more accurate (asymptotic) parameterization due to Best (1950a) (see Fig. 3.1(a), p. 68). As a matter of fact, if $\alpha = 1$ and $\beta = 0$ it is no longer necessary to impose such a power law relationship for $v(D)$. Eq. (3.33) demonstrates that any such relationship will then imply linear relationships between the moments of the raindrop size distribution. Accordingly, under equilibrium conditions, the constraint of Eq. (3.37) can be relaxed to

$$\int_0^\infty x^3 v(x)\,g(x)\,\mathrm{d}x = \frac{10^4}{6\pi}. \tag{3.93}$$

Table 3.7: Peak positions mode_x (mm), peak heights $g(\text{mode}_x)$ ($\text{mm}^{-1}\,\text{m}^{-3}\,(\text{mm}\,\text{h}^{-1})^{-1}$) and shape parameters μ (–) of the three component gamma distributions of an approximation for the general raindrop size distribution $g(x)$ of the three-peak equilibrium distribution (3-PED) (List, 1988). The last two rows are the corresponding values of λ and κ.

Parameter	Distribution 1	Distribution 2	Distribution 3
mode_x	0.259	0.766	1.671
$g(\text{mode}_x)$	269	36.1	11.8
μ	10	13	19
λ	38.6	17.0	11.4
κ	4.36×10^{12}	5.11×10^{8}	1.22×10^{5}

Nevertheless, substitution of the obtained three-peak form for $g(x)$ and the Best (1950a) raindrop terminal fall speed parameterization in this expression still yields a violation of the self-consistency, albeit less pronounced ($S_p = 0.95$). It has been preferred here to stay with the power law relationship for $v(D)$ and adjust the values of κ for the three component distributions (through division by S_p) so as to impose self-consistency. In any case, the difference will be small (see Fig. 3.1(b), p. 68).

Fig. 3.3(a) and (b) show the general raindrop size distribution function $g(x)$ and the corresponding general rain rate density function $h(x)$ for this self-consistent form of the three-peak equilibrium distribution. Since $\alpha = 1$ and $\beta = 0$, the horizontal axis in both cases directly represents the (unaffected) raindrop diameter D (mm). This is a reflection of the fact that in equilibrium rainfall the characteristic raindrop sizes are constants. The vertical axis of Fig. 3.3(a) is simply the raindrop size distribution divided by the rain rate R ($\text{mm}\,\text{h}^{-1}$). This is a reflection of the proportionality between the rainfall integral parameters and indicates that the effect of an increase in R is simply a proportional scaling of $N_V(D, R)$. This is exactly the opposite of that found for the Marshall-Palmer distribution, where only the horizontal axis was affected and not the vertical.

Regarding the general rain rate density function $h(x)$ (Fig. 3.3(b)), the fact that $\beta = 0$ implies that it is completely independent of rain rate, something which List (1988) notes as well. Fig. 3.3(b) shows that the first two relatively pronounced peaks in $g(x)$ have almost entirely disappeared. The resulting form of $h(x)$ is approximately unimodal, the majority of the contribution coming from the third peak in $g(x)$ (about 90% according to List (1988)). This indicates that although the existence of peaks in raindrop size distributions is undoubtedly important for a proper understanding of (warm) rain microphysics, it is much less relevant to applications in radar meteorology, hydrology and telecommunications, where the interest lies typically in the higher order moments of the raindrop size distribution (as discussed in Chapter 2). The main interest of the equilibrium raindrop size distribution here is the fact that it represents a very particular case of the scaling law.

A final remark concerns the Z–R relationship in equilibrium rainfall. List (1988) demonstrates that, for his approximation to the three-peak equilibrium distribution

Table 3.8: Three special cases of variability of the raindrop size distribution: (1) raindrop size controlled; (2) intermediate case (Marshall-Palmer rainfall); (3) raindrop concentration controlled (equilibrium rainfall). Scaling exponents α and β and the associated values of the exponents of power law relationships between ρ_V, D_C and Ω_m on the one hand and R on the other (all dimensionless).

Case	$\gamma_{\rho_V}, \gamma_{D_C}$	$\alpha = \gamma_{\rho_V} - \gamma_{D_C}$	$\beta = \gamma_{D_C}$	$\gamma_{\Omega_m} = \alpha + (m+1)\beta$
		$(\alpha = 1 - (4+\gamma)\beta)$		$(= 1 + [m - (3+\gamma)]\beta)$
$\rho_V = \text{cst.}$	$\gamma_{\rho_V} = 0$	$-\frac{1}{3+\gamma}$	$\frac{1}{3+\gamma}$	$\frac{m}{3+\gamma}$
$\rho_V/D_C = \text{cst.}$	$\gamma_{\rho_V} = \gamma_{D_C}$	0	$\frac{1}{4+\gamma}$	$\frac{m+1}{4+\gamma}$
$D_C = \text{cst.}$	$\gamma_{D_C} = 0$	1	0	1

(Table 3.7), the radar reflectivity factor Z ($\text{mm}^6\,\text{m}^{-3}$) will exhibit a *linear* dependence on R according to

$$Z = 742R. \tag{3.94}$$

It has been demonstrated that this proportionality is a direct consequence of the values of the scaling exponents ($\alpha = 1$, $\beta = 0$). List speculated that the 742 $\text{mm}^6\,\text{m}^{-3}\,(\text{mm}\,\text{h})^{-1}$ is 'a universal constant for steady tropical rain'. Since by definition Z equals Ω_6 with $c_{\Omega_6} = 1$ (Eq. (3.18)), evaluation of Eq. (3.20) shows that

$$C_Z = \int_0^\infty x^6 g(x)\,\mathrm{d}x, \tag{3.95}$$

i.e. that this 'universal constant' must simply be the 6th moment of the three-peak general raindrop size distribution function. Evaluation of Eq. (3.95) using the self-consistent form for $g(x)$ derived on the basis of the power law $v(D)$ relationship gives $C_Z = 753$, reasonably close. Using the self-consistent form derived on the basis of the Best (1950a) $v(D)$ relationship, the one presumably used by List (1988), yields $C_Z = 724$ and *not* 742, as would be expected[22]. In any case, Eq. (3.95) will be used extensively when the implications of the scaling law for Z–R relationships will be discussed (Chapter 6).

3.5.3 A third special case: $\alpha + \beta = 0$

In Sections 3.5.1 and 3.5.2 two special cases of the scaling law have been treated. The Marshall-Palmer distribution corresponds to the case where the spatial and temporal variability of the raindrop size distribution (and hence that of any derived rainfall related variable) is controlled both by that of the raindrop concentration ρ_V and by that of the characteristic raindrop sizes D_C. This happens in such a way that the ratio of ρ_V to D_C is a constant, implying that the exponents of power law relationships

[22]The origin of this discrepancy is not clear, but it might be due to a typographical error in List's (1988) article. Using the original values of κ given in Table 3.7 (not corrected for violation of self-consistency) leads to an even worse agreement: $C_Z = 688$.

between them and any reference variable Ψ are equal, i.e. $\gamma_{\rho_V} = \gamma_{D_C}$ (Eq. (3.3)). Via Eq. (3.5) this implies that the scaling exponent α is zero. Subsequently, in case R is taken to be the reference variable, the self-consistency constraint on the scaling exponents (Eq. (3.38)) yields for the other scaling exponent $\beta = \frac{1}{4+\gamma}$, where γ is the exponent of the power law $v(D)$ relationship (Eq. (2.10)). These values of the scaling exponents finally imply via Eq. (3.53) (or Eq. (3.54)) that the exponent γ_{Ω_m} of a power law relationship between a moment Ω_m of the raindrop size distribution (such as Z) and R must equal $\frac{m+1}{4+\gamma}$. From the latter, it is easy to see that the exponent of a power law relationship between Ω_m and any other moment Ω_n in general follows $\frac{\gamma_{\Omega_m}}{\gamma_{\Omega_n}} = \frac{m+1}{n+1}$. This whole process has been summarized in Table 3.8.

In a similar manner, the equilibrium raindrop size distribution corresponds to the special (limiting) case where the variability of the raindrop size distribution is entirely controlled by that of the raindrop concentration. Since the characteristic raindrop sizes are constants in this case, $\gamma_{D_C} = 0$. This implies $\beta = 0$ and hence $\alpha = 1$, for *any* choice of the rainfall integral variable. The result is that $\gamma_{\Omega_m} = 1$, independent of m and the same holds obviously for $\frac{\gamma_{\Omega_m}}{\gamma_{\Omega_n}}$. Again, the results are summarized in Table 3.8.

It is suggested here that the equilibrium rainfall point indeed represents a limiting case. Negative values of β would namely imply that the characteristic raindrop sizes would *decrease* with increasing rain rates (since, according to Eq. (3.6), $\beta = \gamma_{D_C}$). This seems not very plausible and, moreover, in the literature such values of β have to the best of the author's knowledge never been reported. Perhaps it would be possible to use the stochastic collection equation to rigorously prove this hypothesis, but this has not been attempted here[23].

A *third* special case, one which has not been treated yet, is then of course the case where the variability of the raindrop size distribution is entirely controlled by that of the characteristic raindrop sizes, the raindrop concentration remaining constant. This implies $\gamma_{\rho_V} = 0$ and hence $\alpha + \beta = 0$. For R as the reference variable, the self-consistency constraint on the exponents now implies $\alpha = -\frac{1}{3+\gamma}$ and $\beta = \gamma_{D_C} = \frac{1}{3+\gamma}$ (Fig. 3.2(a) and (b)). The resulting exponent of a power law relationship between Ω_m and R is then $\frac{m}{3+\gamma}$ and that of a power law relationship with another moment necessarily $\frac{m}{n}$, as indicated in Table 3.8. This probably represents another limiting

[23]A possible approach toward tackling this and associated problems might be to substitute an appropriate form of the scaling law (Eq. (3.4)) in the transient (i.e. non-stationary) form of the stochastic collection equation. List and McFarquhar (1990) describe their zero-dimensional (box) model for solving this equation numerically as follows: 'Collisional breakup and the coalescence of raindrops are the only two factors considered in the model describing the time evolution of the drop spectra. The studies are performed using a box model with zero spatial dimensions. All drops that fall out of the bottom of the volume element are immediately reinserted at the top so that mass is conserved and so that there are no effects due to the sedimentation of drops.' In other words, List and McFarquhar use a 'torus' model (top = bottom) and neglect the effects of evaporation and condensation. Then conservation of mass implies that the liquid rainwater content W (mg m^{-3}) should be a conserved quantity. This suggests taking W as a stationary reference variable in the scaling law and let the corresponding scaling exponents $\beta_W(t)$ (and through self-consistency $\alpha_W(t) = 1 - 4\beta_W(t)$) become time-dependent functions. It would then be interesting to study (perhaps analytically) the time evolution of $\beta_W(t)$ due to raindrop interaction.

case. A further decrease of α (and hence increase of β) would namely imply that $\gamma_{\rho V}$ would become negative and consequently that the raindrop concentration would *decrease* with increasing rain rates. This seems rather unlikely, but again no definitive proof of this assertion can be given.

This type of reasoning finally provides some clues as to the physical interpretation of the scaling exponents, something about which Sempere Torres et al. (1994, 1998) have remained rather vague. In a plot of β versus α (such as Fig. 3.2(a)) or a plot of $\beta = \gamma_{D_C}$ versus $\alpha + \beta = \gamma_{\rho V}$ (such as Fig. 3.2(a)) there are apparently three fixed points:

1. $\alpha + \beta = 0$, corresponding to raindrop size controlled variability (indicated by a cross in Fig. 3.2);

2. $\alpha = 0$, corresponding to an intermediate case in which the variability is controlled by raindrop size and raindrop concentration in equal proportions (Marshall-Palmer rainfall; the plus in Fig. 3.2);

3. $\beta = 0$, corresponding to raindrop concentration controlled variability (equilibrium rainfall; the circle in Fig. 3.2).

Hence, moving from the cross in Fig. 3.2(a) and (b) via the plus to the circle, the fractional control by raindrop concentration increases from zero to one, whereas that by characteristic raindrop size decreases from one to zero. At the cross, a doubling of the rain rate will be caused by a doubling of the raindrop volumes, at the circle by a doubling of the raindrop concentration (and hence of the raindrop arrival rate). At all points in between, it will be caused partly by an increase of the raindrop volumes, partly by an increase in the raindrop concentration, with the Marshall-Palmer point as a special case. It will be the challenge of Chapters 4–6 to relate these interpretations to the different types of rainfall (stratiform, convective) and possibly to different rainfall climatologies.

It is already clear that the Marshall-Palmer point can be associated more or less with stratiform conditions and the equilibrium point with persistent tropical rainfall. Hence, the meaning that may be attached to the distance (in the geometrical sense) between any given point on the line $\beta = \frac{1-\alpha}{4+\gamma}$ and the point $(\alpha, \beta) = (1, 0)$ (in Fig. 3.2(a)) or similarly that between any given point on the line $\gamma_{D_C} = \frac{1-\gamma_{\rho V}}{3+\gamma}$ and $(\gamma_{\rho V}, \gamma_{D_C}) = (1, 0)$ (in Fig. 3.2(b)) is that it represents a distance (in the physical sense) from equilibrium[24]. In any case, since different points on this line are associated with different Z–R relationships, a rainfall classification in terms of the values of α and β may also have important practical implications.

[24]It might indicate a *time* from equilibrium as well. List (1988) argued that 'any raindrop distribution will develop with time into an equilibrium distribution regardless of the initial spectrum'. Using a similar model as that described above[23], List et al. (1987) simulate the time evolution of initial Marshall-Palmer distributions ($\beta = 0.21$) toward equilibrium distributions ($\beta = 0$). They show that for high values of W equilibrium is reached within one hour (indicating $\Delta\beta/\Delta t \approx 0.2$ h^{-1}), whereas for low values of W this takes more than two hours ($\Delta\beta/\Delta t \approx 0.1$ h^{-1}). Hence, the value of β might be somehow related to the time a raindrop size distribution has been allowed to evolve, which in turn might be related to the type of rainfall (height of the cloud base, updrafts/downdrafts, etc.).

3.6 Summary and conclusions

A comprehensive general framework for the analysis of raindrop size distributions and their properties has been presented. It is a further extension and generalization of a recently proposed general formulation for the raindrop size distribution. In the presented framework, the general formulation for the raindrop size distribution takes the form of a *scaling law.* This law is consistent with the ubiquitous power law relationships between rainfall related variables. They follow logically from its formulation. Moreover, the scaling law unifies all previously proposed parameterizations for the raindrop size distribution. All can be recast in forms which are consistent with the formulation and as such can be considered as special cases thereof.

In the scaling law formulation, the raindrop size distribution is not only a function of the raindrop diameter, but of a *reference variable* as well. Any rainfall related variable can play the role of reference variable, not necessarily the rain rate historically used for that purpose. The spatial and temporal variability of the reference variable reflects that of the raindrop size distribution. There are two *scaling exponents* associated with the reference variable, one to scale the raindrop diameters and another to scale the corresponding raindrop concentrations. Once these scaling exponents have been estimated, they can be used to scale raindrop size distributions corresponding to different values of the reference variable. The identified curve is a scaled raindrop size distribution, the so-called general raindrop size distribution function, which is in principle independent of the value of the reference variable. The physical interpretation of both the scaling exponents and the general raindrop size distribution function has been clarified. In particular, *the values of the scaling exponents determine whether it is the raindrop concentration or the characteristic raindrop sizes which control the variability of the raindrop size distribution.* A second type of general function has been introduced, the general rain rate density function, which has the advantage of behaving as a probability density function. This will facilitate the parameter estimation process.

Since any reference variable is itself a function of the raindrop size distribution, there exist *self-consistency constraints* both on the scaling exponents and on the general raindrop size distribution function. The constraint on the exponents implies that only one of the two is a free parameter. In case the reference variable is proportional to a moment of the raindrop size distribution, the scaling exponents must be linearly related. The constraint on the general raindrop size distribution function implies that it must satisfy an integral equation. This reduces its number of degrees of freedom by one.

From a practical point of view, the two main advantages of the proposed scaling law procedure over previous approaches are its *robustness* and its *generality.* The robustness of the procedure stems from the fact that all available empirical raindrop size distributions can be used directly to identify the general raindrop size distribution function, thus avoiding the common requirement to calculate average distributions for different classes of the reference variable. The generality of the procedure is due to the fact that *it is no longer necessary to impose an a priori functional form for the raindrop size distribution.* Only after the general raindrop size distribution function has been identified, a suitable parameterization may be selected. This selection will consequently be based on all available information. Expressions have been provided for the self-consistent forms of both types of general functions for all analytical forms of the raindrop size distribution which have been proposed in the literature over the years (exponential, gamma, generalized gamma, Best and lognormal). In this manner, the gap between the scaling law formulation and the traditional analytical parameterizations is bridged explicitly.

Chapter 4

Verification of the scaling law using mean raindrop size distributions[1]

4.1 Introduction

In Chapter 3 a general framework for the analysis of raindrop size distributions and their properties has been introduced. In this framework, the formulation of the raindrop size distribution takes the form of a scaling law. Preliminary evidence for its validity has been provided through the discussion of three special cases. The aim of this chapter is to provide further evidence for the validity of the scaling law formulation by revisiting two classical parameterizations for the *mean* raindrop size distribution, namely those due to Best (1950b) and Laws and Parsons (1943). Neither of these has been treated by Sempere Torres et al. (1994, 1998).

In both parameterizations the rain rate R is used as the reference variable, i.e. as the variable controlling the spatial and temporal variability of the parameterizations. However, because they are not formulated in terms of the raindrop size distribution $N_V(D, R)$ itself, these parameterizations are not directly comparable to the scaling law formulation presented in Chapter 3 (Eq. (3.4), p. 59). The analytical parameterization due to Best (1950b) pertains to the (cumulative) distribution of the liquid rainwater content W over all raindrop diameters, i.e. it is a parameterization of $F_W(D, R)$. The tabulated parameterization due to Laws and Parsons (1943) pertains to the distribution of the rain rate R over D, i.e. it is a parameterization of $F_R(D, R)$. In both cases it will therefore be necessary to recast the parameterization into a form which is consistent with the scaling law for the raindrop size distribution, i.e. to derive the intrinsic $N_V(D, R)$-formulation contained in it. For Laws and Parsons' parameterization, the derivation of the general rain rate density function will of course be straightforward (Eq. (3.67), p. 77).

Although the parameterization proposed by Best has not been very widely used (see Wessels (1967, 1972) for a notable exception), it is of interest here because Best

[1]Adapted version of Uijlenhoet, R., Creutin, J.-D., and Stricker, J. N. M. (1999). Scaling properties of classical parameterizations for the raindrop size distribution. *Q. J. R. Meteorol. Soc.* (submitted).

adjusted his parameterization to raindrop size data from different locations around the world, corresponding to very different climatic settings. Knowledge of the values of the scaling exponents α and β and the shapes of the general raindrop size distribution function $g(x)$ and associated general rain rate density function $h(x)$ for these locations may therefore provide information as to the climatological variability of α, β, $g(x)$ and $h(x)$. This may perhaps lead to clues concerning the type of climatic setting to which the three special cases of the scaling law treated in Chapter 3 (Section 3.5) correspond. An additional argument in favor of a treatment of Best's parameterization is that its typical functional form has recently been found to resemble (at least qualitatively) empirical general raindrop size distribution functions observed in convective rainfall (Sempere Torres et al., 1999). Best's parameterization will be the subject of Section 4.2.

After Marshall and Palmer's (1948) celebrated exponential parameterization, the tabulated parameterization of Laws and Parsons has probably been the most widely used standard family of raindrop size distributions, both in radar meteorology (e.g. Doviak and Zrnić, 1993) and in telecommunications research dealing with microwave signal propagation (e.g. Crane, 1971; Olsen et al., 1978; and references therein). It will therefore be of considerable practical interest to have a consistent analytical parameterization available based on Laws and Parsons' data. An additional motivation for discussing Laws and Parsons' parameterization in this context is that, in its original form, it is a *tabulated* parameterization. In this respect, it resembles the raw raindrop size distribution data which may be gathered with disdrometers and optical spectrometers. It will therefore provide an ideal example for testing the procedures for the estimation of the scaling exponents α and β, for the identification of the general functions $g(x)$ and $h(x)$ and for the adjustment of analytical parameterizations to the empirical $g(x)$ and $h(x)$ which have been outlined in Chapter 3 (Section 3.4). The same methodology may then in a later stage be applied to raw raindrop size distributions. In fact, that is what will be done in Chapter 5. Laws and Parsons' parameterization will be treated in Section 4.3.

An interesting alternative approach towards the parameterization of raindrop size distributions, with a particular emphasis on the description of Laws and Parsons' (1943) data, has been discussed by Spilhaus (1948). Probably, this apparently forgotten approach[2] can be regarded as the first attempt to derive a general raindrop size distribution function, i.e. a drop size distribution which is independent of rain rate. It is therefore a pity that Spilhaus' approach, as is demonstrated in Appendix D, contains some fundamental inconsistencies. It will not be pursued any further here.

[2]Although its general subject is the parameterization of raindrop size distributions, Spilhaus' (1948) article is almost only cited for a relation between the terminal fall speed of drops and their diameter which appeared on the first page (Eq. (D.3), Appendix D).

4.2 Best's parameterization

4.2.1 Functional form

Best (1950b) has examined a substantial set of raindrop size distribution measurements collected in many parts of the world. He has found that the mean cumulative distribution of the liquid rainwater content over all raindrop diameters can often be parameterized with good accuracy according to (using a notation consistent with $N_V(D, R)$)

$$F_W(D, R) = 1 - \exp\left[-\left(\frac{D}{a}\right)^n\right]; \quad a, n > 0; \quad D \geq 0, \tag{4.1}$$

where $F_W(D, R)$ [-] is the 'fraction of liquid water in the air comprised by drops with diameter less than D' [L] for a given rain rate R, a [L] is a scale parameter and n [-] is a shape parameter. This functional form corresponds to the (cumulative) probability distribution function of a Weibull distribution (Mood et al., 1974). Best reports that, on average, the parameter a is related to the rain rate R according to the power law

$$a = AR^p, \tag{4.2}$$

where $A = 1.30$ and $p = 0.232$ if a is expressed in mm and R in mm h^{-1}. Moreover, he finds n to be approximately constant with a mean value of 2.25 (and a standard deviation among different locations of 0.41). Finally, he reports the liquid rainwater content W on average to be related to the rain rate R according to the power law

$$W = CR^r, \tag{4.3}$$

where $C = 67$ and $r = 0.846$ if W is expressed in[3] mg m^{-3} and R in mm h^{-1}. Fig. 4.1(a) shows $F_W(D, R)$ for rain rates of 1, 10 and 100 mm h^{-1}, respectively. It is clear that, for larger rain rates, the proportion of large drops increases and as such their contribution to the liquid rainwater content.

A convenient reformulation of Eq. (4.1) from the point of view of parameter estimation is

$$-\ln\left[1 - F_W(D, R)\right] = \left(\frac{D}{a}\right)^n. \tag{4.4}$$

This form shows that if a raindrop size distribution obeys Best's parameterization, a plot of $-\ln\left[1 - F_W(D, R)\right]$ against D on log-log paper will yield a straight line with slope n and intercept $-n \log a$. Best and Wessels (1967, 1972) have used this method to estimate the a and n parameters for the experimental raindrop size distributions they examined. In Fig. 4.1(b) the $F_W(D, R)$-curves of Fig. 4.1(a) are replotted in this manner. That the obtained lines are parallel to each other stems from the fact that, according to Best's parameterization, n is independent of the rain rate R.

[3] In fact, Best expressed W in units of mm^3 m^{-3}. However, since the density of water ρ_w is to a very close approximation 1000 kg m^{-3} (i.e. 1 mg mm^{-3}) over a wide range of air temperatures and pressures, numerically there is little or no difference with units of mg m^{-3}.

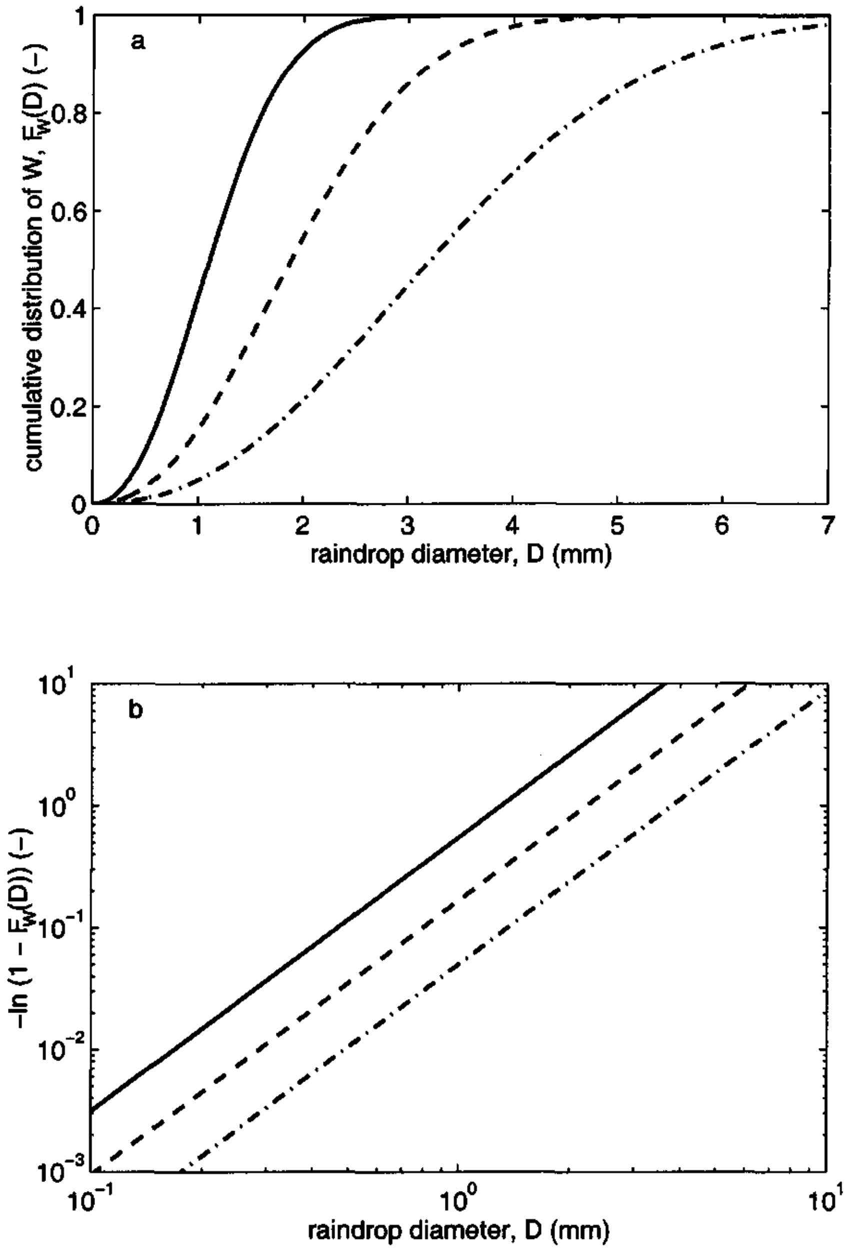


Figure 4.1: (a) Cumulative distributions $F_W(D, R)$ (–) of the liquid rainwater content W ($\mathrm{mg\,m^{-3}}$) over the raindrop diameters D (mm) according to Best's (1950b) parameterization (solid line: 1 $\mathrm{mm\,h^{-1}}$; dashed line: 10 $\mathrm{mm\,h^{-1}}$; dash-dotted line: 100 $\mathrm{mm\,h^{-1}}$). (b) Corresponding curves after the transformation $-\ln[1 - F_W(D, R)]$, with logarithmic scales on both axes.

4.2.2 Characteristic properties

To obtain some idea of the physical meaning of the parameters a and n, a few characteristic properties of Best's parameterization will be discussed. First, as opposed to what is the case for the widely used gamma raindrop size distribution (Ulbrich, 1983), for Best's $F_W(D,R)$-parameterization the median-volume raindrop diameter $D_{0,\mathrm{V}}$ can easily be obtained explicitly. By definition, half of the liquid rainwater content is comprised by raindrops smaller than $D_{0,\mathrm{V}}$, i.e. $F_W(D_{0,\mathrm{V}},R) = 1/2$. This implies

$$D_{0,\mathrm{V}} = a\,(\ln 2)^{1/n} \tag{4.5}$$

(Best, 1950b). In the same manner, Best presents explicit expressions for other quantiles of $F_W(D,R)$. A particular case, not explicitly treated by Best, is the scale parameter a. Eq. (4.1) shows that a is the raindrop diameter for which $F_W(a,R) = 1 - e^{-1}$ (≈ 0.63), independent of n. In other words, for any raindrop size distribution satisfying Best's parameterization, 63% of the liquid rainwater content will be comprised by raindrops smaller than a.

A second characteristic raindrop diameter treated by Best (1950b) is what he calls the 'predominant drop diameter, that is the diameter of the drops which account for the greatest volume of water in the air'. This is the mode of the derivative of $F_W(D,R)$ with respect to D, the liquid rainwater density function $f_W(D,R)$, which can be obtained by setting $\partial^2 F_W(D,R)\,/\partial D^2 = 0$. This yields

$$D_{\mathrm{p,V}} = \begin{cases} a\left(1-\frac{1}{n}\right)^{1/n} & ; \quad n > 1 \\ 0 & ; \quad 0 < n \le 1 \end{cases} . \tag{4.6}$$

Hence, if n approaches infinity $D_{\mathrm{p,V}}$ approaches a. Already for $n = 4.48$, the largest value reported by Best, the ratio of $D_{\mathrm{p,V}}$ to a becomes 0.95.

Another widely used characteristic raindrop diameter, although not treated by Best (1950b), is the so-called volume-weighted mean diameter $D_{\mathrm{m,V}}$ (Ulbrich, 1983; Chapter 2, Section 2.6). By definition, $D_{\mathrm{m,V}}$ is the mean of the distribution of the liquid rainwater content over all raindrop diameters. Hence

$$D_{\mathrm{m,V}} = \int_0^\infty \left[1 - F_W(D,R)\right] \mathrm{d}D = a\Gamma\!\left(1 + \frac{1}{n}\right). \tag{4.7}$$

For $n = 2.25$, the mean value determined by Best (1950b), the ratio of $D_{\mathrm{m,V}}$ to $D_{0,\mathrm{V}}$ is 1.04. For $n = 1.85$ and 4.48, the minimum and maximum values found by Best for different locations, this ratio becomes 1.08 and 0.99, respectively. Hence, for all encountered values of n, $D_{\mathrm{m,V}}$ is a very good approximation to $D_{0,\mathrm{V}}$. The same has been recognized for the case of the gamma raindrop size distribution by Ulbrich (1983).

In general, the difference between the mean and the median of an unimodal distribution is a measure of its skewness (Kendall and Stuart, 1977). Hence, the closeness of $D_{\mathrm{m,V}}$ to $D_{0,\mathrm{V}}$ in this case indicates in advance that the liquid rainwater density function $f_W(D,R)$ must be a more or less symmetrical function of D. It is then natural to look for a measure of its spread (dispersion). Such a measure is the variance

$\sigma^2_{\mathrm{m,V}}$ of the 'mass spectrum' (i.e. $f_W(D,R)$) (Ulbrich, 1983), which in case of Best's parameterization becomes

$$\sigma^2_{\mathrm{m,V}} = a^2 \left[\Gamma\Big(1+\frac{2}{n}\Big) - \Gamma^2\Big(1+\frac{1}{n}\Big)\right]. \tag{4.8}$$

A measure of the *relative* spread of $F_W(D,R)$ is its coefficient of variation $\mathrm{CV}_{\mathrm{m,V}}$, by definition the ratio of $\sigma_{\mathrm{m,V}}$ to $D_{\mathrm{m,V}}$. This yields

$$\mathrm{CV}_{\mathrm{m,V}} = \left[\frac{\Gamma(1+2/n)}{\Gamma^2(1+1/n)} - 1\right]^{1/2}, \tag{4.9}$$

which depends solely on n. For the values $n = 1.85$, $n = 2.25$ and $n = 4.48$ (the minimum, mean and maximum values reported by Best) $\mathrm{CV}_{\mathrm{m,V}}$ becomes 0.561, 0.470 and 0.253, respectively. For reference, exponential distributions have a coefficient of variation equal to one. In other words, 'an increase in the value of n will decrease the spread of the distribution', as noted by Best (1950b). In summary, the parameter a is a measure of the location of $F_W(D)$ and the parameter n a measure of its (relative) dispersion.

4.2.3 Raindrop size distribution

In order to be able to verify whether Best's parameterization satisfies the scaling law formulation for the raindrop size distribution (Eq. (3.4), p. 59), it needs to be recast in a form which is consistent with that formulation. In this case the reference variable is R, so the particular form of the scaling law of interest here is Eq. (3.39) (p. 65). Since Eq. (3.39) is formulated in terms of $N_{\mathrm{V}}(D,R)$, the $N_{\mathrm{V}}(D,R)$ corresponding to Best's $F_W(D,R)$ is needed.

By definition (Table 2.3, p. 39), the liquid rainwater content W ($\mathrm{mg\,m^{-3}}$) is related to $N_{\mathrm{V}}(D,R)$ according to

$$W = \frac{\pi\rho_{\mathrm{w}}}{6} \times 10^{-3} \int_0^\infty D^3 N_{\mathrm{V}}(D,R)\,\mathrm{d}D, \tag{4.10}$$

where $\rho_{\mathrm{w}} = 1000\ \mathrm{kg\,m^{-3}}$ is the density of water. This implies that $F_W(D,R)$ (-) can be expressed in terms of $N_{\mathrm{V}}(D,R)$ according to

$$F_W(D,R) = \frac{\pi\rho_{\mathrm{w}}}{6W} \times 10^{-3} \int_0^D D'^3 N_{\mathrm{V}}(D',R)\,\mathrm{d}D'. \tag{4.11}$$

$F_W(D,R)$ increases from 0 to 1 as D goes from 0 to ∞, as would be expected for a (cumulative) distribution function. The density function $f_W(D,R)$ ($\mathrm{mm^{-1}}$) corresponding to $F_W(D,R)$ can be obtained by taking its derivative with respect to D. This yields for the relationship between $f_W(D,R)$ and $N_{\mathrm{V}}(D,R)$

$$f_W(D,R) = \frac{\partial F_W(D,R)}{\partial D} = \frac{\pi\rho_{\mathrm{w}}}{6W} \times 10^{-3} D^3 N_{\mathrm{V}}(D,R). \tag{4.12}$$

In case of Best's parameterization (Eq. (4.1)), $f_W(D,R)$ is

$$f_W(D,R) = \frac{n}{a}\left(\frac{D}{a}\right)^{n-1} \exp\left[-\left(\frac{D}{a}\right)^n\right]; \quad a,n>0; \quad D \geq 0. \tag{4.13}$$

This functional form corresponds to the probability density function of a Weibull distribution (Mood et al., 1974). Substitution of this result in Eq. (4.12) finally yields for the raindrop size distribution in the air

$$N_{\mathrm{V}}(D,R) = \frac{6\times 10^3 nW}{\pi \rho_{\mathrm{w}} a^4}\left(\frac{D}{a}\right)^{n-4} \exp\left[-\left(\frac{D}{a}\right)^n\right]; \quad a>0; \quad n>3; \quad D\geq 0. \tag{4.14}$$

This is the $N_{\mathrm{V}}(D,R)$-parameterization intrinsically contained in Best's parameterization for $F_W(D,R)$ (Eq. (4.1)). It shows that the probability density function of the raindrop diameters in the air is a special case of the so-called generalized gamma distribution (Stacy, 1962).

Care should be exercised when integrating Eq. (4.14) with respect to D. If D_{min} is taken to be 0, as is usually done, then for a given value of $n > 0$, all moments of $N_{\mathrm{V}}(D,R)$ of orders smaller than or equal to $3-n$ will diverge, i.e. become infinitely large. Hence, only those of orders larger than $3-n$ will be finite, even though for $n>0$ *all* non-negative moments of $f_W(D,R)$ are finite. For example, for the mean value of n reported by Best ($n = 2.25$), the zeroth moment of the raindrop size distribution (i.e. the physically important raindrop concentration) diverges. This implies that the corresponding probability density function of the raindrop diameters in the air does not exist, which is both mathematically undesirable and physically unrealistic. In a later article, dealing with applications of Eq. (4.1) to the size distributions of cloud and fog droplets, Best (1951) proposes a 'solution' to this problem, namely truncating the raindrop size distribution at some minimum diameter D_{min}. However, in practical situations this is often very inconvenient, as D_{min} represents an extra parameter to be estimated from the data at hand. If $D_{\mathrm{min}} = 0$, the requirement that $n > 3$ guarantees the existence of all non-negative moments of $N_{\mathrm{V}}(D,R)$.

4.2.4 Comparison with the Marshall-Palmer distribution

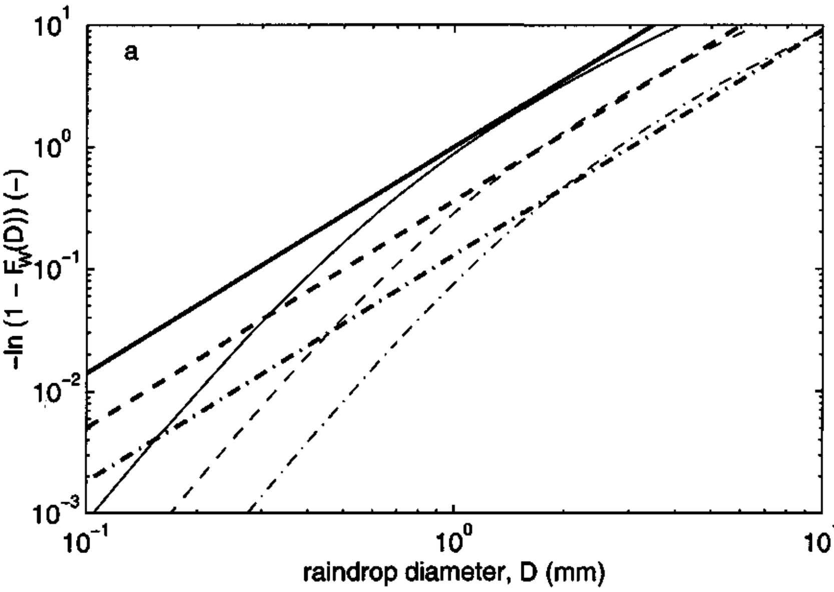
Before considering Eq. (4.14) in the framework of the scaling law formulation, it seems informative to compare Best's parameterization with that which has become the reference parameterization over the years, the exponential Marshall-Palmer (1948) distribution (Eqs. (2.2)–(2.4), p. 25). This will be done in two ways, namely both in terms of $F_W(D,R)$ and in terms of $N_{\mathrm{V}}(D,R)$.

If $D_{\mathrm{min}} = 0$ and $D_{\mathrm{max}} = \infty$ then substituting Eq. (2.2) in Eq. (4.11) yields

$$F_W(D,R) = \Gamma(4,\Lambda D)\,, \tag{4.15}$$

where $\Gamma(\cdot,\cdot)$ is the incomplete gamma function, defined here as

$$\Gamma(s,x) = \frac{1}{\Gamma(s)}\int_0^x y^{s-1}e^{-y}\mathrm{d}y \tag{4.16}$$



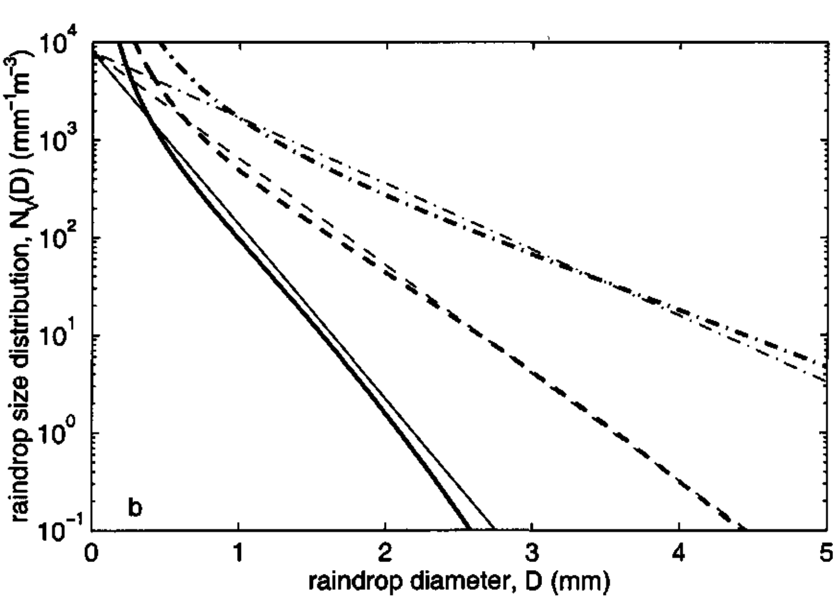


Figure 4.2: (a) Cumulative distributions $F_W(D, R)$ (–) of the liquid rainwater content according to the parameterizations of Best (1950a; bold) and Marshall and Palmer (1948; normal) (solid lines: 1 $\text{mm}\,\text{h}^{-1}$; dashed lines: 10 $\text{mm}\,\text{h}^{-1}$; dash-dotted lines: 100 $\text{mm}\,\text{h}^{-1}$). (b) Corresponding raindrop size distributions $N_V(D, R)$ ($\text{mm}^{-1}\,\text{m}^{-3}$).

(e.g. Abramowitz and Stegun, 1972)[4]. On the basis of Eq. (4.15), Best (1950b) has adjusted his parameterization to Marshall and Palmer's data[5] and reports for the corresponding coefficients the values $n = 1.85$, $A = 1.00$, $p = 0.240$, $C = 72$ and $r = 0.880$ (if a is expressed in mm, W in $\mathrm{mg\,m^{-3}}$ and R in $\mathrm{mm\,h^{-1}}$). In Fig. 4.2(a) this adjustment is compared to $F_W(D,R)$-curves derived for Marshall and Palmer's parameterization for rain rates of 1, 10 and 100 $\mathrm{mm\,h^{-1}}$ (using Eqs. (4.15) and (2.4)). The contribution of small raindrops to the liquid rainwater content is significantly stronger for Best's parameterization than for the Marshall-Palmer parameterization. For diameters in excess of one millimeter, on the other hand, the correspondence is fairly close.

These observations are confirmed by Fig. 4.2(b), where the raindrop size distributions $N_V(D,R)$ corresponding to Best's parameterization (Eq. (4.14)) are compared to Marshall and Palmer's original parameterization (Eq. (2.2)) for the same rain rates. It is exactly this behavior of Best's parameterization at the small diameter end of the raindrop size distribution which may cause the divergence of the low order moments mentioned above.

4.2.5 General raindrop size distribution function

Unconstrained form

Substituting the power law a–R and W–R relationships (Eqs. (4.2) and (4.3)) in the raindrop size distribution corresponding to Best's parameterization (Eq. (4.14)) yields

$$N_V(D,R) = \frac{6 \times 10^3 nCR^{r-4p}}{\pi \rho_w A^4} \left(\frac{D}{AR^p}\right)^{n-4} \exp\left[-\left(\frac{D}{AR^p}\right)^n\right]. \tag{4.17}$$

A comparison with Eq. (3.39) (p. 65) shows that Best's parameterization satisfies the scaling law, if the general raindrop size distribution function is taken to be

$$g(x) = \frac{6 \times 10^3 nC}{\pi \rho_w A^4} \left(\frac{x}{A}\right)^{n-4} \exp\left[-\left(\frac{x}{A}\right)^n\right], \tag{4.18}$$

where $x = R^{-p}D$ is a scaled raindrop diameter, and the scaling exponents are

$$\begin{cases} \alpha = r - 4p \\ \beta = p \end{cases} . \tag{4.19}$$

[4]If s is a positive integer (which is the case here) then successive integrations by parts yield the identity (e.g. Mood et al., 1974)

$$\Gamma(s,x) = 1 - e^{-x} \sum_{j=0}^{s-1} \frac{x^j}{j!}.$$

[5]Rather, Best (1950b) used Marshall and Palmer's (1948) exponential parameterization and the corresponding W–R and Z–R relationships to generate an artificial set of 'empirical' values of $F_W(D,R)$ to which he subsequently adjusted his parameterization (via Eq. (4.4)).

In other words, Best's raindrop size parameterization can be recast in a form which is consistent with the scaling law formulation[6]. For the mean values of p and r reported by Best, $p = 0.232$ and $r = 0.846$, this yields for the scaling exponents $\alpha = -0.082$ and $\beta = 0.232$. These values are quite close to those identified for the Marshall-Palmer raindrop size distribution ($\alpha = 0$ and $\beta = 0.21$, Chapter 3, Section 3.5.1), indicating that Best's mean parameterization could be suitable for stratiform conditions.

Substitution of Eq. (4.18) in Eq. (3.86) (p. 88) yields for the self-consistency coefficient of the prefactor for Best's parameterization

$$S_{\mathrm{p}} = \frac{3.6cCA^{\gamma}}{\rho_{\mathrm{w}}}\Gamma\left(1 + \frac{\gamma}{n}\right). \tag{4.20}$$

Similarly, substitution of Eq. (4.19) in Eq. (3.87) yields for the self-consistency coefficient of the exponent

$$S_{\mathrm{e}} = r + \gamma p. \tag{4.21}$$

For Best's mean values of n, A and C (together with $c = 3.778$ and $\gamma = 0.67$), S_{p} becomes 0.98. Similarly, for Best's mean values of p and r, S_{e} becomes 1.00. In other words, for diameter integration limits of 0 and ∞, Best's parameterization is almost perfectly self-consistent.

In summary, Eqs. (4.18) and (4.19) allow the identification of the general raindrop size distribution function $g(x)$ (from n, A and C) and the associated scaling exponents α and β (from r and p) corresponding to previously published adjustments of Best's parameterization (Eq. (4.1)) to measured raindrop size distributions. The self-consistency of such adjustments can then be evaluated on the basis of Eqs. (4.20) and (4.21).

Self-consistent form

One could of course also force the Best parameterization to be completely self-consistent. For the general raindrop size distribution function $g(x)$ this can be achieved by setting the self-consistency coefficient for the prefactor S_{p} (Eq. (4.20)) equal to one and substituting the result in the derived expression for the general raindrop size distribution function (Eq. (4.18)). The result is

$$g(x) = \frac{10^4 n A^{-(\gamma+n)}}{6\pi c \Gamma(1+\gamma/n)} x^{n-4} \exp\left[-\left(\frac{x}{A}\right)^n\right]. \tag{4.22}$$

In a similar manner, setting the self-consistency coefficient for the exponent S_{e} (Eq. (4.21)) equal to one and substituting the result in Eq. (4.19) yields

$$\left\{\begin{array}{c} \alpha = 1 - (4+\gamma)p \\ \beta = p \end{array}\right. . \tag{4.23}$$

[6]Best himself already recognizes the fact that his raindrop size parameterization can be written independently of rain rate. He notes that 'if we regard the parameter a as determining the scale upon which the drop diameters are measured we see that the drop size distribution is the same for all rates of rainfall'.

Table 4.1: Summary of the statistical properties of the re-parameterized form of Best's raindrop size distribution per unit volume of air. The second column pertains to raindrops present in a volume of air (subscripts V), the third column to those arriving at a surface (subscripts A). The listed properties are: raindrop concentration (m^{-3}) / raindrop arrival rate ($m^{-2}\,s^{-1}$), mean diameter (mm), modal diameter (mm), coefficient of variation of the diameters (–), median-volume diameter (mm) and volume-weighted mean diameter (mm). Note that for the first five properties of the sample volume process ν should exceed 3 and for the corresponding properties of the raindrop arrival process ν should exceed $3 - \gamma$.

Property	Sample volume process	Raindrop arrival process
ρ_V, ρ_A	$N_0 \frac{\Gamma(1-3/\nu)}{\nu \Lambda^{1-3/\nu}}$	$c N_0 \frac{\Gamma[1-(3-\gamma)/\nu]}{\nu \Lambda^{1-(3-\gamma)/\nu}}$
$\mu_{\underline{D}_V}, \mu_{\underline{D}_A}$	$\frac{\Gamma(1-2/\nu)}{\Lambda^{1/\nu}\Gamma(1-3/\nu)}$	$\frac{\Gamma[1-(2-\gamma)/\nu]}{\Lambda^{1/\nu}\Gamma[1-(3-\gamma)/\nu]}$
$\text{mode}_{\underline{D}_V}, \text{mode}_{\underline{D}_A}$	$\begin{cases} \left(\frac{1-4/\nu}{\Lambda}\right)^{1/\nu} & ; \quad \nu \geq 4 \\ 0 & ; \quad \nu < 4 \end{cases}$	$\begin{cases} \left[\frac{1-(4-\gamma)/\nu}{\Lambda}\right]^{1/\nu} & ; \quad \nu \geq 4-\gamma \\ 0 & ; \quad \nu < 4-\gamma \end{cases}$
$\text{CV}_{\underline{D}_V}, \text{CV}_{\underline{D}_A}$	$\left[\frac{\Gamma(1-1/\nu)\Gamma(1-3/\nu)}{\Gamma^2(1-2/\nu)} - 1\right]^{1/2}$	$\left\{\frac{\Gamma[1-(1-\gamma)/\nu]\Gamma[1-(3-\gamma)/\nu]}{\Gamma^2[1-(2-\gamma)/\nu]} - 1\right\}^{1/2}$
$D_{0,V}, D_{0,A}$	$\left(\frac{\ln 2}{\Lambda}\right)^{1/\nu}$	$\left(\frac{0.67+\gamma/\nu}{\Lambda}\right)^{1/\nu}$
$D_{m,V}, D_{m,A}$	$\frac{\Gamma(1+1/\nu)}{\Lambda^{1/\nu}}$	$\frac{\Gamma[1+(1+\gamma)/\nu]}{\Lambda^{1/\nu}\Gamma(1+\gamma/\nu)}$

Obviously, these expressions will both perfectly satisfy the self-consistency constraints posed by Eqs. (3.37) and (3.38) (p. 64). Substituting them in the scaling law (Eq. (3.39)) yields finally

$$\begin{aligned} N_V(D,R) &= \frac{10^4 n A^{-(\gamma+n)}}{6\pi c \Gamma(1+\gamma/n)} R^{1-(\gamma+n)p} D^{n-4} \exp\left(-A^{-n} R^{-pn} D^n\right); \\ & A, p > 0; \quad n > 3; \quad D \geq 0. \end{aligned} \tag{4.24}$$

In contrast to Eq. (4.14), this is a completely self-consistent form of Best's raindrop size distribution parameterization. As a result of the imposed self-consistency, for a given value of n, the coefficients of only one power law relationship between rainfall related variables (in this case of that between a and R) unambiguously determine $N_V(D,R)$. In a slightly different form, Eqs. (4.22) and (4.24) are the self-consistent versions of the general raindrop size distribution function and the general rain rate density function listed in Table 3.4 (p. 81) (with the substitutions $n = \nu$, $A^{-n} = \lambda$ and $p = \beta$).

A concise re-parameterization of Eq. (4.24), with a functional form similar to that introduced for the gamma raindrop size distribution by Ulbrich (1983) (Eq. (3.71), p. 82), is

$$N_V(D,R) = N_0 D^{\nu-4} \exp(-\Lambda D^\nu); \quad N_0, \Lambda > 0; \quad \nu > 3; \quad D \geq 0. \tag{4.25}$$

It follows from Eq. (4.24) and Table 3.4 that this parameterization will only be self-consistent if N_0 ($mm^{-(\nu-3)}\,m^{-3}$) and Λ ($mm^{-\nu}$) are power law functions of the rain rate R (mm h^{-1}) according to

$$N_0 = \frac{10^4}{6\pi c} \frac{\nu\lambda^{1+\gamma/\nu}}{\Gamma(1+\gamma/\nu)} R^{1-(\gamma+\nu)\beta} \tag{4.26}$$

and

$$\Lambda = \lambda R^{-\beta\nu}. \tag{4.27}$$

Table 4.1 summarizes some statistical properties of this parameterization for the raindrop size distribution in terms of the parameters N_0, Λ and μ. It forms an extension of Table 3.5 (p. 85), which lists the corresponding properties for the exponential, gamma and lognormal parameterizations. Unfortunately, no closed form expressions for the median diameter are available in this case. The approximation proposed by Ulbrich (1983) for the median of the gamma distribution ceases to yield satisfactory results for the typical shapes of the Best parameterization. For the median-volume diameter of raindrops arriving at a surface ($D_{0,\mathrm{A}}$), however, it is still valid.

4.2.6 Results and discussion

Best (1950b) has adjusted his parameterization (Eq. (4.1), recast in the form of Eq. (4.4)) to raindrop size distribution datasets from various parts of the world. Table 4.2 lists the values of the scaling exponents α and β and the corresponding values of the self-consistency coefficients S_e and S_p (assuming $c = 3.778$ and $\gamma = 0.67$) which can be obtained on the basis of Eqs. (4.19)–(4.21) and the values of n, A, p, C and r he estimates for the different locations (Table VIII on p. 32 of his article). For comparison, the values corresponding to a fit of Best's parameterization to empirical raindrop size distributions collected in 1968 and 1969 in De Bilt, The Netherlands by Wessels (1972) and colleagues are included as well. These will be discussed at length in Chapter 5. All raindrop size measurements have been obtained using the filter paper method, except those from Washington DC, which have been obtained using the flour method. The latter are the data of Laws and Parsons (1943), which will be analyzed in Section 4.3. Note that there is a small difference between the values of the scaling exponents for Montreal, Canada obtained from Best's adjustment to Marshall and Palmer's (1948) data ($\alpha = -0.08$ and $\beta = 0.24$) and those obtained directly from Marshall and Palmer's original adjustment ($\alpha = 0$ and $\beta = 0.21$). This is probably due to the method Best employed to adjust the Marshall-Palmer distribution to his own parameterization[5].

Table 4.2 shows that the self-consistency constraint on the exponents is satisfied very well in all cases, the deviations being less than $\pm 4\%$ from unity. This can also be seen from Fig. 4.3(a), where the scaling exponents are plotted against each other in a manner analogous to what has been done in Fig. 3.2 (p. 90). All data points closely follow the theoretical self-consistency relationship $\beta = \frac{1-\alpha}{4+\gamma}$. The method used to calculate the error bars around the data point for The Netherlands (the *bootstrap* method) will be explained in Chapter 5. Here it serves to indicate the typical uncertainty associated with the data points as a result of sampling variability.

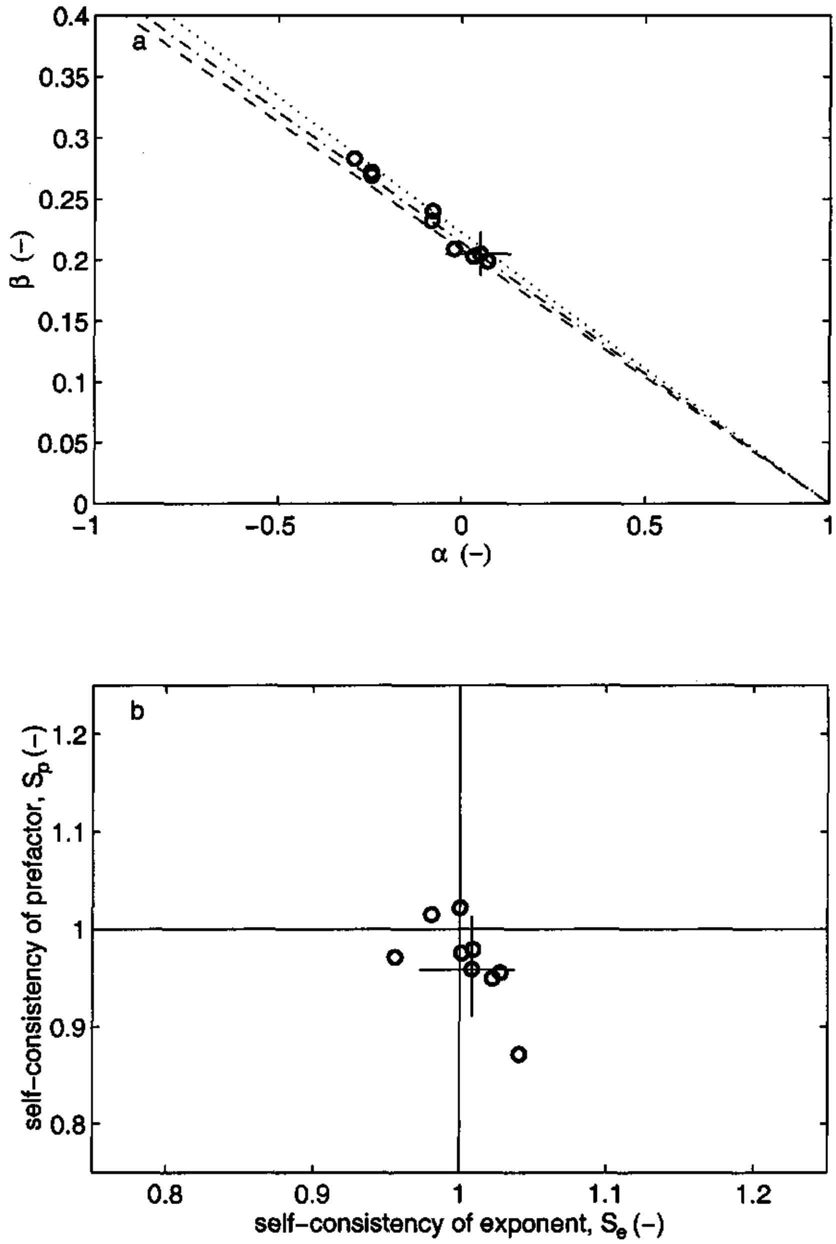


Figure 4.3: (a) Climatological scaling exponents α (–) and β (–) for 7 different locations around the world (Best, 1950a), for the mean distribution of all locations derived by Best and for De Bilt, The Netherlands (Wessels, 1972). Error bars around the Dutch data point indicate 99% confidence limits, estimated from 1000 bootstrap samples. (b) Corresponding values for the self-consistency coefficients of the prefactors S_p and those of the exponents S_e (again with a 99% confidence interval around the Dutch data point).

Table 4.2: Scaling exponents α (–) and β (–) of Best's form of the general raindrop size distribution function $g(x)$ and corresponding self-consistency coefficients S_e (–) and S_p (–) for different locations around the world, based on data reported by Best (1950b) and Wessels (1972).

Location	α (–)	β (–)	S_e (–)	S_p (–)
Hilo (Hawaii, USA)	−0.294	0.283	1.03	0.96
Germany	−0.248	0.272	1.02	0.95
East Hill (UK)	−0.247	0.269	1.01	0.98
Montreal (Canada)	−0.080	0.240	1.04	0.87
mean (Best, 1950b)	−0.082	0.232	1.00	0.98
Shoeburyness (UK)	−0.020	0.209	0.96	0.97
De Bilt (Netherlands)	+0.051	0.205	1.01	0.96
Ynyslas (UK)	+0.033	0.203	0.98	1.02
Washington DC (USA)	+0.071	0.199	1.00	1.02

Fig. 4.3(b) is a graphical representation of the error coefficients associated with the self-consistencies of the prefactors and the exponents (S_p and S_e). That all data points cluster closely around the point with coordinates (1,1) is an indication over their overall self-consistency. From this point of view, the method used by Best (1950b) to estimate the model parameters can be considered quite accurate. The only data point which significantly violates the self-consistency, particularly that with regard to the prefactor, is that associated with Best's re-parameterization of the Marshall-Palmer distribution. In Chapter 3 (Section 3.5.1) the same has been observed with regard to the original formulation of the Marshall-Palmer distribution. The only difference is that for the original Marshall-Palmer distribution, S_p is significantly larger than one (1.16), whereas here it is significantly smaller than one (0.87). This discrepancy is likely to be associated with the fact that in adjusting his parameterization to that of Marshall and Palmer, Best (1950b) forced it to be consistent with the Z–R relationship $Z = 199R^{1.60}$, whereas the original Marshall-Palmer distribution implicitly contains the relationship $Z = 296R^{1.47}$ (Chapter 2, Section 2.7).

On the basis of the values of the scaling exponents given in Table 4.2 and Fig. 4.3(a) roughly three groups of locations can be distinguished: (1) a group with $\beta \approx 0.27$ (Hilo, Germany, East Hill); (2) a group with $\beta \approx 0.24$ (Montreal, the mean of all data); (3) a group with $\beta \approx 0.21$ (Shoeburyness, De Bilt, Ynyslas, Washington DC). In accordance with the reasoning of Chapter 3 (Section 3.5), moving from $\beta \approx 0.27$ to $\beta \approx 0.21$ corresponds to an increase in the proportion of raindrop concentration control on the variability of the raindrop size distribution from 0 to 0.5 and accordingly a decrease of the proportion of diameter control from 1 to 0.5. Because nearly all data points refer to mixtures of different types of rainfall over significant periods of time, it is rather difficult to associate this interpretation of the scaling exponents in terms of different control mechanisms with the physics of the rainfall process for the different locations.

Table 4.3: Parameters κ, λ and ν (where D in mm, $N_V(D,R)$ in $\mathrm{mm^{-1}\,m^{-3}}$ and R in $\mathrm{mm\,h^{-1}}$) of the self-consistent forms of Best's general raindrop size distribution function $g(x)$ and general rain rate density function $h(x)$ for the locations reported by Best (1950b) and Wessels (1972).

Location	κ	λ	ν
Hilo (Hawaii, USA)	1302	1.77	4.48
Germany	128	0.403	2.59
East Hill (UK)	133	0.527	1.99
Montreal (Canada)	292	1.00	1.85
mean (Best, 1950b)	164	0.554	2.25
Shoeburyness (UK)	95.9	0.361	2.29
De Bilt (Netherlands)	215	0.578	2.75
Ynyslas (UK)	230	0.662	2.49
Washington DC (USA)	185	0.600	2.29

There is one exception, however. Best (1950b) clearly states that the data collected in Hilo (Hawaii, USA) correspond to orographic rainfall. That would imply that in orographic rainfall the raindrop concentrations on the average would be approximately constant and that accordingly most spatial and temporal variability of the raindrop size distribution would come from differences in characteristic raindrop sizes. In that sense, it would be the opposite of equilibrium rainfall, where the characteristic raindrop sizes are constant on average and all variability comes from spatial and temporal variations in raindrop concentrations. Clearly, further research is needed to verify this hypothesis for other datasets corresponding to orographic conditions. For the moment, it provides a preliminary interpretation of the third special case of the scaling law discussed in Chapter 3 (Section 3.5) ($\alpha + \beta = 0$) in terms of rainfall typology.

Table 4.3 gives the values of the parameters κ, λ and ν of the general raindrop size distribution function $g(x)$ and the associated general rain rate density function $h(x)$ for the locations treated above. As can be gathered from Eq. (4.24) and the subsequent discussion, these values have been obtained from the parameter values quoted by Best using the relations $\nu = n$, $\lambda = A^{-n}$ and κ according to the expression given in Table 3.4. The latter guarantees self-consistency of the general raindrop size distribution function.

In search for possible dependencies between the parameters (including the scaling exponents), Fig. 4.4(a)–(c) shows scatter plots of β versus λ, β versus ν and λ versus ν, respectively. Because the units of λ ($\mathrm{mm}^{-\nu}\,(\mathrm{mm\,h^{-1}})^{\beta\nu}$) depend on the values of β and ν care must be exercised to avoid interpreting spurious correlations in terms of physical dependencies. However, no clear relations emerge from these scatterplots, indicating that the number of degrees of freedom (the number of free parameters) of the raindrop size distribution cannot be reduced. Incidentally, Hilo is the only location for which the identified value of ν exceeds 3 and as a result the only location

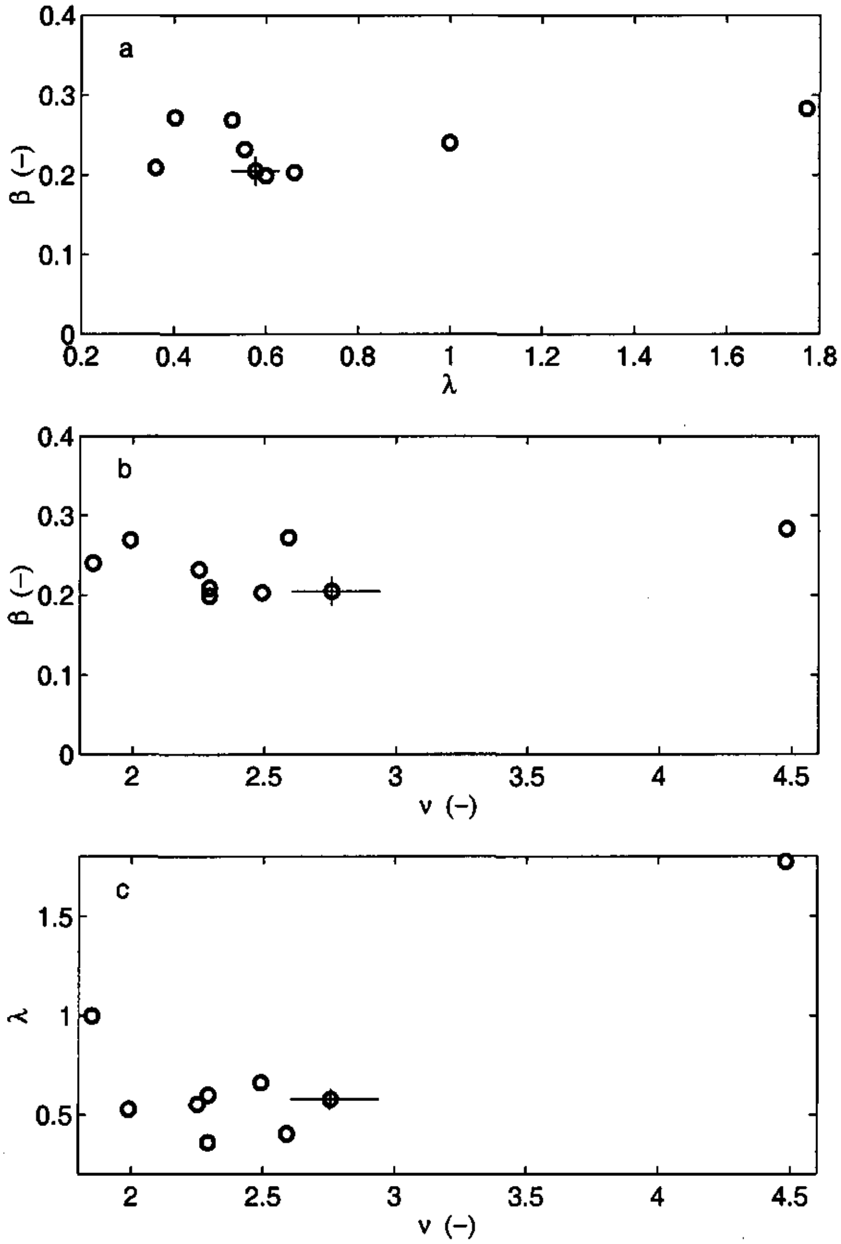


Figure 4.4: (a) Climatological scaling exponent β (–) versus the parameter λ ($\mathrm{mm}^{-\nu}\,(\mathrm{mm\,h^{-1}})^{\beta\nu}$) of the Best (1950b) $g(x)$-parameterization for 7 different locations around the world, for the mean distribution of all locations derived by Best and for De Bilt, The Netherlands (Wessels, 1972). Error bars around the Dutch data point indicate 99% confidence limits, estimated from 1000 bootstrap samples. (b) Idem for β (–) and ν (–). (c) Idem for λ ($\mathrm{mm}^{-\nu}\,(\mathrm{mm\,h^{-1}})^{\beta\nu}$) and ν (–).

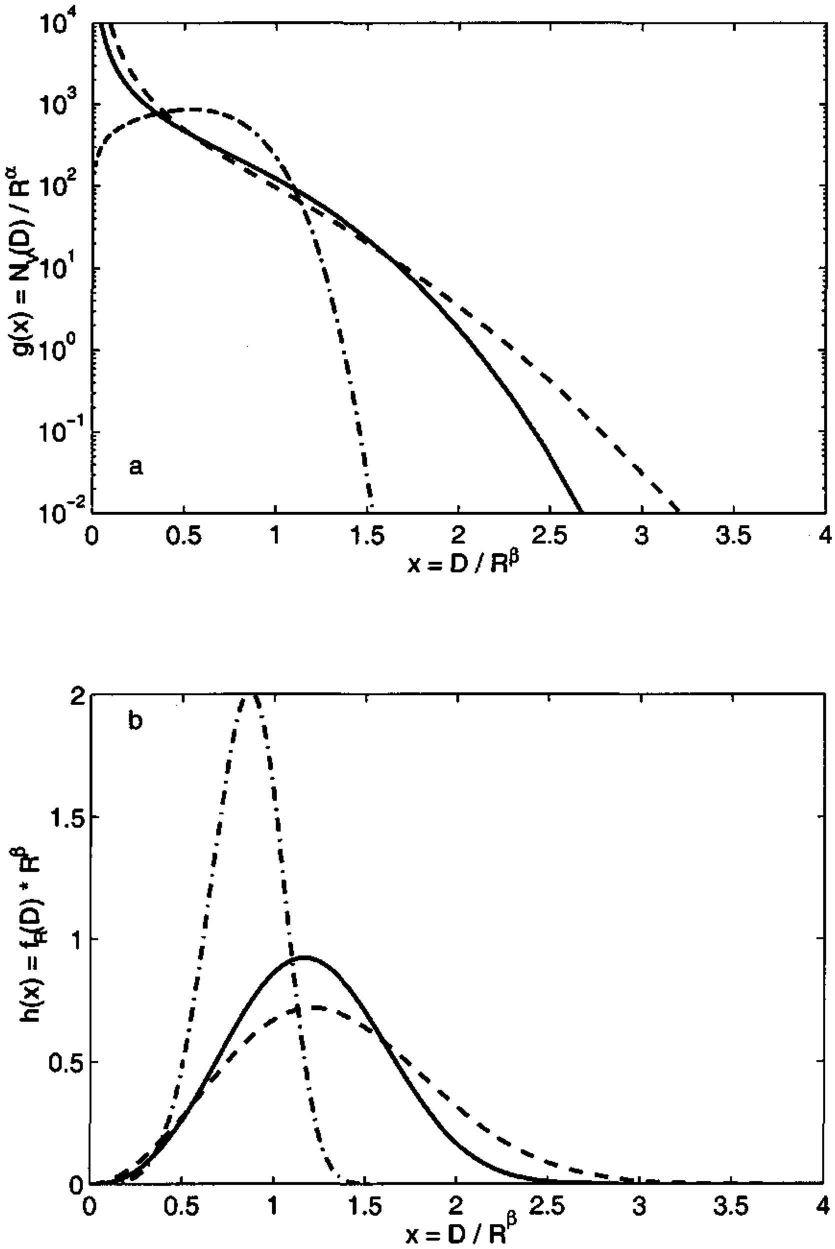


Figure 4.5: (a) Climatological self-consistent general raindrop size distribution functions $g(x)$ $(\mathrm{mm}^{-1}\,\mathrm{m}^{-3}\,(\mathrm{mm\,h}^{-1})^{-\alpha})$, where x $(\mathrm{mm}\,(\mathrm{mm\,h}^{-1})^{-\beta})$ is the scaled raindrop diameter and $\alpha = 1 - 4.67\beta$, for De Bilt, The Netherlands (Wessels, 1972; solid line), for the mean distribution proposed by Best (1950b; dashed line) and for Hilo, Hawaii, USA (Best, 1950b; dash-dotted line). (b) Corresponding self-consistent general rain rate density functions $h(x)$ $(\mathrm{mm}^{-1}\,(\mathrm{mm\,h}^{-1})^{\beta})$.

for which the raindrop concentration will be finite for diameter integration limits of 0 and ∞. All other locations require truncation of the raindrop size distribution at some minimum raindrop diameter $D_{\min}$.

Table 4.3 and Fig. 4.4 show that its orographic rainfall conditions cause Hilo (Hawaii, USA) to be an outlier in comparison with the other locations. Its values of κ, λ and ν (and to a lesser extent that of β) are all significantly higher than those for the other locations. Fig. 4.5 shows how these parameter values translate to the form of the general raindrop size distribution function $g(x)$ and the corresponding general rain rate density function $h(x)$. This figure gives $g(x)$ and $h(x)$ for one location in each of the three groups identified above on the basis of the scaling exponents: Hilo ($\beta \approx 0.27$), the mean of all data ($\beta \approx 0.24$) and De Bilt ($\beta \approx 0.21$). The latter two are not very different, but for Hilo both $g(x)$ and $h(x)$ are much narrower (Eq. (4.9)) and concentrated at smaller scaled raindrop diameters (Eqs. (4.5)–(4.7)).

The fact that mean raindrop sizes in orographic rainfall are generally smaller than those in other types of rainfall (at the same rain rate) is a well-known effect and has been reported before by Cataneo and Stout (1968) and Ulbrich (1983), among others. Note that because for Hilo α approximately equals $-\beta$ ($\alpha + \beta \approx 0$), a change in the rain rate R will scale the horizontal and vertical axes in such a manner that the area under the $g(x)$-curve will remain roughly constant, independent of R. This is a reflection of the fact that all variability in this case is raindrop size controlled, the raindrop concentration remaining constant. Of course, the area under the $h(x)$-curve is by definition constant (one), for any location and any rain rate.

In summary, Best's parameterization for the distribution of the liquid rainwater content over all raindrop diameters contains an intrinsic function for $N_V(D, R)$ which is consistent with the scaling law formulation. This has allowed an estimation of the scaling exponents and an identification for the corresponding general raindrop size distribution functions and general rain rate density functions for all locations for which Best has adjusted his parameterization. Both the estimated scaling exponents and the identified general functions closely satisfy the self-consistency constraints following from the scaling law formulation. For one location (Hilo, Hawaii) it has been possible to relate the values of the scaling exponents and the shapes of the general functions to the type of rainfall (orographic). This has provided a possible interpretation of the third special case of the scaling law treated in Chapter 3 (Section 3.5) ($\alpha + \beta = 0$).

4.3 Laws and Parsons' parameterization

4.3.1 Materials and methods

Laws and Parsons (1943) have carried out measurements of raindrop sizes on the roof of their laboratory in Washington DC during 1938 and 1939. They have used the so-called flour method, which in their case has consisted of exposing pans 10 inch (25.4 cm) in diameter and 1 inch (2.54 cm) deep, filled with calibrated flour, to rainfall 'for intervals ranging from a few minutes to a fraction of a second, depending upon the rain-intensity'. Upon contact with the flour the raindrops captured in

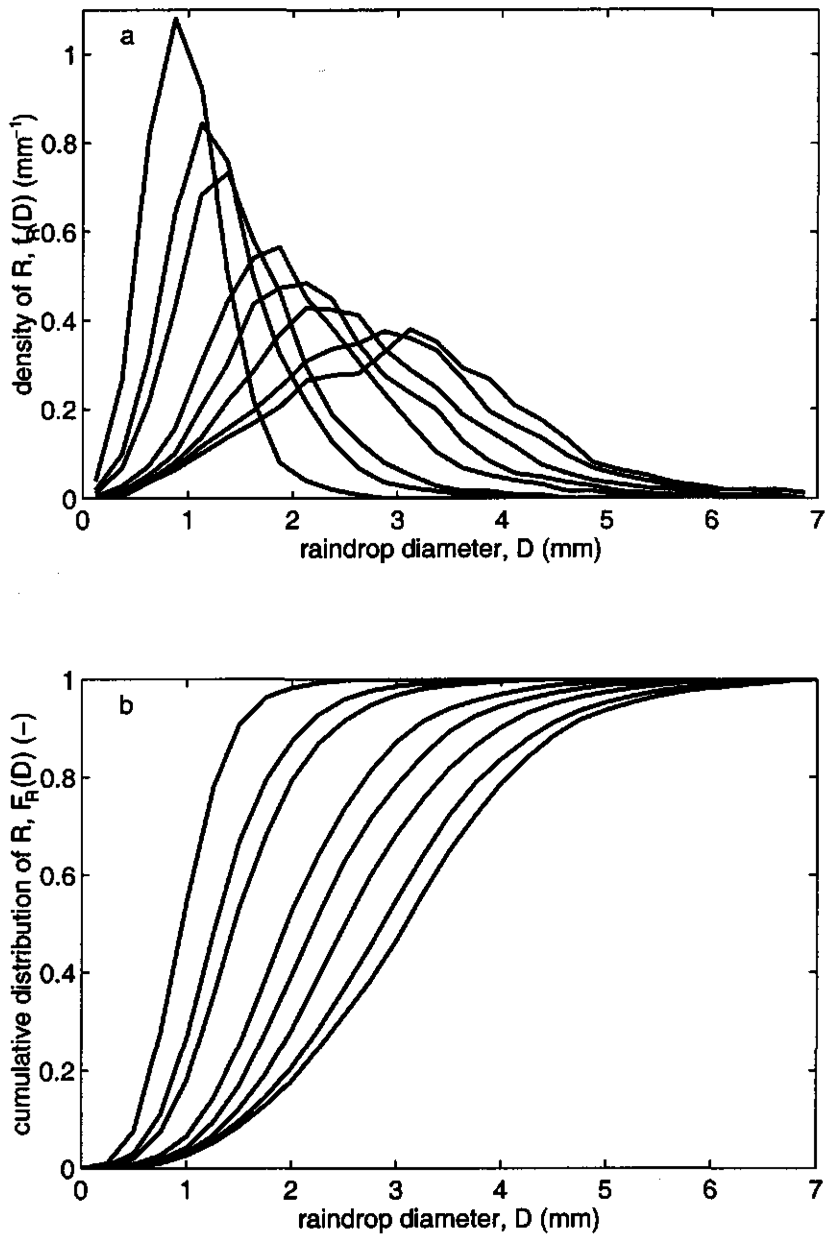


Figure 4.6: (a) Mean rain rate density functions $f_R(D, R)$ (mm^{-1}) for rain rates R of (from left to right) 0.254, 1.27, 2.54, 12.7, 25.4, 50.8, 101.6 and 152.4 mm h^{-1}, obtained in 1938 and 1939 in Washington DC, USA using the flour method (Laws and Parsons, 1943). (b) Corresponding cumulative distribution functions $F_R(D, R)$ (–).

this way tend to produce dough-pellets with sizes related to those of the original raindrops. Laboratory experiments with drops of known sizes have been carried out to establish the calibration curve. After allowing the dough-pellets to dry, they have been separated from the remaining flour and heated in an oven. Subsequently, the pellets have been grouped into different size classes using a set of standard sieves. In this manner, Laws and Parsons have obtained histograms of the fraction of the total rainfall volume per size class captured during a particular time interval (i.e. fraction of the average rain rate during that interval).

Since 'samples of nearly equal intensity displayed wide differences in distribution', Laws and Parsons have found it necessary to classify the obtained empirical histograms into rain rate intervals and derive mean distributions for 8 different rain rates (0.01, 0.05, 0.1, 0.5, 1.0, 2.0, 4.0 and 6.0 inch h^{-1}, i.e. 0.254, 1.27, 2.54, 12.7, 25.4, 50.8, 101.6 and 152.4 mm h^{-1}). In their 1943 article, these mean distributions are presented in the form of a table in which each entry represents the fraction of the rain rate in one of twenty eight raindrop diameter intervals of 0.25 mm width (between 0 and 7 mm). Fig. 4.6(a) shows the corresponding empirical rain rate density functions $f_R(D,R)$ (mm^{-1}) (normalized to unit area) and Fig. 4.6(b) the corresponding (cumulative) distribution functions $F_R(D,R)$ (-). These figures clearly show the well-known shift to larger diameters for increasing rain rates.

4.3.2 Raindrop size distribution and liquid rainwater density function

Laws and Parsons' tabulated parameterization will be analyzed both directly in terms of Best's parameterization (Eqs. (4.1)–(4.4)) and in the general framework posed by the scaling law formulation with rain rate as reference variable (Eq. (3.39), p. 65). Therefore, it has to be recast in forms which are consistent with these formulations. In case of Best's parameterization, this means that the liquid rainwater distribution functions $F_W(D,R)$ (–) corresponding to Laws and Parsons' data are needed. To be able to verify whether Laws and Parsons' data satisfy the scaling law formulation, the corresponding raindrop size distributions $N_V(D,R)$ ($mm^{-1}\,m^{-3}$) should be derived.

Using the definition of the rain rate R in terms of the raindrop size distribution $N_V(D,R)$ in a volume of air (Eq. (3.32), p. 64), the rain rate density function $f_R(D,R)$ (mm^{-1}) can be defined in terms of $N_V(D,R)$ ($mm^{-1}\,m^{-3}$) as

$$f_R(D,R) = \frac{6\pi D^3 v(D)\, N_V(D,R)}{10^4 R}. \tag{4.28}$$

The inversion of this expression yields a relationship between $N_V(D,R)$ and $f_R(D,R)$, namely

$$N_V(D,R) = \frac{10^4 R f_R(D,R)}{6\pi D^3 v(D)}. \tag{4.29}$$

Substitution of this relationship in the definition of $f_W(D,R)$ (mm^{-1}) in terms of

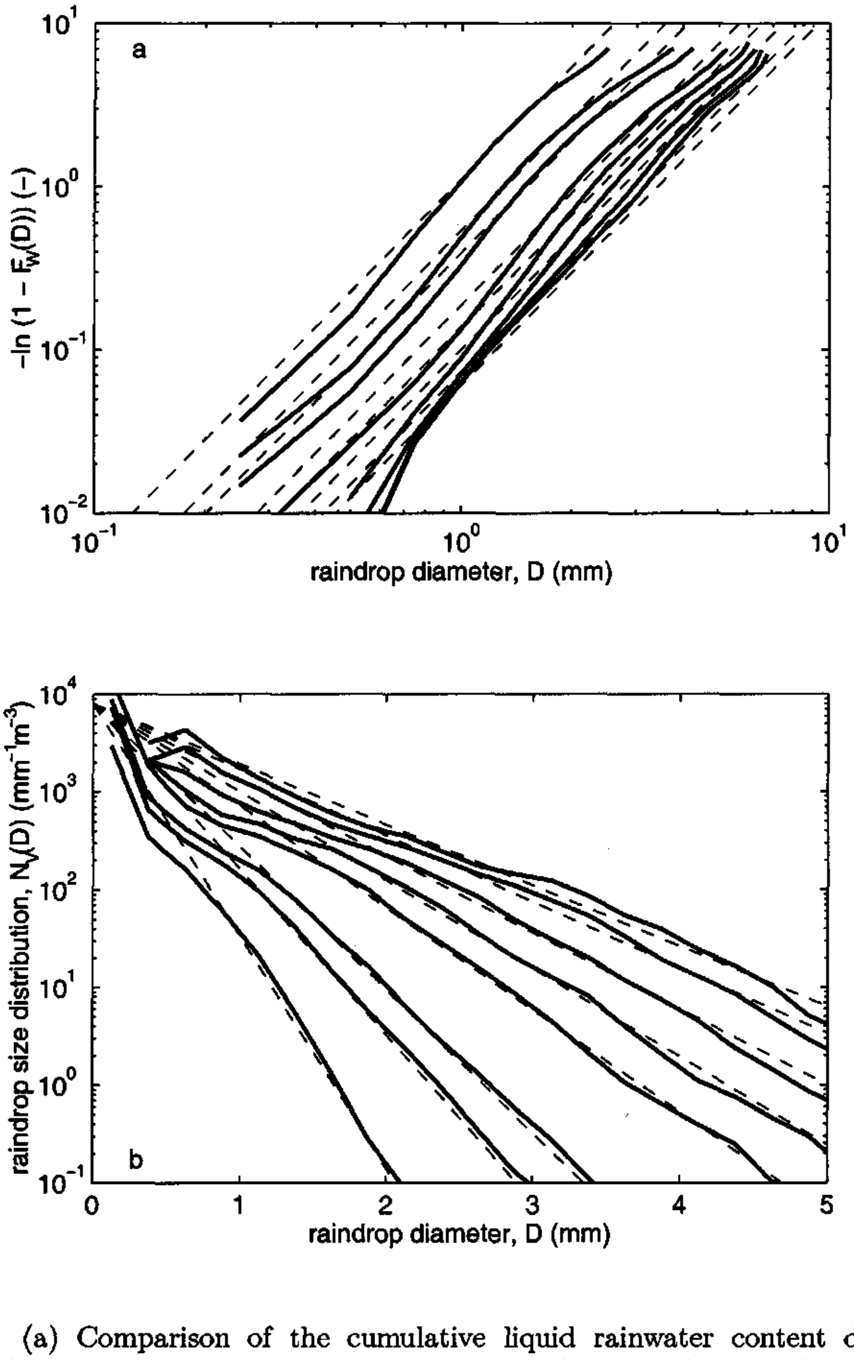


Figure 4.7: (a) Comparison of the cumulative liquid rainwater content distributions $F_W(D, R)$ (–) corresponding to Laws and Parsons' (1943) $f_R(D, R)$-curves (bold lines) with Best's (1950b) analytical parameterization for $F_W(D, R)$ (dashed lines). (b) Comparison of the raindrop size distributions $N_V(D, R)$ ($mm^{-1}\,m^{-3}$) corresponding to Laws and Parsons' $f_R(D, R)$-curves (bold lines) with Marshall and Palmer's (1948) analytical parameterization for $N_V(D, R)$ (dashed lines).

$N_V(D,R)$ (Eq. (4.12)) shows that $f_W(D,R)$ can be expressed in terms of $f_R(D,R)$ as

$$f_W(D,R) = \frac{\rho_w R f_R(D,R)}{3.6 W v(D)}. \tag{4.30}$$

Assuming v ($\mathrm{m\,s^{-1}}$) to be related to D (mm) via a power law (Eq. (2.10), p. 28) with coefficients $c = 3.778$ and $\gamma = 0.67$ and ρ_w to be 1000 $\mathrm{kg\,m^{-3}}$, these relationships have been employed to estimate the raindrop size distributions $N_V(D,R)$ and the (cumulative) liquid rainwater distributions $F_W(D,R)$ from Laws and Parsons' tabulated $f_R(D,R)$-data. The discrete values of D used in these calculations are the mid-points of the twenty eight 0.25 mm wide raindrop diameter intervals. Sheppard's corrections for grouping (Kendall and Stuart, 1977) have not been applied.

Fig. 4.7(a) shows a plot of $-\ln[1 - F_W(D,R)]$ against D on log-log paper for the eight rain rates considered by Laws and Parsons (1943). For reference, the adjustment of Eq. (4.1) to Laws and Parsons' data obtained by Best (1950b) (corresponding to $n = 2.29$, $A = 1.25$ and $p = 0.199$) is shown as well. Best's parameterization provides a reasonable fit to Laws and Parsons' data. Fig. 4.7(b) compares the $N_V(D,R)$-curves estimated from Laws and Parsons' data with the Marshall-Palmer raindrop size distributions (Eq. (2.2), p. 25) for the same rain rates. The latter provides a satisfactory fit in the tails of Laws and Parsons' raindrop size distributions, but deviates for diameters smaller than 1 mm. In this interval, the behavior of Laws and Parsons' distributions (with an excess of small drops as compared to those of Marshall and Palmer for the four lowest rain rates) seems to be more consistent with the shape of Best's $N_V(D,R)$-curves as shown Fig. 4.2(b). Before turning the attention to the identification of the general raindrop size distribution function $g(x)$ for Laws and Parsons' data, this apparent correspondence with Best's parameterization will be investigated in some more detail.

4.3.3 Normalization on the basis of Best's parameterization

For all (cumulative) liquid rainwater distributions $F_W(D,R)$ obtained from Laws and Parsons' rain rate densities $f_R(D,R)$ using Eq. (4.30), the values of the parameters a have been estimated using the fact that for Best's parameterization (Eq. (4.1)) the identity $F_W(a,R) = 1 - e^{-1}$ holds independently of n. The eight values of a thus obtained have been used to determine a power law a–R relationship (Eq. (4.2)) using linear least-squares regression[7] of the logarithm of a on that of R. The estimated values of the prefactor and the exponent are $A = 1.25$ and $p = 0.184$ when a is

[7] The motivation for using linear least-squares regression on the logarithmic values instead of nonlinear (power law) regression on the original values in this context is that Laws and Parsons' raindrop size distributions are *average* distributions. It is likely that those corresponding to the lower rain rates are actually based on a much larger number of *raw* distributions than those corresponding to the higher rain rates. Nonlinear regression would put an unrealistically large weight on those high rain rates in this case. The problem would of course be completely different in case power law relationships would be established based on the raw raindrop size distributions. In that case nonlinear regression seems preferable, the frequency of occurrence of the different rain rates in the sample automatically taking care of the appropriate weighting.

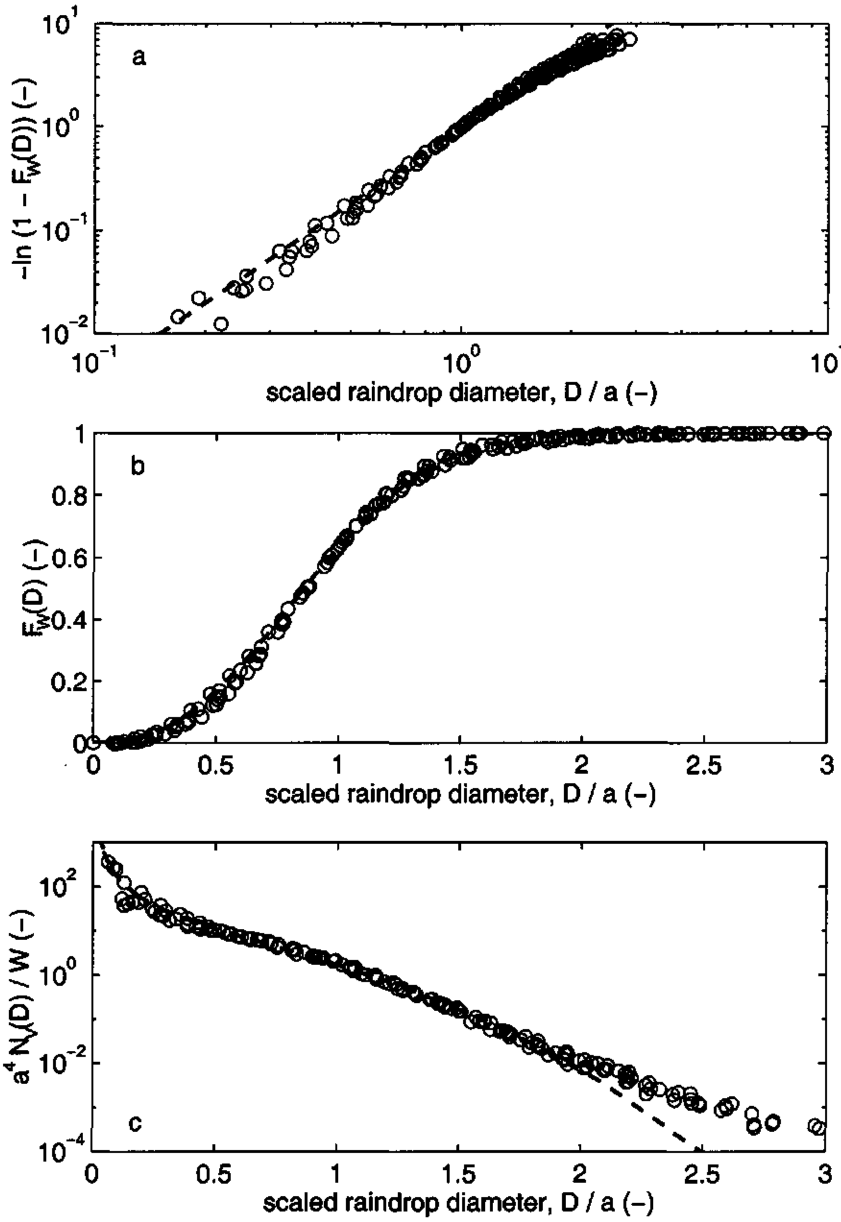


Figure 4.8: (a) Transformed cumulative liquid rainwater content distributions $-\ln[1 - F_W(D, R)]$ (–) corresponding to Laws and Parsons' (1943) data, normalized using a power law relationship between a (mm), the $1 - e^{-1} \approx 63\%$ quantile of $F_W(D, R)$, and R (mm h^{-1}). (b) Corresponding normalized $F_W(D, R)$-curve. (c) Corresponding dimensionless raindrop size distribution $a^4 N_V(D, R) / W$, normalized using a power law relationship between the liquid rainwater content W (here in mm^3 m^{-3}) and R (mm h^{-1}). In all three cases, dashed lines indicate the adjustment of Best's (1950b) parameterization to the data.

expressed in mm and R in $\mathrm{mm\,h^{-1}}$. As a measure for the goodness-of-fit of the obtained a–R relationship, the corresponding *coefficient of determination* r^2 has been calculated[8]. The result is $r^2 = 1.00$, indicating a nearly perfect fit. For comparison, Best reports $A = 1.25$ and $p = 0.199$ for Laws and Parsons' data, i.e. a slightly higher value of p.

Integrating both sides of Eq. (4.30) from $D_{\min}$ to $D_{\max}$ (in this case 0 and 7 mm) yields

$$W = \frac{\rho_{\mathrm{w}} R}{3.6} \int_{D_{\min}}^{D_{\max}} \frac{f_R(D,R)}{v(D)} \mathrm{d}D. \tag{4.31}$$

This equation has been used (again together with a power law $v(D)$ relationship) to estimate the liquid rainwater content W for each of Laws and Parsons' rain rate densities $f_R(D,R)$. The resulting power law W–R relationship (Eq. (4.3)), established on the basis of linear least-squares regression of the logarithm of W on that of R, has coefficients $C = 68.9$ and $r = 0.878$ when W is expressed in $\mathrm{mg\,m^{-3}}$ and R in $\mathrm{mm\,h^{-1}}$ ($r^2 = 1.00$). Best estimates these coefficients as $C = 72$ and $r = 0.867$, quite close to the values obtained here. The small discrepancy can perhaps be attributed to differences in the employed $v(D)$ relationship (Best did not use a power law relationship) and to other differences in computational procedures. In any case, the coefficients determined here have been used in the sequel[9].

Best remarks that 'if we regard the parameter a as determining the scale upon which the drop diameters are measured we see that the drop size distribution is the same for all rates of rainfall'. Indeed, it follows from Eq. (4.1) that if D is rendered dimensionless through dividing by a and $-\ln\left[1 - F_W(D,R)\right]$ is subsequently plotted

[8]The coefficient of determination r^2, sometimes called the *model efficiency* (e.g. Nash and Sutcliffe, 1970), is defined as one minus the ratio of the mean square model error and the sample variance of the observations, i.e.

$$r^2 = 1 - \frac{\sum_i (O_i - M_i)^2}{\sum_i (O_i - \overline{O})^2},$$

where O_i are the observed (measured) values, M_i are the modeled (computed) values and $\overline{O}$ is the mean of the observed values. For instance, in case of an empirical power law a–R relationship, $O_i = a_i$ and $M_i = AR_i^p$ in case the coefficients A and p are estimated using nonlinear (power law) regression and $O_i = \ln a_i$ and $M_i = \ln A + p \ln R_i$ in case they are estimated using linear least-squares regression on the logarithms. The value of r^2 can be interpreted as the fraction of the observed variance explained by the model. $r^2 = 1$ indicates perfect agreement between model and observations, $r^2 = 0$ indicates that the model does not perform better than the mean of the observations and $r^2 < 0$ indicates a complete lack of agreement. The notation r^2 is appropriate because if the model used is the simple linear regression model, then the coefficient of determination reduces to the square of the sample correlation coefficient between the observed and the modeled values.

[9]In a similar manner, power laws have been established which relate the median-volume diameters $D_{0,\mathrm{A}}$ (for the (cumulative) rain rate distribution $F_R(D)$) and $D_{0,\mathrm{V}}$ (for the (cumulative) liquid rainwater distribution $F_W(D)$) to the rain rate R ($\mathrm{mm\,h^{-1}}$). The estimated prefactor and exponent for the $D_{0,\mathrm{A}}$–R relationship are 1.22 and 0.184 ($r^2 = 1.00$), which compares well with the values (converted to the units used here) estimated by Laws and Parsons themselves (1.24 and 0.182). For the $D_{0,\mathrm{V}}$–R relationship, these coefficients are 1.09 and 0.184 ($r^2 = 1.00$). Using Eq. (4.5) and the values for A, p and n found by Best (1950b) for Laws and Parsons' data, the coefficients become 1.07 and 0.199, quite different from the values obtained here.

against D/a on log-log paper then one single *normalized* cumulative liquid rainwater distribution is obtained, a straight line with a slope n independent of R. This is because, according to Best's parameterization, n does not depend on R. Fig. 4.8(a) shows to what extent Laws and Parsons' data satisfy this type of normalization. Indeed, as theory predicts, the data points are more or less concentrated along a straight line. The values of $-\ln[1 - F_W(D,R)]$ for $D/a = 1$ are all very close to 1, almost without any dispersion. This is because the normalization of D has been performed using the established a–R power law, a relationship with virtually no scatter.

According to Best's (1950b) parameterization for $F_W(D,R)$ (Eq. (4.1)), the slope of $-\ln[1 - F_W(D,R)]$ plotted on log-log paper as a function of D/a must equal n. Using the data points plotted in Fig. 4.8(a), n has been estimated using a linear least-squares regression procedure. In order to be consistent with Best's parameterization, the regression line has been forced to go through the point with coordinates $(1,1)$. The resulting slope is $n = 2.43$ ($r^2 = 0.98$). This value is slightly higher than that which Best estimated for Laws and Parsons' data ($n = 2.29$). Fig. 4.8(b) shows the corresponding plot of $F_W(D,R)$ against D/a for $n = 2.43$. The obtained fit is very good ($r^2 = 1.00$). The dispersion of the data around the point with coordinates $(1, 0.63)$ is again negligible.

In an analogous manner, it is possible to go even one step further and normalize the raindrop size distribution $N_V(D,R)$ corresponding to Best's parameterization. Rearranging Eq. (4.14) yields

$$\frac{\rho_w a^4 N_V(D,R)}{10^3 W} = \frac{6n}{\pi}\left(\frac{D}{a}\right)^{n-4} \exp\left[-\left(\frac{D}{a}\right)^n\right], \tag{4.32}$$

which is a dimensionless form of Best's raindrop size distribution. Hence, if the data satisfy Best's parameterization then a plot of $a^4 N_V(D,R)/W$ (which, for the units used here, is numerically the same as the left-hand side of Eq. (4.32)) against D/a will yield one single curve. This curve, defined by the right-hand side of Eq. (4.32), is a normalized raindrop size distribution, independent of rain rate. This type of normalization is in fact very similar to that employed by Sekhon and Srivastava (1971). Apart from their different manner of derivation (being based on the exponential distribution), the only difference with the normalization proposed here is that they use the median-volume diameter $D_{0,V}$ instead of the 63% quantile of the liquid rainwater distribution a.

As has been argued in Chapter 3 (Section 3.3), both the normalization procedure proposed here and that of Sekhon and Srivastava require *two* variables (here a and W, in case of Sekhon and Srivastava $D_{0,V}$ and W). The scaling law formulation, however, predicts that this can (and should) in fact be achieved using only *one* variable, the reference variable. Since power law a–R and W–R relationships have already been established, the rain rate R plays the role of reference variable here. Substitution of the general a–R and W–R relationships (Eqs. (4.2) and (4.3)) into Eq. (4.32) shows that if the normalized raindrop size distribution $a^4 N_V(D,R)/W$ is plotted against the normalized raindrop diameters D/a (provided both a and W are calculated from the rain rate R according to their respective power law relationships) then a scaled version of the general raindrop size distribution function $g(x)$ (Eq. (4.18)) will be obtained.

Fig. 4.8(c) shows the results of this normalization as applied to Laws and Parsons' data. As was the case for the normalization of $F_W(D,R)$ (Fig. 4.8(a) and (b)), the scaling procedure seems to work satisfactorily. The individual curves, which before were distributed over a large domain as a result of their dependence on rain rate (Figs. 4.6 and 4.7), are stacked right on top of each other once the rain rate dependence has been removed. Moreover, the scaled version of Best's general raindrop size distribution function (Fig. 4.8(c)) fits the normalized raindrop size distributions very well, except for scaled raindrop diameters D/a exceeding 2 ($r^2 = 0.87$ on a logarithmic scale). For the values of A and p estimated previously from Laws and Parsons' data ($A = 1.25$ and $p = 0.184$), $D = 2a$ ranges from 1.94 mm at $R = 0.254$ mm h^{-1} to 6.30 mm for $R = 152.4$ mm h^{-1}. These raindrop diameters will contribute hardly anything to the liquid rainwater content at the corresponding rain rates and not much to the radar reflectivity factor either. As a matter of fact, $F_W(2a, R)$ for $n = 2.43$ is more than 0.995, which implies that less than 0.5% of the liquid rainwater content is comprised by drops with diameters exceeding $2a$. Note the particular behavior of the normalized distribution at the small diameter end of the spectrum, typical for Laws and Parsons' data (see Fig. 4.7(b)).

Using Eqs. (4.19)–(4.21) and the estimated values of A, p, C, r and n, the corresponding scaling exponents (α and β) and self-consistency coefficients (S_p and S_e) have been calculated. The results are $\alpha = 0.141$, $\beta = 0.184$, $S_e = 1.00$ and $S_p = 0.98$, respectively. These values of α and β are quite different from those obtained from Best's adjustment (see Table 4.2). The self-consistency constraints on the prefactor and on the exponent, however, remain largely satisfied.

An advantage of the scaling law approach over the normalization procedure used in this section is that the self-consistency is *guaranteed*. Its chief advantage, however, is that it is no longer necessary to impose an a priori functional form for the raindrop size distribution. This will be explored in the next sections.

4.3.4 Estimation of the scaling exponents

In the previous section, the scaling exponents α and β have been estimated using the exponents p and r of power law relationships between two rainfall related variables and the rain rate R, namely those of a and W. However, from a statistical point of view, this is not a very objective and robust method. It has been argued in Chapter 3 (Section 3.4.1) that in principle α and β could be estimated using *any* pair of rainfall related variables. However, since different variables will put their weight on different parts of the drop size distribution, the choice of the pair of variables will influence the results. That is a very undesirable situation, as the optimality and robustness of the applied estimation procedures will determine in the end to what extent differences in the values of the scaling exponents can be related to true physical (meteorological, climatological, or instrument-related) differences between various datasets. In Chapter 3 (Section 3.4.1) two estimation procedures have been presented which intend to

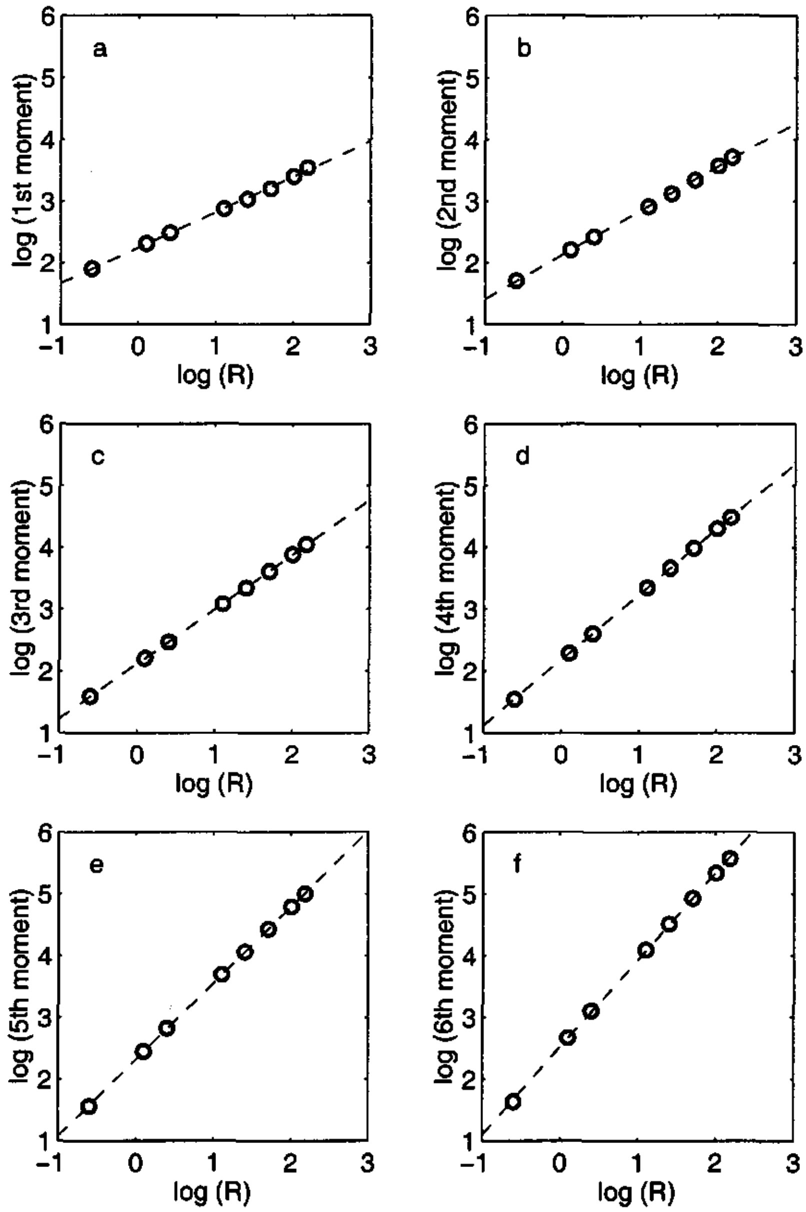
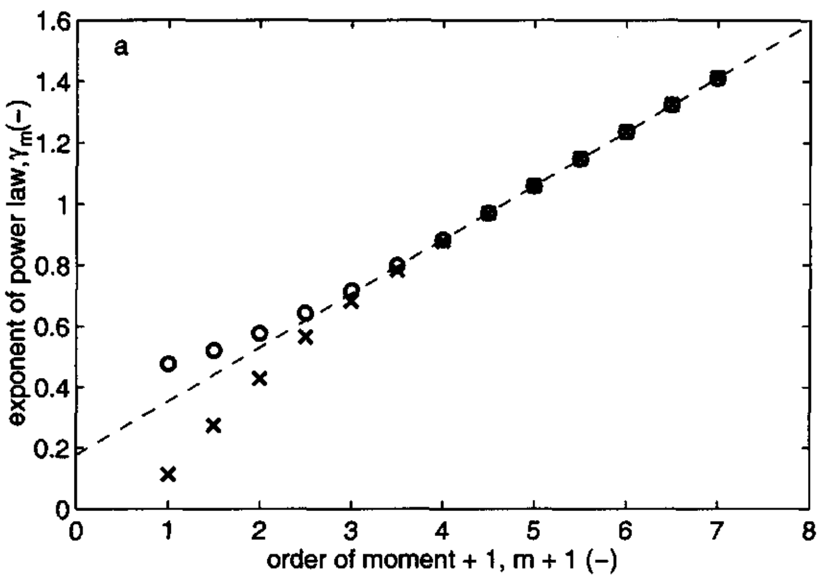


Figure 4.9: Log-log plots of the first six integer moments of the eight mean raindrop size distributions corresponding to Laws and Parsons' (1943) data against the corresponding rain rates ((a)–(f): 1st–6th moments). Dashed lines indicate power law relationships adjusted using linear regression on the logarithmic values.



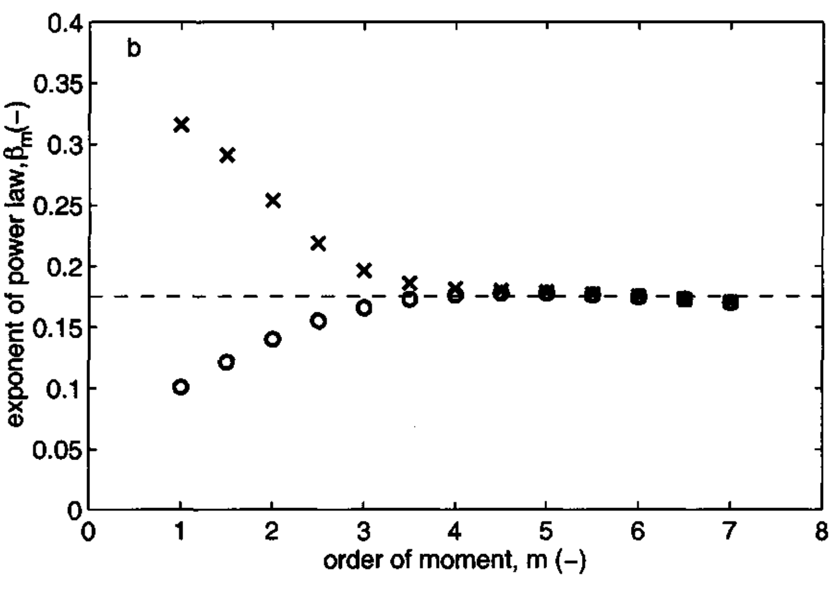


Figure 4.10: (a) Exponents γ_m (–) of power law relationships between the moments of Laws and Parsons' (1943) raindrop size distributions and the corresponding rain rates versus the orders of the moments m plus one. Circles correspond to regressions where the first diameter interval has been taken into account, crosses to regressions where it has been disregarded. Dashed line indicates a linear regression between γ_m and $m+1$ for $m \geq 3$. (b) Idem for the exponents β_m (–) of power law relationships between the weighted mean raindrop diameters and the corresponding rain rates.

overcome this subjectivity problem. The first procedure is based on the exponents of power law relationships between moments Ω_m of the raindrop size distribution and the reference variable Ψ and the second on the exponents of power law relationships between the weighted mean raindrop diameters $\overline{D}_m$ and Ψ. Both have been applied to Laws and Parsons' data, using rain rate as the reference variable.

As an illustration of the first estimation procedure, Fig. 4.9 shows log-log plots of the first six integer moments Ω_m of Laws and Parsons' eight mean raindrop size distributions versus the corresponding rain rates R. The slopes of the associated power laws, which have been adjusted on the basis of linear least-squares regressions on the logarithmic values, clearly exhibit a tendency to increase with the order of the moment m. It should be noted that in calculating the moments, the first raindrop diameter interval (comprising drops with equivalent spherical diameters less than 0.25 mm) has been disregarded. This has been done because, in Laws and Parsons' original table, only the four distributions corresponding to the lowest rain rates show a contribution from the first interval. For the other distributions, Laws and Parsons simply indicate that the contribution of this class to the total rain rate is zero. Since the low order moments are strongly dependent on exactly this part of the raindrop size distribution, the difference between zero and any small fraction in the first diameter interval has an appreciable influence on the calculation of these moments and therefore on the estimation of the exponents of the corresponding power laws.

Fig. 4.10(a) shows a plot of the exponents γ_m (a shorthand notation for γ_{Ω_m}) of the power laws of Fig. 4.9 against the corresponding moment orders m plus one, in a form suggested by Eq. (3.53) (p. 72). Besides the integer order moments, those of half order have been included as well. To illustrate the influence of the first diameter interval on the calculated exponents, both those for which it has been taken into account and those for which it has been disregarded have been plotted. Both sets of exponents deviate quite strongly for moments of order two and lower ($m+1 \leq 3$). For moments of order three and higher however, they coincide and, more importantly, they follow the straight line behavior predicted by the scaling law formulation (Eq. (3.53), p. 72). This set of exponents has been employed in a linear least-squares regression of γ_m on $(m+1)$. The resulting intercept α is 0.177 and the slope β is 0.176 ($r^2 = 1.00$). These values yield for the self-consistency coefficient of the exponent S_e (Eq. (3.87)) a value of 1.00, implying that the regression line perfectly crosses the point $(m+1, \gamma_m) = (4.67, 1)$. Hence, the self-consistency constraint is satisfied perfectly, even though α and β have been estimated independently. This suggests, as noted by Sempere Torres et al. (1994), that self-consistency is 'an implicit property of the experimental data'. In general, it is clear that small changes in the slope β of the regression line will strongly affect the intercept α. Therefore, if it is desired to estimate one scaling exponent from the data at hand and obtain the other by imposing self-consistency, it is recommended to estimate β and then calculate α from Eq. (3.38) on p. 64.

Fig. 4.10(b) shows a plot of the exponents β_m (a shorthand notation for $\gamma_{\overline{D}_m}$) of power law relationships between the weighted mean raindrop diameters $\overline{D}_m = \Omega_m/\Omega_{m-1}$ and the rain rate R against the corresponding moment orders m. These exponents have again been estimated using linear least-squares regressions on the logarithmic values[10]. The scaling law formulation predicts that these exponents should all equal β, independent of the value of m (Eq. (3.28), p. 63). There is again a distinct effect associated with whether or not the first diameter interval is taken into account. But for moments of order four and higher, the two sets of exponents more or less coincide and follow the predicted straight line. The mean value of β_m for $4 \leq m \leq 7$ is 0.175, which can be seen as an estimate of β. The corresponding self-consistent value of α is 0.183. The value of β obtained in this manner corresponds closely to that obtained using the exponents of the moments and is not too different from that obtained above on the basis of Best's parameterization either. Apparently, for Laws and Parsons' data $\alpha \approx \beta$, which implies $\gamma_{\rho_V} \approx 2\gamma_{D_C}$ (Eqs. (3.5) and (3.6), p. 59).

4.3.5 Identification of the general raindrop size distribution function and the general rain rate density function

The procedure outlined in Chapter 3 (Section 3.4.2) shows that, given estimates of the scaling exponents α and β, the general raindrop size distribution function $g(x)$ can be identified by plotting the scaled raindrop size distributions $R^{-\alpha}N_V(D,R)$ against the scaled diameters $R^{-\beta}D$. Similarly, the general rain rate density function $h(x)$ can be identified by plotting the scaled rain rate density functions $R^{\beta}f_R(D,R)$ against the scaled diameters $R^{-\beta}D$ (Chapter 3, Section 3.4.3). The latter is particularly straightforward for the parameterization of Laws and Parsons (1943), because in its original form it is a table of rain rate density functions $f_R(D,R)$ for eight different rain rates. Fig. 4.11 shows the resulting empirical general rain rate density function, obtained using the value of 0.176 estimated for β in the previous section. Although it exhibits some scatter, particularly around its mode, the empirical density function in general closely resembles the unimodal, positively skewed form known from the classical probability density functions of statistical theory. If the data points would have been plotted in a cumulative manner, then the resulting empirical general rain rate distribution function $H(x)$ would have had virtually no scatter (in much the same way as for the normalization based on Best's parameterization shown in Fig. 4.8(b)).

The method employed to adjust analytical parameterizations to the empirical general functions has been to calculate the first two sample moments of the empirical general rain rate density function $h(x)$ and equate them to their theoretical expressions for the different analytical parameterizations presented in Chapter 3 (Tables 3.2–3.4). In statistics, this procedure is known as the *method of moments.* For later comparison, the moments of orders three and four have been calculated as well.

[10]The corresponding estimator for β_m is the least-squares estimator for the slope of a regression line of $\ln \overline{D}_m$ on $\ln R$, i.e.

$$\beta_m = \frac{\mathrm{Cov}\left(\ln \overline{D}_m, \ln R\right)}{\mathrm{Var}\left(\ln R\right)},$$

where Cov and Var denote the sample (co)variance. The fact that $\overline{D}_m = \Omega_m/\Omega_{m-1}$ implies $\ln \overline{D}_m = \ln \Omega_m - \ln \Omega_{m-1}$. This shows that linear regression implies $\beta_m = \gamma_m - \gamma_{m-1}$, in accordance with the scaling law formulation (Eq. (3.28), p. 63). Hence, β_m is simply the local slope of γ_m as a function of m.

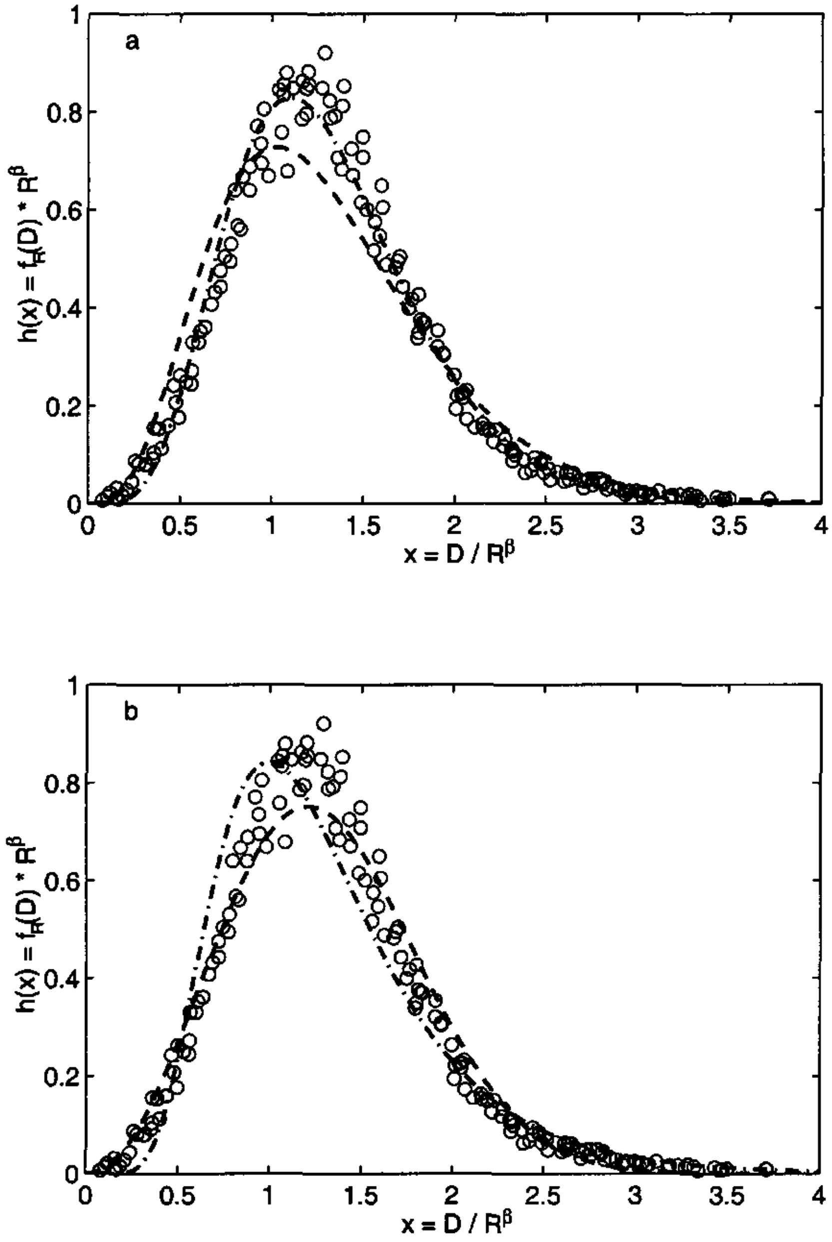


Figure 4.11: Identified general rain rate density function $h(x)$ for Laws and Parsons' (1943) data (circles) and adjusted theoretical parameterizations: (a) exponential $g(x)$ (dashed line; $r^2 = 0.921$) and gamma $g(x)$ (dash-dotted line; $r^2 = 0.977$); (b) Best $g(x)$ (dashed line; $r^2 = 0.966$) and lognormal $g(x)$ (dash-dotted line; $r^2 = 0.924$).

The moments of the empirical rain rate density function have been obtained from[11]

$$\widehat{\mu}'_{x,r} = \frac{\sum_i h_i x_i^r}{\sum_i h_i}, \tag{4.33}$$

where the x_i and h_i are the x- and y-coordinates of the data points in Fig. 4.11. The hat on $\widehat{\mu}'_{x,r}$ indicates that these are the sample moments. Using the expressions given in Table 3.2, the sample moments of orders 1–4 have been employed to calculate the dimensionless coefficients of variation, skewness and kurtosis (peakedness). Since these are dimensionless coefficients, they must in principle be equal to those which can be calculated directly from the eight original $f_R(D,R)$-curves. Hence, for comparison the means and standard deviations of those have been calculated as well. The results are given in Table 4.4. This table shows that the statistics obtained from the empirical general rain rate density function are quite close to those which can be obtained directly from the original rain rate density functions. The small differences can probably be explained by the weighting which is implicitly applied in Eq. (4.33)[11]. The mean and standard deviation of the natural logarithm of x have been calculated as well. These have been used to estimate the parameters of the lognormal distribution[12].

Table 4.5 summarizes the obtained parameter estimates using the method of moments. The parameters of the lognormal distribution have been estimated from the mean and standard deviation of the natural logarithm of x (Table 4.4)[13]. Note the close correspondence of the parameter estimates for the Best parameterization with those given in Table 4.3 ('Washington DC (USA)') based on Best's (1950b) adjustment to Laws and Parsons' data. For comparison with the empirical values, Table 4.5 also provides the values of the coefficients of variation, skewness and kurtosis implied by the estimated parameters. It can be seen that, with the exception of the gamma

[11]Substituting $x_i = R^{-\beta} D_i$ and $h_i = R^{\beta} f_{R,i}$ shows that for a given rain rate R

$$\widehat{\mu}'_{x,r} = \frac{R^{(1-r)\beta} \sum_i f_{R,i} D_i^r}{R^{\beta} \sum_i f_{R,i}} = R^{-r\beta} \widehat{\mu}'_{D,r}.$$

If the data at hand are a combination of empirical rain rate density functions corresponding to different rain rates, as is usually the case, then

$$\widehat{\mu}'_{x,r} = \frac{\sum_j R_j^{(1-r)\beta} \sum_i f_{R,i,j} D_i^r}{\sum_j R_j^{\beta} \sum_i f_{R,i,j}} = \frac{\sum_j R_j^{(1-r)\beta} \widehat{\mu}'_{D,r,j}}{\sum_j R_j^{\beta}} = \frac{\sum_j R_j^{\beta} \widehat{\mu}'_{x,r,j}}{\sum_j R_j^{\beta}}.$$

Hence, $\widehat{\mu}'_{x,r}$ is a weighted mean of the values corresponding to the individual distributions. Only if $\beta = 0$ (equilibrium rainfall) will $\widehat{\mu}'_{x,r}$ represent the arithmetic mean of the individual values.

[12]The values of CV_x and $\sigma_{\ln x}$ given in Table 4.4 do not satisfy the theoretical relationship relating these parameters for the lognormal distribution (Table 3.4), which may indicate a departure of the data from lognormality.

[13]The value of σ estimated from the empirical $h(x)$ (0.433) is very close to that reported by Markowitz (1976) for Laws and Parsons' data (0.432). As a matter of fact, in his Table 1 Markowitz erroneously reports σ for various rain rates to be approximately 0.77 mm. This cannot be correct, however, as σ is a dimensionless parameter. A careful look at Laws and Parsons' data reveals that what he actually means is $\frac{1}{2}\exp(\sigma)$, the factor $\frac{1}{2}$ arising from the fact that he deals with raindrop radii instead of diameters.

Table 4.4: Mean (μ_x) of the scaled raindrop diameters ($x = R^{-\beta}D$, where D in mm and R in mm h^{-1}) with respect to the general rain rate density function $h(x)$, corresponding coefficients of variation (CV_x), skewness (CS_x) and kurtosis (CK_x) and mean ($\mu_{\ln x}$) and standard deviation ($\sigma_{\ln x}$) of $\ln x = \ln D - \beta \ln R$ for Laws and Parsons' (1943) data. Values in the column labeled 'From empirical $h(x)$' have been obtained from the scaled rain rate density function, those in the columns labeled 'From original data' are the means and standard deviations of the values for the 8 original distributions. The value of μ_x in the latter is the prefactor of a power law regression of μ_D on R and the corresponding value of $\mu_{\ln x}$ is the intercept of a linear regression of $\mu_{\ln D}$ on $\ln R$.

Parameter	From empirical $h(x)$	From original data	
		Mean	Standard deviation
μ_x	1.31	1.29	–
CV_x	0.395	0.398	0.015
CS_x	0.684	0.704	0.212
CK_x	0.815	0.836	0.528
$\mu_{\ln x}$	0.181	0.170	–
$\sigma_{\ln x}$	0.433	0.435	0.012

parameterization, the differences are appreciable, indicating that although the adjusted curves will accurately represent the location and dispersion of the empirical general rain rate density function (which is guaranteed by the employed estimation procedure), there may be significant deviations from the overall shape of the empirical general rain rate density function. This is confirmed by Fig. 4.11, which shows the adjusted analytical parameterizations (obtained using the parameter values listed in Table 4.5 and the theoretical expressions given in Tables 3.2–3.4) together with the empirical 'curve'. Visually, the gamma function provides the best fit.

Via the expressions listed in Tables 3.2–3.4 the estimated parameters also define analytical general rain rate density functions $g(x)$. These are shown together with their empirical counterparts in Figs. 4.12 and 4.13. Apparently, the scaling procedure works perfectly for Laws and Parsons' data. There is almost no scatter in the data points, which indicates that the original rain rate dependence of the individual raindrop size distributions has been filtered out entirely by the applied scaling. Table 4.6 provides the coefficients of determination corresponding to the different analytical adjustments to the empirical $g(x)$ and $h(x)$. From these figures it would seem that the exponential parameterization provides the best fit to the empirical $g(x)$. Visually, however, all four parameterizations provide reasonable fits, although each parameterization seems to have a different part of the distribution where it performs best. In particular, there are significant differences between the parameterizations at the small size limit and in the tail of the distribution.

In summary, the raindrop size analysis procedures associated with the scaling law formulation presented in Chapter 3 have been tested on Laws and Parsons' data. Both the estimation of the scaling exponents α and β and the identification of the general functions $g(x)$ and $h(x)$ have demonstrated the power of these procedures

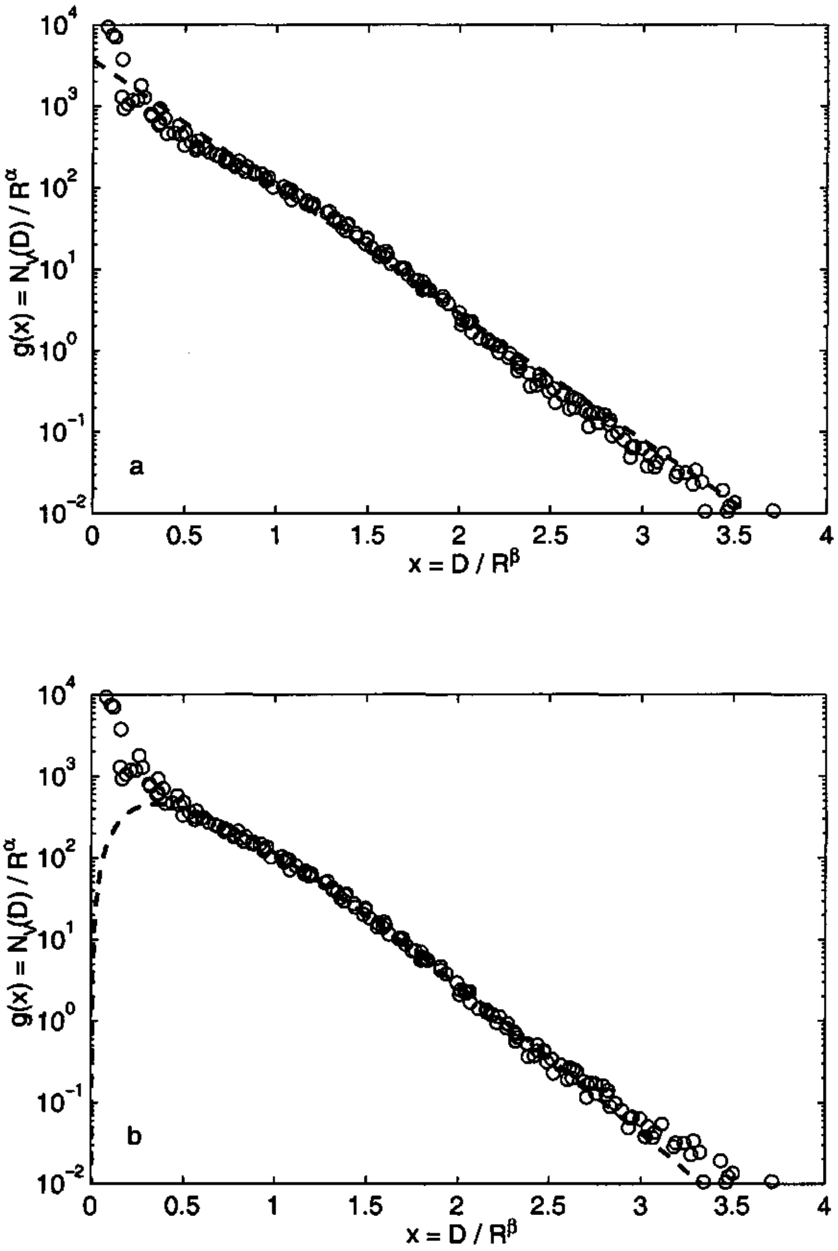


Figure 4.12: Identified general raindrop size distribution function $g(x)$ for Laws and Parsons' (1943) data (circles) and adjusted theoretical parameterizations (dashed lines): (a) exponential ($r^2 = 0.991$); (b) gamma ($r^2 = 0.963$).

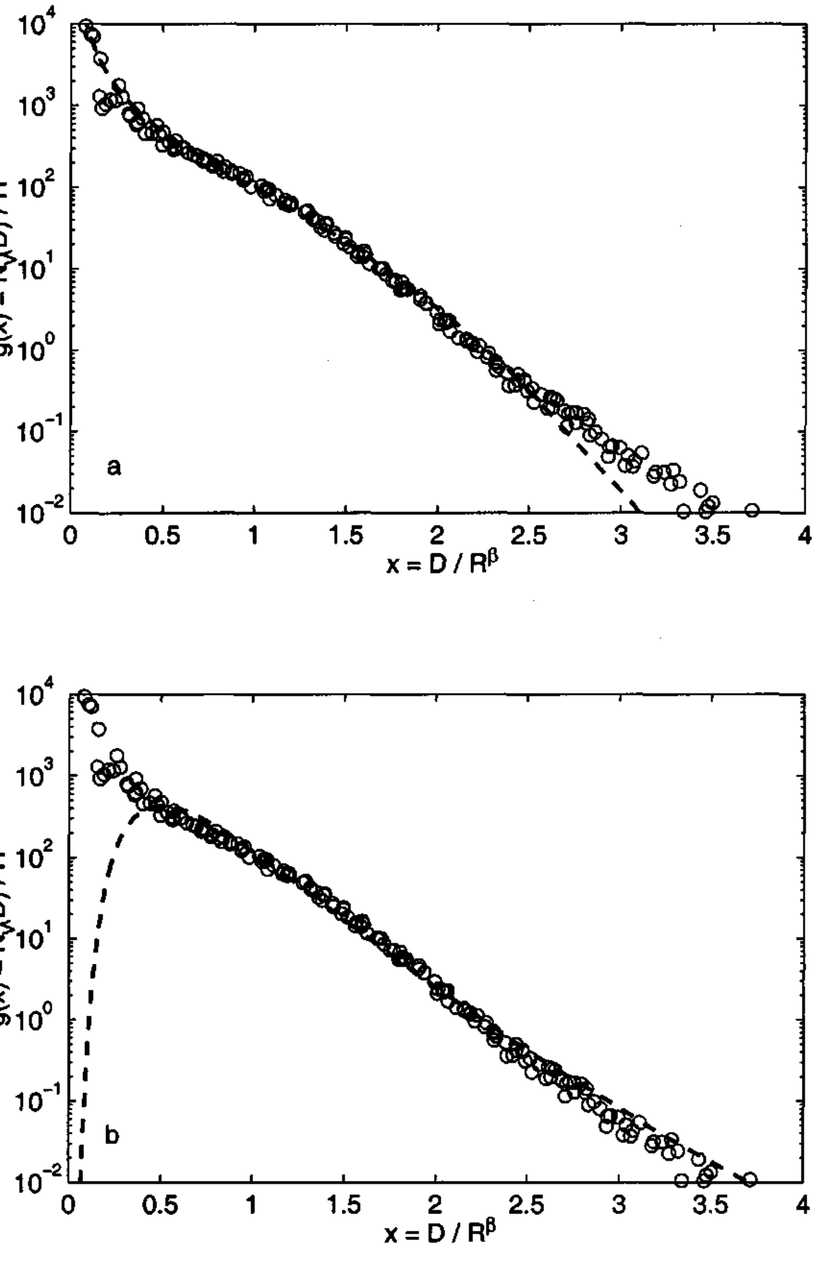


Figure 4.13: Identified general raindrop size distribution function $g(x)$ for Laws and Parsons' (1943) data (circles) and adjusted theoretical parameterizations (dashed lines): (a) Best ($r^2 = 0.942$); (b) lognormal ($r^2 = 0.799$).

Table 4.5: Parameters κ, λ, μ, ν and σ (where D in mm, $N_V(D,R)$ in $\mathrm{mm^{-1}\,m^{-3}}$ and R in $\mathrm{mm\,h^{-1}}$) of different self-consistent forms of the general raindrop size distribution function $g(x)$ and general rain rate density function $h(x)$ for Laws and Parsons' (1943) data. Values in parentheses for the exponential distribution represent the slope λ and corresponding self-consistent intercept κ of a linear regression of $\ln g(x)$ on x. Values in parentheses for the lognormal distribution represent estimates based on μ_x and CV_x instead of $\mu_{\ln x}$ and $\sigma_{\ln x}$. Coefficients of variation (CV_x), skewness (CS_x) and kurtosis (CK_x) represent the values implied by the parameters. That the values of CV_x for the gamma and Best distributions are the same and match the experimental value is because, in contrast with the exponential and lognormal distributions, CV_x has been used as a fitting parameter in the method of moments.

Parameter	Exponential	Gamma	Best	Lognormal
κ	3.66×10^3	1.52×10^4	170	235
	(3.96×10^3)			(192)
λ	3.58	4.90	0.551	–
	(3.64)			
μ	0	1.72	–	−0.507
				(−0.340)
ν	–	–	2.35	–
σ	–	–	–	0.433
				(0.381)
CV_x	0.463	0.395	0.395	0.454
CS_x	0.926	0.791	0.355	1.46
CK_x	1.28	0.938	−0.0833	3.99

and the validity of the scaling law formulation. The concrete result of the presented analyses is a set of four different self-consistent parameterizations for the raindrop size distributions corresponding to Laws and Parsons' data. Substitution of the estimated parameters (Table 4.5) in the expressions of Tables 3.2–3.4 yields (with $\beta = 0.176$ and $\gamma = 0.67$)

$$N_V(D,R) = 3.66 \times 10^3 R^{0.178} \exp\left(-3.58 R^{-0.176} D\right) \tag{4.34}$$

for the exponential parameterization,

$$N_V(D,R) = 1.52 \times 10^4 R^{-0.125} D^{1.72} \exp\left(-4.90 R^{-0.176} D\right) \tag{4.35}$$

for the gamma parameterization,

$$N_V(D,R) = 170 R^{0.468} D^{-1.65} \exp\left(-0.551 R^{-0.414} D^{2.35}\right) \tag{4.36}$$

for the Best parameterization and finally

$$N_V(D,R) = 235 R^{0.354} D^{-1} \exp\left[-2.67 \ln^2\left(1.66 R^{-0.176} D\right)\right] \tag{4.37}$$

for the lognormal parameterization. Note that although Best's parameterization provides a satisfactory fit to the data, it needs to be truncated at some minimum diameter

Table 4.6: Goodness-of-fit, as quantified by the coefficient of determination (r^2), of different theoretical forms for the general raindrop size distribution function $g(x)$ and general rain rate density function $h(x)$ to Laws and Parsons' (1943) data (with and without taking into account the raindrops in the first diameter class). The values for $g(x)$ have been calculated on a logarithmic scale, those for $h(x)$ on a linear scale.

Distribution type	$g(x)$ (logarithmic r^2)		$h(x)$ (linear r^2)	
	with 1st class	without 1st class	with 1st class	without 1st class
Exponential	0.991	0.992	0.921	0.919
Gamma	0.963	0.984	0.977	0.977
Best	0.942	0.934	0.966	0.966
Lognormal	0.799	0.944	0.924	0.929

because $\nu < 3$. Such a truncation will render the raindrop concentration finite without appreciably affecting the higher order moments (liquid rainwater content, rain rate, radar reflectivity factor). Ulbrich (1983) reports for the same data an exponential adjustment with parameters $N_0 = 5.1 \times 10^3 R^{-0.03}$ and $\Lambda = 3.8R^{-0.20}$. Not only are these relationships quite different from the ones implied by Eq. (4.34), they are not self-consistent either, as a comparison with Eqs. (3.69) and (3.70) (p. 82) shows.

The negative exponential distribution provides the most satisfactory results. Although it has only one free parameter, it gives the best adjustment to Laws and Parson's data. This confirms the findings of Sempere Torres et al. (1998), who reported the exponential distribution to be a satisfactory parameterization for general raindrop size distribution functions from a variety of locations. In a sense it also explains why Marshall and Palmer's parameterization has been such a success and still is, more than five decades after its introduction. If the exponential turns out to be the standard form for the general raindrop size distribution function, this implies that a complete parameterization of the raindrop size distribution for a given location requires only two parameters, one scaling exponent (β) and the slope of the exponential function (λ). As mentioned before, it will be a challenge to investigate how these parameters are related to the physics of rainfall (both on weather and climate scales), to the type of measurement device and possibly to each other.

4.4 Summary and conclusions

In search for further evidence of the validity of the general framework for the analysis of raindrop size distributions presented in Chapter 3, the scaling law formulation has been verified experimentally using parameterizations of *mean* raindrop size distributions collected in various climatic settings all over the world.

It has been demonstrated that both Best's analytical parameterization for the distribution of the liquid rainwater content over raindrop size and Laws and Parsons' tabulated parameterization for the distribution of rain rate over raindrop size can be recast in forms which are consistent with the scaling law formulation. This has allowed an identification of the corresponding scaling exponents from previously published adjustments of these parameterizations to measured raindrop size distributions for different types of rainfall in different climatic settings. *The exponents identified in this manner closely satisfy the theoretical self-consistency relationship predicted by the scaling law formulation.* For Best's analytical distributions, these scaling exponents directly lead to analytical parameterizations for the general raindrop size distribution functions and the associated general rain rate density functions.

Interestingly, application of the identified exponents to scale Laws and Parsons' tabulated distributions has also lead to one single general raindrop size distribution function and associated rain rate density function. Both of these are perfectly independent of rain rate, in accordance with the scaling law formulation. Among different analytical descriptions (exponential, gamma, Best and lognormal) of the empirical general raindrop size distribution function, the negative exponential yields the best adjustment.

The obtained results provide further evidence for the scaling law formulation as *the most general description of raindrop size distributions and their properties* consistent with power law relationships between rainfall related variables and as such as a convenient summary of all previously proposed parameterizations in one simple expression.

Chapter 5

Experimental verification of the scaling law using raw raindrop size distributions[1]

5.1 Introduction

In Chapter 4, the data analysis procedures associated with the scaling law formulation developed in Chapter 3 have been applied to two classical raindrop size distribution parameterizations, those due to Best (1950b) and Laws and Parsons (1943). The results have been excellent. Both parameterizations can be recast in forms which are perfectly consistent with the scaling law formulation for the raindrop size distribution. However, in a sense these results have been somewhat fortuitous. That is because the treated parameterizations are only representative for *average* rainfall conditions. In order to arrive at these parameterizations, a large part of the original variability in the raindrop size distribution has been averaged out. It is then not really surprising that the remaining variability can be satisfactorily explained by the variations in only one rainfall related variable, the reference variable (the rain rate in case of both Best's and Laws and Parsons' parameterizations). For instance, the eight distributions of rain rate over raindrop diameter in Laws and Parsons original article (as shown in Fig. 4.6, p. 119) represent mean distributions for eight classes of rain rate. In other words, the variability which must have been present within each rain rate interval has been filtered out by the classification. Indeed, Laws and Parsons remark that 'samples of nearly equal intensity displayed wide differences in distribution'.

The objective of this chapter is to investigate exactly the variability which is generally disregarded in mean (climatological) parameterizations such as those presented in the previous chapter. It will in particular be interesting to see how the scaling law formulation and its data analysis procedures are able to cope with this increased amount of variation. To that end, two types of analyses will be carried out: (1) an event-to-event analysis based on adjustments of Best's parameterization to empirical

[1] Adapted version of Uijlenhoet, R., Wessels, H. R. A., and Stricker, J. N. M. (1999). Experimental verification of a scaling law for the raindrop size distribution. *J. Hydrometeor.* (submitted).

raindrop size distributions for a series 28 rainfall events collected in The Netherlands during 1968 and 1969 (Wessels, 1972); (2) a global (climatological) analysis based in principle on the same dataset, but now employing the *raw* raindrop size distributions. The first analysis will provide insight into the variability of the scaling exponents and the corresponding shapes of the general raindrop size distribution functions from one event to the next. Since it will be based on one adjusted parameterization per event, the variability *within* each event will in principle be disregarded. However, some information regarding this variability will be obtained by re-sampling from the original data according to the so-called *bootstrap* method. The second analysis will provide one set of scaling exponents and one general raindrop size distribution function for the entire data set. Again, the four classical parameterizations for the raindrop size distribution (exponential, gamma, Best and lognormal) will be adjusted to them. Since this analysis will be based on the raw data, it will provide insight as to what extent one single reference variable is able to explain all variability present in the data. As such, this analysis will explore the limits of the scaling law formulation.

Section 5.2 will introduce the raindrop size distribution measurements and the associated data processing. In Section 5.3 the event-to-event analysis based on Best's parameterization will be presented. Section 5.4 will discuss the global (climatological) analysis. Finally, Section 5.5 will present the summary and conclusions of this chapter.

5.2 Materials and methods

From January 1968 to March 1969 Wessels and coworkers have carried out measurements of raindrop size distributions at the Royal Netherlands Meteorological Institute (KNMI) in De Bilt, The Netherlands (Wessels, 1972). They use an ingenious type of device in which filter paper coated with a water-soluble dye (sometimes referred to as 'blotting paper') is transported automatically underneath a 20 cm^2 exposure slit (Wessels, 1967). The height of the 2 cm $\times$ 10 cm slit is 70 cm above ground level. The device is positioned such that the slit is parallel to the prevailing wind direction as much as possible.

The sizes of the stains left by the raindrops on the filter paper are related to their equivalent spherical diameters. The calibration curve has been determined on the basis of laboratory experiments using drops of known sizes. A total number of 454, 976 raindrops have been manually counted, sized and classified into 534 histograms (the empirical raindrop size distributions), each consisting of raindrop counts in 24 diameter classes of 0.2 mm width (from 0 to 4.8 mm). Since it has been found impossible to resolve stains on the filter paper corresponding to drops with diameters less than about 0.08 mm, the center of the first class has been taken to be 0.14 mm. The 534 time intervals identified on the filter paper have been (manually) chosen in such a way that (1) they represent periods during which the rainfall properties (rain rate, size distribution) have remained more or less stationary and (2) they consist of a reasonable number of raindrops (at least of the order of 100, the average sample size being 852). This procedure has resulted in intervals of variable length, ranging from 1 minute for the highest rain rates (where the temporal variability is the limiting

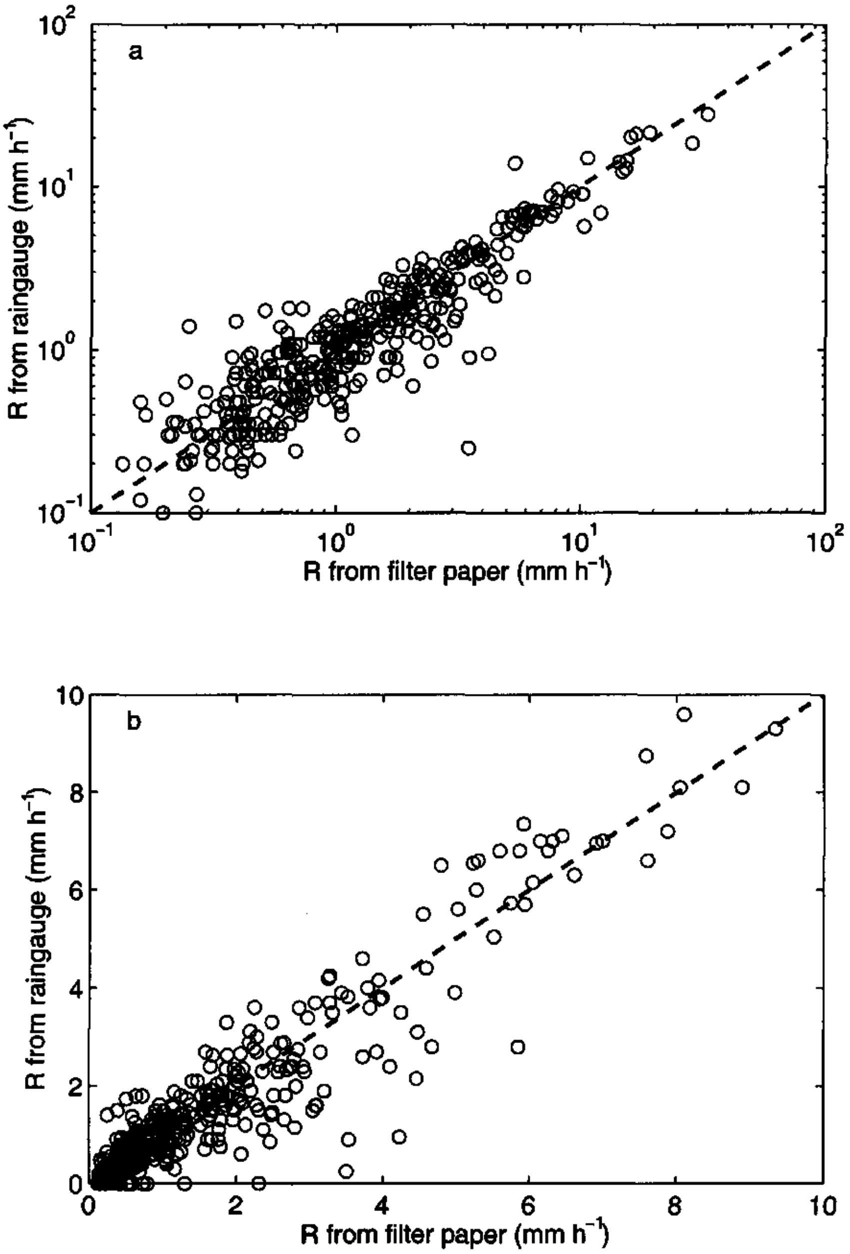


Figure 5.1: Scatterplot of rain rates measured with a 200 cm^2 raingauge versus those measured with a 20 cm^2 semi-automatic filter paper device for 446 time intervals during 28 rainfall events in 1968 and 1969 in De Bilt, The Netherlands (Wessels, 1972). (a) Logarithmic representation. (b) Linear representation.

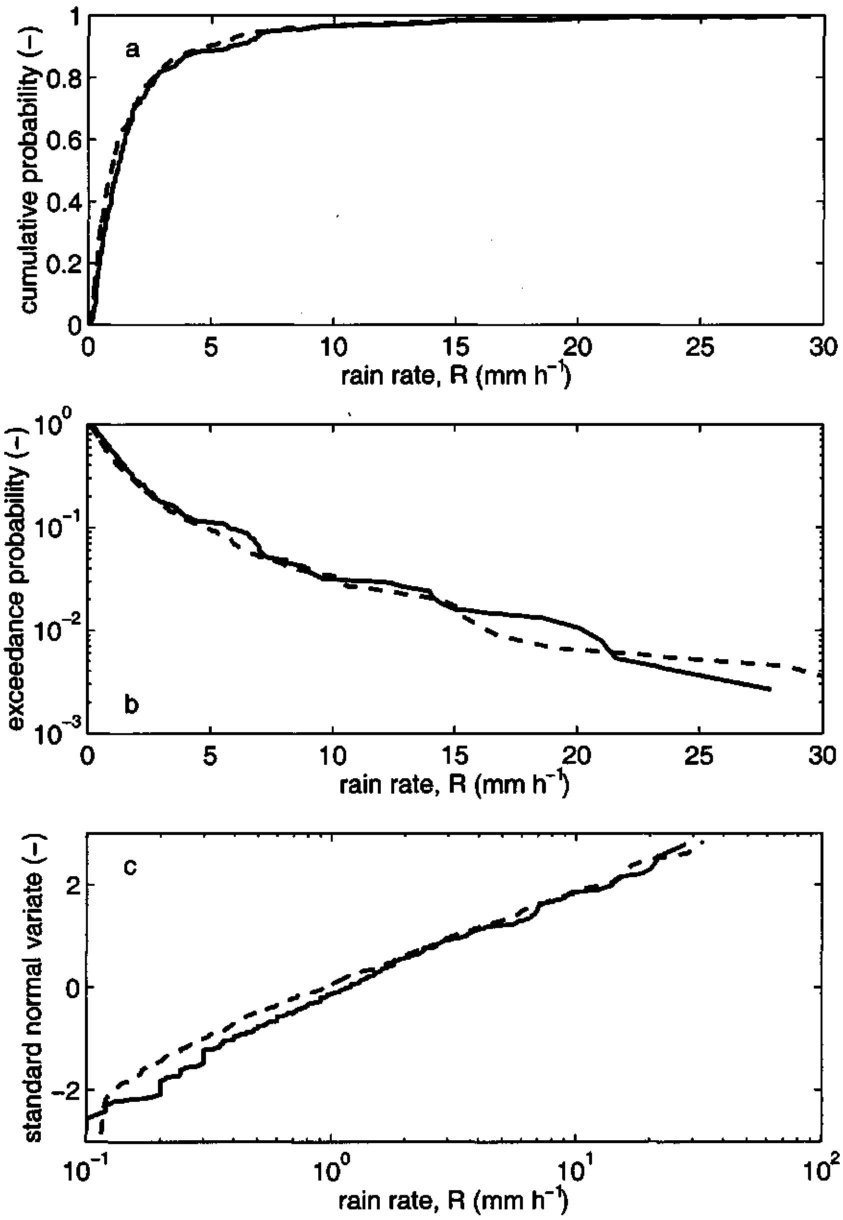


Figure 5.2: Three different representations of the cumulative frequency of rain rates exceeding 0.1 $\mathrm{mm\,h^{-1}}$ measured with a 200 $\mathrm{cm^2}$ raingauge (solid lines) and with a 20 $\mathrm{cm^2}$ semi-automatic filter paper device (dashed lines) for 28 rainfall events in 1968 and 1969 in De Bilt, The Netherlands (Wessels, 1972): (a) empirical cumulative probability distribution function; (b) empirical exceedance probabilities; (c) empirical cumulative probabilities on lognormal probability paper.

factor) to more than 10 minutes for the lowest rain rates (where the sample size is the limiting factor)[2].

The raw raindrop counts have been converted to estimates of raindrop size distributions per unit volume of air according to

$$\widehat{N}_V(D_i) = \frac{n_i}{v(D_i)\, A \Delta t \Delta D_i}, \tag{5.1}$$

where $\widehat{N}_V(D_i)$ ($\mathrm{mm^{-1}\,m^{-3}}$) is the estimated value of the raindrop size distribution for the ith diameter interval, D_i (mm) is the central diameter of that interval, n_i (–) the measured number of raindrops in interval i, $v(D_i)$ ($\mathrm{m\,s^{-1}}$) the terminal fall speed corresponding to D_i, A ($\mathrm{m^2}$) the exposure area (here 2×10^{-3} $\mathrm{m^2}$), Δt the exposure time (s) and ΔD_i (mm) the width of interval i. The various rainfall integral variables of interest can now be estimated using discretized versions of the expressions given in Tables 2.3 (p. 39) and 2.4 (p. 43). Sheppard's corrections for grouping (Kendall and Stuart, 1977) have not been applied. For the particular case of rain rate the appropriate conversion is

$$\widehat{R} = 6\pi \times 10^{-4} \sum_{i=1}^{24} \frac{n_i D_i^3}{A \Delta t}, \tag{5.2}$$

where $\widehat{R}$ ($\mathrm{mm\,h^{-1}}$) is the estimated rain rate. Since the rain rate is a flux variable, this conversion is particularly simple and does not involve the raindrop terminal fall speed.

A comparison of the rain rates calculated according to Eq. (5.2) with those measured by a 200 $\mathrm{cm^2}$ raingauge (height: 40 cm; resolution: 0.05 mm) installed at 10 m distance shows that there are no systematic differences between the two for 446 time periods with rain rates exceeding 0.1 $\mathrm{mm\,h^{-1}}$ (Fig. 5.1). The slope and intercept of a linear regression of the former on the latter are 0.975 (–) and 0.178 ($\mathrm{mm\,h^{-1}}$), respectively ($r^2 = 0.87$). The mean, standard deviation and maximum rain rate estimated from the filter paper measurements are 2.00 $\mathrm{mm\,h^{-1}}$, 3.31 $\mathrm{mm\,h^{-1}}$ and 33.0 $\mathrm{mm\,h^{-1}}$. The corresponding values for the raingauge are 2.22 $\mathrm{mm\,h^{-1}}$, 3.33 $\mathrm{mm\,h^{-1}}$ and 27.9 $\mathrm{mm\,h^{-1}}$. Wessels (1972) reports systematic deviations only during periods of high wind speeds, when the filter paper measurements have a tendency to overestimate the raingauge measurements.

Fig. 5.2(a)–(c) shows three different representations of the cumulative frequencies (i.e. the sample probability distribution functions) of the rain rates estimated with both instruments (for the same 446 time periods as in Fig. 5.1). Obviously, the correspondence between the two devices is again quite good, although the low rain rates seem to be a little bit better represented in the filter paper measurements than in

[2]Although this might seem a strange procedure compared to the fixed time bases of the disdrometers and optical spectrometers used nowadays, a similar approach has been employed to collect the experimental raindrop size distributions from which Marshall and Palmer (1948) have derived their famous parameterization. Describing their measurements, Marshall et al. (1947) remark that 'the time of exposure varied from 30 seconds in very light rain to 3 seconds in heavy rain'. Because the area of their filter papers is much larger (24 cm diameter) they are able to use shorter exposure times than Wessels.

the raingauge measurements. This is probably associated with the lower resolution of the raingauge. If the measurements would be samples from an exponential distribution then the empirical probability distributions in Fig. 5.2(b) would roughly be straight lines. However, this does not seem to be the case. Similarly, if the data would be samples from a lognormal distribution then the curves in Fig. 5.2(c) would approximately be straight lines, which is more or less true. Wessels (1972) has also compared the empirical probability distribution function obtained from the filter paper measurements with that from three consecutive years (1968, 1969 and 1970) of raingauge measurements. He reports a close correspondence. This indicates that the available set of raindrop size distributions can roughly be assumed climatologically representative for the De Bilt.

5.3 Event-to-event analysis

5.3.1 Estimation of the scaling exponents

Using the method proposed by Best (1950b), i.e. by plotting $-\ln[1 - F_W(D)]$ (with $F_W(D)$ from Eq. (4.11), p. 106) against D on log-log paper, Wessels (1972) has been able to adjust Best's parameterization (Eq. (4.1), p. 103) to 521 of the 534 experimental raindrop size distributions. Apart from the values of a (mm) and n (–), he has calculated the values of the rain rate R (mm h^{-1}) and the liquid rainwater content W (mg m^{-3}) for these distributions as well. For an event-to-event analysis of power law relationships, he has selected 28 rainfall events with at least 6 raindrop size distributions per event, comprising a total of 476 experimental distributions. The same set of distributions has been used for the current analysis. The values of a, R and W determined by Wessels for these distributions have been used to estimate the coefficients A, p, C and r of the power law a–R and W–R relationships (Eqs. (4.2) and (4.3), p. 103) for each of the 28 events via nonlinear (power law) least-squares regression. A mean value of n has been determined for each event as a weighted average (using R as weight) of the individual n-values determined by Wessels (1972). In this manner, 28 values of A, p, C, r and n have been obtained, one for each of the selected rainfall events.

With Eq. (4.19) (p. 109), the values of p and r can be used directly to obtain the corresponding values of the scaling exponents α and β for each rainfall event. Fig. 5.3(a) shows a plot of the obtained 28 (α, β)-pairs. The error bars indicate estimates of the 68% confidence limits on α and β. These would correspond to plus and minus one standard deviation from their mean values if their sampling distributions would be normal (Mood et al., 1974). These confidence limits actually represent the 16% and 84% quantiles of the empirical sampling distributions of α and β, estimated from 1000 so-called *bootstrap* samples in each case (Efron and Tibshirani, 1993)[3]. The data points in Fig. 5.3(a) themselves are not the means of the bootstrap

[3]The *bootstrap* method is a so-called nonparametric statistical method, i.e. it does not make any distributional assumption (Efron and Tibshirani, 1993). In this case, the method works as follows. Suppose an (α, β)-pair is calculated from a sample of n empirical raindrop size distributions. (1)

samples, but the values obtained directly from the original samples (which is not exactly the same).

Fig. 5.3(a) shows clearly that the uncertainty associated with α tends to be larger than that associated with β. As has been noted in Chapter 4, if a self-consistent (α, β)-pair is required, it seems therefore wiser to estimate β from the data and obtain α from the self-consistency constraint on the exponent (Eq. (3.38), p. 64), i.e. to consider β as the single free scaling exponent. A large part of the uncertainty in α and β must be due to the limited number of raindrop size distributions per event, i.e. due to sampling variability. Note that 8 of the 28 events consist of less than 10 experimental distributions. Nevertheless, on average the 28 (α, β)-pairs cluster quite closely around the theoretically predicted curves, an indication that the self-consistency constraint on the exponent is largely satisfied.

This is confirmed by Fig. 5.3(b), which shows the self-consistency coefficients of the prefactors S_p (Eq. (4.20) on p. 110, with $c = 3.778$ and $\gamma = 0.67$) plotted against those of the exponents S_e (Eq. (4.21)) for the 28 values of A, p, C, r and n determined by Wessels. Again, the 68% confidence intervals are obtained using the bootstrap method (1000 samples). For all events, the self-consistency is satisfied to within $\pm 20\%$, for the majority of the cases even to within $\pm 10\%$ (both for the prefactors and for the exponents). There seems to be a slight tendency towards underestimation of the exponents. This might be associated with the procedure used to estimate the coefficients of the power law a–R and W–R relationships (nonlinear (power law) least-squares regression). Moreover, the prefactors may have been affected by the fact that possible truncation effects have been neglected. In general, however, the results are satisfactory.

In the same manner as the individual values of A, p, C, r and n have been determined for each of the 28 selected rainfall events, global values of these coefficients for the entire set of 476 experimental raindrop size distributions have been calculated. With a, R and W expressed in the same units as above, the estimated coefficients of the global power law a–R relationship become $A = 1.22$ and $p = 0.205$ ($r^2 = 0.66$) and those of the global power law W–R relationship $C = 68$ and $r = 0.871$ ($r^2 = 0.99$). The weighted mean value of n is 2.75 (with a standard deviation on the estimated mean of 0.03), respectively. The estimated coefficients for the a–R relationship correspond closely to those reported by Wessels (1972) for the same data set ($A = 1.21$ and $p = 0.21$). These values, however, are both smaller than the mean values obtained by Best (1950b). That the estimated power law W–R relationship provides an almost perfect fit to the data is not surprising in view of the fact that W and R are proportional to the 3rd and the $(3 + \gamma)$th moment of the raindrop size

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Associate with each distribution the probability $1/n$. (2) Draw n new distributions *independently and with replacement* from the original sample. This new sample is called the bootstrap sample. (3) Compute α and β for the bootstrap sample using the method indicated in the text. (4) Repeat steps (2) and (3) a large number of times (in this case 1000) each time using an independent new bootstrap sample. (5) Sort the 1000 values of α and β thus obtained (in ascending or descending order) and select the 159th and 842nd values. These constitute bootstrap estimates of the 68.26% confidence intervals on α and β (if the distributions of α and β would be normal, these would correspond to intervals of length 2σ).

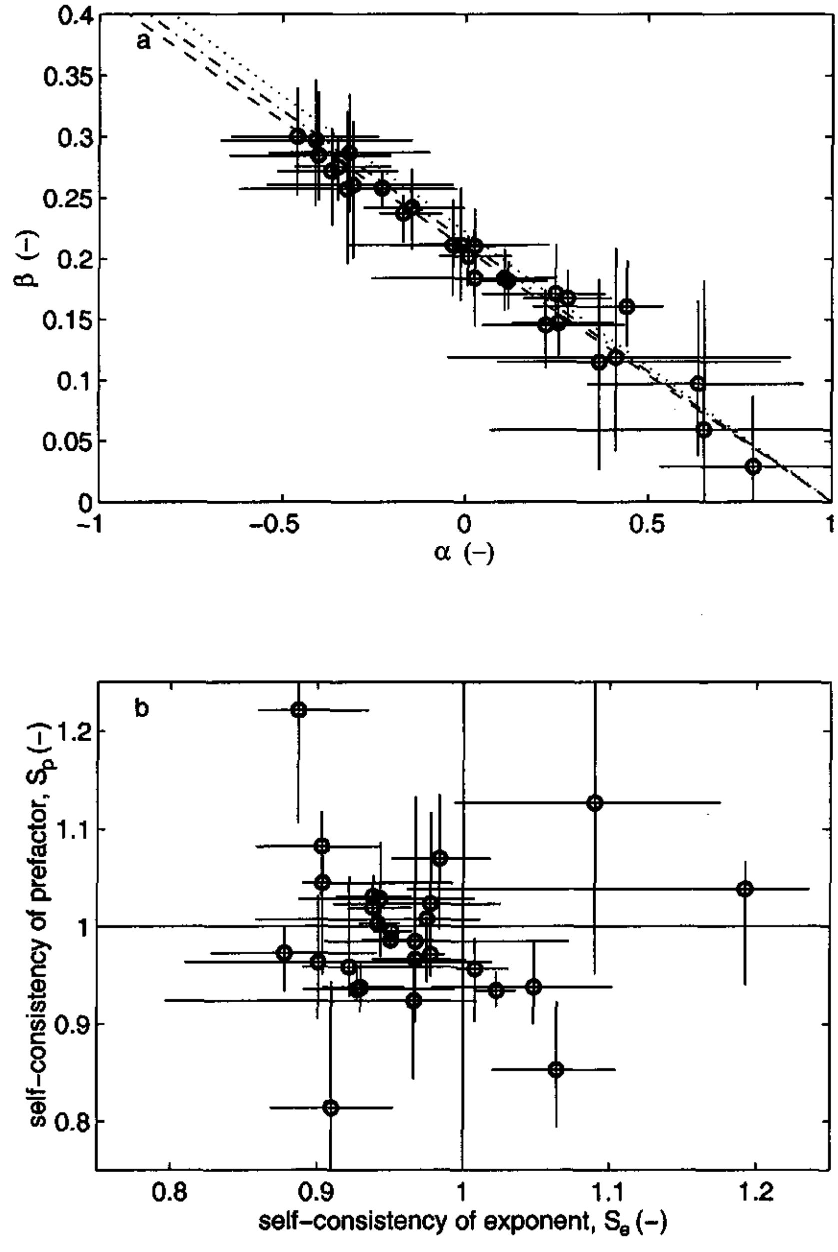


Figure 5.3: (a) Scaling exponents α (–) and β (–) for 28 rainfall events in 1968 and 1969 in De Bilt, The Netherlands (based on data reported by Wessels (1972)). Error bars indicate 68% confidence limits, estimated from 1000 bootstrap samples in each case. Straight lines represent the theoretical self-consistency relationship between the scaling exponents. (b) Corresponding values for the self-consistency coefficients of the prefactors S_p and those of the exponents S_e (again with 68% confidence intervals).

distribution, respectively.

From the estimated global values for the coefficients A, p, C, r and n, the scaling exponents and self-consistency coefficients can again be found from Eqs. (4.19)–(4.21) (p. 109). As a matter of fact, these have already been encountered in Chapter 4. In Fig. 4.3(a) and (b) (p. 113) they have been plotted for comparison with the values for the locations treated by Best (1950b). In contrast to Fig. 5.3, the error bars around the Dutch data point in Fig. 4.3 represent the minimum and maximum values encountered in 1000 bootstrap samples. Hence, they correspond approximately to the 99.74% confidence interval. These would be of length 6σ if the corresponding sampling distributions would be normal. Again, the uncertainty associated with α tends be larger than that associated with β.

If Fig. 5.3 is compared to Fig. 4.3, there are three points which directly draw the attention. First of all, the data points in Fig. 5.3(a) are distributed along a significantly larger portion of the three reference lines than in Fig. 4.3(a). Secondly, the self-consistency errors in Fig. 5.3(b) are much larger than in Fig. 4.3(b). Finally, as can be seen from the lengths of the error bars, the uncertainty associated with each point in Fig. 5.3 is much larger than that in Fig. 4.3. The fundamental difference between both figures is of course that in Fig. 5.3 the data points represent different rainfall events collected at one location, whereas in Fig. 4.3 they represent average values for different locations. Hence, Fig. 5.3 is more indicative of the variability associated with different rainfall (weather) types within a certain climatology, whereas Fig. 4.3 is more indicative of that due to different rainfall climatologies. What is surprising is that the data points in Fig. 5.3 cover practically the entire range from purely raindrop size controlled variability to purely raindrop concentration controlled variability. Obviously, part of this spread can be explained in terms of sampling variability, as indicated by the lengths of the error bars. However, even if this is taken into account, there appear to remain physically significant differences between the events.

5.3.2 Identification of the general raindrop size distribution functions and the general rain rate density functions

Using the parameters A and n estimated for the 28 rainfall events, the parameters κ, λ and ν of the corresponding self-consistent general raindrop size distribution functions can be obtained from $\nu = n$, $\lambda = A^{-n}$ (Eq. (4.22), p. 110) and κ according to the expression given in Table 3.4 (p. 81). To investigate possible dependencies between these parameters and the scaling exponents, Fig. 5.4(a)–(c) shows scatter plots of β versus λ, β versus ν and λ versus ν, respectively. The error bars are again estimates of the 68% confidence intervals based on 1000 bootstrap samples. Just as for the climatological analysis presented in Fig. 4.4 (p. 116), there are no clear relations between these parameters. This is again an indication that the number of free parameters cannot be reduced. This contrasts a similar type of event-to-event analysis (based on exponential general raindrop size distribution functions however) with two clearly distinguishable groups of data points, one corresponding to stratiform events and the

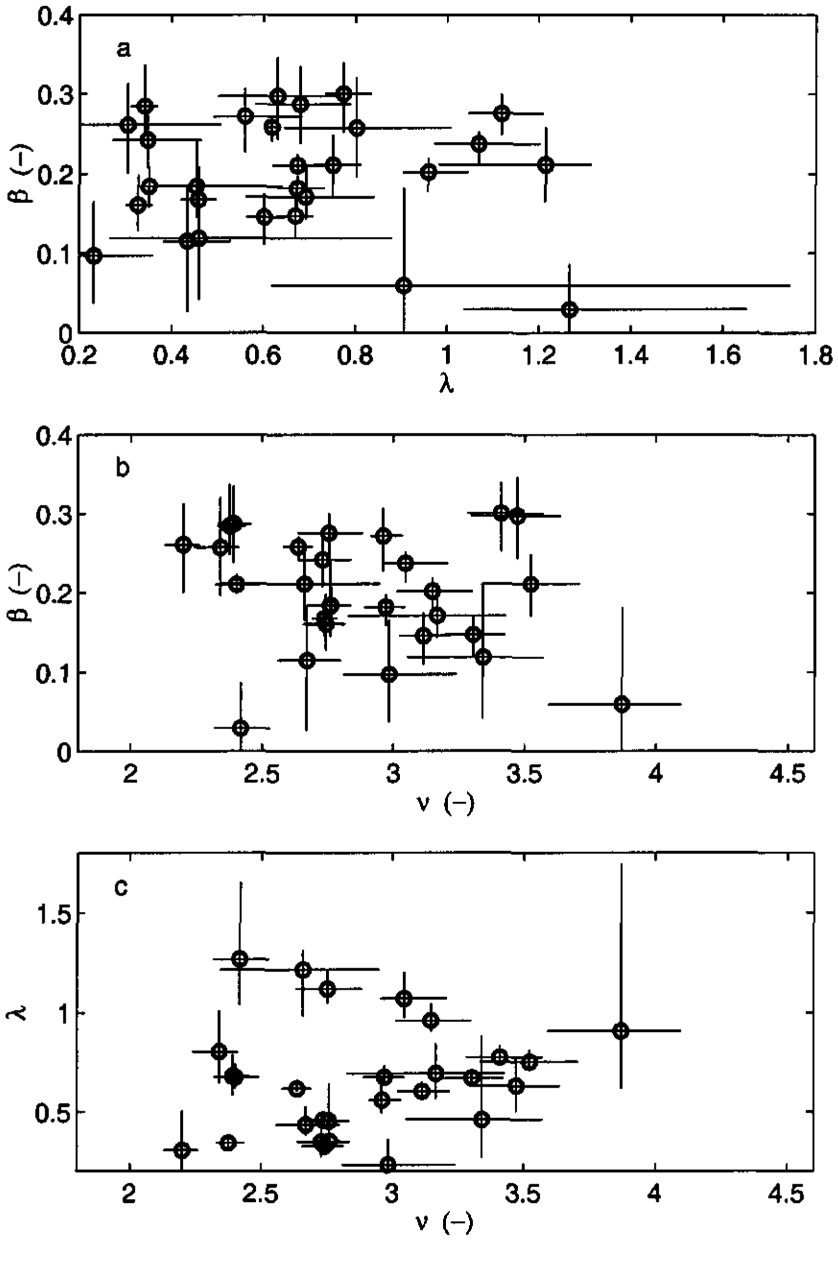


Figure 5.4: (a) Scaling exponent β (–) versus the parameter λ ($\mathrm{mm}^{-\nu}\,(\mathrm{mm\,h^{-1}})^{\beta\nu}$) of the Best (1950b) $g\,(x)$-parameterization for 28 rainfall events in 1968 and 1969 in De Bilt, The Netherlands (based on data reported by Wessels (1972)). Error bars indicate 68% confidence limits, estimated from 1000 bootstrap samples in each case. (b) Idem for β (–) and ν (–). (c) Idem for λ ($\mathrm{mm}^{-\nu}\,(\mathrm{mm\,h^{-1}})^{\beta\nu}$) and ν (–).

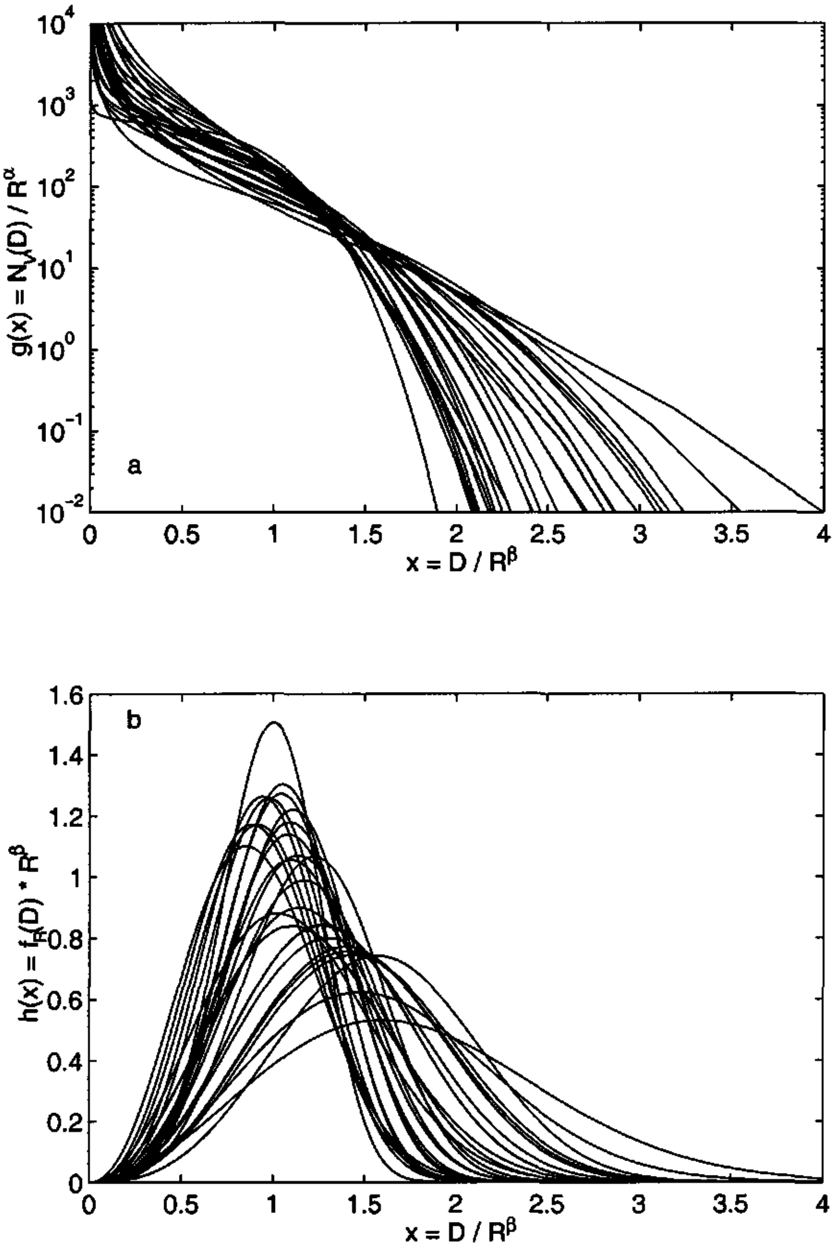


Figure 5.5: (a) Self-consistent general raindrop size distribution functions $g(x)$ ($\mathrm{mm}^{-1}\,\mathrm{m}^{-3}\,(\mathrm{mm\,h}^{-1})^{-\alpha}$), where x ($\mathrm{mm}\,(\mathrm{mm\,h}^{-1})^{-\beta}$) is the scaled raindrop diameter and $\alpha = 1 - 4.67\beta$, for 28 rainfall events in 1968 and 1969 in De Bilt, The Netherlands (based on data reported by Wessels (1972)). (b) Corresponding self-consistent general rain rate density functions $h(x)$ ($\mathrm{mm}^{-1}\,(\mathrm{mm\,h}^{-1})^{\beta}$).

other to convective events (Salles et al., 1999). Nevertheless, although part of the observed variability in the parameters will be related to sampling fluctuations (as indicated by the lengths of the error bars), just as in Fig. 5.3 there seem to remain physically significant differences between the rainfall events.

The corresponding parameter values based on the global estimates for A, p and n have been plotted in Fig. 4.4 (p. 116). The error bars associated with the Dutch data point represent again 99% confidence intervals estimated from 1000 bootstrap samples. A comparison of Fig. 5.4 with Fig. 4.4 shows that although the inter-event variability in the value of the scaling exponent β is larger than the climatological variability, this is less true for the parameters λ and ν (although it makes quite a difference whether or not the data point corresponding to Hilo, Hawaii in Fig. 4.4 is taken into account). Obviously, the sampling variability associated with the climatological parameter estimates will be less strong.

Fig. 5.5(a) and (b) shows what these parameter values mean in terms of the general raindrop size distribution functions and the general rain rate density functions for the 28 rainfall events. When this figure is compared to Fig. 4.5 (p. 117), which shows the global functions for De Bilt, then it is clear that there exists a strong inter-event variability. Since this is more than just sampling variability, it would be interesting to investigate to what extent the scaling exponents and the parameters of the general raindrop size distribution functions are related to certain meteorological quantities. Indeed, Wessels (1972) has investigated the possible relations between on the one hand the parameters A, p and n of Best's parameterization and on the other hand:

1. rainfall type (drizzle, widespread rain, showers, thunderstorm);
2. synoptic weather type (warm front, cold front, etc.);
3. atmospheric stability;
4. height of the 0°C isotherm (which is related to the precipitation formation process);
5. mean relative humidity during rainfall (which might indicate the possible evaporation of small droplets);
6. mean wind speed at 10 m height (which might indicate possible size sorting effects).

However, quite disappointingly, Wessels has not found any relation between these quantities and the values of the parameters A, p and n. Since p equals the scaling exponent β and A and n determine the shape of the general functions $g(x)$ and $h(x)$, this would imply that the observed event-to-event variability in the scaling exponents and the general functions would have no relation whatsoever with any meteorological variability. This seems hard to believe and is therefore a subject which needs further investigation. Perhaps other types of meteorological quantities need to be considered, such as those which can be derived from radar data (e.g. Steiner et al., 1995). Sempere Torres et al. (1999) show some preliminary but nevertheless promising results in this direction. They have been able to relate the shape of the general raindrop size distribution function to the type of rainfall (convective, stratiform, transition) using a pre-classification based on volume-scan radar data.

5.4 Climatological analysis

Although some preliminary global results have already been presented, they have been based on Wessels' adjustments of Best's parameterization to the Dutch rainfall data and their main purpose has been to provide a comparison with the climatological parameterizations proposed by Best (1950b) for various other locations (Chapter 4). In this section, a climatological analysis of the Dutch rainfall data will be carried out based on the *raw* raindrop size distributions. To provide a comparison with the analyses presented in Chapter 4 for Laws and Parsons' (1943) *average* raindrop size distributions, again two analyses will be presented. The first analysis will be a normalization procedure based on Best's parameterization and the second a general analysis, independent of any a priori assumption regarding the form of the raindrop size distribution, based on the scaling law formulation. The analyses are based on the 446 empirical raindrop size distributions corresponding to rain rates exceeding 0.1 $\mathrm{mm\,h^{-1}}$.

5.4.1 Normalization on the basis of Best's parameterization

Fig. 5.6 shows the resulting normalized forms of the cumulative liquid rainwater content distribution $F_W(D)$ and the raindrop size distribution $N_V(D)$. These have been obtained using the values of the scale parameter a (the $1 - e^{-1} \approx 63\%$ quantile of $F_W(D)$) and the liquid rainwater content W calculated for each raindrop size distribution separately. Hence, this is a two-parameter normalization, similar to that used by Sekhon and Srivastava (1971). A comparison with Fig. 4.8 (p. 123) demonstrates clearly that the normalization for the Dutch data works less satisfactory than for Laws and Parsons' data. However, this is not surprising given the fact that the former are raw raindrop size distributions and the latter mean raindrop size distributions for different classes of rain rate. This immediately shows two general advantages of applying normalizations to raw data: (1) it avoids the subjective grouping of the data into classes of rain rate by using all available data at once; (2) it reveals much more clearly the limitations of any parameterization for the raindrop size distribution (i.e. it does not hide any of the variability present in the data).

In accordance with Best's parameterization (Eq. (4.4), p. 103), the slope of the (linear) regression line in Fig. 5.6(a) is an estimate for the parameter n. This yields $n = 2.65$ ($r^2 = 0.94$ on a logarithmic scale), which is quite close to the value estimated in Section 5.3 using Wessels' (1972) fits of n to the individual distributions (2.75). The resulting cumulative liquid rainwater distribution $F_W(D)$ (Eq. (4.1)) is given on a linear scale in Fig. 5.6(b) ($r^2 = 0.98$). Fig. 5.6(c) shows the corresponding normalized raindrop size distribution (Eq. (4.32), p. 125) ($r^2 = 0.87$ on a logarithmic scale). The fact that there is no scatter at $D/a = 1$ in Fig. 5.6(a) and (b) is because a has been

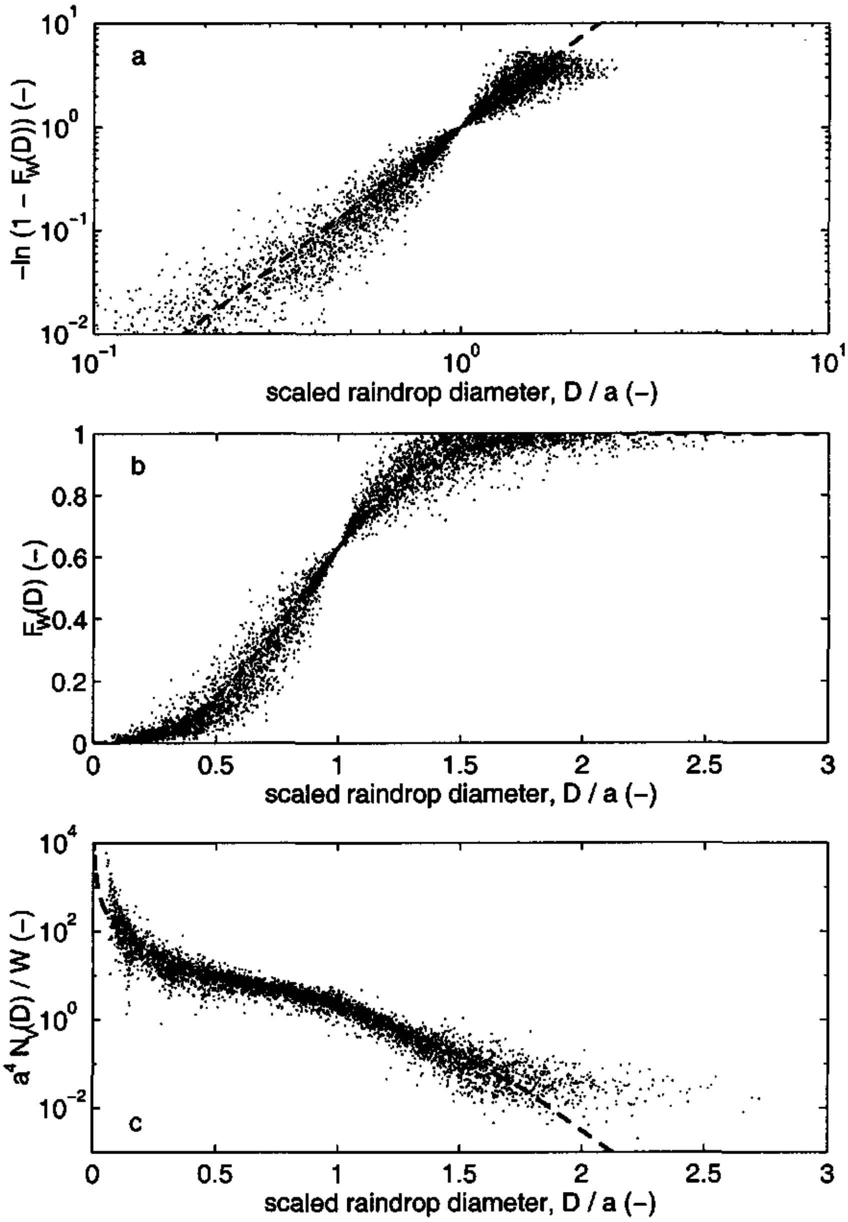


Figure 5.6: (a) Transformed cumulative liquid rainwater content distributions $-\ln[1 - F_W(D, R)]$ (–) corresponding to Wessels' (1972) data, normalized using a (mm), the $1 - e^{-1} \approx 63\%$ quantile of $F_W(D, R)$. (b) Corresponding normalized $F_W(D, R)$–curve. (c) Corresponding dimensionless raindrop size distribution $a^4 N_V(D, R)\,/W$, normalized using the liquid rainwater content W (here in $\mathrm{mm^3\,m^{-3}}$). Dashed lines indicate the adjustment of Best's (1950b) parameterization to the data.

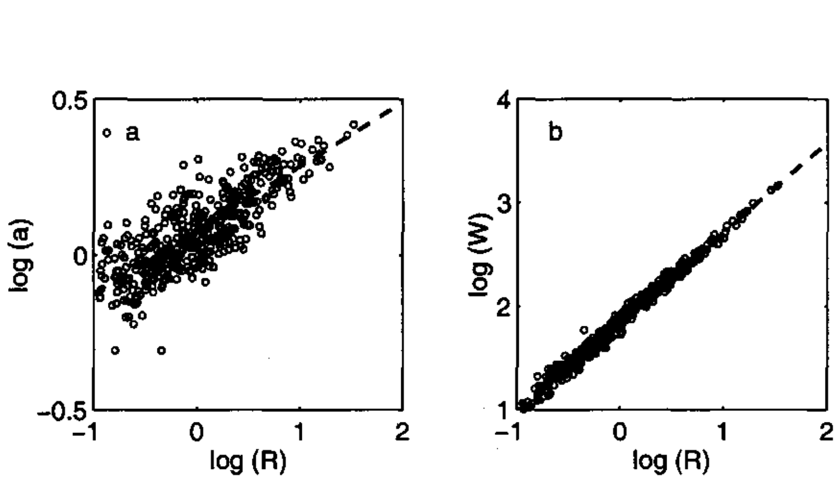


Figure 5.7: (a) Scatterplot of a (mm) versus R (mm h^{-1}) for 446 empirical raindrop size distributions corresponding to rain rates exceeding 0.1 mm h^{-1} collected in 1968 and 1969 in De Bilt, The Netherlands (Wessels, 1972), and power law relationship $a = 1.20R^{0.204}$ (dashed line) adjusted using nonlinear regression ($r^2 = 0.657$). (b) Idem for W (mg m^{-3}) versus R, with power law relationship $W = 70.0R^{0.869}$ ($r^2 = 0.992$).

calculated as the 63% quantile of $F_W(D)$ for each empirical raindrop size distribution separately. In a similar manner, the liquid rainwater contents W used to obtain the normalization in 5.6(c) have been determined for each distribution separately.

One step in the direction of the scaling law formulation is to perform the normalization not using a and W determined for each distribution separately, but on the basis of power law a–R and W–R relationships. The two-parameter normalization then reduces to a one-parameter normalization. Fig. 5.7 shows the empirical power law relationships and corresponding regression lines for the 446 raindrop size distributions considered here. The coefficients (adjusted using nonlinear (power law) regression) are $A = 1.20$ and $p = 0.204$ for the a–R relationship ($r^2 = 0.66$) and $C = 70.0$ and $r = 0.869$ for the W–R relationship ($r^2 = 0.99$). These values are close to those estimated in Section 5.3 using Wessels' (1972) values of a, R and W for the individual distributions. Although the W–R relationship provides an almost perfect fit, the a–R relationship does not. This is obviously going to affect the normalization results.

Fig. 5.8 shows the corresponding normalizations. Comparing these with those given in Fig. 5.6 shows immediately that the amount of scatter about the adjusted parameterizations has increased significantly. This is not surprising in view of the fact that one rainfall related variable R will obviously be able to explain a smaller fraction of the natural variability than two rainfall related variables (a and W). Nevertheless, the use of only one variable as explanatory variable (i.e. reference variable) has been the starting point of any of the parameterizations for the raindrop size distribution which have been encountered before (Marshall-Palmer, Best, Laws-Parsons). Moreover, it is the basis of any of the ubiquitous power law relationships of radar meteorology. That one variable is not able to explain all spatial and temporal vari-

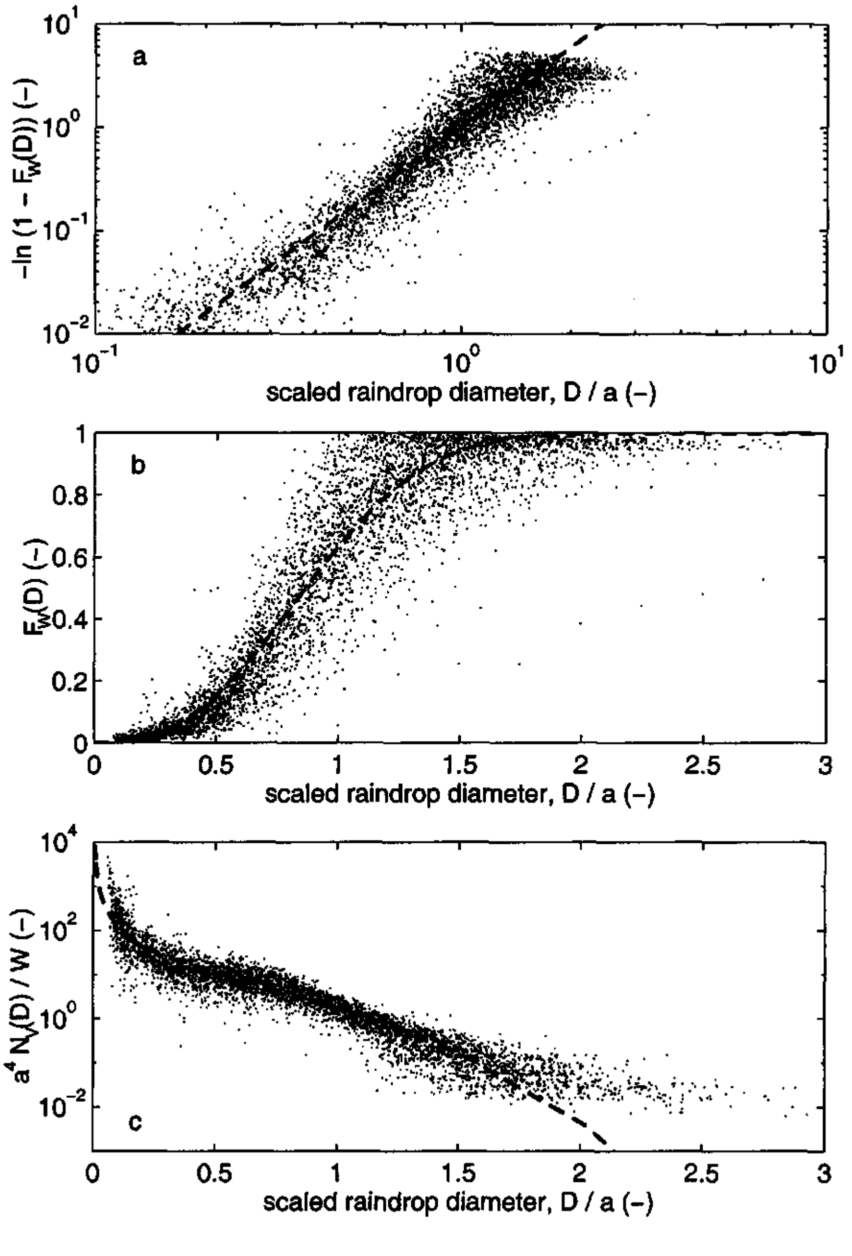


Figure 5.8: Same as Fig. 5.6, only the scaling of the diameters and raindrop size distributions is not performed using the values of a and W for each individual distribution, but those obtained using global power law a–R and W–R relationships adjusted to the entire dataset (Fig. 5.7).

ability of the raindrop size distribution has been used as an argument in favor of multi-parameter radar (e.g. Atlas et al., 1984).

Nevertheless, as can be seen from Fig. 5.8(a)–(c), the rain rate R is still able to explain a significant amount of the variability present in the data. The slope of the linear regression line in Fig. 5.8(a) now yields a value of 2.58 for n ($r^2 = 0.89$), slightly lower than that obtained from Fig. 5.6(a). The coefficients of determination r^2 for the adjusted parameterizations in Fig. 5.8(b) and (c) are 0.91 and 0.76, respectively, both reduced with respect to 5.6(b) and (c). The fact that a is now a function of R and has not been determined for each distribution separately, as in Fig. 5.6(a) and (b), makes that the scatter is no longer absent in Fig. 5.8(a) and (b) for $D/a = 1$. The estimated values of the parameters A, p, C, r and n can be used to calculate the corresponding scaling exponents and self-consistency coefficients from Eqs. (4.19)–(4.21) (p. 109). This yields $\alpha = 0.053$, $\beta = 0.204$, $S_e = 1.01$ and $S_p = 0.98$. These values are close to those resulting from the analysis in Section 5.3 (Table 4.2, p. 114).

In the next two sections, the procedures developed in Chapter 3 to estimate the scaling exponents α and β and identify the general raindrop size distribution function $g(x)$ and the associated general rain rate density function $h(x)$ will be applied to the Dutch rainfall data. Recall that the main advantage of the scaling law approach over an approach such as that treated in this section is that it is no longer necessary to impose a particular a priori functional form for the raindrop size distribution.

5.4.2 Estimation of the scaling exponents

Entirely analogous to the analysis of Laws and Parsons' (1943) data presented in Chapter 4 (Section 4.3), two methods to estimate the scaling exponents will be applied to the Dutch rainfall data. The first is based on power law relationships between the moments Ω_m of the raindrop size distribution and a reference variable Ψ (again the rain rate R), the second on power law relationships between the weighted mean raindrop diameters $\overline{D}_m$ and R. Fig. 5.9 shows log-log plots of the first six integer moments of the 446 raindrop size distributions considered here versus the corresponding rain rates R. The regression lines indicated in the figure have been adjusted using linear least-squares regression on the logarithmic values[4]. There are two aspects which draw the attention: (1) in correspondence with Fig. 4.9 (p. 127), there is a clear tendency of the slopes of the regression lines to increase with the order of the moment m; (2) in contrast to Fig. 4.9, there is an appreciable amount of scatter about these regression lines. That there is virtually no scatter about the regression lines in Fig. 4.9 is because Laws and Parsons' data represent *average* raindrop size distributions for different classes of rain rate. The analysis in Fig. 5.9, however, is performed on raw raindrop size distributions.

There is a well-defined tendency in the amounts of scatter about the regression lines as well, both visually and as indicated by the r^2-values. The further the order of a moment is away from 3.67 (that corresponding to R), the more pronounced is the

[4]Although nonlinear (power law) regression seems preferable in this context, convergence problems for the moments of orders 0–1 have been encountered. In order to obtain consistent results, linear regression on the logarithmic values has therefore been applied to *all* moments.

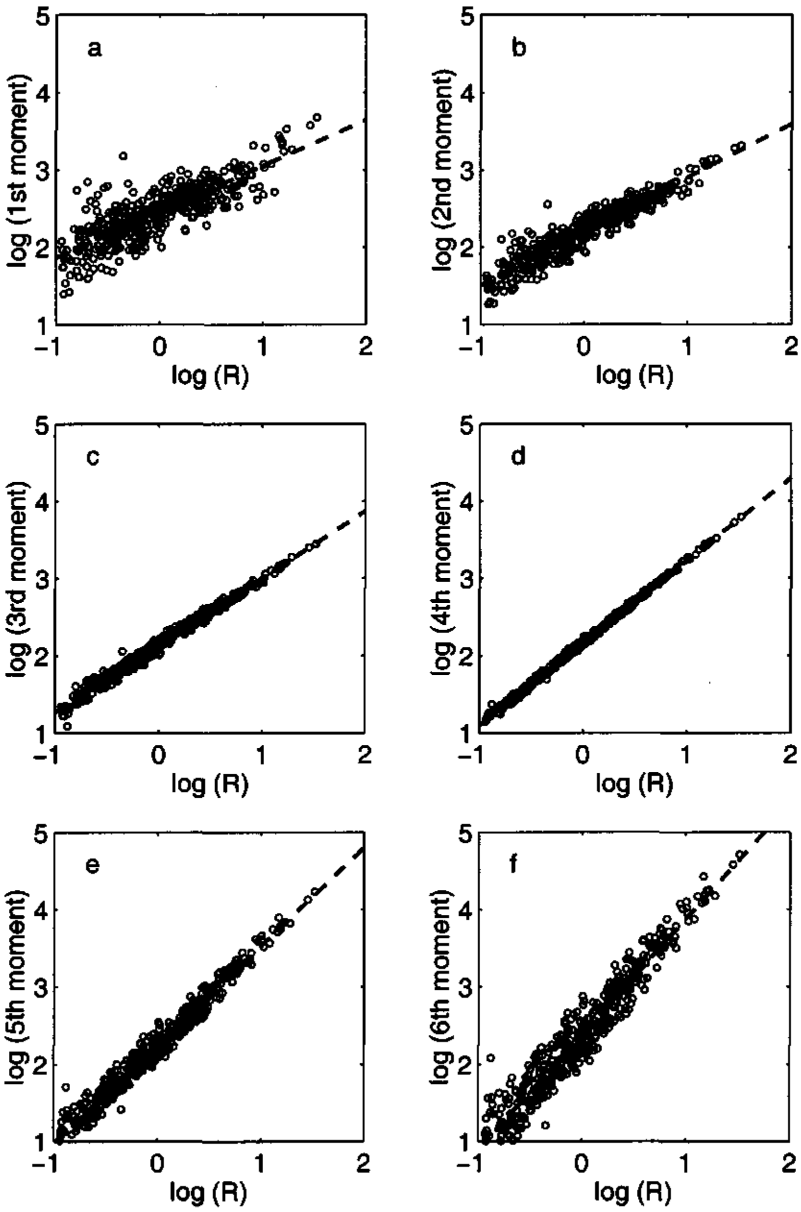


Figure 5.9: Log-log plots of the first six integer moments and the corresponding rain rates for 446 raindrop size distributions collected in 1968 and 1969 in De Bilt, The Netherlands (Wessels, 1972) ((a)-(f): 1st-6th moments). Dashed lines indicate power law relationships adjusted using linear regression on the logarithmic values ($r^2 = 0.669$, 0.880, 0.986, 0.998, 0.970 and 0.930, respectively).

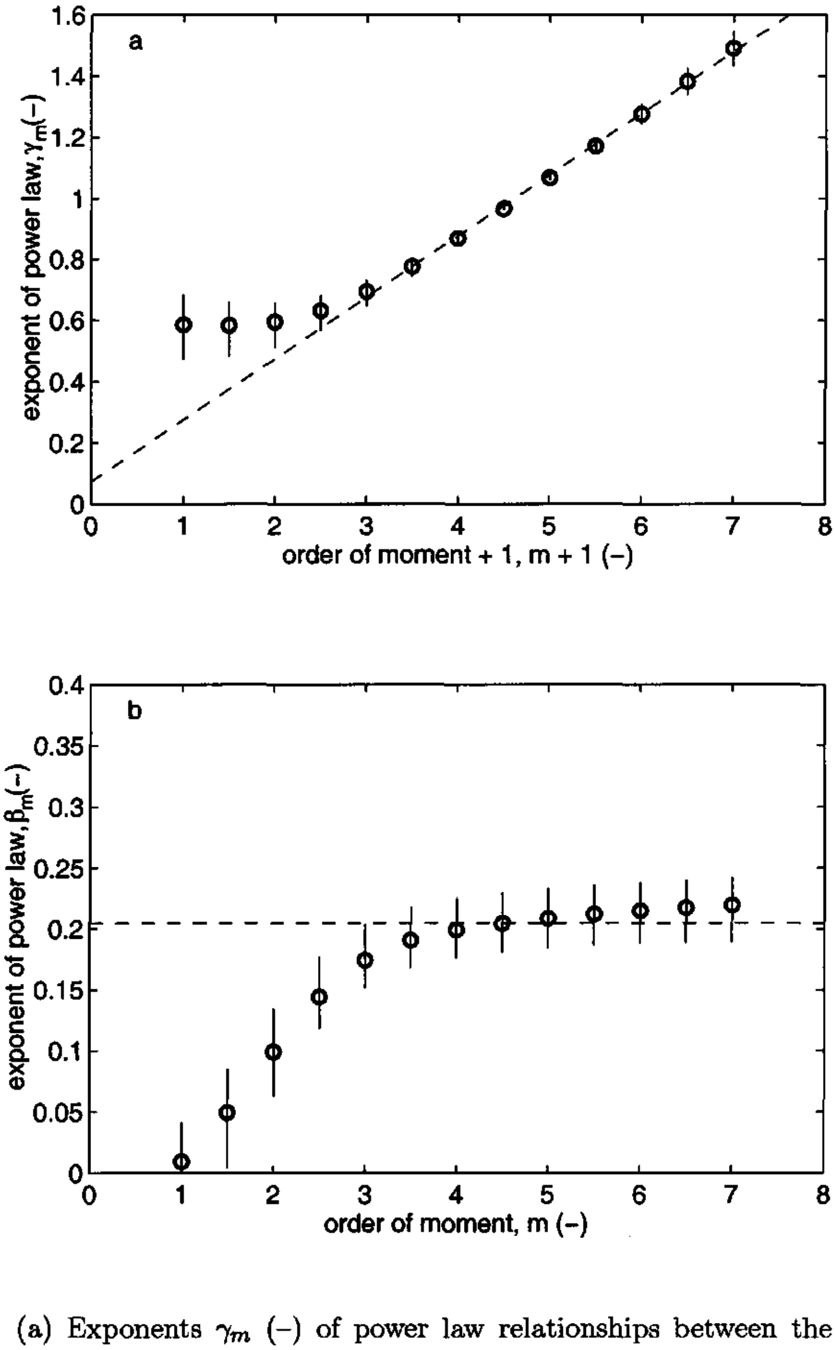


Figure 5.10: (a) Exponents γ_m (–) of power law relationships between the moments of Wessels' (1972) 446 raindrop size distributions and the corresponding rain rates versus the orders of the moments m plus one. Dashed line indicates linear regression between γ_m and $m + 1$ for $m \geq 2$. Error bars indicate 99% confidence limits on the exponents, estimated from 250 bootstrap samples. (b) Idem for the exponents β_m (–) of power law relationships between the weighted mean raindrop diameters and the corresponding rain rates.

scatter. For example, for the fourth moment (Fig. 5.9(d)), there is hardly any scatter, whereas for the first and the sixth moments (Fig. 5.9(a) and (f)) there are appreciable amounts of scatter. This simply indicates that the further the order of a moment is away from that corresponding to the rain rate R, the less able is R to explain the variability associated with that moment. This observation indicates a fundamental limitation of the scaling law formulation, and as such of any raindrop size distribution parameterization based on only one explanatory rainfall variable (reference variable). Fig. 5.9 clearly shows that the power laws between rainfall related variables in general are *statistical* in nature, not *deterministic*.

Fig. 5.10(a) shows a plot of the slopes γ_m of the regression lines of Fig. 5.9 versus the orders of the corresponding moments m plus one. As an indication for the (sampling) uncertainty associated with each point, error bars representing estimates of the 99% confidence intervals on the exponents (obtained from 250 bootstrap samples) have been included. It is clear that the uncertainty in the exponents of orders 2–5 is negligible, but that for lower orders (0–2) and higher orders (5–6) it becomes appreciable. Moreover, the exponents of orders less than two deviate from the straight line behavior predicted by the scaling law formulation (in much the same way as the circles in Fig. 4.10(a), p. 128). Therefore, those of orders 2–6 have been employed in a linear regression of γ_m on $(m+1)$. The resulting intercept α is 0.072 and the corresponding slope β is 0.201 ($r^2 = 1.00$). For the self-consistency coefficient of the exponent S_e (Eq. (3.87), p. 88) this yields 1.01, implying an almost perfect consistency. As an indication of the (sampling) uncertainties in these coefficients, 99% confidence limits have been estimated on the basis of 250 bootstrap samples. For α these limits are -0.014 and 0.171, for β they are 0.179 and 0.219. As noted before, the uncertainty associated with α is appreciably larger than that associated with β.

Fig. 5.10(b) shows a plot of the exponents β_m of power law relationships between the weighted mean raindrop diameters $\overline{D}_m$ and the rain rate R versus the order of the moment m. The exponents have again been obtained using linear regression on the logarithmic values. The error bars indicate 99% confidence limits estimated from 250 bootstrap samples. In this case the uncertainty remains significant for moments of all orders, although it is stronger for the lower order moments. For these moments, there is also a clear deviation from the horizontal straight line behavior predicted by the scaling law formulation. Qualitatively, the effect is the same as that which has been observed in Fig. 4.10(b) (p. 128), but in this case it is much stronger. For instance, the data point for $m = 1$ indicates the virtual absence of correlation between the natural logarithm of the mean raindrop diameters and that of the rain rate. This is likely to be associated with sampling effects, e.g. underestimation of the number of small drops. A reliable estimate of the scaling exponent β can be obtained by taking the mean of the exponents β_m for $m \geq 3$. This yields $\beta = 0.205$, with a corresponding self-consistent value of α equal to 0.043. This value of β is approximately the same as that estimated from Fig. 5.10(a) (0.201). The associated 99% confidence limits estimated from 250 bootstrap samples are now 0.182 and 0.223. It can be concluded that the values of the scaling exponents α and β for De Bilt, The Netherlands are close to those corresponding to the Marshall-Palmer distribution ($\alpha = 0$, $\beta = 0.21$).

5.4.3 Identification of the general raindrop size distribution function and the general rain rate density function

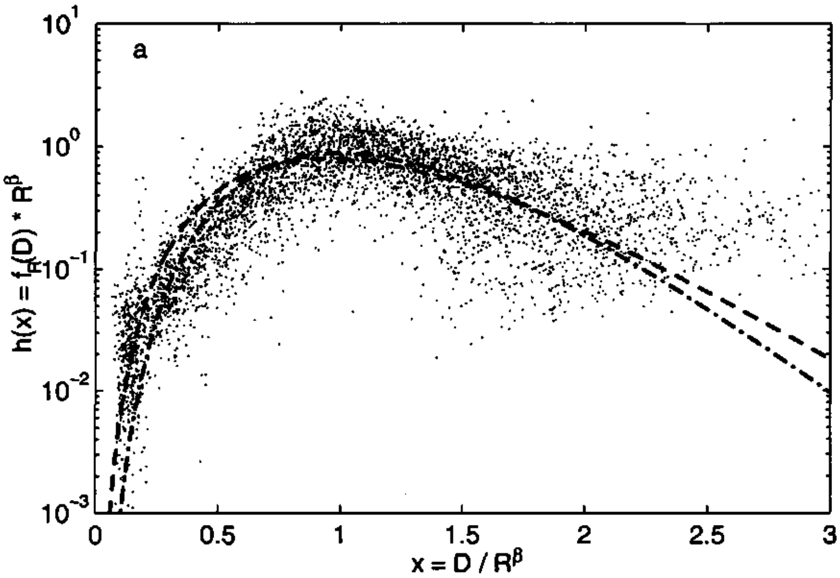
Fig. 5.11 shows the empirical general rain rate density function $h(x)$ which has been identified for the Dutch data by plotting the scaled rain rate density functions $R^{\beta} f_R(D)$ corresponding to each of the 446 measured raindrop size distributions versus the scaled raindrop diameters $R^{-\beta}D$ (for $\beta = 0.201$). The obtained scaling is far from perfect, i.e. there remains a significant amount of scatter in the data points. This indicates that the rain rate alone is not able to explain all variability in the experimental data. Nevertheless, there is still a relatively clear unimodal probability density function discernible. Its empirical moments will be used to estimate the parameters of various analytical functions which will be adjusted to the data.

The experimental data points in Fig. 5.11 have been used to estimate the sample moments of orders 1–4 from Eq. (4.33) (p. 132). Using the expressions given in Table 3.2, these have been employed to estimate the coefficients of variation, skewness and kurtosis of the empirical general rain rate density function. Table 5.1 summarizes the statistics estimated from the empirical $h(x)$. For comparison, the corresponding statistics have been calculated directly from the 446 empirical rain rate density functions as well. In contrast to what has been found for Laws and Parsons' (1943) data (Table 4.4, p. 133), there are significant differences between the dimensionless coefficients of variation, skewness and kurtosis estimated from the empirical $h(x)$ and the means of those estimated directly from the original data, particularly for CS_x and CK_x. This is a result of (1) the temporal variability of these coefficients from one experimental raindrop size distribution to the next and (2) the particular weighting which is implicitly involved in calculating them from the empirical $h(x)$ (Footnote 11, p. 132).

As a matter of fact, a basic assumption in the derivation of the scaling law has been that all spatial and temporal variability arises either as a result of fluctuations in the raindrop concentration or as a result of fluctuations in the characteristic raindrop diameters (or due to a combination of these two). The dimensionless shape coefficients are assumed to remain constant. That this is not the case for the Dutch data is shown in Fig. 5.12, which plots these coefficients against each other for the 446 empirical raindrop size distributions considered here. The lines in this figure indicate the theoretical relationships between these coefficients for the gamma, Best and lognormal distributions (given in Tables 3.3–3.4). Note that a significant amount of the variability and dependence between the empirical coefficients may be due to sampling fluctuations[5]. It is therefore difficult to attach a meaning to the (lack of) correspondence between the 'clouds' of data points and the theoretical relationships. They serve here mainly to indicate the appreciable amount of variability which exists in these coefficients from one experimental raindrop size distribution to the next.

Table 5.2 summarizes the parameters of four different analytical distributions estimated from the sample values of μ_x and CV_x ($\mu_{\ln x}$ and $\sigma_{\ln x}$ in case of the lognormal

[5]For instance, the coefficient of kurtosis (peakedness) of a rain rate density function $f_R(D)$ is a function of its 4th moment, which in turn is proportional to the 7.67th moment of the corresponding raindrop size distribution $N_V(D)$.



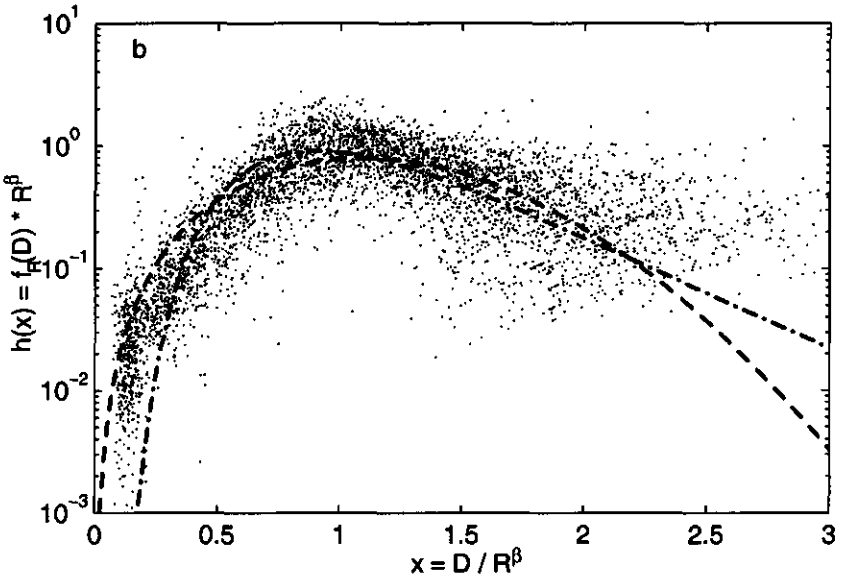


Figure 5.11: Identified general rain rate density function $h(x)$ for Wessels' (1972) data (dots) and adjusted theoretical parameterizations: (a) exponential $g(x)$ (dashed line; $r^2 = 0.455$) and gamma $g(x)$ (dash-dotted line; $r^2 = 0.502$); (b) Best $g(x)$ (dashed line; $r^2 = 0.474$) and lognormal $g(x)$ (dash-dotted line; $r^2 = 0.471$).

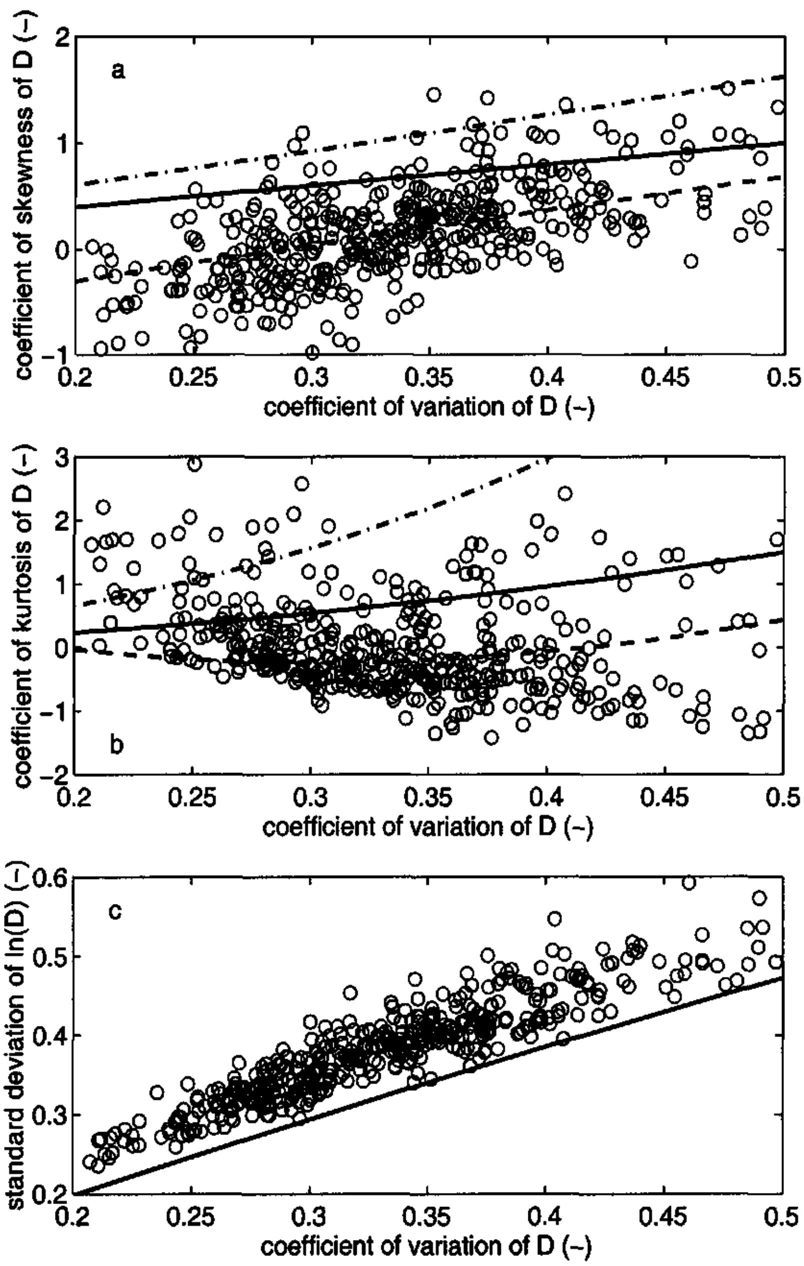


Figure 5.12: Relationships between the dimensionless shape coefficients of 446 empirical rain rate density functions corresponding to rain rates exceeding 0.1 mm h^{-1} collected in 1968 and 1969 in De Bilt, The Netherlands (Wessels, 1972), and comparison with theoretical relationships. (a) CV_D versus CS_D (solid line: gamma; dashed line: Best; dash-dotted line: lognormal). (b) CV_D versus CK_D (idem). (c) CV_D versus $\sigma(\ln D)$ (solid line: lognormal). Note that all coefficients pertain to the scaled raindrop diameters x as well.

Table 5.1: Mean (μ_x) of the scaled raindrop diameters ($x = R^{-\beta}D$, where D in mm and R in mm h^{-1}) with respect to the general rain rate density function $h(x)$, corresponding coefficients of variation (CV_x), skewness (CS_x) and kurtosis (CK_x) and mean ($\mu_{\ln x}$) and standard deviation ($\sigma_{\ln x}$) of $\ln x = \ln D - \beta \ln R$ for Wessels' (1972) data. Values in the column labeled 'From empirical $h(x)$' have been obtained from the scaled rain rate density function, those in the columns labeled 'From original data' are the means and standard deviations of the values for the 446 original distributions. The value of μ_x in the latter is the prefactor of a power law regression of μ_D on R and the corresponding value of $\mu_{\ln x}$ is the intercept of a linear regression of $\mu_{\ln D}$ on $\ln R$.

Parameter	From empirical $h(x)$	From original data	
		Mean	Standard deviation
μ_x	1.20	1.18	–
CV_x	0.410	0.334	0.061
CS_x	0.952	0.143	0.456
CK_x	1.85	0.0199	0.794
$\mu_{\ln x}$	0.0905	0.0947	–
$\sigma_{\ln x}$	0.442	0.384	0.064

distribution) and the theoretical expressions given in Tables 3.2–3.4 (method of moments). As a measure of the uncertainty in these parameter estimates, 99% confidence limits estimated from 250 bootstrap samples are provided as well. Note that the parameter estimates obtained for the Best parameterization are different from those reported in Table 4.3 (p. 115). The latter have been based on Wessels' (1972) adjustments of Best's parameterization to the raw raindrop size distributions and not on the raw distributions themselves. The value for ν obtained here happens to be exactly the mean value for ν proposed by Best (1950b).

Table 5.3 compares the values of the coefficients of variation, skewness and kurtosis implied by the estimated parameters with the values obtained directly from the empirical $h(x)$. The appreciable differences between the theoretical and the empirical values of CS_x and CK_x indicate that the method of moments does not guarantee a good adjustment to the overall shape of the distribution. It merely equates the first two sample moments (only the first in case of the exponential $g(x)$–parameterization) with the first two theoretical moments. Fig. 5.11 shows how well the four analytical expressions defined by the parameters in Table 5.2 describe the experimental data. It is hard to make any judgment based on a visual inspection of this figure, but it seems clear that the lognormal distribution provides a rather poor fit, particularly for the smaller scaled raindrop diameters.

Fig. 5.13 shows the empirical general raindrop size distribution function $g(x)$ for the Dutch data. It has been identified by plotting the scaled raindrop size distributions $R^{-\alpha}N_V(D)$ corresponding to each of the 446 measured raindrop size distributions versus the scaled raindrop diameters $R^{-\beta}D$ (for $\beta = 0.201$). In the same figure the four analytical expressions for $g(x)$ defined by the parameters of Table 5.2 have been plotted. Table 5.4 gives the coefficients of determination corresponding to the

Table 5.2: Parameters κ, λ, μ, ν and σ (where D in mm, $N_V(D,R)$ in $\mathrm{mm^{-1}\,m^{-3}}$ and R in $\mathrm{mm\,h^{-1}}$) of different self-consistent forms of the general raindrop size distribution function $g(x)$ and general rain rate density function $h(x)$ for Wessels' (1972) data. Values in parentheses for the exponential distribution represent the slope λ and corresponding self-consistent intercept $\ln\kappa$ of a linear regression of $\ln g(x)$ on x. Values in parentheses for the lognormal distribution represent estimates based on μ_x and CV_x instead of $\mu_{\ln x}$ and $\sigma_{\ln x}$. The 'mean' values represent the parameters estimated directly from the empirical $h(x)$. The 'minimum' and 'maximum' values indicate 99% confidence limits, estimated from 250 bootstrap samples.

Distribution type	Parameter	minimum	mean	maximum
Exponential	κ	4.93×10^3	5.51×10^3	6.23×10^3
		(2.44×10^3)	(2.86×10^3)	(3.41×10^3)
	λ	3.81	3.91	4.01
		(3.28)	(3.40)	(3.53)
Gamma	κ	8.11×10^3	1.81×10^4	2.98×10^4
	λ	4.26	4.99	5.44
	μ	0.51	1.29	1.72
Best	κ	208	222	240
	λ	0.664	0.703	0.743
	ν	2.05	2.25	2.34
Lognormal	κ	316	339	373
		(255)	(279)	(305)
	μ	−0.664	−0.626	−0.597
		(−0.541)	(−0.468)	(−0.428)
	σ	0.433	0.442	0.456
		(0.381)	(0.394)	(0.420)

Table 5.3: Coefficients of variation (CV_x), skewness (CS_x) and kurtosis (CK_x) implied by the parameters of the different distribution types adjusted to Wessels' (1972) data . That the values of CV_x for the gamma and Best distributions are the same and match the experimental value is because, in contrast with the exponential and lognormal distributions, CV_x has been used as a fitting parameter in the method of moments.

Parameter	From $h(x)$	Exponential	Gamma	Best	Lognormal
CV_x	0.410	0.463	0.410	0.410	0.464
CS_x	0.952	0.926	0.819	0.399	1.49
CK_x	1.85	1.28	1.01	−0.0341	4.21

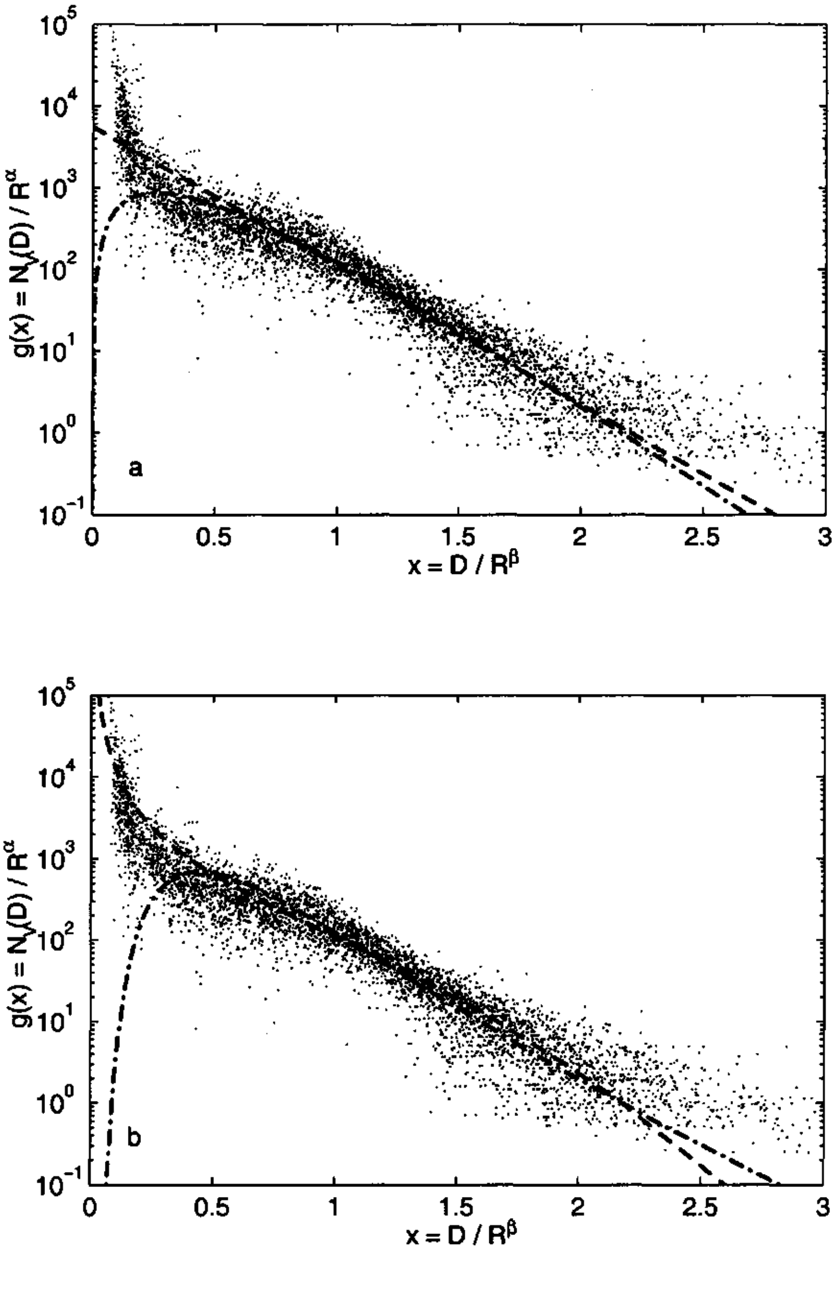


Figure 5.13: Identified general raindrop size distribution function $g(x)$ for Wessels' (1972) data (dots) and adjusted theoretical parameterizations: (a) exponential (dashed line; $r^2 = 0.871$) and gamma (dash-dotted line; $r^2 = 0.817$); (b) Best (dashed line; $r^2 = 0.825$) and lognormal (dash-dotted line; $r^2 = 0.390$).

Table 5.4: Goodness-of-fit, as quantified by the coefficient of determination (r^2), of different theoretical forms for the general raindrop size distribution function $g(x)$ and general rain rate density function $h(x)$ to Wessels' (1972) data. The values for $g(x)$ have been calculated on a logarithmic scale, those for $h(x)$ on a linear scale.

Distribution type	$g(x)$ (logarithmic r^2)	$h(x)$ (linear r^2)
Exponential	0.871	0.455
Gamma	0.817	0.502
Best	0.825	0.474
Lognormal	0.390	0.471

various analytical adjustments to the empirical $g(x)$ and $h(x)$. It is again the negative exponential function for $g(x)$ which provides the best adjustment, notwithstanding the fact that it has a parameter less than the other distributions. The behavior of the empirical $g(x)$ for small scaled raindrop diameters is similar to that found for Laws and Parsons' data (Fig. 4.12 and 4.13)[6].

Substitution of the estimated parameters (Table 5.2) in the expressions given in Tables 3.2–3.4 yields (with $\beta = 0.201$ and $\gamma = 0.67$) four different, but all self-consistent, climatological parameterizations for raindrop size distributions in The Netherlands. They are

$$N_V(D,R) = 5.51 \times 10^3 R^{0.0613} \exp\left(-3.91 R^{-0.201} D\right) \tag{5.3}$$

for the exponential parameterization,

$$N_V(D,R) = 1.81 \times 10^4 R^{-0.198} D^{1.29} \exp\left(-4.99 R^{-0.201} D\right) \tag{5.4}$$

for the gamma parameterization,

$$N_V(D,R) = 222 R^{0.413} D^{-1.75} \exp\left(-0.703 R^{-0.452} D^{2.25}\right) \tag{5.5}$$

for the Best parameterization and finally

$$N_V(D,R) = 339 R^{0.262} D^{-1} \exp\left[-2.56 \ln^2\left(1.87 R^{-0.201} D\right)\right] \tag{5.6}$$

for the lognormal parameterization. Note that the Best parameterization should again be truncated at some minimum diameter because $\nu < 3$. Which of these four parameterizations would be the most suitable in a given situation is something which is difficult to judge at this point. The exponential is probably a good candidate on the average.

[6]Rogers et al. (1991) report a similar behavior in a modeling study of the temporal evolution of raindrop size distributions in steady light rain.

5.5 Summary and conclusions

The scaling law formulation and its analysis procedures have been verified experimentally on the basis of raindrop size distributions collected with the filter paper technique at the Royal Netherlands Meteorological Institute in De Bilt, The Netherlands. Two types of analyses have been carried out: (1) an *event-to-event analysis* based on Wessels' (1972) adjustments of Best's parameterization to 476 raindrop size distributions for a series of 28 rainfall events; (2) a *climatological analysis* based on 446 raw raindrop size distributions. Both types of analysis have yielded satisfactory results in the sense that it has been possible to estimate the scaling exponents and identify the general raindrop size distribution function and the general rain rate density function.

Although re-sampling of the distributions according to the bootstrap method has indicated that there is an appreciable amount of uncertainty associated with the estimates of the scaling exponents and the parameters of the general functions for each of the 28 rainfall events, they closely satisfy the self-consistency constraints following from the scaling law formulation. Moreover, there seems to be more inter-event variability in the exponents and parameters than can be explained solely on the basis of sampling uncertainties. However, quite disappointingly, an effort to try to relate this variability to differences in various meteorological quantities (type of rainfall, synoptic weather type, atmospheric stability, height of the 0°C isotherm, relative humidity and wind speed) has failed. This suggests that these quantities are not appropriate indicators for the type of rainfall and that *one has to look for other manners to classify different rainfall regimes, perhaps based on the use of radar data.*

The climatological analysis based on the raw raindrop size distribution data has indicated that although there is an appreciable amount of scatter about the mean curves, it is still possible to obtain consistent estimates of the scaling exponents and reasonably accurate fits to the general raindrop size distribution function and general rain rate density function. As such, *this analysis confirms the power of the scaling law formulation as a manner to obtain climatological parameterizations for the raindrop size distribution.* Four different self-consistent parameterizations have been adjusted to the Dutch raindrop size distributions (exponential, gamma, Best and lognormal). The exponential parameterization seems to provide the best adjustment.

The remaining scatter about the mean parameterizations is an indication of the fact that not all observed variability can be explained by one single reference variable (in this case the rain rate). This should not be interpreted as a weak point of the scaling law in particular. The use of one single rainfall related variable as explanatory (reference) variable has formed the basis of all previously proposed parameterizations for the raindrop size distribution (Marshall-Palmer, Best, Laws-Parsons) and, moreover, of the ubiquitous power law relationships of radar meteorology. There is a remaining amount of variability associated in part with sampling fluctuations and in part with other sources of natural variability. This suggests that it would be useful to further *extend the scaling law formulation in such a manner that it would be able to cope with this excess variability.* A first approach could then be to recognize that the power law relationships between rainfall related variables are not deterministic in nature, but statistical. This leads to a statistical interpretation for the scaling exponents, as is demonstrated in Appendix E. A second approach could be the inclusion of an additional reference variable in the scaling law. In this manner, each rainfall related variable would become a function of two others. As a matter of fact, this type of approach has formed the basis of multi-parameter radar methods.

Chapter 6

Implications of the scaling law formulation for radar reflectivity – rain rate relationships[1]

6.1 Introduction

The most fundamental conversion in radar remote sensing of rainfall is that from radar reflectivity factor Z ($\text{mm}^6\,\text{m}^{-3}$) to rain rate R ($\text{mm}\,\text{h}^{-1}$). This is but one aspect of the much larger problem of the use of weather radar for the reliable and accurate estimation of the spatial and temporal distribution of rainfall over an area. The crucial step in tackling that problem is the conversion of (equivalent) radar reflectivities measured aloft to rain rates at the ground. In an ideal situation, i.e. one in which all other possible error sources are negligible, the main uncertainty in rainfall estimates by (conventional, i.e. single parameter) weather radar will be due to uncertainty in the Z–R relationship. In practice, this means a situation where a non-attenuated, pencil beam weather radar is observing nearby homogeneous rainfall close to the ground. In reality, these requirements are hardly ever met. Therefore, in any practical situation the uncertainty in the Z–R relationship will provide a lower bound to the uncertainties associated with radar rainfall estimation. That alone seems reason enough to merit a careful treatment.

Establishing radar reflectivity factor–rain rate relationships has captured the attention of radar meteorologists since the early days of weather radar more than five decades ago. From the point of view of instrumentation, there exist two approaches. Either they are established using a combination of weather radar and raingauge measurements (e.g. Wilson and Brandes, 1979) or they are established on the basis of measurements of raindrop size distributions (in the air or at the ground) (e.g. Marshall and Palmer, 1948). From the point of view of methodology there exist basically three approaches. The classical approach is that of a regression analysis to estimate the coefficients of power law Z–R relationships (e.g. Marshall and Palmer, 1948).

[1]Partly based on Uijlenhoet, R. (1999). Raindrop size distributions and radar reflectivity–rain rate relationships for radar hydrology. *Hydrol. Earth Syst. Sci.* (accepted for publication).

Statistically speaking, this amounts to approximating the expected value of R for a given value of Z (i.e. the conditional mean of R)[2].

A more recent approach, but one which has gained rapid recognition, is a nonparametric method called the Probability Matching Method. It has been re-introduced into the field of radar meteorology by Calheiros and Zawadzki (1987) and has been generalized more recently by Rosenfeld et al. (1993). The method amounts to matching the empirical (sample) cumulative distribution functions of Z and R. It has the advantage over regression methods that it no longer requires synchronous measurements (or rather estimates) of Z and R. There has recently been quite some discussion about the advantages and disadvantages of the two methods (regression and probability matching). The question does not yet seem to be entirely solved. The reader is referred to Krajewski and Smith (1991), Haddad and Rosenfeld (1997) and Rosenfeld and Amitai (1998) for discussions on various aspects of the problem.

A third approach is based on the explicit recognition of the fact that Z and R are related to each via the raindrop size distribution. Therefore, any parameterization for the raindrop size distribution with one explanatory (reference) variable *implies* a particular Z–R relationship. For the particular case of Marshall and Palmer's (1948) exponential parameterization this has already been demonstrated in Chapter 2 (Section 2.7, Eq. (2.65) and subsequent discussion)[3]. This is the approach which will be followed in this chapter. However, the treatment here will be much more general since it will be based on the scaling law for the raindrop size distribution presented in Chapter 3. Therefore, it is no longer necessary to make any a priori assumption regarding the functional form of the raindrop size distribution.

In Section 6.2 the implications of the scaling law formulation for the functional form of radar reflectivity–rain rate relationships will be presented. In Section 6.3 the resulting methodology will be used to derive such Z–R relationships from the various raindrop size distribution parameterizations which have been developed in Chapters 4 and 5 (i.e. those obtained from the data of Best (1950b), Laws and Parsons (1943) and Wessels (1972)). Battan's (1973) classical list of 69 empirical Z–R relationships will be revisited in the light of the scaling law formulation in Section 6.4 in an effort to obtain from them mean raindrop size distribution parameterizations for different types of rainfall. In Section 6.5 the methodology developed in the previous sections will be used to shed some light on a curious but widely used relationship between two parameters of the gamma raindrop size distribution established by Ulbrich (1983) on the basis of Battan's Z–R relationships. Finally, the summary and conclusions of this chapter will be discussed in Section 6.6.

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[2]Only in the particular case where Z and R are jointly lognormally distributed (i.e. $\log Z$ and $\log R$ have a bivariate normal distribution) the conditional mean of R given Z really reduces to the classical power law relationship.

[3]In their 1948 article, Marshall and Palmer provide *two* Z–R relationships, one ($Z = 220R^{1.60}$) based on a regression analysis of Z on R and another ($Z = 296R^{1.47}$) which follows analytically from their parameterization for the raindrop size distribution.

6.2 Radar reflectivity–rain rate relationships and the scaling law formulation

By definition, the radar reflectivity factor Z ($\mathrm{mm^6\,m^{-3}}$) is the sixth moment of the raindrop size distribution in a volume of air $N_V(D)$ ($\mathrm{mm^{-1}\,m^{-3}}$) (Table 2.3, p. 39). Or, in terms of the notation employed in Eq. (3.18) (p. 62), Z is Ω_6 with $c_Z = 1$. If the reference variable Ψ in Eq. (3.18) is taken to be the rain rate R then Eqs. (3.19)–(3.21) imply that Z must be related to R according to the power law

$$Z = C_Z R^{\gamma_Z} \tag{6.1}$$

with prefactor

$$C_Z = \int_0^\infty x^6 g(x)\,\mathrm{d}x \tag{6.2}$$

(where $x = R^{-\beta} D$ is the scaled raindrop diameter) and exponent

$$\gamma_Z = \alpha + 7\beta. \tag{6.3}$$

Since R is the reference variable, the self-consistency constraint on the scaling exponent α is given by $\alpha = 1 - (4 + \gamma)\,\beta$ (Eq. (3.38), p. 64), where γ is the exponent of the power law raindrop terminal fall speed–diameter relationship. Substitution of this constraint in Eq. (6.3) yields

$$\gamma_Z = 1 + (3 - \gamma)\,\beta, \tag{6.4}$$

which for $\gamma = 0.67$ (Atlas and Ulbrich, 1977) reduces to

$$\gamma_Z = 1 + 2.33\beta. \tag{6.5}$$

Hence, the prefactor of the Z–R relationship is simply the sixth moment of the general raindrop size distribution function $g(x)$ and the exponent is uniquely determined by the value of the scaling exponent β.

As a matter of fact, these relations have already been encountered before in a different context (Footnote 11, p. 72 and Eq. (3.95), p. 95). They demonstrate that the value of the exponent γ_Z of the Z–R relationship (and of any other power law relationship between rainfall related variables) is independent of the shape of the scaled raindrop size distribution. Information regarding the shape of the scaled raindrop size distribution is entirely contained in the prefactor C_Z. Using the definition of the general rain rate density function $h(x)$ in terms of the general raindrop size distribution function $g(x)$ (Eq. (3.66), p. 77), C_Z can also be written as

$$C_Z = \frac{10^4}{6\pi c}\int_0^\infty x^{3-\gamma} h(x)\,\mathrm{d}x \tag{6.6}$$

Specific expressions for the prefactor C_Z for the five analytical forms of $g(x)$ and $h(x)$ presented in Tables 3.2–3.4 (p. 79–81) (the exponential, gamma, generalized gamma, Best and lognormal forms) can now be obtained by substituting these functions in Eqs. (6.2) or (6.6). Table 6.1 summarizes the results. This table can be seen as an extension of Tables 3.2–3.4.

Table 6.1: Theoretical expressions for the prefactors C_Z of power law Z–R relationships (where D in mm, $N_V(D,R)$ in $\mathrm{mm^{-1}\,m^{-3}}$, R in $\mathrm{mm\,h^{-1}}$, $v(D) = cD^\gamma$ in $\mathrm{m\,s^{-1}}$ and Z in $\mathrm{mm^6\,m^{-3}}$) for different self-consistent forms of the general raindrop size distribution function $g(x)$ (where x is the scaled raindrop diameter $R^{-\beta}D$) or the corresponding general rain rate density function $h(x)$.

	$g(x)$	C_Z
Definition	$N_V(x,1)$	$\int_0^\infty x^6 g(x)\mathrm{d}x = \frac{10^4}{6\pi c}\int_0^\infty x^{3-\gamma}h(x)\mathrm{d}x$
Exponential	$\kappa\exp(-\lambda x)$	$\frac{10^4}{6\pi c}\frac{\Gamma(7)}{\lambda^{3-\gamma}\Gamma(4+\gamma)}$
Gamma	$\kappa x^\mu \exp(-\lambda x)$	$\frac{10^4}{6\pi c}\frac{\Gamma(7+\mu)}{\lambda^{3-\gamma}\Gamma(4+\gamma+\mu)}$
Generalized gamma	$\kappa x^\mu \exp(-\lambda x^\nu)$	$\frac{10^4}{6\pi c}\frac{\Gamma[(7+\mu)/\nu]}{\lambda^{(3-\gamma)/\nu}\Gamma[(4+\gamma+\mu)/\nu]}$
Best	$\kappa x^{\nu-4}\exp(-\lambda x^\nu)$	$\frac{10^4}{6\pi c}\frac{\Gamma(1+3/\nu)}{\lambda^{(3-\gamma)/\nu}\Gamma(1+\gamma/\nu)}$
Lognormal	$\kappa x^{-1}\exp\left[-\frac{1}{2}\left(\frac{\ln x-\mu}{\sigma}\right)^2\right]$	$\frac{10^4}{6\pi c}\exp\left[(3-\gamma)\mu + \frac{1}{2}(3-\gamma)(9+\gamma)\sigma^2\right]$

6.3 Radar reflectivity–rain rate relationships from raindrop size distribution parameterizations

6.3.1 Best's data

On the basis of the values of the parameters of the self-consistent parameterizations for the raindrop size distribution derived in Chapters 4 and 5, specific Z–R relationships are easily obtained. For instance, the values of the scaling exponents β in Table 4.2 (p. 114) and those of the parameters of the self-consistent form of the Best general raindrop size distribution function in Table 4.3 (p. 115) yield the Z–R coefficients listed in Table 6.2. These values differ little from those derived by Best (1950b) himself for the various locations (Table VIII on p. 32 of his article). However, his manner of derivation does not guarantee self-consistency, whereas the scaling law approach does.

Fig. 6.1(a) shows a plot of the exponents γ_Z versus the prefactors C_Z. The two reference lines in the figure correspond to the Marshall-Palmer Z–R relationship ($Z = 200R^{1.6}$) and the Z–R relationship consistent with the Marshall-Palmer (1948) raindrop size distribution and the Atlas and Ulbrich (1977) raindrop terminal fall speed parameterization ($Z = 237R^{1.50}$). Fig. 6.1(b) shows the coefficients $C_R = (1/C_Z)^{1/\gamma_Z}$ and $\gamma_R = 1/\gamma_Z$ of the corresponding power law R–Z relationships. Note that this conversion introduces a certain dependence between the coefficients, a negative correlation to be precise, to which more attention will be paid later in this section. Fig. 6.2(a) finally is a plot of the climatological Z–R relationships for the three locations (De Bilt, The Netherlands; the mean of Best's data; Hilo, Hawaii) for

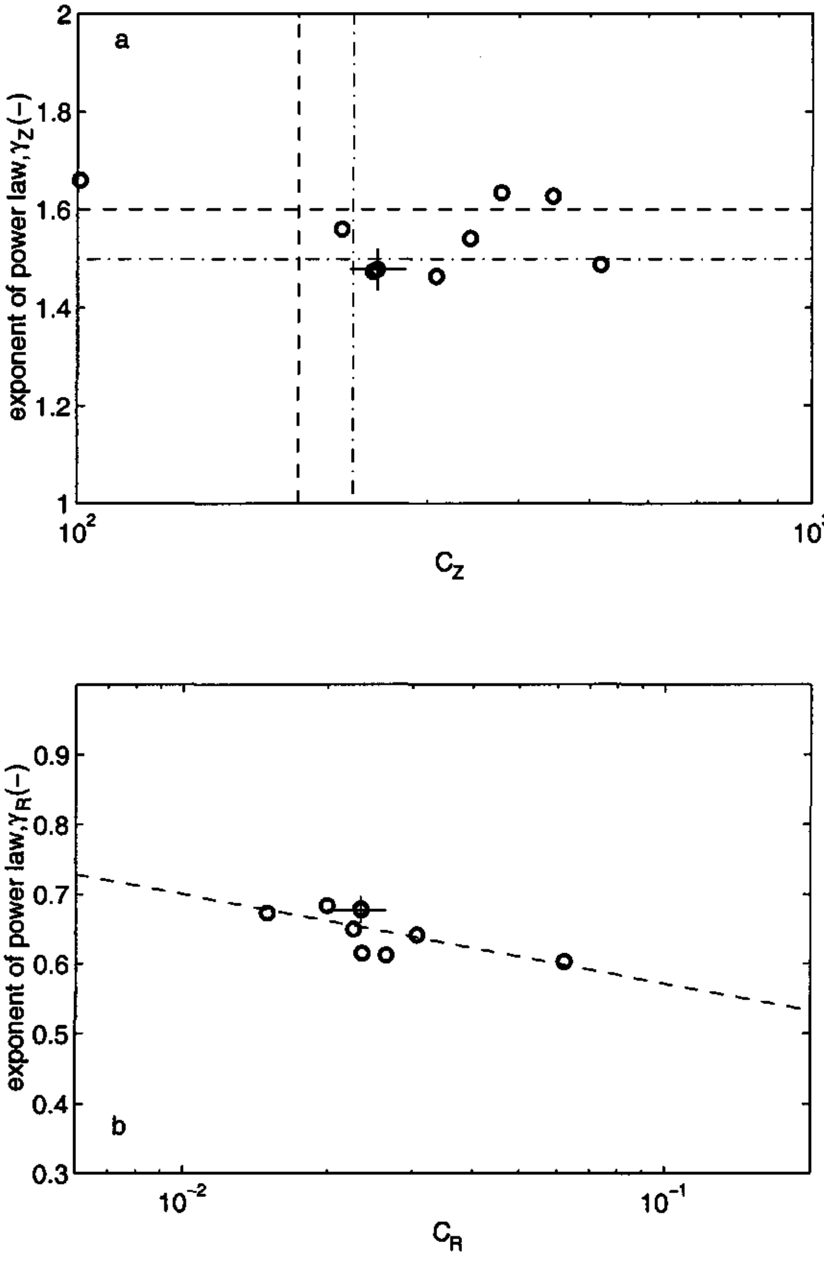

Figure 6.1: (a) Coefficients C_Z ($\mathrm{mm^6\,m^{-3}\,(mm\,h^{-1})^{-\gamma_Z}}$) and γ_Z (–) of climatological power law Z–R relationships for 7 different locations around the world (Best, 1950a), for the mean distribution of all locations derived by Best and for De Bilt, The Netherlands (Wessels, 1972). Error bars around the Dutch data point indicate 99% confidence limits, estimated from 1000 bootstrap samples. Dashed line corresponds to $Z = 200R^{1.6}$ (Marshall et al., 1955), dash-dotted line to $Z = 237R^{1.50}$ (Marshall and Palmer, 1948). (b) Coefficients of corresponding R–Z relationships (again with 99% confidence interval around the Dutch data point) and regression line (dashed) of γ_R on $\log C_R$.

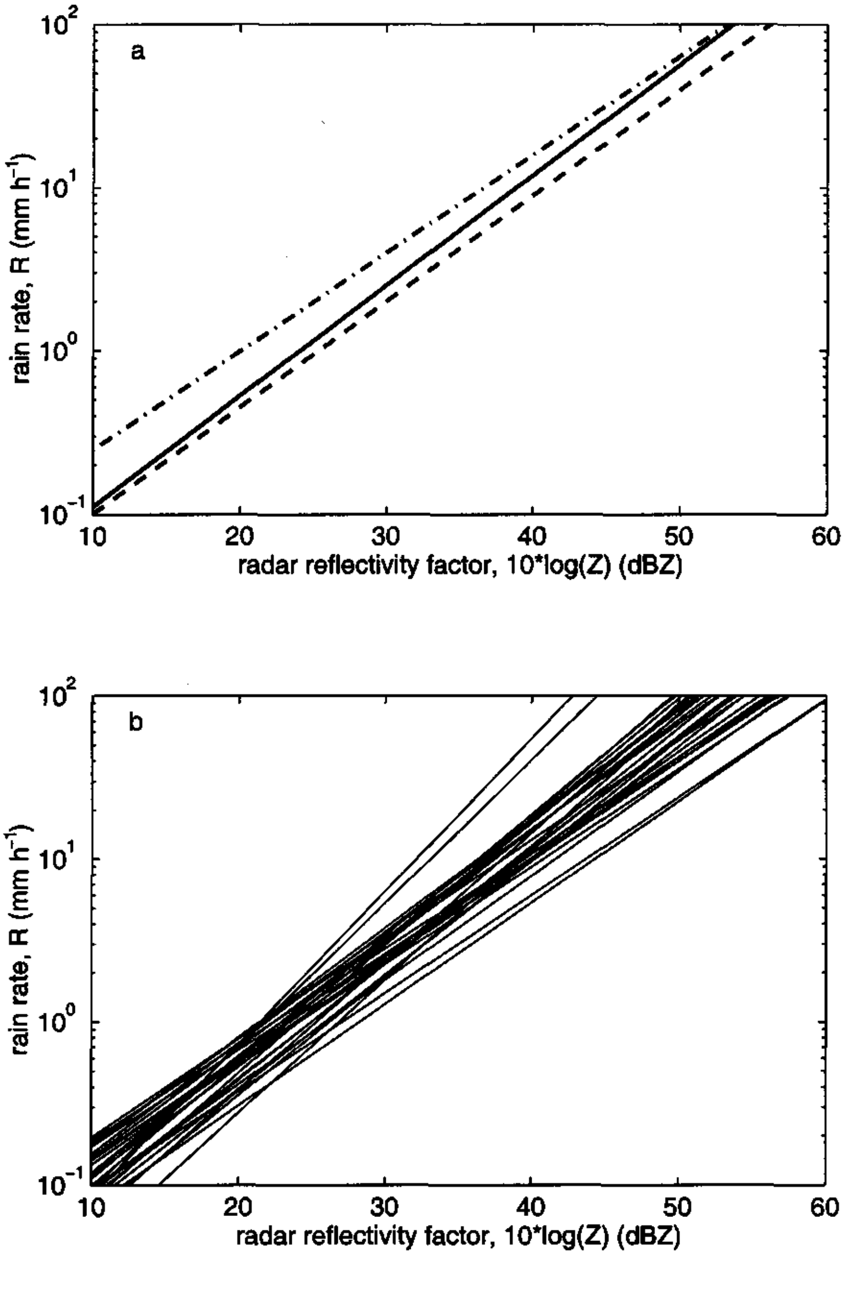

Figure 6.2: (a) Climatological Z–R relationships for De Bilt, The Netherlands (Wessels, 1972; solid line), for the mean distribution proposed by Best (1950a; dashed line) and for Hilo, Hawaii, USA (Best, 1950a; dash-dotted line). (b) Relationships between radar reflectivity factor $10 \log Z$ (dBZ) and rain rate R (mm h^{-1}) for 28 rainfall events in 1968 and 1969 in De Bilt, The Netherlands (based on data reported by Wessels, 1972).

Table 6.2: Prefactors (C_Z) and exponents (γ_Z) of Z–R relationships (with R in $\mathrm{mm\,h^{-1}}$ and Z in $\mathrm{mm^6\,m^{-3}}$) obtained from self-consistent forms of Best's general raindrop size distribution function $g(x)$ (or the corresponding general rain rate density function $h(x)$) for the locations reported by Best (1950a) and Wessels (1972).

Location	C_Z	γ_Z
Hilo (Hawaii, USA)	101	1.66
Germany	379	1.63
East Hill (UK)	445	1.63
Montreal (Canada)	229	1.56
mean (Best, 1950b)	343	1.54
Shoeburyness (UK)	517	1.49
De Bilt (Netherlands)	256	1.48
Ynyslas (UK)	253	1.47
Washington DC (USA)	308	1.46

which the self-consistent $g(x)$ and $h(x)$ have been encountered in Fig. 4.5 (p. 117).

The coefficients given in Table 6.2 for De Bilt, The Netherlands are close to those obtained from a regression analysis on the Dutch data by Wessels (1972) ($C_Z = 260$, $\gamma_Z = 1.43$). The small value for the prefactor for Hilo (Hawaii) is typical for orographic rainfall (e.g. Cataneo and Stout, 1968) and is associated with the fact that the general raindrop size distribution $g(x)$ for this location is narrow and concentrated at small scaled raindrop diameters (Fig. 4.5, p. 117). The large values of the prefactors C_Z for East Hill and particularly for Shoeburyness are caused by large values of the prefactors A of the corresponding power law a–R relationships (Eq. (4.2), p. 103). This indicates liquid rainwater content distributions which are weighted towards larger raindrops, something which may be related to thunderstorm rainfall (e.g. Joss and Waldvogel, 1969; Sekhon and Srivastava, 1971; Battan, 1973). On the other hand, Waldvogel (1974) and Huggel et al. (1996) associate distributions with large raindrops with widespread rainfall without any convective activity and a very pronounced bright band[4]. From Best's (1950a) description of the East Hill and Shoeburyness data it does not become clear which of these explanations is justified in this case.

[4]This confirms the findings of Pruppacher and Klett (1978), who, in a discussion of various model results regarding collisional breakup in rainfall, argue that 'in precipitation from "warm" clouds, where no ice particles are present, the raindrop size distribution is likely to be limited to drops of diameters less than 2 to 3 mm, as a result of collisional breakup. On the other hand, in precipitation from "cold" clouds, which do contain ice particles, raindrops larger than 3 mm in diameter may be present if the melting level in the atmosphere is relatively close to the ground such that the ice particles have sufficient time to melt, but insufficient time to change their size spectrum by collisional breakup'.

Table 6.3: Prefactors (C_Z) and exponents (γ_Z) of Z–R relationships (with R in mm h^{-1} and Z in mm^6 m^{-3}) obtained from different self-consistent forms of the general raindrop size distribution function $g(x)$ (or the corresponding general rain rate density function $h(x)$) for Laws and Parsons' (1943) data.

Parameter	Exponential	Gamma	Best	Lognormal
C_Z	351	326	325	357
γ_Z	1.41	1.41	1.41	1.41

6.3.2 Laws and Parsons' data

The value of β estimated in Chapter 4 (Section 4.3) for Laws and Parsons' (1943) tabulated raindrop size distribution parameterization ($\beta = 0.176$) and the corresponding values of the parameters of the four self-consistent analytical forms for the general raindrop size distribution function $g(x)$ (Table 4.5, p. 136) yields the Z–R coefficients summarized in Table 6.3. Since the value of the exponent γ_Z is uniquely determined by the value of the scaling exponent β (Eq. (6.5)), γ_Z is equal for all four parameterizations. The prefactors are all somewhat higher and the exponents somewhat lower than those obtained on the basis of Best's (1950a) adjustment to Laws and Parsons' data (Table 6.2, 'Washington DC (USA)').

The differences between the prefactors for the different parameterizations are not very significant. Specifically, the values for the exponential and lognormal parameterizations are almost equal, as are those for the gamma and Best parameterizations. Recall that the parameters of $g(x)$ and $h(x)$ have been adjusted on the basis of the method of moments, using the first (mean) and second moment (variance) of $h(x)$. These are proportional to the 4.67th and 5.67th moment of $g(x)$, respectively. The latter is very close to the 6th moment of $g(x)$, i.e. to C_Z and it is therefore not surprising to find that the prefactors in Table 6.3 are relatively close. As a matter of fact, these could have been forced to be equal through use of the 6th moment of $g(x)$, i.e. the 2.33th moment of $h(x)$, in the method of moments employed to estimate the parameters.

6.3.3 Dutch rainfall data

Event-to-event analysis

The values of the scaling exponents β and the parameters λ and ν of the self-consistent general raindrop size distributions functions for the 28 rainfall events in De Bilt, The Netherlands to which Wessels (1972) adjusted Best's parameterization have been obtained in Chapter 5 (Fig. 5.4, p. 148 and Fig. 5.5, p. 149). Fig. 6.3(a) shows the corresponding prefactors C_Z (calculated using the expression for the Best parameterization given in Table 6.1) and exponents γ_Z, Fig. 6.2(b) the corresponding 28 Z–R relationships. As can be seen, there is an appreciable amount of inter-event variability in the Z–R relationships, something which has been noted by Smith and Krajewski (1993)

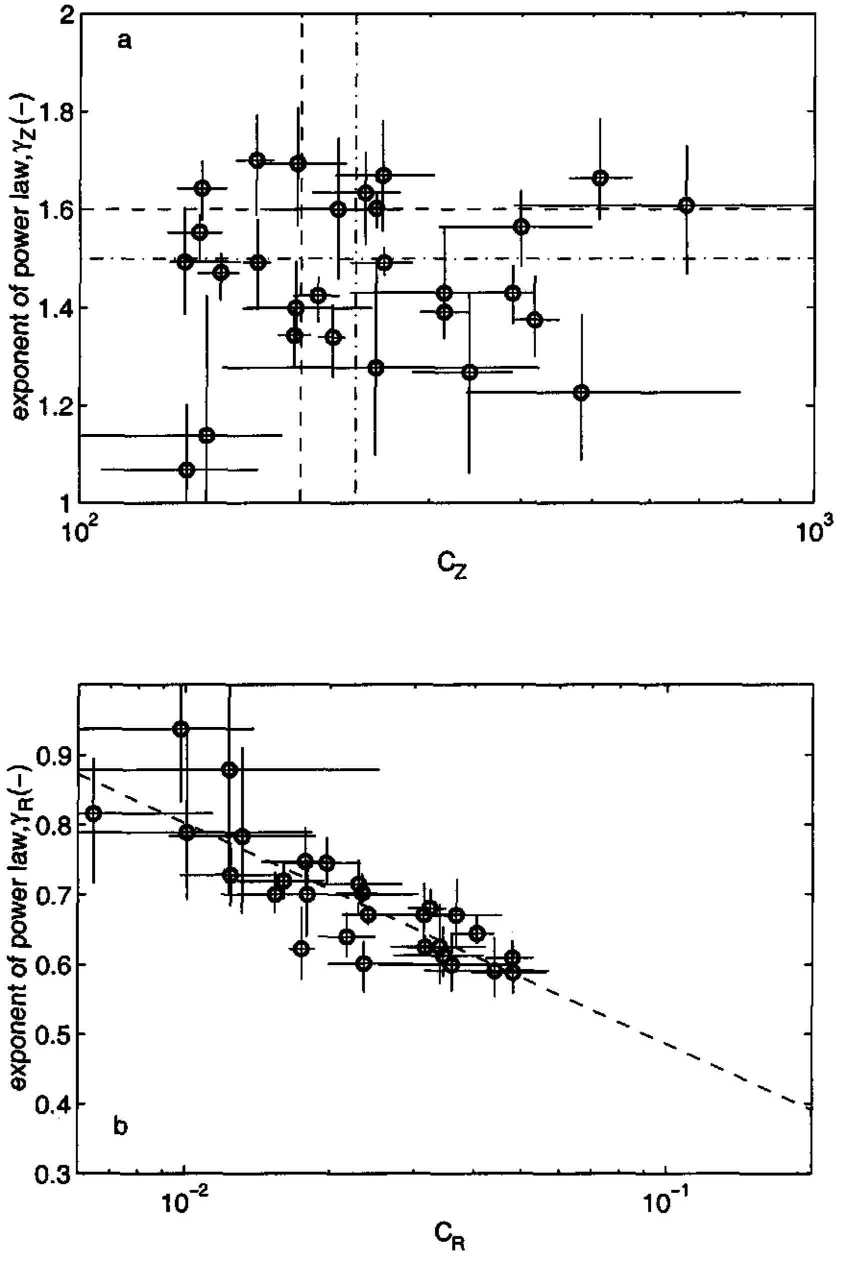

Figure 6.3: (a) Coefficients C_Z ($\mathrm{mm^6\,m^{-3}\,(mm\,h^{-1})^{-\gamma_Z}}$) and γ_Z (–) of power law Z–R relationships for 28 rainfall events in 1968 and 1969 in De Bilt, The Netherlands (Wessels, 1972). Error bars indicate 68% confidence limits, estimated from 1000 bootstrap samples in each case. Dashed line corresponds to $Z = 200R^{1.6}$ (Marshall et al., 1955), dash-dotted line to $Z = 237R^{1.50}$ (Marshall and Palmer, 1948). (b) Coefficients of corresponding R–Z relationships (again with 68% confidence intervals) and regression line (dashed) of γ_R on $\log C_R$.

among others as well. As indicated by the lengths of the error bars in Fig. 6.3(a), this cannot solely be explained in terms of sampling fluctuations. It is therefore a pity that Wessels (1972) has not been able to relate this variability to changes in any of a series of important meteorological parameters (see the discussion in Chapter 5, Section 5.3).

Rain rate–radar reflectivity relationships and spurious correlations

For hydrological and meteorological applications, R–Z relationships are often more appropriate than Z–R relationships. This is because the rain rate R is the quantity which needs to be determined from the radar-estimated reflectivity factor Z. The coefficients $C_R = (1/C_Z)^{1/\gamma_Z}$ and $\gamma_R = 1/\gamma_Z$ of the power law R–Z relationships corresponding to the Z–R relationships for the 28 rainfall events have been plotted in Fig. 6.3(b). Just as in Fig. 6.1(b), the data points exhibit a pronounced negative correlation, even though the coefficients of the original Z–R relationships seem to be more or less uncorrelated. A similar effect has been observed by Smith and Krajewski (1993) in a study of storm-to-storm variability of the coefficients of R–Z relationships in North Carolina (USA). How can this be explained?

In Appendix F it is shown that if $\underline{C}_Z$ and $\underline{\gamma}_Z$ are supposed to be two independent random variables, then the square of the correlation coefficient between $\log \underline{C}_R = -\underline{\gamma}_R \log \underline{C}_Z$ and $\underline{\gamma}_R = \underline{\gamma}_Z^{-1}$ is given by

$$\rho^2 = \frac{\mathrm{CV}^2\left(\underline{\gamma}_Z^{-1}\right)}{\mathrm{CV}^2\left(\underline{\gamma}_Z^{-1}\right) + \mathrm{CV}^2\left(\log \underline{C}_Z\right) + \mathrm{CV}^2\left(\underline{\gamma}_Z^{-1}\right)\mathrm{CV}^2\left(\log \underline{C}_Z\right)}, \qquad (6.7)$$

where CV denotes the coefficient of variation (i.e. the ratio of the standard deviation to the mean). It is also shown that the sign of ρ, the actual correlation coefficient, is negative as long as $\mathrm{E}[\log \underline{C}_Z]$ is positive, i.e. as long as the geometric mean of $\underline{C}_Z$ exceeds one (which is generally the case). Eq. (6.7) shows that even independent fluctuations in the prefactors and exponents of Z–R relationships are enough to cause (negative) correlations between the prefactors and exponents of R–Z relationships. This is of course not really surprising, given the fact that C_R depends both on C_Z and on γ_Z. Eq. (6.7) quantifies this effect.

For the 28 rainfall events collected in 1968 and 1969 in De Bilt, The Netherlands, $\log C_Z$ and γ_Z are found to be virtually uncorrelated, with a sample correlation coefficient r of only 0.0867. The sample geometric mean of C_Z is 246, well above one, and therefore the correlation between $\log C_R$ and γ_R will be negative. The sample coefficients of variation of $\log C_Z$ and γ_Z^{-1} are 0.0785 and 0.1265, respectively. Substituting these values in Eq. (6.7) yields a correlation coefficient of -0.85, which is quite close to the actual sample correlation coefficient between $\log C_R$ and γ_R (-0.82) and as such demonstrates the validity of Eq. (6.7).

Appendix F also shows that if Z and R would be expressed in SI units (i.e. in m^3 and $\mathrm{m\,s}^{-1}$, respectively), then the corresponding prefactors and exponents of both Z–R and R–Z relationships would be strongly *positively* correlated. All this serves to show that the observed (negative) correlation between $\log C_R$ and γ_R is in fact a *spurious* correlation, i.e. an apparent correlation between variables which may just as well be uncorrelated (Haan, 1977). The magnitude and sign of the correlation is completely determined by the employed units for Z and R. Care should therefore be exercised when a physical meaning is attributed to such correlations.

Climatological analysis

Table 6.4: Prefactors (C_Z) and exponents (γ_Z) of Z–R relationships (with R in $\mathrm{mm\,h^{-1}}$ and Z in $\mathrm{mm^6\,m^{-3}}$) for De Bilt, The Netherlands, obtained from (1) different self-consistent forms of the general raindrop size distribution function $g(x)$ (or the corresponding general rain rate density function $h(x)$) and (2) different types of least-squares regression. The 'min' and 'max' values indicate 99% confidence limits, estimated from 250 bootstrap samples.

Method	Specific form	C_Z			γ_Z		
		min	mean	max	min	mean	max
Scaling law	Exponential	265	286	302	1.41	1.47	1.53
	Gamma	247	270	289	1.41	1.47	1.53
	Best	246	269	287	1.41	1.47	1.53
	Lognormal	269	295	315	1.41	1.47	1.53
Regression	$\log Z - \log R$	224	241	255	1.43	1.49	1.55
	$\log R - \log Z$	225	241	255	1.54	1.60	1.66
	$Z - R$	186	285	396	1.31	1.47	1.60
	$R - Z$	193	258	303	1.45	1.53	1.71

The value of β estimated in Chapter 5 (Section 5.4) for the 446 raw raindrop size distributions collected by Wessels (1972) and colleagues ($\beta = 0.201$) and the corresponding values of the parameters of the four self-consistent analytical forms for the general raindrop size distribution function $g(x)$ (Table 5.2, p. 163) lead to the Z–R coefficients given in Table 6.4. The exponent γ_Z is again equal for all four parameterizations. Moreover, as is the case for the Z–R relationships obtained for Laws and Parsons' parameterization (Table 6.3) the values for the exponential and lognormal parameterizations are almost equal, as are those of the gamma and Best parameterizations. Those for the Best parameterization are also quite close to those given in Table 6.2 for De Bilt, The Netherlands, which are based on Wessels' (1972) adjustment of Best's parameterization to the raw data. Table 6.4 also provides estimates of the 99% confidence limits on the coefficients, based on 250 bootstrap samples. It can be seen that the prefactors are more sensitive to sampling fluctuations than the exponents. Note that because the sampling fluctuations in C_Z and γ_Z will be correlated, their confidence intervals will not be independent.

For comparison, the coefficients of the climatological Z–R relationship for De Bilt have also been determined on the basis of four different least-squares regression analyses: twice linear regression on the logarithmic values ($\log Z$ on $\log R$ and $\log R$ on $\log Z$) and twice nonlinear (power law) regression on the linear values (Z on R and R

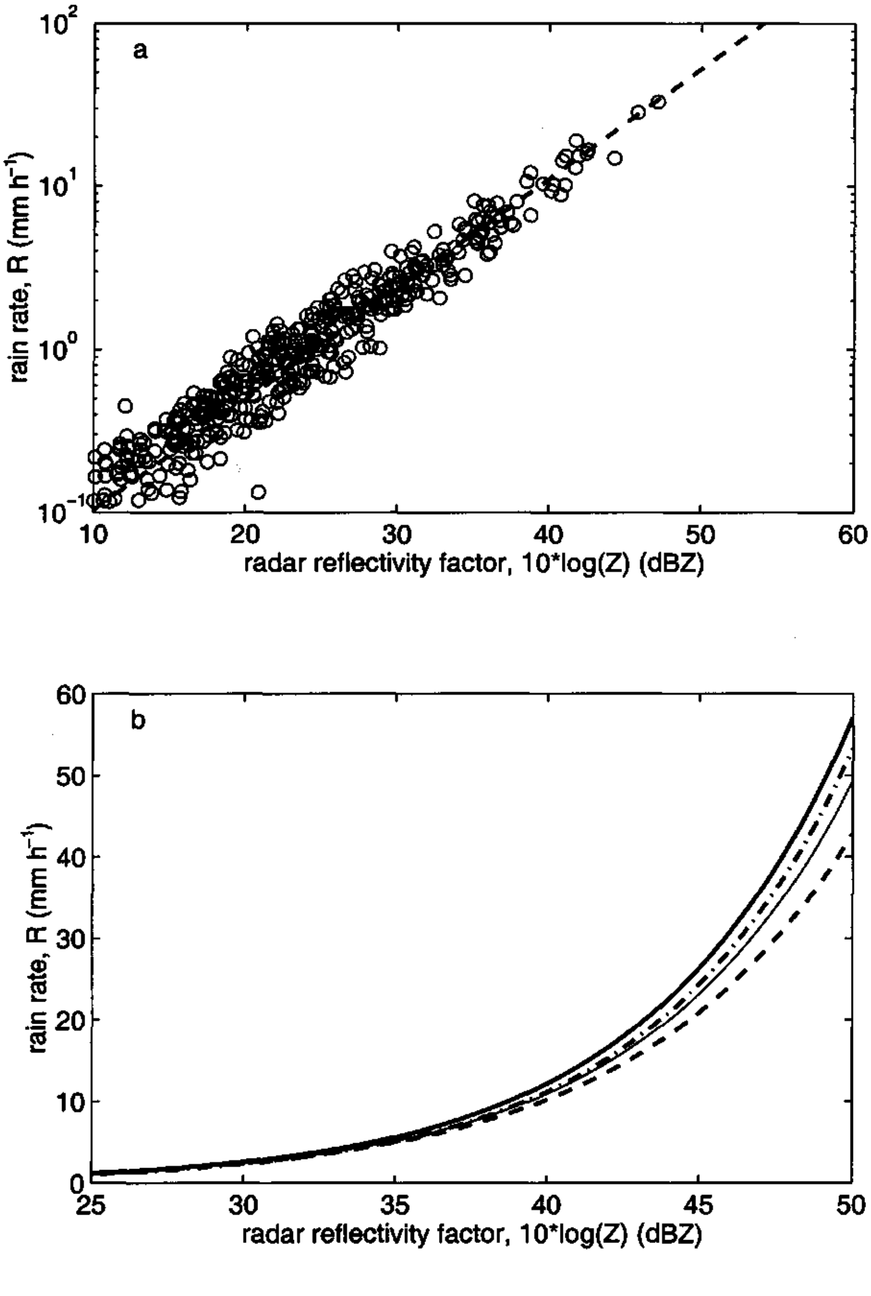

Figure 6.4: (a) Scatterplot of rain rate R ($\mathrm{mm\,h^{-1}}$) versus radar reflectivity factor Z ($\mathrm{mm^6\,m^{-3}}$) (expressed in logarithmic units) calculated from 446 raindrop size distributions collected in 1968 and 1969 in De Bilt, The Netherlands (Wessels, 1972). Dashed line is $Z = 285R^{1.47}$, obtained from nonlinear (power law) regression of Z on R. (b) Comparison of different power law relationships adjusted to the data in (a): $Z = 241R^{1.49}$ (linear regression of $\log Z$ on $\log R$, bold line); $Z = 241R^{1.60}$ (linear regression of $\log R$ on $\log Z$, dashed line); $Z = 285R^{1.47}$ (nonlinear regression of Z on R, dash-dotted line); $Z = 258R^{1.53}$ (nonlinear regression of R on Z, solid line).

on Z). On the average, the prefactors are somewhat lower and the exponents somewhat higher than those obtained from the scaling law approach[5]. Note that power law regression of Z on R yields almost exactly the same coefficients as those which follow from the self-consistent exponential parameterization adjusted to the data. The sampling uncertainties associated with the coefficients obtained using linear regression on the logarithms are comparable to those obtained from the raindrop size distribution parameterizations. This is perhaps not surprising, since the value of the scaling exponent β (which determines γ_Z via Eq. (6.5)) has been obtained using linear regression on the logarithms as well (Chapter 5, Section 5.4). The uncertainties associated with the coefficients obtained from the power law regression procedures are significantly greater than those associated with the other coefficients. Fig. 6.4(a) is a scatterplot of Z versus R for the 446 empirical raindrop size distributions and Fig. 6.4(b) gives a comparison between the four different regression-based Z–R relationships. Both figures reflect the fundamental uncertainty associated with mean Z–R relationships.

6.4 Battan's radar reflectivity–rain rate relationships revisited

In Section 6.3 the methodology developed in Section 6.2 has been applied to derive Z–R relationships from the different parameterizations for the raindrop size distribution presented in Chapters 4 and 5. In this section, the reverse will be done. In an effort to obtain more information on the manner in which the parameters of raindrop size distribution parameterizations depend on the type of rainfall, Z–R relationships published in the literature will be used to derive the raindrop size distribution parameterizations which correspond to them. Specifically, the classical list of Z–R relationships compiled by Battan (1973) will be used for this purpose. This list (Battan's Table 7.1 on his p. 90–92) provides a total of 69 Z–R relationships derived from raindrop size distribution measurements for different types of rainfall in many parts of the world. Ulbrich (1983) argues that 'the observed variations in [C_Z] and [γ_Z] are [...] not due to measurement errors nor are they induced by correlations between the errors involved in measuring Z and R' and that as a result 'these variations in [C_Z] and [γ_Z] are due to real physical differences between the types of rainfall to which the Z–R relations apply'. Although this is perhaps stated somewhat boldly (the analyses in Section 6.3 have for instance shown that there can be a pronounced effect of the manner in which the coefficients C_Z and γ_Z are adjusted to the data), Ulbrich's

[5]There is an ongoing debate whether it would be preferable to use R as the independent variable or Z. From the point of view of rain rate estimation, the choice for Z might seem logical, as that is the variable estimated using weather radar. However, from the point of view of sampling variability, the choice for R is preferable. Z is the sixth moment of the raindrop size distribution and depends strongly on the scarcely sampled large raindrops. In order to avoid the asymmetry associated with ordinary regression procedures (where the coefficients depend on the choice of the independent variable), certain researchers prefer to minimize the sum of the squared distances *normal* to the regression line (Amayenc, 1999, personal communication). However, that has not been pursued here.

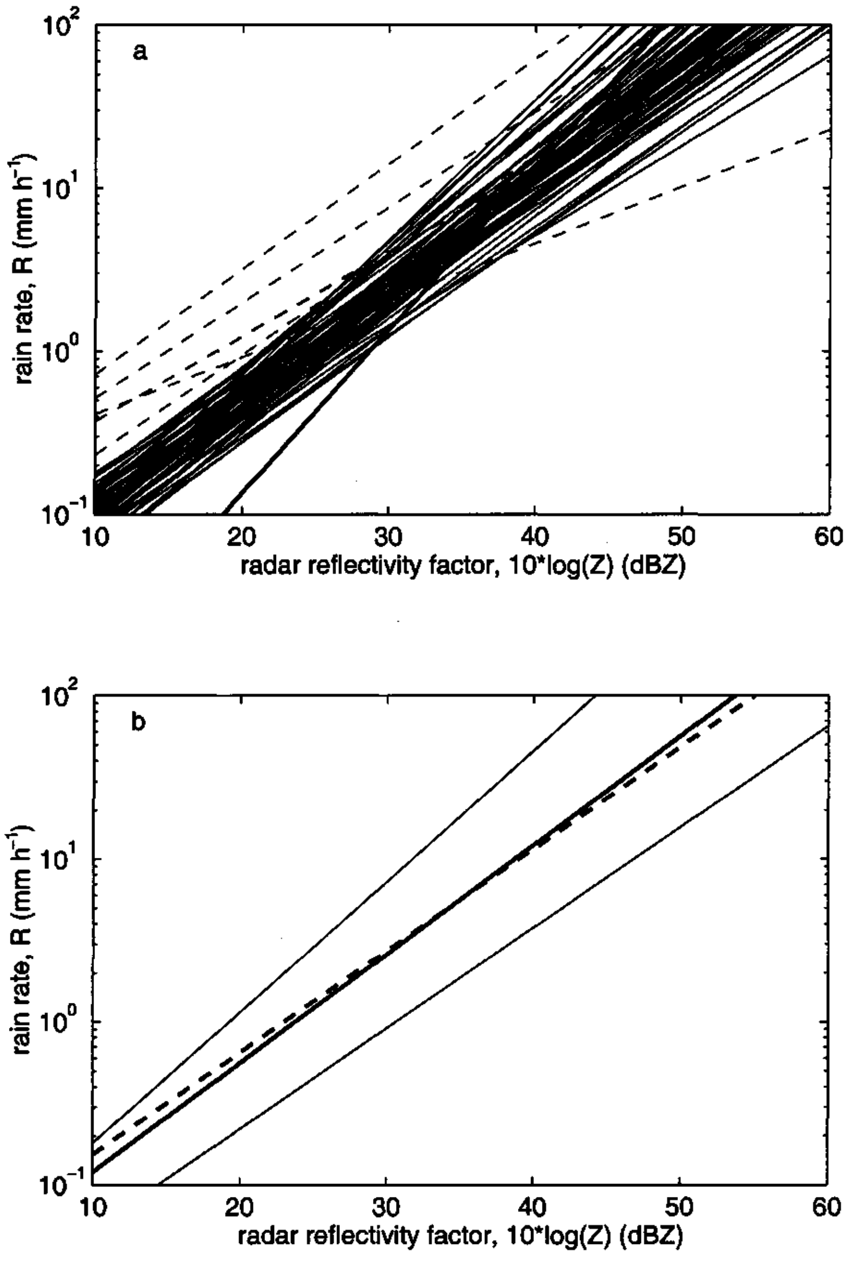

Figure 6.5: (a) The 69 Z–R relationships quoted by Battan (1973; p. 90–92), including five deviating relationships (dashed lines), four of which have prefactors C_Z significantly smaller than 100 and one of which has an exponent γ_Z as high as 2.87. The bold line indicates the linear relationship $Z = 742R$ (List, 1988). (b) The mean of Battan's relationships, $Z = 238R^{1.50}$ (bold line), the reference relationship $Z = 200R^{1.6}$ (Marshall et al., 1955; dashed line) and the envelope of 64 of Battan's 69 Z–R relationships (solid lines).

arguments will be taken as the starting point for the analysis which follows.

6.4.1 Presentation and discussion of the radar reflectivity–rain rate relationships

Fig. 6.5(a) shows a plot of Battan's 69 Z–R relationships. For reference, the linear Z–R relationship proposed by List (1988) for equilibrium rainfall conditions is included as well (Eq. (3.94), p. 95). Whereas Fig. 6.4 has provided an idea of the uncertainty associated with the Z–R relationship *within* a given rainfall climatology, Fig. 6.5(a) provides an idea of the variability associated with Z–R relationships *between* different climatologies. The latter is even more clearly demonstrated in Fig. 6.5(b), which shows the envelope of 64 of Battan's Z–R relationships. Note that five strongly deviating relationships, four with prefactors C_Z significantly smaller than 100 (the smallest two, corresponding to orographic rainfall in Hawaii, have values of only 16.6 and 31) and one with an exponent γ_Z as high as 2.87 have not been taken into account in calculating the envelope[6]. Also shown in Fig. 6.5(b) is the mean of all 69 Z–R relationships, obtained by taking the geometric mean of all prefactors and the arithmetic mean of all exponents[7]. Interestingly, the coefficients of this mean relationship ($Z = 238R^{1.50}$) are almost exactly the same as those which follow from the exponential raindrop size distribution with Marshall and Palmer's (1948) value for N_0 and Atlas and Ulbrich's (1977) raindrop terminal fall speed parameterization ($Z = 237R^{1.50}$, see Eq. (2.65) on p. 51 and subsequent discussion). As Fig. 6.5(b) shows, this mean relationship is not very different from the Marshall-Palmer Z–R relationship either.

From the remarks provided by Battan in his table, it is possible to associate 25 of the 69 Z–R relationships unambiguously with a particular type of rainfall. Using the same stratification as Ulbrich (1983), 4 of the relationships have been identified as pertaining to orographic rainfall, 5 to thunderstorm rainfall, 10 to widespread or stratiform rainfall and 6 to showers[8]. For the other 44 relationships no unambiguous identification has been possible, either because they correspond to mixtures of different types of rainfall or because Battan has not indicated a rainfall type at all. Fig. 6.6(a) shows a plot of the exponents γ_Z versus the prefactors C_Z for the different types of rainfall, similar to Fig. 6.1(a) for Best's data and Fig. 6.3(a) for the Dutch data. If Fig. 6.6(a) is indicative for the climatological variability of the coefficients

[6]For $\gamma_Z = 2.87$, Eq. (6.5) yields $\beta = 0.803$. The self-consistency constraint on α (Eq. (3.38), p. 64) then implies $\alpha = -2.75$. Substituting these values in Eqs. (3.5) and (3.6) (p. 59) finally gives $\gamma_{\rho_V} = -1.95$ and $\gamma_{D_C} = 0.803$. This indicates that for $\gamma_Z = 2.87$, the raindrop concentration ρ_V would *decrease* almost proportionally to the square of the rain rate R, i.e. a doubling of R would roughly correspond to a 75% reduction of ρ_V (and a 75% increase in the characteristic diameters D_C to compensate). This seems very unlikely.

[7]Although this is a rather ad hoc method, it has some theoretical justification in that it is the same as taking the arithmetic mean of the coefficients of the linear $\log Z$–$\log R$ relationships.

[8]Rogers and Yau (1996) make a useful distinction between continuous rain and showers, by approximating the former as 'a steady-state process, in which cloud quantities may vary with height but are constant with time at any given height' and the latter as 'systems in which the cloud properties vary with time but are constant with height at any given time'.

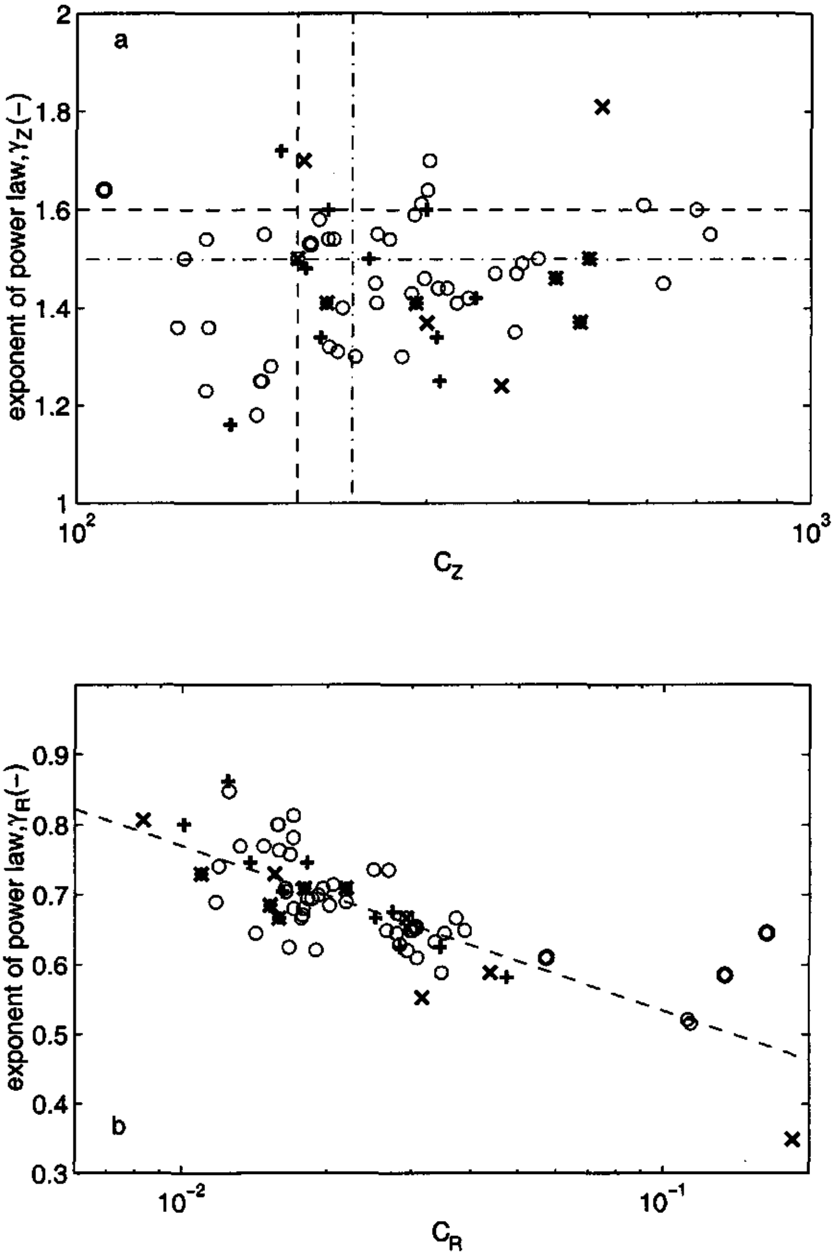

Figure 6.6: (a) Coefficients C_Z ($\mathrm{mm^6\,m^{-3}\,(mm\,h^{-1})^{-\gamma_Z}}$) and γ_Z (–) of 69 power law Z–R relationships quoted by Battan (1973), stratified according to rainfall type: orographic (bold circles); thunderstorm (bold stars); widespread/stratiform (bold plusses); showers (bold crosses); no unambiguous identification possible (circles). Dashed line corresponds to $Z = 200R^{1.6}$ (Marshall et al., 1955), dash-dotted line to $Z = 237R^{1.50}$ (Marshall and Palmer, 1948). (b) Coefficients of corresponding R–Z relationships and regression line (dashed) of γ_R on $\log C_R$.

of Z–R relationships and Fig. 6.3(a) for the event-to-event variability within a given climatology, then it would seem that both types of variability are comparable. This also follows from a comparison of 6.5(b) with Fig. 6.2(b).

There is again no significant correlation between $\log C_Z$ and γ_Z, the sample correlation coefficient for the 69 coordinate pairs being only -0.2146. Moreover, it is almost impossible to unambiguously distinguish the different types of rainfall, i.e. to associate them with different non-overlapping regions in the $(\log C_Z, \gamma_Z)$-parameter space. For orographic rainfall the prefactors tend to be lower and the exponents higher than the average. For thunderstorm rainfall the opposite seems true, with higher prefactors and lower exponents than the average. For widespread/stratiform rainfall, the values are in between these two extremes on the average. This is confirmed by the 'typical' Z–R relationships provided by Battan for these three types of rainfall. For showers, the coefficients are highly variable and, as Ulbrich states, 'no general statement can be made about [their] range of values'.

Fig. 6.6(b) shows the prefactors versus the exponents of the R–Z relationships implied by Battan's Z–R relationships. As in Fig. 6.3(b), these coefficients exhibit again a clear (spurious) negative correlation. The sample coefficients of variation of $\log C_Z$ and γ_Z^{-1} are now 0.1140 and 0.1213, respectively, and Eq. (6.7) predicts a correlation coefficient of -0.73 (negative since the sample geometric mean of C_Z is 238, larger than one). Notwithstanding the weak correlation which exists between $\log C_Z$ and γ_Z (assumed to be independent in Eq. (6.7)), this is still reasonably close to the actual sample correlation coefficient between $\log C_R$ and γ_R (-0.78). This again confirms the validity of Eq. (6.7).

6.4.2 Implications for raindrop size distribution parameterizations

Since parameterizations for the raindrop size distribution imply coefficients of Z–R relationships via Eq. (6.5) and the expressions given in Table 6.1, the prefactors C_Z and the exponents γ_Z of Z–R relationships can in principle be employed to infer the parameters of raindrop size distributions. It will be clear that for a given value of γ_Z (–), Eq. (6.5) provides a direct estimate of the scaling exponent β (–). For the prefactors the situation is a little more restrictive, since for each value of C_Z only one raindrop size distribution parameter can be estimated. The only one-parameter distribution mentioned in Table 6.1 is the exponential distribution.

A given value of C_Z ($\mathrm{mm^6\,m^{-3}\,(mm\,h^{-1})^{-\gamma_Z}}$) implies an estimate of the parameter λ ($\mathrm{mm^{-1}\,(mm\,h^{-1})^{\beta}}$) of that distribution via

$$\lambda = \left[\frac{10^4}{6\pi c}\frac{\Gamma(7)}{\Gamma(4+\gamma)}\right]^{1/(3-\gamma)} C_Z^{-1/(3-\gamma)}, \tag{6.8}$$

which reduces to

$$\lambda = 44.3 C_Z^{-0.429} \tag{6.9}$$

for $c = 3.778$ ($\mathrm{m\,s^{-1}\,mm^{-\gamma}}$) and $\gamma = 0.67$ (–). Using the corresponding expression for κ ($\mathrm{mm^{-1}\,m^{-3}\,(mm\,h^{-1})^{-\alpha}}$, where $\alpha = 1-(4+\gamma)\beta$) given in Table 3.2, p. 79, Eq. (6.8)

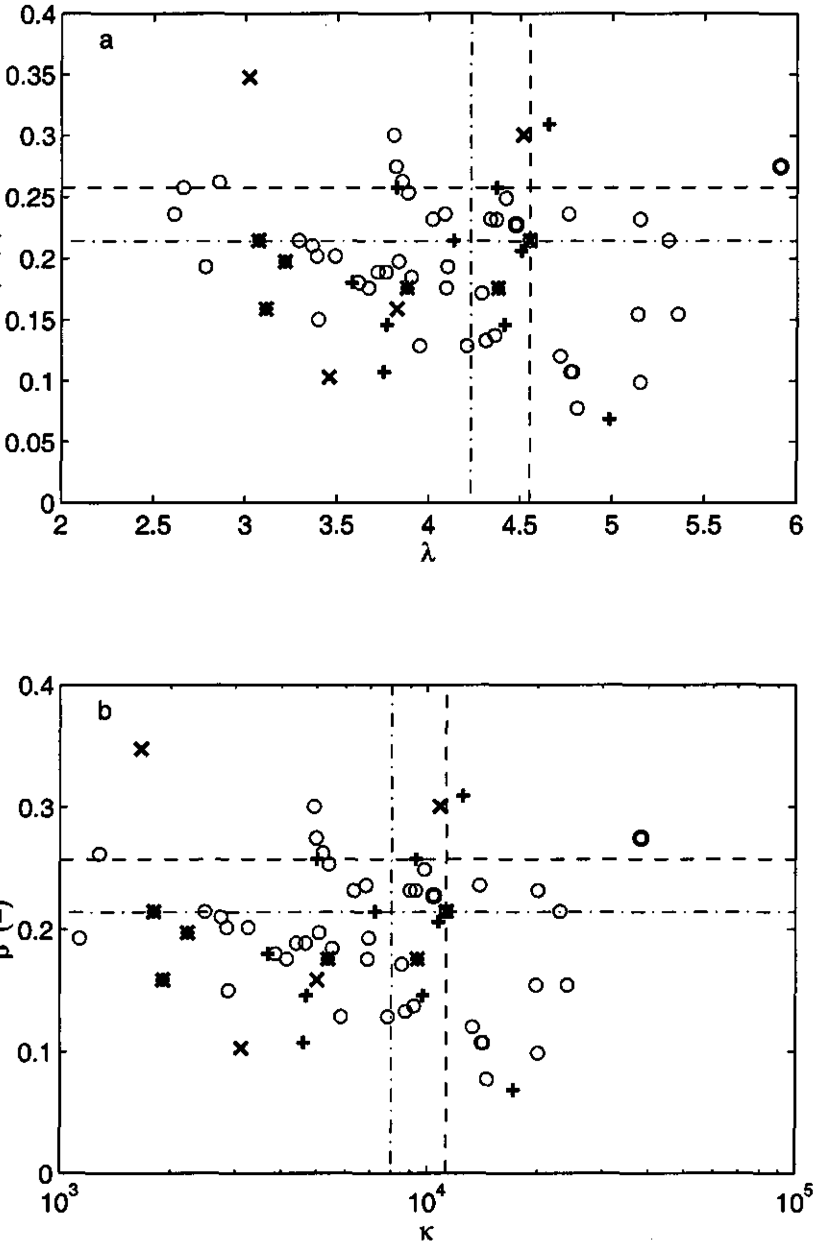

Figure 6.7: (a) Parameters λ ($\mathrm{mm}^{-1}\,(\mathrm{mm\,h}^{-1})^{\beta}$) and β (–) of exponential parameterizations for $g\,(x)$ obtained from Battan's (1973) 69 Z–R relationships (bold circles: orographic; bold stars: thunderstorm; bold plusses: widespread/stratiform; bold crosses: showers; circles: no unambiguous identification possible). Dashed line corresponds to $Z = 200R^{1.6}$ (Marshall et al., 1955), dash-dotted line to $Z = 237R^{1.50}$ (Marshall and Palmer, 1948). (b) Idem for the parameters κ ($\mathrm{mm}^{-1}\,\mathrm{m}^{-3}\,(\mathrm{mm\,h}^{-1})^{-\alpha}$, where $\alpha = 1 - 4.67\beta$) and β.

implies

$$\kappa = \frac{10^4}{6\pi c}\frac{1}{\Gamma(4+\gamma)}\left[\frac{10^4}{6\pi c}\frac{\Gamma(7)}{\Gamma(4+\gamma)}\right]^{(4+\gamma)/(3-\gamma)} C_Z^{-(4+\gamma)/(3-\gamma)}, \tag{6.10}$$

which for $c = 3.778$ and $\gamma = 0.67$ reduces to

$$\kappa = 4.62 \times 10^8 C_Z^{-2.00}. \tag{6.11}$$

In order to be able to employ C_Z for the identification of any of the other parameterizations, one or more parameters have to be assumed constant[9]. The two coefficients of Z–R relationships in principle provide no further information about these additional parameters, which in general may also differ from one location to the next. Therefore, the analysis which follows has been restricted to the exponential distribution. An approach to relate the coefficients of Z–R relationships to the parameters of the gamma distribution will be discussed in Section 6.5.

Eqs. (6.5), (6.9) and (6.11) have been employed to estimate the values of the scaling exponents β and the parameters λ and κ of exponential raindrop size distributions on the basis of the coefficients of Battan's 69 Z–R relationships. Fig. 6.7 shows the resulting scatter plots of β versus λ and β versus κ for the previously identified types of rainfall. That Fig. 6.7(a) and (b) mimic each other closely is a result of the fact that $\log \kappa$ and λ are nearly linearly related for the range of values of these parameters considered here. It is again not easy to distinguish between the different types of rainfall in these scatter plots, but orographic rainfall seems to be associated with larger values of β, κ and λ, thunderstorm rainfall with smaller values of β, κ and λ and widespread/stratiform rainfall with values in between.

This is confirmed by Table 6.5, which shows the mean values and associated standard deviations of the exponent β and the parameter λ for the different types of rainfall. Clearly, the standard deviation of β in case of showers and that of λ in case of orographic rainfall are of such a magnitude compared to the corresponding means, that the mean results should be interpreted with care. This holds in fact for all statistics in Table 6.5, as the sample sizes are extremely small. These results merely indicate some tendencies, and not more than that.

Nevertheless, the results seem more or less consistent with what has been encountered before. Recall that β (–) can be interpreted physically as an indicator for the proportion of diameter control in the variability of the raindrop size distribution (Chapter 3, Section 3.5.3) and that λ ($\mathrm{mm}^{-1}\,(\mathrm{mm\,h}^{-1})^{\beta}$) represents the inverse mean raindrop diameter for $R = 1\ \mathrm{mm\,h}^{-1}$ (Eq. 3.70, p. 82). Then it is seen that orographic rainfall is associated with almost pure diameter control ($\alpha \approx -\beta$) and small mean raindrop diameters, confirming the results obtained in Chapter 4 (Section 4.2) for Hilo, Hawaii. For thunderstorm rainfall exactly the opposite is the case, with a relatively large proportion of raindrop concentration control (i.e. closer to equilibrium than orographic rainfall) and large mean raindrop diameters. The values for widespread/stratiform rainfall fall again in between these two extremes. The mean

[9] Recall that the exponential parameterization is a special case of the gamma parameterization for $\mu = \text{constant} = 0$.

Table 6.5: Summary statistics for the exponential $g(x)$-model as applied to Battan's (1973) 69 Z–R relationships, stratified according to rainfall type. The category 'Rest' contains all relationships for which an unambiguous identification of rainfall type is impossible, 'nr.' denotes the number of relationships in each category and 's.d.' is the standard deviation of the corresponding parameter. The parameter κ and the coefficients C_Z and γ_Z of the corresponding Z–R relationships are those implied by the mean values of the parameters β and λ in each category.

Rainfall type	nr.	β		λ		κ	C_Z	γ_Z
		mean	s.d.	mean	s.d.			
Orographic	4	0.261	0.036	8.44	4.01	2.02×10^5	47.4	1.61
Thunderstorm	5	0.185	0.022	3.53	0.58	3.44×10^3	361	1.43
Widespread/ stratiform	10	0.189	0.074	4.20	0.46	7.74×10^3	241	1.44
Showers	6	0.321	0.252	4.15	0.90	7.32×10^3	248	1.75
Rest	44	0.202	0.068	4.21	0.97	7.85×10^3	240	1.47
All	69	0.213	0.099	4.40	1.57	9.63×10^3	216	1.50

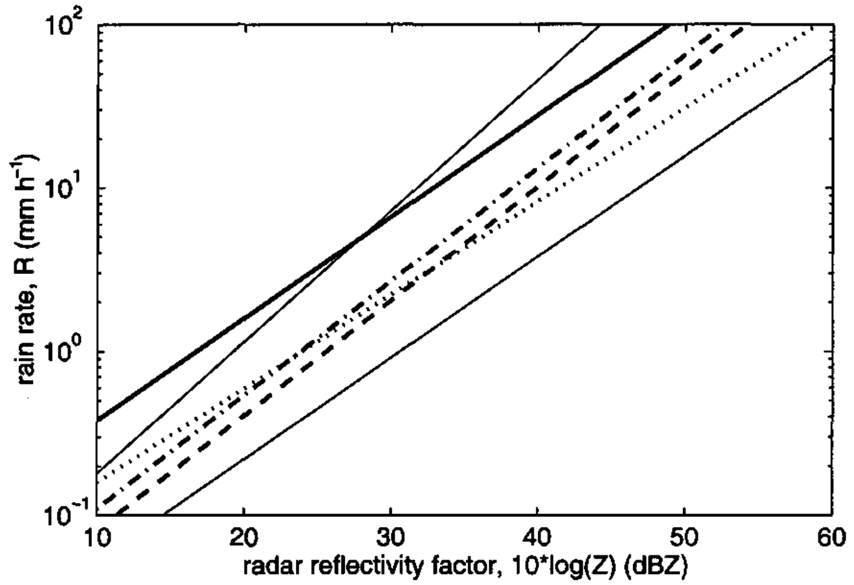

Figure 6.8: Mean Z–R relationships implied by the exponential $g(x)$-model for different rainfall types (based on Battan's (1973) 69 Z–R relationships): orographic (bold line); thunderstorm (dashed line); widespread/stratiform (dash-dotted line); showers (dotted line). The solid lines indicate the envelope of the majority of Battan's Z–R relationships.

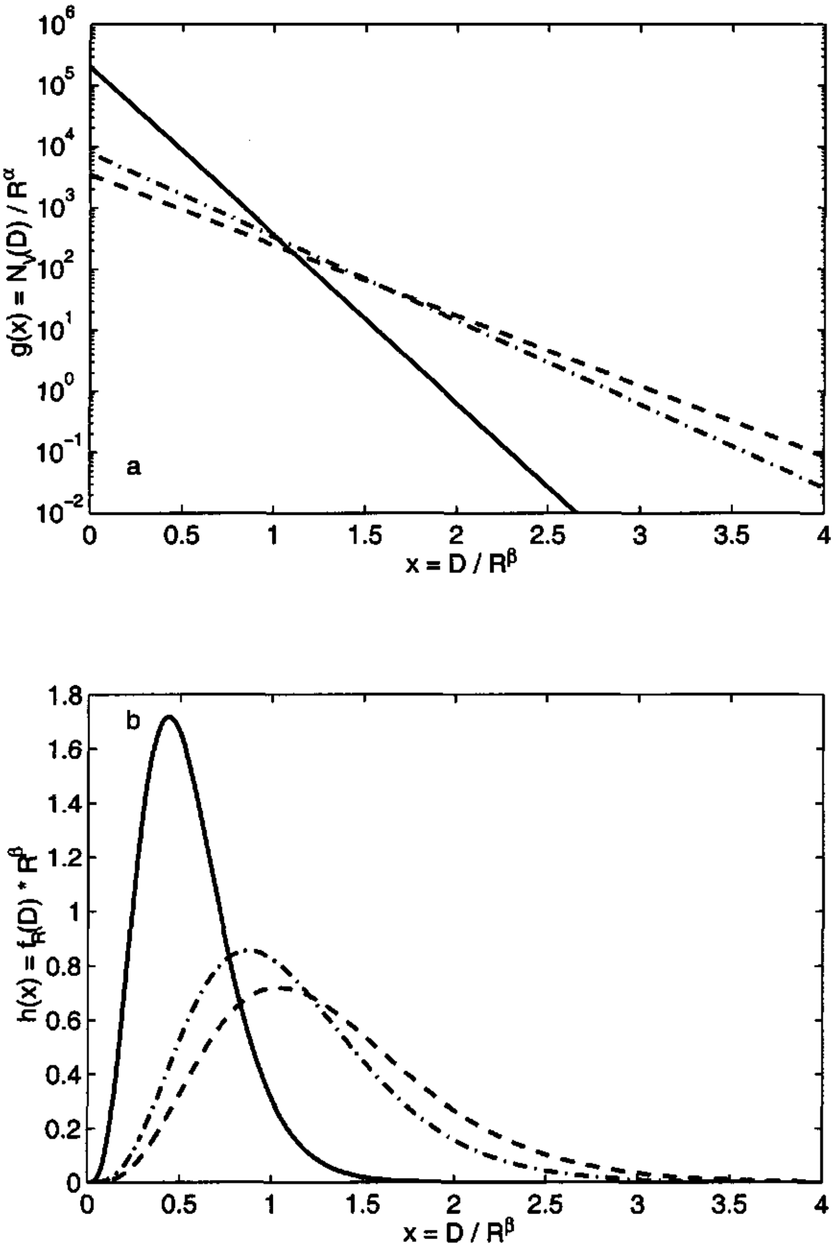

Figure 6.9: (a) Exponential models of the mean general raindrop size distribution functions $g(x)$ ($\mathrm{mm}^{-1}\,\mathrm{m}^{-3}\,(\mathrm{mm\,h}^{-1})^{-\alpha}$), where x ($\mathrm{mm}\,(\mathrm{mm\,h}^{-1})^{-\beta}$) is the scaled raindrop diameter and $\alpha = 1 - 4.67\beta$, for different types of rainfall (based on Battan's (1973) Z–R relationships): orographic (solid line); thunderstorm (dashed line); widespread/stratiform, showers (dash-dotted line). (b) Corresponding general rain rate density functions $h(x)$ ($\mathrm{mm}^{-1}\,(\mathrm{mm\,h}^{-1})^{\beta}$).

value of β for showers (0.321) is suspect. Via Eq. (3.38) (p. 64) it implies $\alpha = -0.499$. Substituting these values in Eqs. (3.5) and (3.6) (p. 59 then yields $\gamma_{\rho_V} = -0.178$. This suggests that the raindrop concentration in case of showers would *decrease* with increasing rain rates, something which seems unlikely.

Table 6.5 also gives the values of the parameters κ and those of the prefactors C_Z (from the expression for the exponential parameterization in Table 6.1) and exponents γ_Z of Z–R relationships as implied by the mean values of β and λ for the different types of rainfall. The Z–R relationships derived in this manner for orographic, thunderstorm and widespread/stratiform rainfall correspond quite closely to those provided by Battan as being 'typical' for these types of rainfall. Fig. 6.8 shows a plot of the obtained Z–R relationships for the four types of rainfall considered. For reference, the envelope of 64 of Battan's 69 Z–R relationships is indicated as well. Fig. 6.9 gives the corresponding general raindrop size distribution functions $g(x)$ and general rain rate density functions $h(x)$. Those for widespread/stratiform rainfall and showers are indistinguishable. Again, qualitatively, the plotted functions have the behavior one would expect for these types of rainfall. This serves to show that the scaling law formulation does not only allow to derive consistent Z–R relationships from raindrop size distribution parameterizations, but consistent raindrop size distribution parameterizations from Z–R relationships as well. In the general framework provided by the scaling law, they are two sides of the same coin.

6.5 Ulbrich's N_0–μ relationship revisited

6.5.1 General relationships implied by the scaling law formulation

Suppose one wants to relax the hypothesis of an exponential raindrop size distribution to a form with more than one parameter. Then the correspondence between a Z–R relationship on the one hand and a raindrop size distribution parameterization on the other will only be unique if the additional parameters are assumed to be constant or if all parameters are assumed to be uniquely related to each other. Several approaches in this direction have been proposed in the literature over the years, in particular for the gamma parameterization. Therefore, the analysis here will be restricted to the latter.

The expression for the prefactor C_Z ($\mathrm{mm^6\,m^{-3}\,(mm\,h^{-1})^{-\gamma_Z}}$) for the gamma parameterization given in Table 6.1 can be inverted to yield for λ (expressed in units of $\mathrm{mm^{-1}\,(mm\,h^{-1})^{\beta}}$)

$$\lambda = \left[\frac{10^4}{6\pi c}\frac{\Gamma(7+\mu)}{\Gamma(4+\gamma+\mu)}\right]^{1/(3-\gamma)} C_Z^{-1/(3-\gamma)} \tag{6.12}$$

Using the corresponding expression for κ ($\mathrm{mm^{-(1+\mu)}\,m^{-3}\,(mm\,h^{-1})^{-(1-(4+\gamma+\mu)\beta)}}$) given

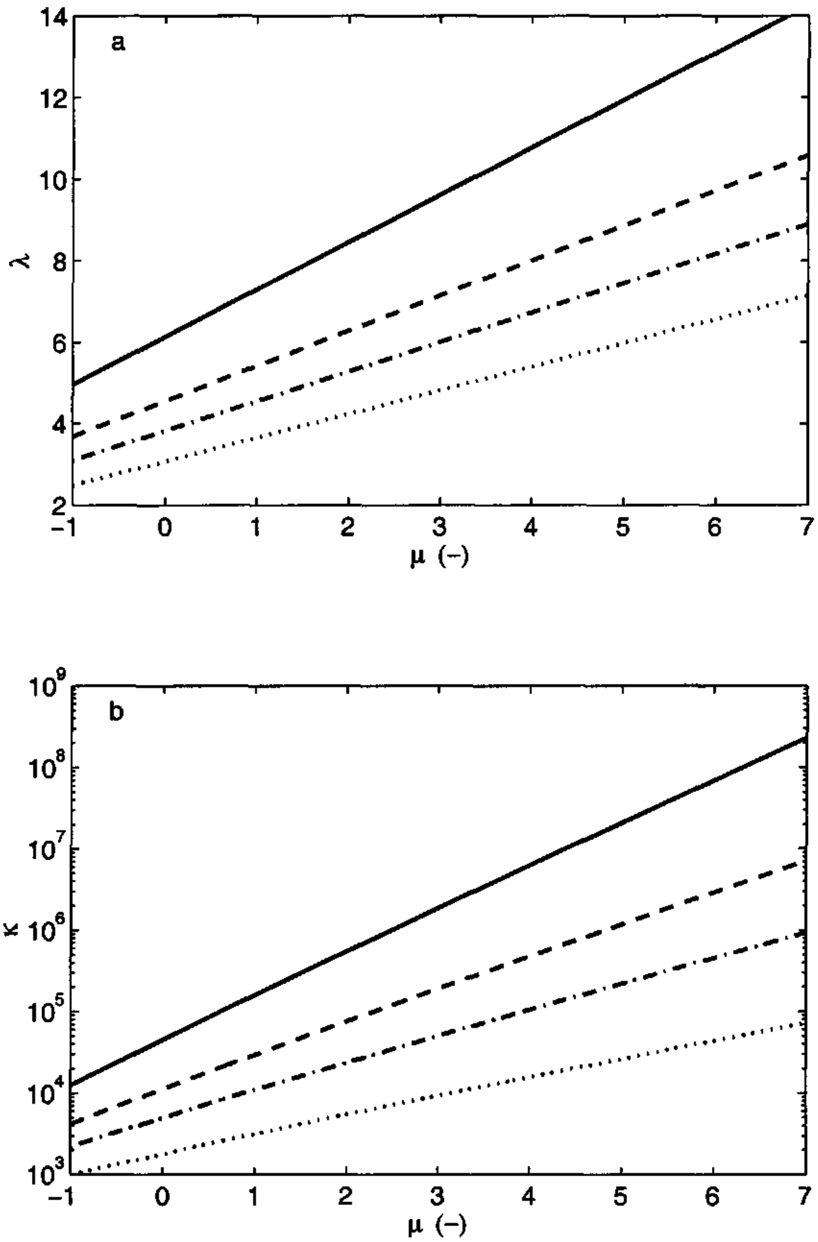

Figure 6.10: (a) Theoretical dependence of the parameter λ ($\mathrm{mm}^{-1}\,(\mathrm{mm\,h}^{-1})^{\beta}$) of the gamma model for $g(x)$ on the parameter μ (–) for given values of the prefactor C_Z of the corresponding Z–R relationship (solid: 100; dashed: 200; dash-dotted: 300; dotted: 500). (b) Idem for the parameter κ ($\mathrm{mm}^{-1}\,\mathrm{m}^{-3}\,(\mathrm{mm\,h}^{-1})^{-(1-(4.67+\mu)\beta)}\,\mathrm{mm}^{-\mu}$).

in Table 3.3, p. 80, Eq. (6.12) implies

$$\kappa = \frac{10^4}{6\pi c}\frac{1}{\Gamma(4+\gamma+\mu)}\left[\frac{10^4}{6\pi c}\frac{\Gamma(7+\mu)}{\Gamma(4+\gamma+\mu)}\right]^{(4+\gamma+\mu)/(3-\gamma)} C_Z^{-(4+\gamma+\mu)/(3-\gamma)}. \tag{6.13}$$

For $\mu = 0$, these expression reduce to Eqs. (6.8) and (6.10) derived for the exponential parameterization. In general however, λ and κ will not only depend on the value of C_Z but on that of μ (–) as well. Fig. 6.10 shows in what manner the parameters λ and κ are related to μ for different values of the prefactor C_Z. Apparently, for the range of values considered here, λ and $\log\kappa$ depend both approximately linearly on μ. Eqs. (6.12) and (6.13) can in principle be employed to estimate, for a particular value of the parameter μ, the parameters λ and κ implied by Battan's Z–R relationships. Above, this has been done for the special case when $\mu = 0$. In general, however, μ will differ from one location to another and its value cannot be obtained from the coefficients of the Z–R relationship.

6.5.2 Ulbrich's approach

By a slightly different approach, Ulbrich (1983) has been able to obtain estimates of μ for all of Battan's Z–R relationships. Using his parameterization for the gamma distribution (Eq. (3.71), p. 82), he obtains expressions for Z and R (and in general any moment of the raindrop size distribution) in terms of N_0 ($\mathrm{mm}^{-(1+\mu)}\,\mathrm{m}^{-3}$), Λ (mm^{-1}) and μ (–). Subsequently, by eliminating[10] Λ, Ulbrich derives a general power law Z–R relationship with an exponent which depends solely on μ and a prefactor which depends on both N_0 and μ. In this manner, he is able to employ the exponents γ_Z of Battan's 69 Z–R relationships to estimate the corresponding parameters μ. From them and the prefactors C_Z he estimates the corresponding values of N_0. Ulbrich subsequently plots $\log N_0$ versus μ for Battan's 69 Z–R relationships and 11 additional power law relationships between other pairs rainfall integral variables (his Fig. 6) and finds an almost perfect linear relationship between the two. A linear regression analysis reveals that $\ln N_0$ (with N_0 expressed in $\mathrm{cm}^{-(1+\mu)}\,\mathrm{m}^{-3}$) is related to μ according to the equation

$$N_0 = C_{N_0} \exp\left(\gamma_{N_0}\mu\right), \tag{6.14}$$

with $C_{N_0} = 6 \times 10^4$, $\gamma_{N_0} = 3.2$ and 'the linear correlation coefficient between $\ln N_0$ and μ for these data greater than 0.98'. If Eq. (6.14) would represent a true physical relation between N_0 and μ, it would have important practical implications for radar remote sensing of rainfall, because it would reduce the number of degrees of freedom of the gamma raindrop size distribution from three to two. Indeed, during the past decade Eq. (6.14) has found wide application in radar meteorology, notably in feasibility studies concerning polarimetric weather radar (see Illingworth and Blackman, 1999 and references therein).

[10] As a matter of fact, Ulbrich (1983) eliminates the median-volume raindrop diameter $D_0 = \frac{3.67+\mu}{\Lambda}$ (mm) instead of the inverse mean diameter Λ (mm^{-1}), but the final result is identical.

How can Ulbrich's approach be understood in terms of the framework developed in this thesis? Since Ulbrich derives one value of N_0 and one value of μ for each Z–R relationship, he implicitly assumes these two variables to be independent of the rain rate R. This is a result of the fact that he eliminates Λ, thereby implicitly assuming that all variation with R arises as a result of variation in Λ. From Eq. (3.72) (p. 83) and Table 3.6 it follows directly that the assumption of a rain rate-independent N_0 implies that

$$1 - (4 + \gamma + \mu)\,\beta = 0, \tag{6.15}$$

or

$$\mu = \beta^{-1} - (4 + \gamma) \tag{6.16}$$

and that consequently

$$N_0 = \kappa. \tag{6.17}$$

It has already been established that the scaling exponent β is related to the exponent γ_Z according to Eq. (6.4) and that the parameter κ is related to the prefactor C_Z according to Eq. (6.13). Substitution of these expressions in Eqs. (6.16) and (6.17) yields

$$\mu = \frac{7 - (4 + \gamma)\,\gamma_Z}{\gamma_Z - 1} \tag{6.18}$$

and

$$N_0 = \left\{ \frac{\Gamma(7 + \mu)}{C_Z\left[6\pi \times 10^{-4} c \Gamma(4 + \gamma + \mu)\right]^{\gamma_Z}} \right\}^{1/(\gamma_Z - 1)}, \tag{6.19}$$

in units of $\mathrm{mm}^{-(1+\mu)}\,\mathrm{m}^{-3}$. These expressions[11] relate μ to γ_Z and N_0 to C_Z and γ_Z. As a matter of fact, they equal Ulbrich's Eqs. (22) and (23) for the special case when the moments involved in the power law relationship are Z and R.

Fig. 6.11 shows plots of λ and $\kappa = N_0$ versus μ (obtained from C_Z and γ_Z using Eqs. (6.12), (6.18) and (6.19)) corresponding to Battan's 69 Z–R relationships. For reference, the curves of Fig. 6.10 have been re-plotted in this figure. It is clear that a significant number of data points correspond to values of μ smaller or equal than -1. For non-truncated raindrop size distributions, these are not permissible, as they correspond to diverging raindrop concentrations ρ_V (Table 3.5). This already indicates one weak point of the approach adopted by Ulbrich.

Another problem associated with his approach is the following. The square of the sample linear correlation coefficient between $\ln \kappa$ ($= \ln N_0$) and μ for the data points in Fig. 6.11(b) (i.e. the coefficient of determination r^2 of a linear regression of $\ln \kappa$ on μ) is only 0.776, whereas Ulbrich (1983) finds $r^2 \approx 0.96$. The origin of this discrepancy is that Ulbrich does not express N_0 in units of $\mathrm{mm}^{-(1+\mu)}\,\mathrm{m}^{-3}$, but in units of $\mathrm{cm}^{-(1+\mu)}\,\mathrm{m}^{-3}$. The corresponding values can easily be obtained from the ones which have already been plotted in Fig. 6.11(b) by multiplying them with $10^{1+\mu}$.

[11] Sempere Torres et al. (1994) use the parameters of gamma distributions adjusted in this manner by Ulbrich (1983) to some of Battan's (1973) Z–R relationships as a means to verify the self-consistency of the scaling exponents α and β. However, Ulbrich's approach guarantees self-consistency and it is therefore not surprising that Sempere Torres et al. find that the obtained exponents satisfy the self-consistency relationship ($\alpha = 1 - (4 + \gamma)\,\beta$) perfectly.

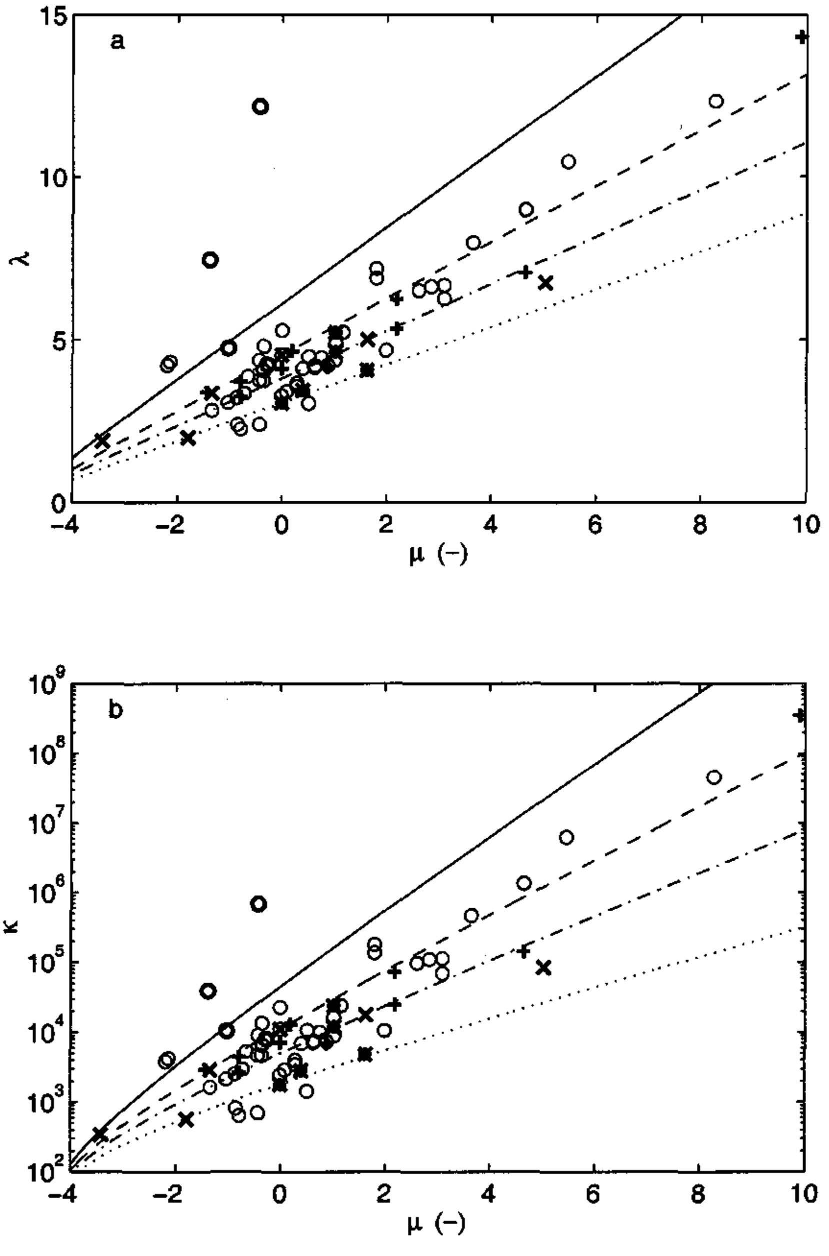

Figure 6.11: (a) Dependence of the parameter λ ($\mathrm{mm}^{-1}\,(\mathrm{mm\,h}^{-1})^{\beta}$, where $\beta = (4.67 + \mu)^{-1}$ in this case) of the gamma model for $g(x)$ on μ (–) for Battan's (1973) 69 Z–R relationships, as implied by Ulbrich's (1983) assumption of a rain rate-independent N_0 (which therefore equals κ). (b) Idem for the parameter $\kappa = N_0$ ($\mathrm{mm}^{-(1+\mu)}\,\mathrm{m}^{-3}$). Bold circles: orographic; bold stars: thunderstorm; bold plusses: widespread/stratiform; bold crosses: showers; circles: no unambiguous identification possible.

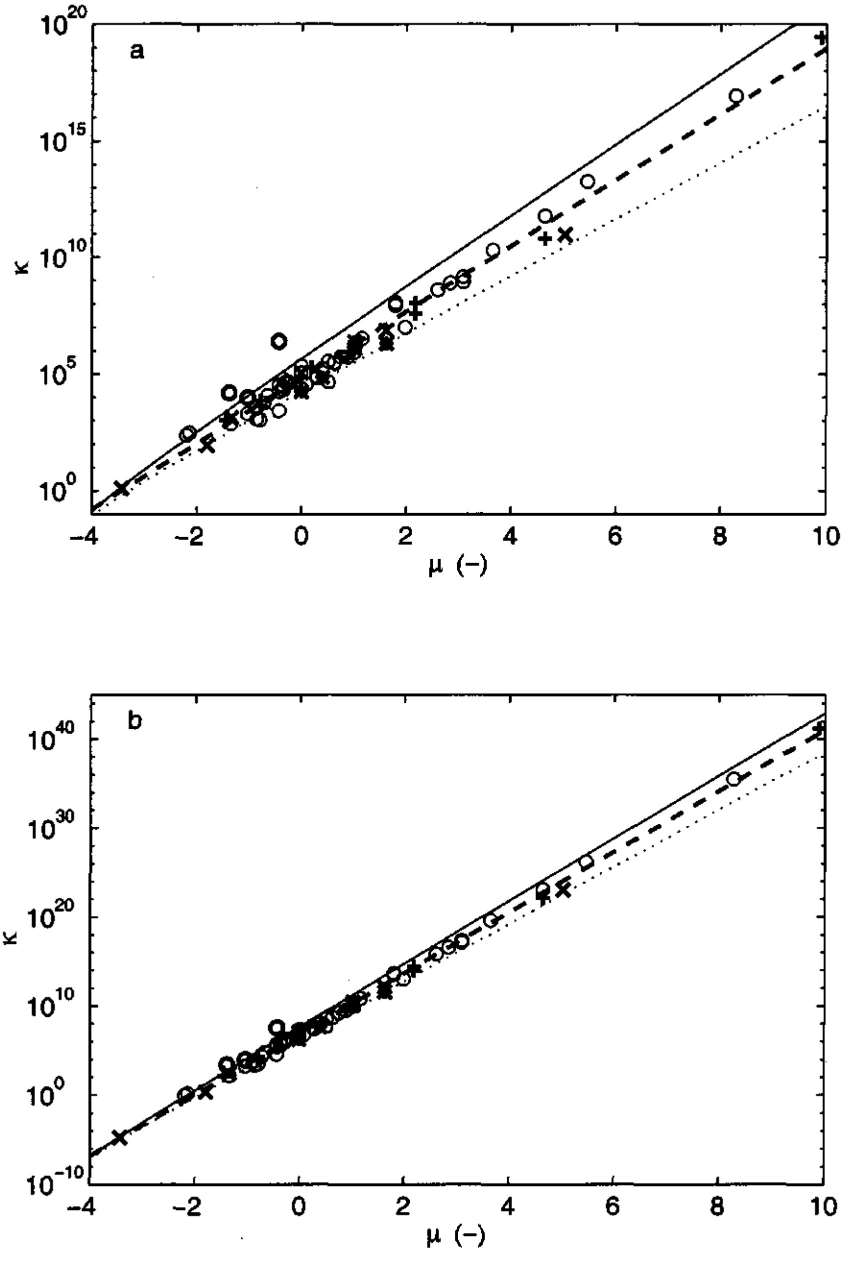

Figure 6.12: (a) Dependence of $\kappa = N_0$ (expressed in units of $\text{cm}^{-(1+\mu)}\,\text{m}^{-3}$) on μ for Battan's (1973) 69 Z–R relationships, as implied by Ulbrich's (1983) assumption of a rain rate-independent N_0. Solid line: theoretical κ–μ relationship for $C_Z = 100$; dashed line: linear regression of $\ln\kappa$ on μ; dotted line: theoretical κ–μ relationship for $C_Z = 500$. (b) Idem for κ expressed in units of $\text{m}^{-(1+\mu)}\,\text{m}^{-3} = \text{m}^{-(4+\mu)}$.

Table 6.6: Dependence on the units of $\kappa = N_0$ of the parameters C_{N_0} and γ_{N_0} and the associated coefficients of determination r^2 of regression relationships of the form $N_0 = C_{N_0} \exp(\gamma_{N_0}\mu)$ for Battan's (1973) 69 Z–R relationships, as implied by Ulbrich's (1983) assumption of a rain rate-independent N_0. As suggested by Ulbrich, the parameters and r^2-values have been obtained from linear regressions of $\ln N_0$ on μ.

Units of $\kappa = N_0$	C_{N_0}	γ_{N_0}	r^2
$\mathrm{mm}^{-(1+\mu)}\,\mathrm{m}^{-3}$	7.00×10^3	0.950	0.776
$\mathrm{cm}^{-(1+\mu)}\,\mathrm{m}^{-3}$	7.00×10^4	3.25	0.976
$\mathrm{m}^{-(1+\mu)}\,\mathrm{m}^{-3} = \mathrm{m}^{-(4+\mu)}$	7.00×10^6	7.86	0.996

The result is shown in Fig. 6.12(a). Visually, the linear dependence of $\ln N_0$ and μ has become significantly stronger. This is confirmed by the corresponding coefficient of determination, which now equals 0.976, approximately equal to that reported by Ulbrich. As a matter of fact, this correlation can be made to increase even more, when N_0 is expressed in SI units ($\mathrm{m}^{-(4+\mu)}$). This corresponds to a further multiplication of the N_0-values by $100^{1+\mu}$. Fig. 6.12(b) shows the result. The dependence of $\ln N_0$ on μ has become almost perfect, without any scatter. The corresponding coefficient of determination has increased to 0.996. Table 6.6 summarizes the results. Also given are the coefficients of a regression relation of the form of Eq. (6.14).

6.5.3 The N_0–μ relationship and spurious correlation

In Appendix G, Ulbrich's approach is analyzed from a theoretical point of view. Two concrete results are obtained. First, it is demonstrated that a first order Taylor series expansion of $\ln N_0$ (= $\ln \kappa$ with κ according to Eq. (6.13)) about $\mu = 0$ yields an expression of the form of Eq. (6.14) with coefficients which are numerically related to the prefactor C_Z of the Z–R relationship according to

$$C_{N_0} = 4.62 \times 10^8 C_Z^{-2.00} \tag{6.20}$$

and

$$\gamma_{N_0} = 3.25 - 0.429 \ln C_Z, \tag{6.21}$$

if $c = 3.778\ \mathrm{m\,s^{-1}\,mm^{-\gamma}}$ and $\gamma = 0.67$ (Atlas and Ulbrich, 1977) and N_0 is expressed in units of $\mathrm{mm}^{-(1+\mu)}\,\mathrm{m}^{-3}$. Note that Eq. (6.20) equals Eq. (6.11), which is logical because for $\mu = 0$ Eq. (6.14) reduces to $N_0 = \kappa = C_{N_0}$. Eqs. (6.20) and (6.21) provide a theoretical explanation for the approximately straight line behavior observed in Fig. 6.10(b) and Fig. 6.11(b) where κ has been plotted against μ on semi-logarithmic paper.

The effect of expressing N_0 in $\mathrm{cm}^{-(1+\mu)}\,\mathrm{m}^{-3}$ instead of in $\mathrm{mm}^{-(1+\mu)}\,\mathrm{m}^{-3}$ is a multiplication of C_{N_0} by 10 and an increase of γ_{N_0} with $\ln 10$. For the typical mean value of $C_Z = 250$, the resulting values are $C_{N_0} = 7.39 \times 10^4$ and $\gamma_{N_0} = 3.18$, quite close to Ulbrich's (1983) empirically determined values (6×10^4 and 3.2). For $C_Z = 277$ the values become almost perfectly equal to those of Ulbrich ($C_{N_0} = 6.02 \times 10^4$ and

$\gamma_{N_0} = 3.14$). $C_Z = 277$ happens to be exactly the (arithmetic) mean value of the prefactors of all 69 Z–R relationships quoted by Battan (1973).

In other words, a first order Taylor series expansion of $\ln N_0$ about $\mu = 0$ shows that the strong positive correlation between $\ln N_0$ (with N_0 expressed in $\mathrm{cm}^{-(1+\mu)}\,\mathrm{m}^{-3}$) and μ found by Ulbrich (1983) can be explained purely on theoretical grounds. It is the result of Ulbrich's implicit assumption that N_0 is independent of rain rate. A closer look at the functional form of the gamma raindrop size distribution ($N_V(D) = N_0 D^\mu \exp(-\Lambda D)$) reveals that when D is expressed in units of cm (a consequence of Ulbrich's units for N_0) then high values of μ correspond to very low values of D^μ, because typical equivalent spherical raindrop diameters are much smaller than 1 cm. In order to compensate for this effect and ensure that total integrals over $N_V(D)$ (such as the rain rate R) remain within physically realistic ranges, the value of N_0 then has to be very high. Exactly the opposite effect is at work for negative values of μ.

A second result of Appendix G is that it is demonstrated that if the units of D are changed such that its original value changes to $D' = s^{-1}D$ and consequently the original value of N_0 changes to $N_0' = s^{1+\mu}N_0$ then the parameters of the corresponding N_0'–μ relationship become

$$C_{N_0'} = sC_{N_0} \tag{6.22}$$

for the prefactor,

$$\gamma_{N_0'} = \gamma_{N_0} + \ln s \tag{6.23}$$

for the exponent and

$$\rho^2_{N_0'} = \frac{\rho^2_{N_0}\left(\gamma_{N_0} + \ln s\right)^2}{\rho^2_{N_0}\left(\gamma_{N_0} + \ln s\right)^2 + \left(1 - \rho^2_{N_0}\right)\gamma^2_{N_0}} \tag{6.24}$$

for the square of the correlation coefficient (coefficient of determination) between $\ln N_0'$ and μ. Eqs. (6.22) and (6.23) can of course be obtained directly from Eq. (6.14) via multiplication with $s^{1+\mu}$. Eq. (6.24) is interesting in that it provides a priori estimates of the change of the correlation coefficient with changes in the units of N_0. Note that $\rho_{N_0'}$ remains positive as long as the slope of the regression line ($\gamma_{N_0'}$) remains positive, i.e. as long as $\gamma_{N_0} + \ln s > 0$. It becomes zero when $s = \exp(-\gamma_{N_0})$. Obviously, $\rho^2_{N_0'}$ reduces to $\rho^2_{N_0}$ for $s = 1$.

The parameter values of Ulbrich's (1983) regression line are $C_{N_0} = 6 \times 10^4$, $\gamma_{N_0} = 3.2$ and $\rho^2_{N_0} = 0.96$ (with N_0 expressed in $\mathrm{cm}^{-(1+\mu)}\,\mathrm{m}^{-3}$). Hence, if the units of N_0 were to be changed to $\mathrm{mm}^{-(1+\mu)}\,\mathrm{m}^{-3}$ (which corresponds to $s = 0.1$) then C_{N_0} would become 6×10^3, γ_{N_0} would be reduced to 0.9 and $\rho^2_{N_0}$ would be reduced to 0.66 (corresponding to a correlation coefficient of approximately 0.8). This is comparable to what has been estimated from Fig. 6.11(b). Similarly, Eqs. (6.22)–(6.24) are able to perfectly predict the values given in Table 6.6. Starting with N_0 expressed in any of the indicated units, these equations can be used to obtain the values of C_{N_0}, γ_{N_0} and $\rho^2_{N_0}$ for N_0 expressed in one of the other units. As the strong positive correlation between $\ln N_0$ and μ found by Ulbrich (1983) can be explained entirely in terms of the particular units he employed for N_0, it is a spurious one. As a matter of fact, it can be made to disappear entirely and even rendered negative by a simple change of units.

In short, the parameter N_0, with units which depend on the value of the parameter μ, is not a very suitable concentration parameter in the gamma raindrop size distribution. Alternative parameters, such as the raindrop concentration (Chandrasekar and Bringi, 1987) or the recently proposed parameters N_L (Illingworth and Blackman, 1999; Illingworth and Johnson, 1999) and N_0^* (Dou et al., 1999; Testud et al., 1999), all parameters with units independent of the value of μ, are preferable[12].

6.6 Summary and conclusions

A new method for establishing power law Z–R relationships has been presented. It is based on the scaling law formulation for the raindrop size distribution. It has been demonstrated that once a self-consistent parameterization for the raindrop size distribution has been established for a particular location, the coefficients of the Z–R relationship follow naturally. They are two sides of the same coin. The exponent of the Z–R relationship is uniquely determined by the value of the scaling exponent β, its prefactor is a function of the parameters of the general raindrop size distribution function (or general rain rate density function). Therefore, *the dependence of the Z–R relationship on the shape of the (scaled) raindrop size distribution is entirely contained in the prefactor.*

Specific expressions have been presented for the exponential, gamma, generalized gamma, Best and lognormal parameterizations for the general raindrop size distribution function. These have subsequently been used to derive Z–R relationships using the parameterizations for the raindrop size distribution obtained in Chapters 4 and 5. The results show that besides a strong climatological variability, Z–R relationships exhibit an even more pronounced inter-event variability (within one rainfall climatology). These observations are consistent with estimates of these variabilities reported in the literature. This suggests that *climatological Z–R relationships are probably of little practical use in the radar estimation of rainfall.* One should be able to distinguish between different types of rainfall, perhaps on the basis of the parameters of the scaling law for the raindrop size distribution. The strong negative dependence observed between the prefactors and exponents of power law R–Z relationships has been shown to be the result of a spurious correlation and should therefore be interpreted with care.

[12]The parameters N_L and N_0^* (which are the same) are related to the concept of the *normalized gamma distribution*. They are defined as the equivalent N_0 of an exponential raindrop size distribution with the same liquid rainwater content W and median-volume raindrop diameter $D_{0,V}$ as the gamma distribution under consideration. Using the definitions of W and $D_{0,V}$ for the exponential distribution given in Chapter 2, it can be shown that N_L and N_0^* are related to N_0 according to

$$N_L = N_0^* = N_0 \frac{\Gamma(4+\mu)}{\Gamma(4)} \left(\frac{3.67}{3.67+\mu}\right)^4 \Lambda^{-\mu}.$$

The 69 Z–R relationships reported by Battan (1973) have been used to estimate the parameters of the corresponding exponential forms of the general raindrop size distribution function. The Z–R data have been stratified according to rainfall type (orographic, thunderstorm, widespread/stratiform and showers) and a mean parameterization has been derived for each type of rainfall. *The obtained functional forms are consistent with the type of rainfall to which they pertain.* It has been demonstrated that if the prefactors and exponents of Z–R relationships are used to estimate the parameters of other forms than the exponential distribution, assumptions have to be made regarding the values of the additional parameters.

One such an approach is that of Ulbrich (1983), who assumes the parameter N_0 of the gamma raindrop size distribution to be independent of rain rate. It has been shown that the widely used exponential N_0–μ relationship he obtains on the basis of an analysis Battan's Z–R relationships is in fact a spurious relationship. It is the result of the fact that the units of N_0 depend on the value of μ. It is therefore recommended to *abandon* N_0 *as concentration parameter* in the gamma raindrop size distribution and replace it in favor of some other parameter.

Chapter 7

Experimental verification of the Poisson homogeneity hypothesis in stationary rainfall[1]

7.1 Introduction

7.1.1 Background

It has been explained in Chapters 1 and 2 that the concept of the raindrop size distribution, used extensively throughout this thesis, is only a useful and valid concept if raindrops, at least over some minimum scale, are distributed homogeneously in space and time. In analogy with the terminology used for transport phenomena in porous media, this minimum spatial scale could be called the *representative elementary volume* of rainfall. If rainfall were not homogeneous over some minimum scale, it would be impossible to define a representative elementary volume and consequently the raindrop size distribution would intrinsically depend on the size of the reference volume to which it pertains. Traditionally however, the raindrop size distribution has been defined *independent* of any notion of spatial or temporal scale, thus implicitly assuming homogeneity at the local scale.

Because both the radar reflectivity factor Z and the rain rate R are defined in terms of the raindrop size distribution (Chapter 1), local homogeneity is a fundamental assumption in radar meteorology as well. In this application, the homogeneity hypothesis is perhaps even stronger than for the concept of the raindrop size distribution as such, because typical radar sample volumes can be as large as 1 km^3. Besides via the raindrop size distribution, the homogeneity assumption also appears in connection with the fluctuation statistics of the backscattered signal ("echo") received from a radar sample volume filled with hydrometeors (in this case raindrops). In particular, it forms the basis of the classical Rayleigh probability distribution for

[1]Adapted version of Uijlenhoet, R., Stricker, J. N. M., Torfs, P. J. J. F., and Creutin, J.-D. (1999). Towards a stochastic model of rainfall for radar hydrology: testing the Poisson homogeneity hypothesis. *Phys. Chem. Earth (B)*, 24:747–755.

the sample-to-sample fluctuations of the echo voltage (i.e. the exponential distribution for those of the echo power). This can be considered one of the cornerstones of radar meteorology (Marshall and Hitschfeld, 1953; Wallace, 1953; Atlas, 1964). Even though the theory of radar echo fluctuations has been generalized to be able to cope with inhomogeneities at larger scales, inducing non-Rayleigh statistics as the combined result of antenna motion and target variability, the assumption of homogeneity at the local scale has not been abandoned (Smith, 1966; Rogers, 1971; Jameson and Kostinski, 1996). The distribution of the sample-to-sample fluctuations determines in what manner the variance of the signal decreases under local averaging and therefore the accuracy with which the mean echo power can be estimated. It is the mean echo power which determines the radar reflectivity factor via the weather radar equation (Eq. (1.1)).

The homogeneity hypothesis is not only fundamental to the concept of the raindrop size distribution and to the principle of weather radar, it has been widely used in the study of sampling fluctuations in rainfall observations as well (e.g. Sasyo, 1965; Cornford, 1967; Cornford, 1968; Joss and Waldvogel, 1969; de Bruin, 1977; Gertzman and Atlas, 1977; Stow and Jones, 1981; Wirth et al., 1983; Wong and Chidambaram, 1985; Chandrasekar and Bringi, 1987; Hosking and Stow, 1987; Chandrasekar and Gori, 1991; Smith et al., 1993; Bardsley, 1995). Most of these investigations have simply assumed rainfall to be homogeneous without testing the hypothesis. Although there is indeed some theoretical justification for using the "law of rare events" as a model of raindrop arrivals at the ground, the choice of the Poisson model seems to have been made mainly because of its mathematical tractability.

Due to the small collector areas (typically 20–200 cm^2) and the short accumulation periods (typically 1–60 s) nowadays commonly employed by surface raindrop measurement devices such as disdrometers and optical spectrometers, the numbers of raindrops in samples collected with such instruments are typically *not* large from a statistical point of view. As a result, the observed rain rate fluctuations must be due 'both to statistical sampling errors and to real fine-scale physical variations which are not readily separable from the statistical ones' (Gertzman and Atlas, 1977) (see Fig. 1.2). As mentioned in Chapter 1, the terminology generally adopted for these two types of fluctuations is *sampling fluctuations* and *natural variability*, respectively. It would be of practical importance to be able to distinguish between both sources of variability, because the parameters of raindrop size distributions and the coefficients of Z–R relationships should represent the properties of the type of rainfall to which they pertain as much as possible and the properties of the raindrop sampling device from which they are derived as little as possible. It is therefore necessary to investigate to what extent rainfall fluctuations observed with different types of instruments reflect the properties of the rainfall process itself and to what extent they are merely instrumental artefacts.

Homogeneity (in a statistical sense) implies that the sampled numbers of raindrops in fixed volumes and time intervals obey Poisson statistics. Recently, several investigations have questioned the validity of the Poisson homogeneity hypothesis in rainfall because it would be unable to cope with the spatial and temporal *clustering* of raindrops observed in reality. Two groups of investigations can be distinguished: (1)

those which propose to generalize the restrictive homogeneous Poisson process (which has a constant mean) to a Poisson process with randomly varying rate of occurrence (i.e. to a so-called doubly stochastic Poisson process or Cox process) (e.g. Smith, 1993a; Kostinski and Jameson, 1997; Jameson and Kostinski, 1998, 1999a; Kostinski and Jameson, 1999a; Jameson et al., 1999); (2) those which propose to abandon the Poisson process framework altogether and replace it with a (multi-)fractal framework (e.g. Lovejoy and Schertzer, 1990; Zawadzki, 1995; Lavergnat and Golé, 1998). The cited doubly stochastic Poisson process models tend to produce clustering of raindrops on certain distinct, predefined spatial and/or temporal scales. The implications of this type of rainfall behavior for sample-to-sample radar echo fluctuations are discussed by Jameson and Kostinski (1999b). (Multi-)fractal processes on the other hand are associated with clustering of raindrops on *all* scales. If rainfall would indeed exhibit such a strong clustering behavior, the implications for radar remote sensing of rainfall would be profound, as pointed out by Lovejoy and Schertzer (1990). For instance, there would no longer be a simple proportionality between the expected number of raindrops in a radar sample volume and the size of that sample volume. Due to increased coherent scattering, it would affect the sample-to-sample echo fluctuations as well. In short, it would essentially be necessary to revise the currently accepted theory of weather radar.

(Multi-)fractal models have originally been used to describe turbulence. Since rainfall is intimately related to the (turbulent) wind field in the atmosphere, it seems natural to use the same approach for modeling rainfall (e.g. de Lima, 1998). However, Fabry (1996) argues that, since raindrops are not passive tracers of the wind field, the analogy between wind and rain may break down at the smallest spatial and temporal scales. The fact that raindrops have different sizes and therefore different fall speeds would tend to filter out the scaling properties of the wind field at those scales. A "white noise" (i.e. homogeneous) regime would be the result.

Additionally, it has recently been demonstrated that one of the strongest empirical arguments in favor of the (multi-)fractal hypothesis at the raindrop scale available to date (the results reported by Lovejoy and Schertzer (1990)) may not be as convincing as it seems (Jameson and Kostinski, 1998). Lovejoy and Schertzer (1990) report on a box counting analysis of blotting paper observations of the spatial distribution of raindrops. They find evidence for the scaling behavior of raindrops in space. However, first of all the limited size of their sample (comprising only 452 raindrop stains) questions the statistical significance of their results. Moreover, since the sizes of the raindrops are not taken into account in their analysis, it remains unclear whether the reported scaling behavior is exhibited to the same extent by raindrops of all sizes. Perhaps the deviation from homogeneity is largely restricted to particular raindrop sizes. Thirdly, Jameson and Kostinski (1998) present the results of a numerical simulation experiment intended to mimic Lovejoy and Schertzer's box counting analysis. They find exactly the same fractal dimension as Lovejoy and Schertzer, even though their simulation is based on uniformly distributed raindrops, consistent with the Poisson hypothesis. This indicates that the fractal dimension reported by Lovejoy and

Schertzer may have been a mere sampling artifact[2].

7.1.2 Objectives

The objective of this chapter is to investigate experimentally whether the raindrop arrival process at the ground can at times be considered a homogeneous Poisson process or whether it systematically exhibits clustering (or possibly even scaling) behavior. Kostinski and Jameson (1997) find indications for Poisson behavior during 'a time of unusually constant flux'. The same authors argue that 'evidence of non-clustering, Poissonian structure conflicts with any ubiquitous fractal description of rain' (Jameson and Kostinski, 1998). It would not conflict with the doubly stochastic Poisson process description of rain, however. The latter contains the homogeneous Poisson process as a limiting case. In view of these arguments, this chapter will report on the analysis of a stationary dataset with mostly sampling fluctuations and very little natural variability. Acceptation of the Poisson homogeneity hypothesis would then automatically imply a rejection of the (multi-)fractal hypothesis (at least at the raindrop scale).

Section 7.2 will present the available dataset and will review some properties of the homogeneous Poisson process which are relevant to the data analysis which follows. In Section 7.3 the results of two types of analysis will be presented: (1) a global analysis taking into account all raindrops regardless of their size; (2) a spectral analysis in which a distinction between the raindrops in the different diameter classes is made. Finally, Section 7.4 will present the summary and conclusions of this chapter.

7.2 Materials and methods

7.2.1 Rainfall data

The available dataset consists of raindrop counts in 16 diameter intervals of 0.21 mm width for 1066 consecutive time intervals of about 10 s duration, i.e. almost 3 h in total. The data have been collected as part of the NERC Special Topic HYREX, a large hydrological radar experiment organized in the United Kingdom, at the Bridge Farm Orchard site on 14 February 1995. The instrument used is an Illingworth-Stevens Paired-Pulse Optical Disdrometer, which has an area presented to the rain of 50 cm^2 (Illingworth and Stevens, 1987). Although the first diameter interval actually comprises all raindrops smaller than 0.72 mm and the last diameter interval all raindrops larger than 3.65 mm, the limits of these intervals are simply taken to be 0.51 mm and 3.86 mm, respectively, in accordance with the class widths of the other intervals. This will not affect the results of the current data analysis, as it will be restricted to the raindrop counts themselves and does not involve the calculation of rainfall integral variables. Since the first diameter interval is at the resolution limit of the instrument

[2]That this is indeed the case can be demonstrated analytically (Uijlenhoet, R. (1999). An explanation for the apparent fractal dimension of homogeneously distributed raindrops. *J. Atmos. Sci.* (submitted)).

and the last interval in general contains only very few raindrops, these diameter intervals are often disregarded in practice anyway (e.g. Hall and Calder, 1993). Rain rates calculated using the observed raindrop counts vary from 0 to 9 $\mathrm{mm\,h^{-1}}$. The average wind speed during the event amounts approximately 3 $\mathrm{m\,s^{-1}}$.

As mentioned in Section 7.1, observed rain rate fluctuations are caused both by sampling fluctuations and by natural variability, which are not readily separable from each other. That is why experimental studies intended to test the Poisson homogeneity hypothesis in rain are often bound to fail. Unless of course there are strong indications that the amount of natural variability present in a particular time series is negligible as compared to the amount of sampling fluctuations. This rare situation happens to be the case in the dataset at hand during a period of 35 min. This period contains 210 consecutive 10 s raindrop size distributions (comprising a total of 6281 raindrops) and is roughly characterized by uncorrelated fluctuations around a constant mean rain rate of about 3.5 $\mathrm{mm\,h^{-1}}$. Fig. 7.1(a) shows the time series of the total raindrop arrival rate for this period, i.e. including all diameter intervals. The 5 min mean raindrop arrival rate drawn in this figure does not display any systematic changes (trends) during this period, which is an indication that the time series may be considered approximately stationary.

Fig. 7.2(a) gives the empirical autocorrelation function calculated from the 210 raindrop count observations. If the true autocorrelation is zero then the sample autocorrelation is known to be approximately normally distributed with mean $\mu = -1/(n-1)$ and variance $\sigma^2 = (n-2)/(n-1)^2$, provided the number of observations n from which the autocorrelation is calculated is large in comparison to the number of time lags considered (e.g. Haan, 1977). In this case $n = 210$, resulting in $\mu \approx -0.005$ and $\sigma \approx 0.07$. This asymptotic property has been used to define $\mu \pm 2\sigma$ as approximate 95% confidence limits for the autocorrelation function. Fig. 7.2(a) shows that although the raindrop arrival rate generally displays very little autocorrelation, it is probably not negligible for the first (10 s) time lag. Nevertheless, the amount of natural variability present in this dataset seems small enough to allow an analysis of the raindrop count fluctuations for the purpose of verifying the homogeneity hypothesis.

7.2.2 The homogeneous Poisson process

If the stochastic process of raindrops arriving at a disdrometer is indeed a homogeneous Poisson process with rate parameter ρ_A ($\mathrm{m^{-2}\,s^{-1}}$), the probability that the instrument with receptor area A ($\mathrm{m^2}$) will catch n $(0, 1, 2, \cdots)$ (–) raindrops in a time interval of length t (s) is

$$\Pr\{\underline{n}(t) = n\} = e^{-A\rho_\mathrm{A}t}\frac{(A\rho_\mathrm{A}t)^n}{n!}. \tag{7.1}$$

This is the frequency function of a Poisson distribution with parameter $A\rho_\mathrm{A}t$ (–) (e.g. Mood et al., 1974). The random variable $\underline{n}(t)$ denotes the number of raindrops caught by the raingauge in an interval of length t. For a homogeneous Poisson process, $\underline{n}(t)$

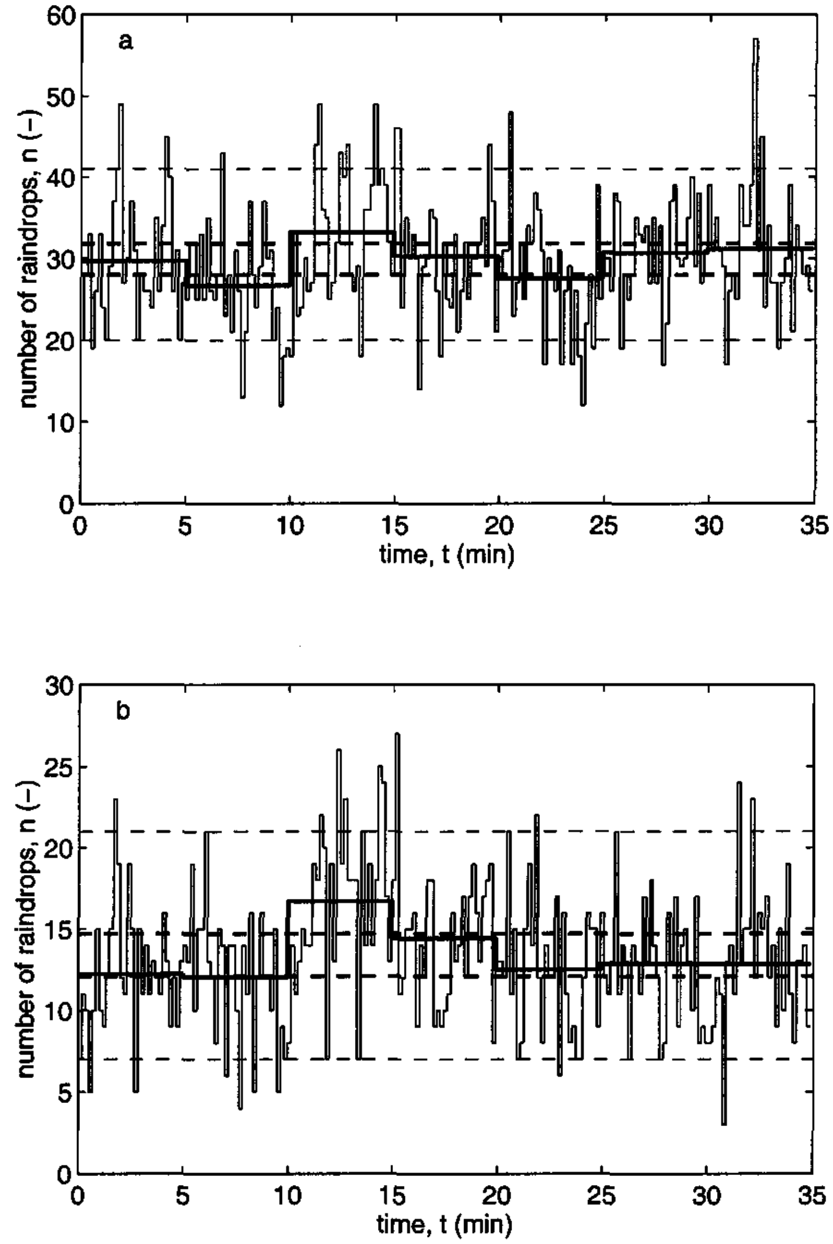

Figure 7.1: Time series of 10 s raindrop counts (thin lines) and 5 min mean raindrop counts (bold lines) together with respective 95% confidence intervals (dashed lines). (a) All raindrop diameters. (b) Raindrop diameters larger than 1.14 mm.

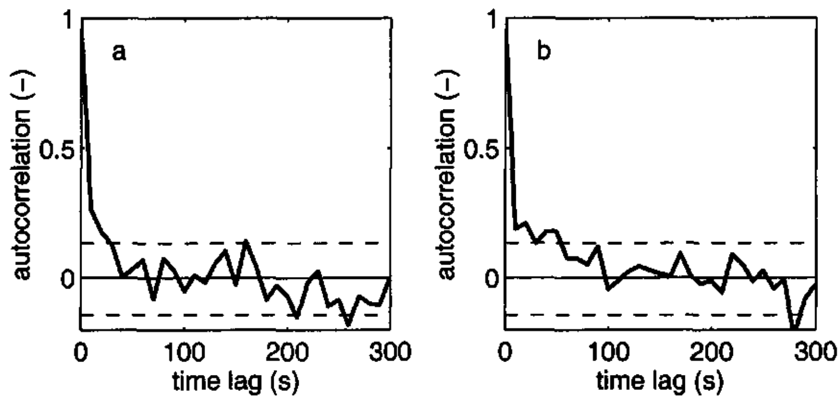

Figure 7.2: Empirical autocorrelation functions of 10 s raindrop counts (solid lines) together with approximate 95% confidence limits (dashed lines). (a) All raindrop diameters. (b) Raindrop diameters larger than 1.14 mm.

has the same probability distribution for any interval of length t. Moreover, for non-overlapping intervals these distributions are independent (e.g. Cox and Isham, 1980). The mean of $\underline{n}(t)$ is equal to its variance and is proportional to t according to

$$\mathrm{E}\left[\underline{n}(t)\right] = \mathrm{Var}\left[\underline{n}(t)\right] = A\rho_{\mathrm{A}}t. \tag{7.2}$$

This property of the Poisson process can be used to define a so-called *dispersion index* as the ratio of Var[$\underline{n}(t)$] to E[$\underline{n}(t)$] (e.g. Cox and Isham, 1980). For a homogeneous Poisson process this index obviously equals one for any duration t. Significant deviations from one observed in real data can then be interpreted as indications for deviations from homogeneous Poisson behavior.

A useful property of the homogeneous Poisson process is that the sum of M independent random variables $\underline{n}_i(t)$ (for $i = 1, 2, \cdots, M$) each distributed according to Eq. (7.1) follows a Poisson distribution with parameter $MA\rho_{\mathrm{A}}t$ (e.g. Mood et al., 1974). This property directly leads to an exact expression for the frequency function of the sample mean (i.e. its sampling distribution) in samples from a homogeneous Poisson process (again for $n = 0, 1, 2, \cdots$), namely

$$\begin{aligned} \Pr\left\{\frac{1}{M}\sum_{i=1}^{M}\underline{n}_i(t) = \frac{n}{M}\right\} &= \Pr\left\{\sum_{i=1}^{M}\underline{n}_i(t) = n\right\} \\ &= e^{-MA\rho_{\mathrm{A}}t}\frac{(MA\rho_{\mathrm{A}}t)^n}{n!}. \end{aligned} \tag{7.3}$$

This expression can be used to obtain an estimate of the confidence interval about the sample mean at a given level of significance. Confidence intervals may aid to distinguish between significant deviations from homogeneous Poisson behavior and sampling fluctuations.

In a similar manner, the exact sampling distributions of the sample frequencies in samples from a homogeneous Poisson process may be obtained. Consider a random sample of M intervals from the Poisson distribution defined by Eq. (7.1). To calculate the sampling fluctuations around $\Pr\{\underline{n}(t) = n\}$, it must be recognized that the sample at hand can be interpreted as a sequence of M independent Bernoulli trials, where for each trial $\Pr\{\underline{n}(t) = n\}$ is the probability of success and $\Pr\{\underline{n}(t) \neq n\} = 1 - \Pr\{\underline{n}(t) = n\}$ the probability of failure. The probability that out of a total of M intervals, m ($\leq M$) intervals contain exactly n raindrops is then found to be governed by the binomial probability distribution

$$\Pr\{\underline{m}(M,n) = m\} = \binom{M}{m} [p(n)]^m [1 - p(n)]^{M-m}. \tag{7.4}$$

Here, the random variable $\underline{m}(M,n)$ denotes the number of intervals out of a total of M containing exactly n raindrops and $p(n)$ is a shorthand notation for $\Pr\{\underline{n}(t) = n\}$. By the same token, the sampling distribution of the sample *cumulative* frequencies follows Eq. (7.4), with $p(n)$ now representing $\Pr\{\underline{n}(t) \leq n\}$. Again, these sampling distributions may be employed to estimate confidence intervals.

7.3 Results and Discussion

7.3.1 Global analysis

Eq. (7.1) has been used to calculate 95% confidence limits for the raindrop count time series in Fig. 7.1(a). These can be found as the 0.025 and 0.975 quantiles of the cumulative Poisson distribution with mean $A\rho t = 6281/210 = 29.9$ raindrops. Fig. 7.1(a) shows that the data exceed these confidence limits 33 times in total, 14 times the upper confidence limit and 19 times the lower limit. At the 95% confidence level one would expect this to happen on the average only about 10 times in 210 observations ($0.05 \times 210 = 10.5$) if these would form a random sample drawn from the Poisson population with mean 29.9. This indicates that the total raindrop count is more dispersed (i.e. more heavily fluctuating) than would be expected on the basis of homogeneous Poisson behavior.

In an entirely analogous manner, Eq. (7.3) has been employed to calculate 95% confidence limits for the 5 min mean raindrop counts shown in Fig. 7.1(a). The result of this exercise is seen to be 3 exceedances out of 7 observations, again indicating a more erratic behavior than expected for a homogeneous Poisson process.

Fig. 7.3 shows the series of Poisson dispersion indices calculated over consecutive 5 min intervals. According to Hosking and Stow (1987), the asymptotic distribution of the dispersion index calculated from a random sample of n observations drawn from a Poisson distribution has mean $\mu = 1$ and standard deviation $\sigma = [2/(n-1)]^{1/2}$. In this case, each 5 min interval contains 30 basic 10 s intervals (i.e. $n = 30$), which yields $\sigma \approx 0.26$. Analogous to what has been done for the autocorrelation function, $\mu \pm 2\sigma$ has been defined as the approximate 95% confidence limits for the Poisson dispersion index. Once again the data seem to indicate overdispersion with respect to a homogeneous Poisson process.

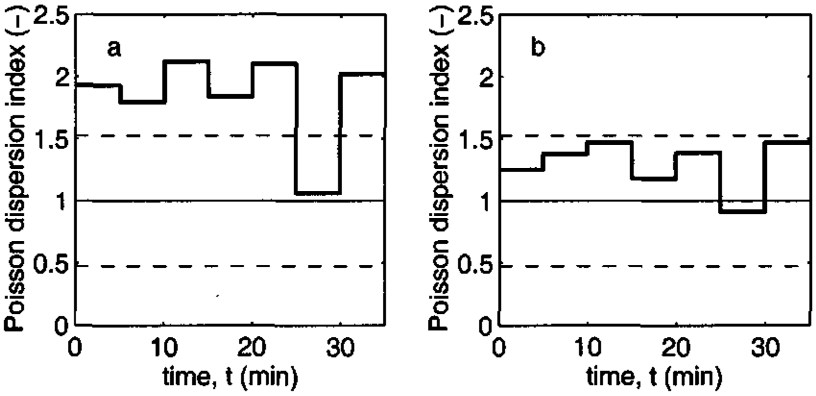

Figure 7.3: Time series of 5 min Poisson dispersion indices of 10 s raindrop counts (solid lines) together with approximate 95% confidence limits (dashed lines). (a) All raindrop diameters. (b) Raindrop diameters larger than 1.14 mm.

Finally, a χ^2 test has been carried out to compare the actual number of observations (out of a total of 210) in 61 different classes (from 0 to 60 raindrops) with the theoretically expected number on the basis of a Poisson distribution with the same mean. The resulting value of the χ^2 goodness-of-fit statistic is found to be 1832. This is two orders of magnitude larger than the 0.95 quantile of a χ^2 distribution with 59 degrees of freedom (e.g. Mood et al., 1974). The conclusion can only be that the hypothesis that the experimental data can be considered a random sample from a Poisson distribution is rejected.

7.3.2 Spectral analysis

The question remains whether the observed deviations from homogeneity are restricted to particular raindrop diameter intervals. To this end, the raindrop count fluctuations have been analyzed for each diameter interval separately. A generalization of a property of the homogeneous Poisson process mentioned in Section 7.2.2 is that the sum of M independent Poisson distributed random variables $\underline{n}_i(t)$ (for $i = 1, 2, \cdots, M$) with *different* means $A\rho_{\mathrm{A},i}t$ follows a Poisson distribution with parameter $A\sum_{i=1}^{M}\rho_{\mathrm{A},i}t$ (e.g. Cox and Isham, 1980). This means that if the raindrop counts in each diameter interval separately behave according to Poisson statistics, the total raindrop count over all intervals together will behave according to Poisson statistics as well.

With this in mind, the empirical frequency function calculated from the 210 observations has been compared for each raindrop diameter interval with the theoretical frequency function expected for a homogeneous Poisson process with the same mean. Fig. 7.4 shows the results for the first 6 intervals, corresponding to diameters from 0.51 mm to 1.77 mm. The error bars in this figure represent 95% confidence limits, calculated using Eq. (7.4). Fig. 7.4 also provides the mean raindrop count, the value

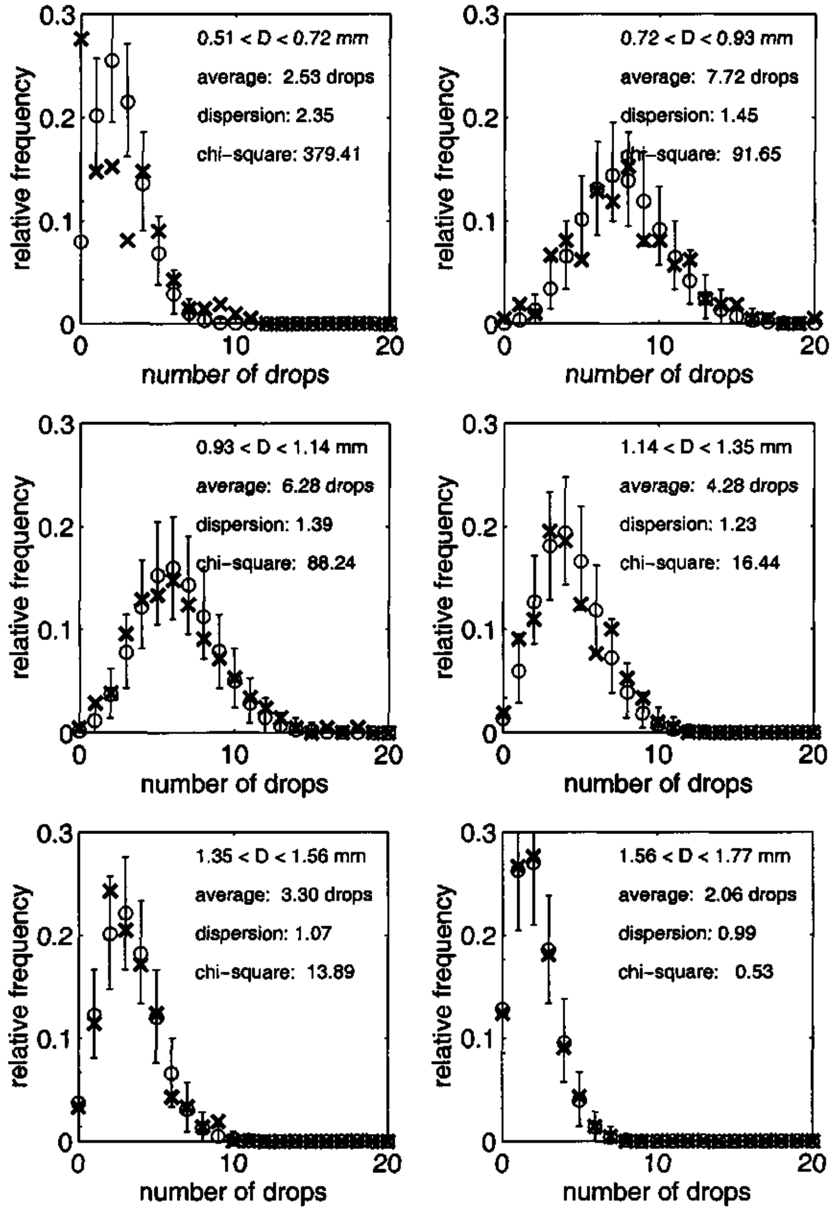

Figure 7.4: Empirical (crosses) and theoretical Poisson (circles) frequency functions of raindrop counts for diameters between 0.51 and 1.77 mm diameter (24 degrees of freedom). Error bars indicate 95% confidence limits. Also indicated are the average number of raindrops per 10 s interval, the Poisson dispersion index and the χ^2 goodness-of-fit statistic.

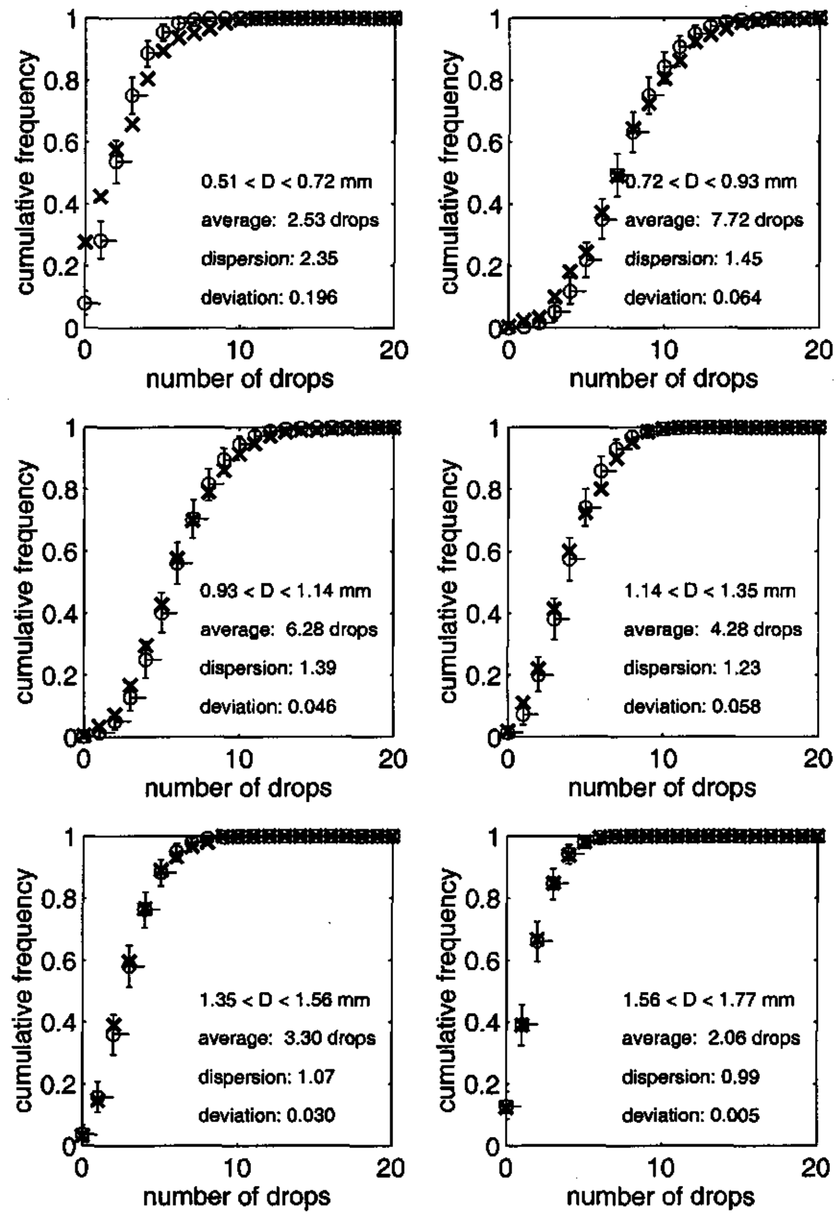

Figure 7.5: Empirical (crosses) and theoretical Poisson (circles) cumulative frequency functions of raindrop counts for diameters between 0.51 and 1.77 mm diameter. Error bars indicate 95% confidence limits. Also indicated are the average number of raindrops per 10 s interval, the Poisson dispersion index and the maximum absolute deviation between the empirical and the theoretical cumulative frequency function.

of the Poisson dispersion index and the value of the χ^2 goodness-of-fit statistic for each diameter interval. Fig. 7.5 gives the corresponding results for the empirical *cumulative* frequency function. Again, the 95% confidence limits have been calculated using Eq. (7.4). This figure also provides the maximum absolute deviation between the empirical and the theoretical cumulative frequency function for each diameter interval.

A visual inspection of Figs. 7.4 and 7.5 reveals that only the first diameter interval shows major deviations from Poisson behavior. For all other intervals the relative frequencies more or less correspond to what can be expected on the basis of Poisson statistics. The fit with the Poisson frequency function becomes nearly perfect for the last diameter intervals[3]. A closer look at the values of the Poisson dispersion indices may give these observations a more quantitative basis. For $n = 210$ observations, the value of the standard error for this index (which is by definition equal to the standard deviation of its sampling distribution $[2/(n-1)]^{1/2}$) becomes about 0.1. This means roughly that there is a probability of only 5 percent that fluctuations outside the range between 0.8 and 1.2 are the result of pure sampling effects. All diameter intervals containing raindrops with diameters larger than 1.14 mm fall roughly within this range. The 0.95 quantiles of χ^2 distributions with 24 degrees of freedom is found to be 36.4 (e.g. Mood et al., 1974). Again, the hypothesis that the raindrop counts can be considered random samples from Poisson distributions is only rejected for the first three diameter intervals, containing raindrops with diameters less than 1.14 mm. Figs. 7.1(b), 7.2(b) and 7.3(b) show the time series of raindrop counts, the empirical autocorrelation function and the time series of Poisson dispersion indices for all raindrops larger than 1.14 mm. These figures confirm the validity of the Poisson homogeneity hypothesis for raindrops of this size.

These findings are in close agreement with those of Hosking and Stow (1987). In a first case study of a time series of total raindrop counts, they find strong deviations from Poisson behavior toward clustering of raindrops as well. Moreover, a second, more detailed case study indicates that clustering occurs predominantly for the smallest raindrops, confirming the observations of this chapter. The main difference between their results and the ones reported here is that they find the transition from clustering to Poisson behavior to occur at about 0.5 mm diameter, as opposed to 1.14 mm for the dataset analyzed here. It is not clear what the cause for this discrepancy is. It could be attributed either to specific instrumental effects or to different local environmental conditions. As for the reason why deviations from Poisson behavior seem to be found mainly for the smallest raindrops in the first place, Hosking and Stow (1987) conclude after a careful analysis of several possible causes that 'the precise mechanism for drop clustering remains obscure'. The results obtained here do not seem to shed any more light on this matter. In any case, they seem to be incompatible with any (multi-)fractal description of rainfall at the considered spatial and temporal scales, except perhaps for the smallest raindrops.

[3]The results for the last 10 diameter intervals (1.77–3.86 mm) are not shown here as they provide little extra information.

7.4 Summary and conclusions

The classical Poisson homogeneity hypothesis in rainfall, a fundamental hypothesis in radar meteorology, has been tested using a unique extraordinary stationary 35 min time series of 10 s raindrop size distributions collected with a 50 cm^2 optical disdrometer. The rain rates calculated from the distributions indicate roughly uncorrelated fluctuations around a constant mean rain rate of about 3.5 $mm\,h^{-1}$.

Two types of analyses of the raindrop counts have been carried out, a global analysis taking into account all raindrops regardless of their size and a "spectral" analysis considering the raindrop counts in the 16 diameter intervals of 0.21 mm width separately. The first type of analysis reveals that even for the more or less stationary time series under consideration *the total raindrop arrival rate is overdispersed with respect to the homogeneous Poisson process.* The second type of analysis demonstrates that this rejection of the homogeneity hypothesis can be attributed entirely to raindrops with diameters smaller than 1.14 mm. Although these raindrops account for 66% of the raindrop concentration in the air and 55% of the raindrop arrival rate at the ground, they only account for 14% of the rain rate and 2% of the radar reflectivity factor (on the basis of the mean raindrop size distribution during the experiment). In other words, although clustering may be a significant phenomenon for the smallest raindrops, the analyzed data seem to indicate that *for moderate rain rates the arrival rate fluctuations of the raindrops which contribute most to rain rate and radar reflectivity factor behave according to Poisson statistics.*

Chapter 8

Summary and conclusions

A comprehensive general framework for the description and analysis of the microstructure of rainfall has been presented. The microstructure of rainfall has been parameterized in terms of the raindrop size distribution. It is the raindrop size distribution which determines both the macroscopic physical properties of rainfall and the relationships between them. Several of these, such as the rain rate R and the radar reflectivity factor Z, have a direct relevance for radar meteorology (radar remote sensing of rainfall) and hydrology (land surface processes).

A rainfall parameterization based on the exponential raindrop size distribution

As an *example* of how the definitions of rainfall related variables in terms of the raindrop size distribution naturally lead to power law relationships, a rainfall parameterization based on the widely used exponential distribution has been presented. First of all, it has been explained that there exist two fundamentally different forms of the raindrop size distribution, namely that *per unit volume of air* and that *per unit surface area and per unit time.*

Subsequently, it has been shown how various hydrologically and meteorologically relevant rainfall variables are related to both these forms of the raindrop size distribution. Three groups of rainfall related variables have been considered, namely *properties of individual raindrops* (size, speed, volume, mass, momentum and kinetic energy), *rainfall integral variables* (raindrop concentration, raindrop arrival rate, liquid rainwater content, rain rate, rainfall pressure, rainfall power and radar reflectivity factor) and *characteristic raindrop sizes* (median-volume diameter, volume-weighted mean diameter and mean-volume diameter). In the treatment of these variables, the importance of the distinction between the properties of raindrops present in a volume of air and those of raindrops arriving at a surface has been emphasized. For the rainfall integral variables, this has lead to a distinction between *state variables*, representing concentrations, and *flux (or rate) variables*, representing flux densities.

Finally, it has been demonstrated how the coefficients of power law relationships between such rainfall variables are determined by the parameters of both forms of the raindrop size distribution, i.e. by the parameters N_0 and Λ of the exponential raindrop size distribution and the coefficients c and γ of the power law relationship between raindrop terminal fall speed and equivalent spherical diameter. Six different consistent sets of power law relationships between the rainfall related variables and rain rate have been derived, based on different assumptions regarding the rain rate dependence of N_0 and Λ. Special attention has been paid to the *internal consistency* of the different sets of power law relationships.

A general framework for the analysis of raindrop size distributions and their properties

Although the widely used exponential distribution can be considered the "null hypothesis" of radar meteorology, it is but one possible analytical form for the raindrop size distribution. There does not seem to be any physical reason why raindrop size distributions observed in nature should necessarily follow the exponential form. Bearing this in mind, the previously presented rainfall parameterization based on the exponential distribution has been generalized to be able to cope with *any* functional form for the raindrop size distribution. In the resulting general framework, the formulation for the raindrop size distribution takes the form of a *scaling law.* This law is consistent with the ubiquitous power law relationships between rainfall related variables. They follow logically from its formulation. Moreover, the scaling law unifies all previously proposed parameterizations for the raindrop size distribution. All can be recast in forms which are consistent with the formulation and as such can be considered as special cases thereof.

In the scaling law formulation, the raindrop size distribution is not only a function of the raindrop diameter, but of a *reference variable* as well. Any rainfall related variable can play the role of reference variable, not necessarily the rain rate historically used for that purpose. The spatial and temporal variability of the reference variable reflects that of the raindrop size distribution. There are two *scaling exponents* associated with the reference variable, one to scale the raindrop diameters and another to scale the corresponding raindrop concentrations. Once these scaling exponents have been estimated, they can be used to scale raindrop size distributions corresponding to different values of the reference variable. The identified curve is a scaled raindrop size distribution, the so-called general raindrop size distribution function, which is in principle independent of the value of the reference variable. The physical interpretation of both the scaling exponents and the general raindrop size distribution function has been clarified. In particular, *the values of the scaling exponents determine whether it is the raindrop concentration or the characteristic raindrop sizes which control the variability of the raindrop size distribution* (as shown in Fig. 3.2, p. 90). A second type of general function has been introduced, the general rain rate density function, which has the advantage of behaving as a probability density function. This will facilitate the parameter estimation process.

Since any reference variable is itself a function of the raindrop size distribution, there exist *self-consistency constraints* both on the scaling exponents and on the general raindrop size distribution function. The constraint on the exponents implies that only one of the two is a free parameter. In case the reference variable is proportional to a moment of the raindrop size distribution, the scaling exponents must be linearly related. The constraint on the general raindrop size distribution function implies that it must satisfy an integral equation. This reduces its number of degrees of freedom by one.

From a practical point of view, the two main advantages of the proposed scaling law procedure over previous approaches are its *robustness* and its *generality*. The robustness of the procedure stems from the fact that all available empirical raindrop size distributions can be used directly to identify the general raindrop size distribution function, thus avoiding the common requirement to calculate average distributions for different classes of the reference variable. The generality of the procedure is due to the fact that *it is no longer necessary to impose an a priori functional form for the raindrop size distribution*. Only after the general raindrop size distribution function has been identified, a suitable parameterization may be selected. This selection will consequently be based on all available information. Expressions have been provided for the self-consistent forms of both types of general functions for all analytical forms of the raindrop size distribution which have been proposed in the literature over the years (exponential, gamma, generalized gamma, Best and lognormal). In this manner, the gap between the scaling law formulation and the traditional analytical parameterizations is bridged explicitly.

Experimental verification of the scaling law using mean raindrop size distributions

In search for further evidence of its validity, the scaling law formulation has been verified experimentally using parameterizations of *mean* raindrop size distributions collected in various climatic settings all over the world.

It has been demonstrated that both Best's (1950b) analytical parameterization for the distribution of the liquid rainwater content over raindrop size and Laws and Parsons' (1943) tabulated parameterization for the distribution of rain rate over raindrop size can be recast in forms which are consistent with the scaling law formulation. This has allowed an identification of the corresponding scaling exponents from previously published adjustments of these parameterizations to measured raindrop size distributions for different types of rainfall in different climatic settings. *The exponents identified in this manner closely satisfy the theoretical self-consistency relationship predicted by the scaling law formulation.* For Best's analytical distributions, these scaling exponents directly lead to analytical parameterizations for the general raindrop size distribution functions and the associated general rain rate density functions.

Interestingly, application of the identified exponents to scale Laws and Parsons' tabulated distributions has also lead to one single general raindrop size distribution function and associated rain rate density function. Both of these are perfectly independent of rain rate, in accordance with the scaling law formulation. Among different analytical descriptions (exponential, gamma, Best and lognormal) of the empirical general raindrop size distribution function, the negative exponential yields the best adjustment.

The obtained results provide further evidence for the scaling law formulation as *the most general description of raindrop size distributions and their properties* consistent with power law relationships between rainfall related variables and as such as a convenient summary of all previously proposed parameterizations in one simple expression.

Experimental verification of the scaling law using raw raindrop size distributions

The successful verification of the scaling law using the parameterizations for the mean raindrop size distribution has provided an important indication for its validity and usefulness. However, the ultimate test has been provided by a direct confrontation with raw (empirical) raindrop size distributions.

In particular, the scaling law formulation and its analysis procedures have been verified experimentally on the basis of raindrop size distributions collected with the filter paper technique at the Royal Netherlands Meteorological Institute in De Bilt, The Netherlands. Two types of analyses have been carried out: (1) an *event-to-event analysis* based on Wessels' (1972) adjustments of Best's parameterization to 476 raindrop size distributions for a series of 28 rainfall events collected during a period of more than a year; (2) a *climatological analysis* based on 446 raw raindrop size distributions from all events together. Both types of analysis have yielded satisfactory results in the sense that it has been possible to estimate the scaling exponents and identify the general raindrop size distribution function and the general rain rate density function.

Although re-sampling of the distributions according to the bootstrap method has indicated that there is an appreciable amount of uncertainty associated with the estimates of the scaling exponents and the parameters of the general functions for each of the 28 rainfall events, they closely satisfy the self-consistency constraints following from the scaling law formulation. Moreover, there seems to be more inter-event variability in the exponents and parameters than can be explained solely on the basis of sampling uncertainties. However, quite disappointingly, an effort to try to relate this variability to differences in various meteorological quantities (type of rainfall, synoptic weather type, atmospheric stability, height of the 0°C isotherm, relative humidity and wind speed) has failed. This suggests that these quantities are not appropriate indicators for the type of rainfall and that *one has to look for other manners to classify different rainfall regimes, perhaps based on the use of radar data.*

The climatological analysis based on the raw raindrop size distribution data has indicated that although there is an appreciable amount of scatter about the mean curves, it is still possible to obtain consistent estimates of the scaling exponents and reasonably accurate fits to the general raindrop size distribution function and general rain rate density function. As such, *this analysis confirms the power of the scaling law formulation as a manner to obtain climatological parameterizations for the raindrop size distribution.* Four different self-consistent parameterizations have been adjusted to the Dutch raindrop size distributions (exponential, gamma, Best and lognormal). The exponential parameterization again seems to provide the best adjustment.

Implications of the scaling law formulation for radar reflectivity – rain rate relationships

As an example of the application of the scaling law formulation, a new method for establishing power law Z–R relationships has been presented. It has been demonstrated that once a self-consistent parameterization for the raindrop size distribution has been established for a particular location, the coefficients of the Z–R relationship follow naturally. They are two sides of the same coin. The exponent of the Z–R relationship is uniquely determined by the value of the scaling exponent β, its prefactor is a function of the parameters of the general raindrop size distribution function (or general rain rate density function). Therefore, *the dependence of the Z–R relationship on the shape of the (scaled) raindrop size distribution is entirely contained in the prefactor.*

Specific expressions have been presented for the exponential, gamma, generalized gamma, Best and lognormal parameterizations for the general raindrop size distribution function. These have subsequently been used to derive Z–R relationships using the parameterizations for the raindrop size distribution obtained in Chapters 4 and 5. The results show that besides a strong climatological variability, Z–R relationships exhibit an even more pronounced inter-event variability (within one rainfall climatology). These observations are consistent with estimates of these variabilities reported in the literature. This suggests that *climatological Z–R relationships are probably of little practical use in the radar estimation of rainfall.* One should be able to distinguish between different types of rainfall, perhaps on the basis of the parameters of the scaling law for the raindrop size distribution. The strong negative dependence observed between the prefactors and exponents of power law R–Z relationships has been shown to be the result of a spurious correlation and should therefore be interpreted with care.

The 69 Z–R relationships reported by Battan (1973) have been used to estimate the parameters of the corresponding exponential forms of the general raindrop size distribution function. The Z–R data have been stratified according to rainfall type (orographic, thunderstorm, widespread/stratiform and showers) and a mean parameterization has been derived for each type of rainfall. *The obtained functional forms are consistent with the type of rainfall to which they pertain.* It has been demonstrated that if the prefactors and exponents of Z–R relationships are used to estimate the parameters of other forms than the exponential distribution, assumptions have to be made regarding the values of the additional parameters.

One such an approach is that of Ulbrich (1983), who assumes the parameter N_0 of the gamma raindrop size distribution to be independent of rain rate. It has been shown that the widely used exponential N_0–μ relationship he obtains on the basis of an analysis Battan's Z–R relationships is in fact a spurious relationship. It is the result of the fact that the units of N_0 depend on the value of μ. It is therefore recommended to *abandon N_0 as concentration parameter* in the gamma raindrop size distribution and replace it in favor of some other parameter.

Experimental verification of the Poisson homogeneity hypothesis in stationary rainfall

Both the established theory of weather radar and the concept of the raindrop size distribution (on which the scaling law is based) rely on the assumption that, at least over certain minimum scales, raindrops are homogeneously distributed in space and time. This so-called Poisson homogeneity hypothesis is generally difficult to verify experimentally due to the strong natural variability of the rainfall process on many scales. However, using a unique extraordinary stationary 35 min time series of 10 s raindrop size distributions collected with a 50 cm^2 optical disdrometer, it has been possible to do just this.

The rain rates calculated from the distributions indicate roughly uncorrelated fluctuations around a constant mean rain rate of about 3.5 $mm\,h^{-1}$. Two types of analyses of the raindrop counts have been carried out, a global analysis taking into account all raindrops regardless of their size and a "spectral" analysis considering the raindrop counts in the 16 diameter intervals of 0.21 mm width separately. The first type of analysis reveals that even for the more or less stationary time series under consideration *the total raindrop arrival rate is overdispersed with respect to the homogeneous Poisson process.* The second type of analysis demonstrates that this rejection of the homogeneity hypothesis can be attributed entirely to raindrops with diameters smaller than 1.14 mm. Although these raindrops account for 66% of the raindrop concentration in the air and 55% of the raindrop arrival rate at the ground, they only account for 14% of the rain rate and 2% of the radar reflectivity factor (on the basis of the mean raindrop size distribution during the experiment). In other words, although clustering may be a significant phenomenon for the smallest raindrops, the analyzed data seem to indicate that *for moderate rain rates the arrival rate fluctuations of the raindrops which contribute most to rain rate and radar reflectivity factor behave according to Poisson statistics.*

Perspectives

Smith (1993) states that 'the study of drop-size distributions, with its roots in both land-surface processes [e.g. interception, erosion, infiltration and surface runoff] and atmospheric remote sensing [radar meteorology], provides an important element to an integrated program of hydrometeorological research'. It has been the aim of this thesis to contribute to such a program by providing the hydrometeorological community with a consistent framework for treating raindrop size distributions and related rainfall properties, i.e. for treating the microstructure of rainfall.

Where to go from here? The research described in this thesis has shown that there are a couple of points which merit further attention. First, there is the observation of the remaining residual scatter about the mean curves when the scaling law is applied to raw raindrop size distribution data (Chapter 5). This is an indication of the fact that not all observed variability can be explained by one single reference variable (in this case the rain rate). This should not be interpreted as a weak point of the scaling law in particular. The use of one single rainfall related variable as explanatory (reference) variable has formed the basis of all previously proposed parameterizations for the raindrop size distribution (Marshall-Palmer, Best, Laws-Parsons) and, moreover, of the ubiquitous power law relationships of radar meteorology. The remaining amount of variability can be associated in part with sampling fluctuations and in part with sources of natural variability. This suggests that it would be useful to further *extend the scaling law formulation in such a manner that it would be able to cope with this excess variability.* A first approach could then be to recognize that the power law relationships between rainfall related variables are not deterministic in nature, but statistical. This leads to a statistical interpretation for the scaling exponents, as is demonstrated in Appendix E. A second approach could be the inclusion of an additional reference variable in the scaling law. In this manner, each rainfall related variable would become a function of two others. As a matter of fact, this type of approach has formed the basis of multi-parameter radar methods.

A second possible extension of the scaling law formulation as presented in this thesis might be the relaxation of the assumption of power law relationships between rainfall related variables altogether. It should be possible to *generalize the scaling law in such a manner that it is able to cope with other than power law relationships.* In this manner, both the shape of the raindrop size distribution and the shape of the relationships between rainfall related variables could be left to follow from the experimental data at hand. It should be noted, however, that the empirical evidence for power law relationships between such variables is very strong (at least in a statistical sense), as has been shown in Chapter 2.

A third possible extension of the scaling law, one which is probably relatively easy to realize, would be to formulate a scaling law for the Doppler velocity spectrum, using the radar reflectivity factor Z as the reference variable. The scaling law as presented in this thesis is a convenient summary of raindrop size distributions. However, since the Doppler velocity spectrum by definition is the distribution of the total reflectivity over the fall speeds of all particles in the radar sample volume, in still air there will exist a direct relationship between the raindrop size distribution and the Doppler velocity spectrum. This notion could allow one to *reformulate the scaling law in terms of the Doppler velocity spectrum.* This might provide interesting possibilities for the analysis of data collected with vertically pointing Doppler radars. A possible product of such analyses might be vertical profiles of the scaling exponents and general distribution functions. Such profiles might contribute to an improved understanding of the physical processes which shape the vertical structure of precipitation.

A final point of attention is that of *sampling fluctuations.* Any surface measurement of raindrop size distributions and consequently any derived rainfall property will be subject to sampling fluctuations (e.g. Smith et al., 1993). This is because the sample sizes employed to estimate raindrop size distributions and their properties are typically *not* large from a statistical point of view (Chapter 7). Hence, if the objective is to relate the values of the scaling exponents and the shapes of the general distribution functions to the natural variability of rainfall, it should be quantified in advance to what extent their estimation is affected by sampling fluctuations. A first step towards tackling this problem has been provided by the results of Chapter 7, where it has been demonstrated that, at least for stationary rainfall conditions, the

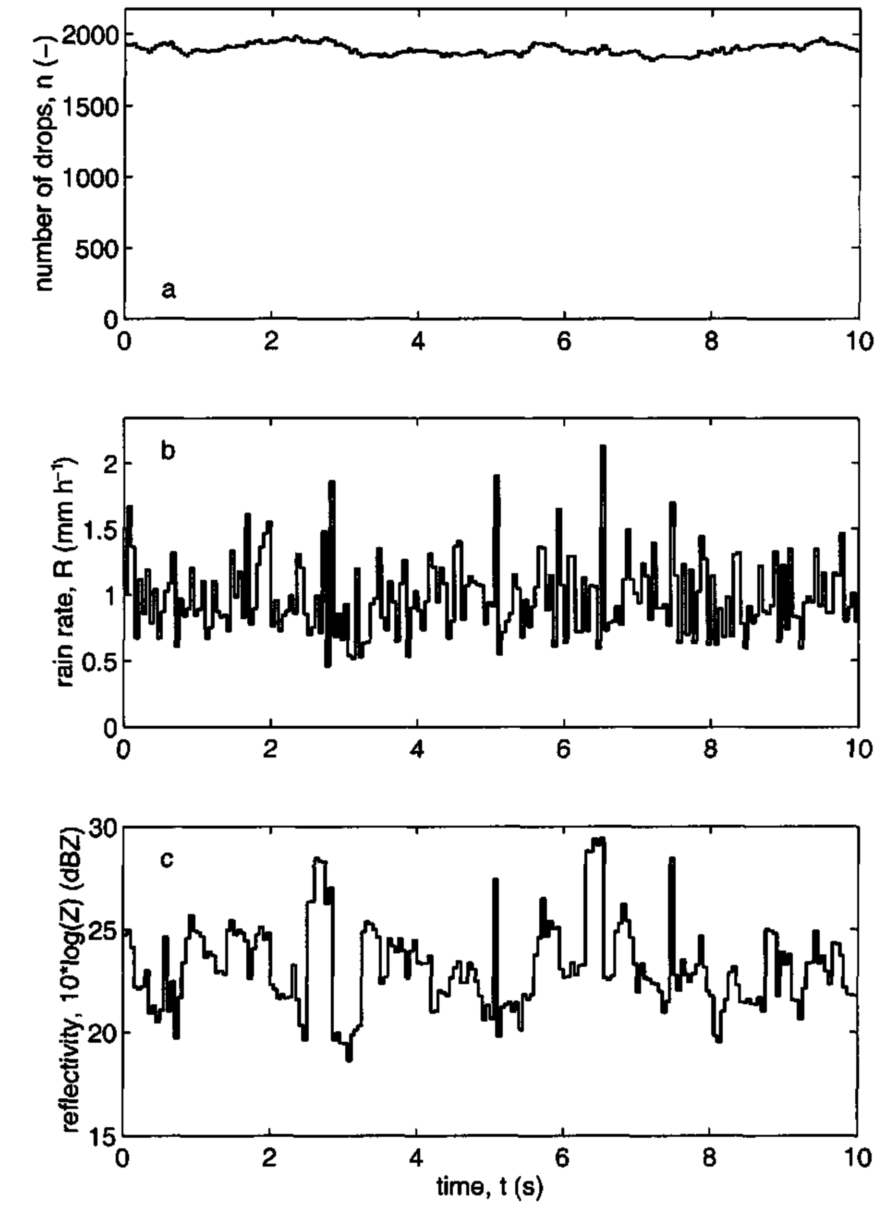

Figure 8.1: Simulation of the temporal evolution of rainfall integral variables in a 1 m^3 sample volume on the basis of the Poisson homogeneity hypothesis: (a) number of raindrops n (–) or raindrop concentration ρ_V (m^{-3}); (b) rain rate R ($mm\,h^{-1}$); (c) radar reflectivity factor Z ($mm^6\,m^{-3}$). Mean rain rate $R = 1$ $mm\,h^{-1}$, Marshall-Palmer raindrop size distribution $N_V(D, R)$, raindrop diameter resolution $\Delta D = 0.1$ mm, maximum raindrop diameter $D_{max} = 6$ mm, time step $\Delta t = 0.05$ s.

raindrop arrival rate fluctuations may be assumed to behave according to Poisson statistics. Preferably, a stochastic model of rainfall to study the sampling problem in its entire complexity should incorporate both sampling fluctuations and natural variability.

It should be noted that, with the smaller and smaller sample volumes employed, (Doppler) *radar observations of precipitation will be subject to sampling fluctuations as well.* For example, suppose the rectangular reference volume indicated in Fig. 1.1 (p. 12) is a radar sample volume of 1 m^3. Fig. 8.1 shows what the temporal (sampling) fluctuations in the raindrop concentration ρ_V (m^{-3}), the rain rate R ($mm\,h^{-1}$) and the radar reflectivity factor Z ($mm^6\,m^{-3}$) might look like for this hypothetical sample volume, assuming a Marshall-Palmer raindrop size distribution and a constant (mean) rain rate of 1 $mm\,h^{-1}$. This simulation has been based on an adapted version of von Smoluchowski's (1916) stochastic model of density fluctuations for intermittent observations (e.g. Fürth, 1918, 1919; Chandrasekhar, 1943; Smith, 1993a). Although ρ_V remains roughly constant, R and particularly Z are observed to fluctuate appreciably. Note the differences in correlation structure between these three rainfall integral variables as well. Again, in a practical situation, a first estimate of the magnitude and speed of these fluctuations may be obtained on the basis of the Poisson homogeneity hypothesis.

Samenvatting en conclusies

De microstructuur van regen, in het bijzonder het concept van de druppelgrootteverdeling, bepaalt de samenhang tussen alle fysische (mechanische en elektromagnetische) eigenschappen van regen. Bovendien bepaalt de druppelgrootteverdeling, althans gedeeltelijk, het discrete karakter van regen wanneer de traditionele continuümbenadering wordt verlaten. Hoewel er sinds de aanvang van het 'moderne' wetenschappelijke onderzoek op dit gebied, ongeveer honderd jaar geleden, veel individuele bijdragen zijn geleverd, ontbreekt tot op heden een algemeen kader voor het bestuderen van druppelgrootteverdelingen. Dit proefschrift beoogt een dergelijk samenhangend raamwerk voor de beschrijving van de microstructuur van regen te geven.

De concrete doelstelling van dit proefschrift is een parametrisatie voor de microstructuur van regen te ontwikkelen ten behoeve van toepassingen in de radarmeteorologie en hydrologie. De term 'parametrisatie' betekent in dit verband dat de belangrijkste aspecten van de microstructuur van regen worden gevat in een beperkt aantal parameters, zoals de concentratie van regendruppels en hun karakteristieke grootte. De variabiliteit van deze parameters in tijd en ruimte bepaalt dan de variabiliteit van iedere afgeleide grootheid en bepaalt bovendien het karakter van de relaties tussen zulke grootheden. Dit zal leiden tot een verbeterd begrip van zowel de (on)mogelijkheden van radar remote sensing van regen als van de interacties tussen regenval en landoppervlak en uiteindelijk tot verbeterde schattingen van de daarmee verband houdende processen.

Een parametrisatie van regen gebaseerd op de exponentiële druppelgrootteverdeling

Als *voorbeeld* van de manier waarop de definities van regenvariabelen in termen van de druppelgrootteverdeling op natuurlijke wijze tot machtsrelaties leiden, wordt in Hoofdstuk 2 een parametrisatie van regen gebaseerd op de veel gebruikte exponentiële druppelgrootteverdeling gepresenteerd. Allereerst wordt uitgelegd dat er twee fundamenteel verschillende vormen van de druppelgrootteverdeling bestaan, namelijk enerzijds die *per eenheid van volume* en anderzijds die *per eenheid van oppervlak en per eenheid van tijd.*

Vervolgens wordt getoond hoe verschillende hydrologisch en meteorologisch relevante regenvariabelen zijn gerelateerd aan deze beide vormen van de druppelgrootteverdeling. Drie groepen regenvariabelen worden onderscheiden, namelijk *eigenschappen van individuele druppels* (grootte, snelheid, volume, massa, impuls en kinetische energie), *integrale regenvariabelen* (druppelconcentratie, druppelfluxdichtheid, vochtgehalte, regenintensiteit, druk, kinetische energie fluxdichtheid en radar reflectiviteit) en *karakteristieke druppelgrootten* (de diameter corresponderend met de mediaan van het vochtgehalte, de volume-gewogen gemiddelde diameter en de diameter corresponderend met het gemiddelde druppelvolume). Bij de behandeling van deze variabelen wordt het belang benadrukt van het onderscheid tussen de eigenschappen van druppels die aanwezig zijn in een volume lucht en de eigenschappen van druppels die aankomen op een oppervlak. Voor de integrale regenvariabelen leidt dit tot een onderscheid tussen *toestandsvariabelen*, die concentraties voorstellen, en *fluxvariabelen*, die fluxdichtheden voorstellen.

Tenslotte wordt gedemonstreerd hoe de coëfficiënten van machtsrelaties tussen regenvariabelen worden bepaald door de parameters van beide vormen van de druppelgrootteverdeling, dat wil zeggen door de parameters N_0 en Λ van de exponentiële druppelgrootteverdeling en de coëfficiënten c en γ van de machtsrelatie tussen de terminale valsnelheid en de equivalente sferische diameter van regendruppels. Op die manier worden zes verschillende consistente groepen van machtsrelaties tussen regenvariabelen en de regenintensiteit afgeleid, gebaseerd op verschillende veronderstellingen betreffende de afhankelijkheid van N_0 en Λ van de regenintensiteit. Speciale aandacht wordt besteed aan de interne consistentie van de verschillende groepen machtsrelaties.

Een algemeen kader voor de analyse van druppelgrootteverdelingen en hun eigenschappen

Hoewel de veel gebruikte exponentiële verdeling beschouwd kan worden als de 'nul hypothese' van de radarmeteorologie, is het slechts één mogelijke analytische vorm voor de druppelgrootteverdeling. Er lijkt vooralsnog geen enkele fysische reden te bestaan om aan te nemen dat druppelgrootteverdelingen zoals die in de natuur worden waargenomen noodzakelijkerwijs een exponentiële vorm moeten hebben. Met dit in gedachten wordt de in Hoofdstuk 2 gepresenteerde parametrisatie van regen gebaseerd op de exponentiële druppelgrootteverdeling in Hoofdstuk 3 veralgemeend tot een parametrisatie die compatibel is met *iedere* willekeurige vorm van de druppelgrootteverdeling. In het resulterende algemene raamwerk neemt de formulering van de druppelgrootteverdeling de vorm aan van een *schaalwet* die consistent is met de alomtegenwoordige machtsrelaties tussen regenvariabelen. Dergelijke relaties volgen op natuurlijke wijze uit de formulering. Bovendien verenigt de schaalwet alle voorheen voorgestelde parametrisaties voor de druppelgrootteverdeling in één formulering. Deze parametrisaties kunnen allemaal worden herschreven in een vorm die consistent is met de schaalwet en kunnen dientengevolge als speciale gevallen daarvan beschouwd worden.

In de formulering als schaalwet is de druppelgrootteverdeling niet alleen een functie van de druppeldiameter maar ook van een zogenaamde *referentievariabele*. Iedere regenvariabele kan in principe de rol van referentievariabele op zich nemen, niet alleen de regenintensiteit die veelal voor dat doel wordt gebruikt. De variabiliteit van de referentievariabele in tijd en ruimte weerspiegelt de variabiliteit van de druppelgrootteverdeling als geheel. Met de referentievariabele zijn twee zogenaamde *schaalexponenten* verbonden, één om de druppeldiameters te schalen en een ander om de corresponderende druppelconcentraties te schalen. Indien deze schaalexponenten eenmaal zijn bepaald, kunnen zij gebruikt worden om druppelgrootteverdelingen die corresponderen met verschillende waarden van de referentievariabele te schalen. De op die manier verkregen curve is een geschaalde druppelgrootteverdeling, de zogenaamde algemene druppelgrootteverdelingsfunctie, die in principe onafhankelijk is van de waarde van de referentievariabele. De fysische interpretatie van beide schaalexponenten en van de algemene druppelgrootteverdelingsfunctie wordt opgehelderd. Het blijkt dat *de waarden van de schaalexponenten bepalen of het de druppelconcentratie is dan wel de karakteristieke druppelgrootte die de variabiliteit van de druppelgrootteverdeling stuurt* (zoals wordt getoond in Fig. 3.2, p. 90). Een tweede type algemene functie wordt geïntroduceerd, de zogenaamde algemene regenintensiteitsdichtheidsfunctie. Die heeft het voordeel zich als een kansdichtheidsfunctie te gedragen, hetgeen het schatten van de parameters vergemakkelijkt.

Aangezien iedere referentievariabele zelf weer een functie is van de druppelgrootteverdeling, leidt de eis van interne consistentie ertoe dat er *beperkingen* gelden zowel ten aanzien van de schaalexponenten als ten aanzien van de algemene druppelgrootteverdelingsfunctie. De beperking ten aanzien van de exponenten heeft tot gevolg dat slechts één van beide een vrije parameter is. Indien, zoals gebruikelijk, de referentievariabele evenredig is met een moment van de druppelgrootteverdeling, zijn de schaalexponenten lineair afhankelijk van elkaar. De beperking ten aanzien van de algemene druppelgrootteverdelingsfunctie heeft de vorm van een integraalvergelijking. Deze reduceert het aantal vrijheidsgraden van de functie met één.

Vanuit praktisch oogpunt zijn de twee belangrijkste voordelen van de voorgestelde schalingsprocedure ten opzichte van de bestaande aanpak dat de procedure *robuust* en *algemeen* is. Robuust in de zin van dat alle beschikbare empirische druppelgrootteverdelingen direct gebruikt kunnen worden om de algemene druppelgrootteverdelingsfunctie te bepalen. Op die manier vervalt de gebruikelijke eis gemiddelde verdelingen voor verschillende klassen van de referentievariabele te berekenen. Algemeen in de zin van dat *het niet langer noodzakelijk is een a priori functionele vorm voor de druppelgrootteverdeling op te leggen.* Een geschikte parametrisatie hoeft pas gekozen te worden nadat de algemene druppelgrootteverdelingsfunctie is bepaald. Deze keuze zal dientengevolge gebaseerd zijn op alle beschikbare informatie. Uitdrukkingen voor de consistente vormen van beide typen algemene functies worden gepresenteerd voor alle analytische vormen van de druppelgrootteverdeling die tot op heden zijn voorgesteld in de literatuur (exponentieel, gamma, gegeneraliseerde gamma, Best en lognormaal). Op die manier wordt de formulering van de druppelgrootteverdeling als schaalwet verenigd met de traditionele analytische parametrisaties.

Experimentele verificatie van de schaalwet op basis van gemiddelde druppelgrootteverdelingen

Op zoek naar nieuw bewijsmateriaal voor het in Hoofdstuk 3 gepresenteerde algemene raamwerk, wordt de formulering van de druppelgrootteverdeling als schaalwet in

Hoofdstuk 4 experimenteel geverifieerd op basis van parametrisaties van *gemiddelde* druppelgrootteverdelingen. De hieraan ten grondslag liggende experimentele verdelingen zijn verzameld in verschillende klimatologische omstandigheden over de gehele wereld.

Gedemonstreerd wordt dat zowel de analytische parametrisatie van Best (1950b) voor de verdeling van het vochtgehalte over de druppelgrootten als de getabelleerde parametrisatie van Laws en Parsons (1943) voor de verdeling van de regenintensiteit over de druppelgrootten herschreven kunnen worden in vormen die consistent zijn met de schaalwet. Daardoor is het mogelijk gebleken voorheen gepubliceerde aanpassingen van deze parametrisaties aan gemeten druppelgrootteverdelingen voor verschillende soorten regen in verschillende klimatologische omstandigheden te gebruiken om de corresponderende schaalexponenten te schatten. *De op deze manier verkregen schaalexponenten blijken nauw te voldoen aan de theoretische relatie die volgt uit de schaalwet en de eis van interne consistentie.* Voor de analytische verdelingen van Best leiden de geschatte schaalexponenten direct tot analytische parametrisaties voor de algemene druppelgrootteverdelingsfuncties en de daaraan gekoppelde algemene regenintensiteitsdichtheidsfuncties.

Interessant is dat het gebruik van de geschatte exponenten om de getabelleerde verdelingen van Laws en Parsons te schalen tot één enkele algemene druppelgrootteverdelingsfunctie en een corresponderende algemene regenintensiteitsdichtheidsfunctie leidt. Beide functies zijn volkomen onafhankelijk van de regenintensiteit, in overeenstemming met de formulering als schaalwet. Van verschillende analytische beschrijvingen (exponentieel, gamma, Best en lognormaal) voor de empirische algemene druppelgrootteverdelingsfunctie blijkt de exponentiële het best te voldoen.

De verkregen resultaten vormen nieuw bewijsmateriaal voor de schaalwet als *de meest algemene beschrijving van druppelgrootteverdelingen en hun eigenschappen*, consistent met machtsrelaties tussen regenvariabelen. Als zodanig vormt de schaalwet een praktische samenvatting van alle tot op heden voorgestelde parametrisaties in één eenvoudige uitdrukking.

Experimentele verificatie van de schaalwet op basis van ruwe druppelgrootteverdelingen

De succesvolle verificatie van de schaalwet op basis van parametrisaties van gemiddelde druppelgrootteverdelingen in Hoofdstuk 4 vormt een belangrijke indicatie voor de geldigheid en bruikbaarheid ervan. Echter, de ultieme test wordt gevormd door een directe confrontatie met ruwe (empirische) druppelgrootteverdelingen.

In Hoofdstuk 4 worden de schaalwet en de daarmee corresponderende analyse procedures experimenteel geverifieerd op basis van druppelgrootteverdelingen verzameld met de filterpapiertechniek op het KNMI in de Bilt. Twee soorten analyses worden uitgevoerd: (1) een *analyse per bui* gebaseerd op Wessels' (1972) aanpassingen van de parametrisatie van Best aan 476 druppelgrootteverdelingen voor een reeks van 28 buien verzameld gedurende een periode van ruim een jaar; (2) een *klimatologische analyse* gebaseerd op 446 ruwe druppelgrootteverdelingen van alle buien tezamen. Beide analyses geven bevredigende resultaten. Het blijkt mogelijk de schaalexponenten te schatten en de algemene druppelgrootteverdelingsfunctie en algemene regenintensiteitsdichtheidsfunctie te bepalen.

Hoewel het herbemonsteren van de verdelingen volgens de 'bootstrap' methode aangeeft dat de schattingen van de schaalexponenten en de parameters van de algemene functies voor ieder van de 28 regenbuien tamelijk onzeker zijn, voldoen zij nauw aan de beperkingen die volgen uit de schaalwet en de eis van interne consistentie. Bovendien blijkt er meer variabiliteit per bui te bestaan in de exponenten en parameters dan verklaard kan worden op basis van steekproefonzekerheden alleen. Enigszins teleurstellend is echter dat het niet mogelijk blijkt deze variabiliteit te verklaren aan de hand van verschillen in een aantal meteorologische grootheden (soort regen, synoptisch weertype, atmosferische stabiliteit, hoogte van de 0°C isotherm, relatieve vochtigheid en windsnelheid). Dit suggereert dat deze grootheden geen geschikte indicatoren zijn voor het type regen en dat *er gezocht dient te worden naar alternatieve manieren om regenregimes te classificeren, wellicht gebaseerd op het gebruik van radargegevens.*

De klimatologische analyse gebaseerd op de ruwe druppelgrootteverdelingen geeft aan dat hoewel er een aanzienlijke hoeveelheid verstrooiing rond de gemiddelde curves is, het nog steeds mogelijk blijkt consistente schattingen van de schaalexponenten te verkrijgen alsmede redelijk nauwkeurige aanpassingen van de algemene druppelgrootteverdelingsfunctie en de algemene regenintensiteitsdichtheidsfunctie. Als zodanig *bevestigt deze analyse de kracht van de schaalwet als een manier om klimatologische parametrisaties voor de druppelgrootteverdeling af te leiden.* Vier verschillende consistente parametrisaties worden aangepast aan de Nederlandse druppelgrootteverdelingen (exponentieel, gamma, Best en lognormaal). Wederom geeft de exponentiële parametrisatie het beste resultaat.

Implicaties van de schaalwet voor radar reflectiviteit – regenintensiteitsrelaties

Als voorbeeld van de toepassing van de schaalwet voor de druppelgrootteverdeling wordt een nieuwe methode voor het bepalen van Z–R machtsrelaties gepresenteerd. Hoofdstuk 6 laat zien dat indien een consistente parametrisatie voor de druppelgrootteverdeling is afgeleid voor een bepaalde locatie, de coëfficiënten van de corresponderende Z–R relatie daaruit rechtstreeks volgen. Parametrisatie en Z–R relatie zijn aldus direct gekoppeld aan elkaar. De exponent van de Z–R relatie wordt ondubbelzinnig bepaald door de waarde van de vrije schaalexponent, de prefactor is een functie van de parameters van de algemene druppelgrootteverdelingsfunctie (of van de algemene regenintensiteitsdichtheidsfunctie). Met andere woorden, *alle informatie over de manier waarop de Z–R relatie afhangt van de vorm van de (geschaalde) druppelgrootteverdeling is volledig geconcentreerd in de prefactor.*

Specifieke uitdrukkingen worden afgeleid voor de exponentiële, gamma, gegeneraliseerde gamma, Best en lognormale vorm voor de algemene druppelgrootteverdelingsfunctie. Deze worden vervolgens gebruikt om Z–R relaties af te leiden op basis van de parametrisaties voor de druppelgrootteverdeling die in Hoofdstukken 4 en 5 zijn verkregen. De resultaten laten zien dat behalve een duidelijke klimatologische variabiliteit, Z–R relaties een nog sterkere variabiliteit per bui (binnen één klimatologische situatie) kennen. Deze bevindingen zijn consistent met schattingen van deze variabiliteit die in de literatuur worden gerapporteerd. Dit suggereert dat *klimatologische Z–R relaties van een beperkt praktisch nut zullen zijn ten behoeve van radar remote sensing van regen.* Het is daarom noodzakelijk verschillende soorten regen te onderscheiden, wellicht gebaseerd op de parameters van de schaalwet voor de druppelgrootteverdeling. Overigens wordt aangetoond dat de sterke negatieve afhankelijkheid die in het algemeen gevonden wordt tussen de prefactoren en de exponenten van Z–R machtsrelaties is toe te schrijven aan een schijncorrelatie en dientengevolge voorzichtig geïnterpreteerd dient te worden.

De 69 Z–R relaties die door Battan (1973) bij elkaar zijn gebracht, worden gebruikt om de parameters te schatten van de corresponderende exponentiële druppelgrootteverdelingen. De Z–R gegevens worden gegroepeerd naar regentype (orografisch, onweer, gelijkmatig/stratiform en buiig) en voor ieder regentype wordt een gemiddelde parametrisatie afgeleid. *De aldus verkregen functionele vormen zijn consistent met het type regenval waar zij betrekking op hebben.* Gedemonstreerd wordt dat indien de prefactoren en exponenten van Z–R relaties gebruikt worden om de parameters te schatten van andere vormen dan de exponentiële druppelgrootteverdeling, veronderstellingen met betrekking tot de waarden van de additionele parameters gemaakt dienen te worden.

Een dergelijke benadering wordt gevolgd door Ulbrich (1983), die aanneemt dat de parameter N_0 van de gamma druppelgrootteverdeling onafhankelijk is van de regenintensiteit. Aangetoond wordt dat de veel gebruikte exponentiële N_0–μ relatie die door Ulbrich wordt gevonden op basis van een analyse van Battan's Z–R relaties in feite een schijnafhankelijkheid is. Het is het gevolg van het feit dat de eenheid van N_0 afhangt van de waarde van μ. *Het wordt daarom afgeraden N_0 nog langer als concentratieparameter in de gamma druppelgrootteverdeling te gebruiken.* N_0 dient vervangen te worden door een parameter met een eenheid die onafhankelijk is van de waarde van μ.

Experimentele verificatie van de Poisson homogeniteitshypothese in stationaire regenval

Zowel de gevestigde theorie van weerradar als het concept van de druppelgrootteverdeling (waarop de schaalwet is gebaseerd) steunt op de veronderstelling dat, tenminste over zekere minimum schalen, regendruppels homogeen verdeeld zijn in tijd en ruimte. Deze zogenaamde Poisson homogeniteitshypothese is in het algemeen lastig experimenteel te verifiëren als gevolg van de sterke natuurlijke variabiliteit van het regenproces. In Hoofdstuk 7 echter wordt een poging hiertoe gedaan, gebruikmakend van een unieke, uitzonderlijk stationaire 35 minuten lange tijdreeks van over 10 seconden geaggregeerde druppelgrootteverdelingen.

De regenintensiteiten berekend uit de druppelgrootteverdelingen geven aan dat er sprake is van ruwweg ongecorreleerde fluctuaties rond een constante gemiddelde regenintensiteit van ongeveer 3.5 millimeter per uur. Twee soorten analyses van de aantallen regendruppels worden uitgevoerd, een globale analyse waarbij alle druppels worden meegenomen, ongeacht hun grootte, en een 'spectrale' analyse waarbij de aantallen druppels in 16 diameter intervallen van 0.21 millimeter breedte apart worden geanalyseerd. De eerste analyse toont aan dat zelfs voor de min of meer stationaire tijdreeks die voorhanden is *de totale druppelfluxdichtheid een sterkere spreiding vertoont dan op basis van een homogeen Poisson proces verwacht mag worden.* De tweede analyse laat zien dat deze verwerping van de homogeniteitshypothese volledig kan worden toegerekend aan regendruppels met diameters kleiner dan 1.14 millimeter. Hoewel deze druppels 66% van de druppelconcentratie in de lucht vertegenwoordigen en 55% van de druppelfluxdichtheid aan de grond, nemen zij slechts 14% van de regenintensiteit en 2% van de radar reflectiviteit voor hun rekening (op basis van de gemiddelde druppelgrootteverdeling gedurende het experiment). Met andere woorden, hoewel clusteren een significant verschijnsel kan zijn voor de kleinste regendruppels, lijken de geanalyseerde gegevens erop te wijzen dat *voor beperkte regenintensiteiten de fluctuaties in de fluxdichtheid van de druppels die het meest bijdragen aan de regenintensiteit en de radar reflectiviteit de Poisson statistiek volgen.*

Epiloog

Smith (1993) merkt op dat 'de bestudering van druppelgrootteverdelingen, met hun wortels zowel in de processen aan het landoppervlak [zoals interceptie van regen door vegetatie en gebouwen, bodemerosie als gevolg van de inslag van regendruppels, infiltratie van regenwater in de bodem en oppervlakkige afvoer] als in de remote sensing van de atmosfeer [zoals radarmeteorologie], een belangrijk element vormt voor een geïntegreerd programma van hydrometeorologisch onderzoek'. Het is de bedoeling van dit proefschrift geweest een bijdrage te leveren aan een dergelijk onderzoeksprogramma door de hydrometeorologische gemeenschap te voorzien van een consistent raamwerk voor het bestuderen van druppelgrootteverdelingen en de daaraan gekoppelde eigenschappen van regen, kortom: zijn microstructuur. Tot slot worden in Hoofdstuk 8 nog een aantal suggesties voor toekomstig onderzoek gedaan.

Appendix A

The method of derived distributions

Suppose the probability density function of a certain non-negative random variable $\underline{D}$ is given by $f_{\underline{D}}(D)$. If another non-negative random variable $\underline{\omega}$ is related to $\underline{D}$ via the power law relationship $\underline{\omega}(\underline{D}) = c_\omega \underline{D}^{\gamma_\omega}$ (where c_ω and γ_ω are both positive coefficients), how will its probability density function $f_{\underline{\omega}}(\omega)$ be related to $f_{\underline{D}}(D)$? This is a special case of a classical problem in statistics, which can be solved via the *method of derived distributions* (e.g. Mood et al., 1974).

The fact that $\omega(D) = c_\omega D^{\gamma_\omega}$ is a monotonically increasing function of D implies

$$\Pr\{\underline{\omega} \leq \omega\} = \Pr\{\underline{D} \leq D(\omega)\}, \tag{A.1}$$

where

$$D(\omega) = \left(\frac{\omega}{c_\omega}\right)^{1/\gamma_\omega}. \tag{A.2}$$

In terms of the probability density functions of $\underline{\omega}$ and $\underline{D}$ this becomes

$$\int_0^\omega f_{\underline{\omega}}(x)\,\mathrm{d}x = \int_0^{D(\omega)} f_{\underline{D}}(x)\,\mathrm{d}x. \tag{A.3}$$

Taking derivatives with respect to ω on both sides of this equality (using Leibniz's rule for differentiating an integral) gives the general relationship

$$f_{\underline{\omega}}(\omega) = f_{\underline{D}}(D(\omega))\left|\frac{\mathrm{d}D(\omega)}{\mathrm{d}\omega}\right|, \tag{A.4}$$

where the absolute value sign $|\,|$ is introduced to ensure that $f_{\underline{\omega}}(\omega)$ remains positive even if the derivative of $D(\omega)$ with respect to ω is negative. In case of a power law $D(\omega)$ relationship, the absolute value sign can be neglected and one simply obtains

$$f_{\underline{\omega}}(\omega) = \frac{1}{c_\omega\gamma_\omega}\left(\frac{\omega}{c_\omega}\right)^{1/\gamma_\omega - 1} f_{\underline{D}}\left[\left(\frac{\omega}{c_\omega}\right)^{1/\gamma_\omega}\right]; \quad c_\omega, \gamma_\omega > 0; \quad \omega \geq 0, \tag{A.5}$$

which is the desired result.

Appendix B

General relationships between the probability density functions of raindrop terminal fall speed in a volume and at a surface

Consider the general relationship between the size distribution of raindrops present in a volume of air $N_V(D)$ (mm^{-1} m^{-3}) and the size distribution of those arriving at a surface $N_A(D)$ (mm^{-1} m^{-2} s^{-1})

$$\begin{cases} N_A(D) = v(D)\, N_V(D) \\ N_V(D) = v(D)^{-1}\, N_A(D) \end{cases}, \tag{B.1}$$

where $v(D)$ denotes the relationship between the terminal fall speed v (m s^{-1}) of a raindrop in still air and its equivalent spherical diameter D (mm). The validity of this formulation is strictly limited to the case where (1) the fall speed of each raindrop is a constant which is entirely determined by its size (and not by other factors such as wind, turbulence or the interaction with other raindrops) and (2) the raindrop size distributions do not depend on time or location (i.e. in the case of stationary rainfall).

In terms of the corresponding probability density functions this implies (Smith, 1993)

$$\begin{cases} \rho_A f_{\underline{D}_A}(D) = v(D)\, \rho_V f_{\underline{D}_V}(D) \\ \rho_V f_{\underline{D}_V}(D) = v(D)^{-1}\, \rho_A f_{\underline{D}_A}(D) \end{cases}, \tag{B.2}$$

where ρ_A (m^{-2} s^{-1}) is the raindrop arrival rate and ρ_V (m^{-3}) the raindrop concentration. Transformations to the corresponding probability density functions of raindrop terminal fall speed in the air $\underline{v}_V$ and at the ground $\underline{v}_A$ yields

$$\begin{cases} \rho_A f_{\underline{v}_A}(v) = v \rho_V f_{\underline{v}_V}(v) \\ \rho_V f_{\underline{v}_V}(v) = v^{-1} \rho_A f_{\underline{v}_A}(v) \end{cases}. \tag{B.3}$$

Integrating both expressions on either side of the equality sign between zero and infinity gives

$$\begin{cases} \rho_{\mathrm{A}} = \rho_{\mathrm{V}} \mathrm{E}\left[\underline{v}_{\mathrm{V}}\right] \\ \rho_{\mathrm{V}} = \rho_{\mathrm{A}} \mathrm{E}\left[\underline{v}_{\mathrm{A}}^{-1}\right] \end{cases}, \tag{B.4}$$

where E[] is the expectation operator. This implies

$$\frac{\rho_{\mathrm{A}}}{\rho_{\mathrm{V}}} = \mathrm{E}\left[\underline{v}_{\mathrm{V}}\right] = \mathrm{E}\left[\underline{v}_{\mathrm{A}}^{-1}\right]^{-1}. \tag{B.5}$$

In other words, the *arithmetic* mean of the fall speeds of raindrops in a volume of air is equal to the *harmonic* mean of the fall speeds of those arriving at a surface.

A second relationship can be obtained by multiplying both sides of the first equation in (B.3) with v and subsequently integrating the result between zero and infinity. This gives

$$\rho_{\mathrm{A}} \mathrm{E}\left[\underline{v}_{\mathrm{A}}\right] = \rho_{\mathrm{V}} \mathrm{E}\left[\underline{v}_{\mathrm{V}}^2\right], \tag{B.6}$$

which is equivalent to

$$\mathrm{E}\left[\underline{v}_{\mathrm{A}}\right] = \frac{\mathrm{E}\left[\underline{v}_{\mathrm{V}}^2\right]}{\mathrm{E}\left[\underline{v}_{\mathrm{V}}\right]}. \tag{B.7}$$

Using the definition of the variance of the raindrop terminal fall speeds in the air ($\mathrm{Var}[\underline{v}_{\mathrm{V}}]$), this gives

$$\mathrm{E}\left[\underline{v}_{\mathrm{A}}\right] = \mathrm{E}\left[\underline{v}_{\mathrm{V}}\right] + \frac{\mathrm{Var}\left[\underline{v}_{\mathrm{V}}\right]}{\mathrm{E}\left[\underline{v}_{\mathrm{V}}\right]}, \tag{B.8}$$

which in terms of the corresponding coefficient of variation $\mathrm{CV}_{\underline{v}_{\mathrm{V}}}$ (the ratio of the standard deviation to the mean) corresponds to

$$\mathrm{E}\left[\underline{v}_{\mathrm{A}}\right] = \left(1 + \mathrm{CV}_{\underline{v}_{\mathrm{V}}}^2\right) \mathrm{E}\left[\underline{v}_{\mathrm{V}}\right]. \tag{B.9}$$

This equation immediately shows that the mean fall speed of raindrops arriving at a surface will generally be larger than that of raindrops present in a volume of air. Only if there is no variability whatsoever in the fall speeds ($\mathrm{CV}_{\underline{v}_{\mathrm{V}}} = 0$), both means will be equal (an obvious result). Similar relations between the higher order moments of the distributions of $\underline{v}_{\mathrm{V}}$ and $\underline{v}_{\mathrm{A}}$ are easy to obtain, but have little practical relevance.

Eq. (B.8) is a well-known result in traffic flow theory, where $\underline{v}_{\mathrm{V}}$ represents the speed of cars present at a certain moment on a particular stretch of highway and $\underline{v}_{\mathrm{A}}$ the speed of cars passing a particular point on that highway during a certain period of time (e.g. Gerlough and Huber, 1975). The analogy with the problem of falling raindrops will be clear.

Appendix C

Consistent sets of power law relationships

Table C.1: Power law relationships of the mean properties of raindrops present in a volume of air (diameter (mm), terminal fall speed ($\mathrm{m\,s^{-1}}$), volume ($\mathrm{mm^3}$), momentum ($\mathrm{kg\,m\,s^{-1}}$) and kinetic energy (J), respectively) with rain rate ($\mathrm{mm\,h^{-1}}$) for six different consistent sets of power law relationships.

Set	$\mu_{\underline{D}_V} \times 10$	$\mu_{\underline{v}_V}$	$\mu_{\underline{V}_V} \times 10^2$	$\mu_{\underline{M}_V} \times 10^7$	$\mu_{\underline{E}_V} \times 10^7$
N_0, Λ	$2.44R^{0.210}$	$1.02R^{0.160}$	$4.56R^{0.630}$	$1.42R^{0.790}$	$2.55R^{0.950}$
N_0, v	$2.36R^{0.214}$	$1.30R^{0.143}$	$4.15R^{0.642}$	$1.47R^{0.786}$	$2.90R^{0.929}$
N_0, Z	$2.31R^{0.229}$	$2.13R^{0.086}$	$3.85R^{0.686}$	$1.51R^{0.771}$	$3.05R^{0.857}$
Λ, v	$2.44R^{0.210}$	$1.33R^{0.141}$	$4.56R^{0.630}$	$1.65R^{0.771}$	$3.32R^{0.911}$
Λ, Z	$2.44R^{0.210}$	$3.60R^{0.030}$	$4.56R^{0.630}$	$2.11R^{0.660}$	$4.89R^{0.690}$
v, Z	$2.20R^{0.258}$	$1.24R^{0.173}$	$3.33R^{0.773}$	$1.12R^{0.945}$	$2.10R^{1.12}$

Table C.2: Power law relationships of the mean properties of raindrops arriving at a surface (diameter (mm), terminal fall speed ($\mathrm{m\,s^{-1}}$), volume ($\mathrm{mm^3}$), momentum ($\mathrm{kg\,m\,s^{-1}}$) and kinetic energy (J), respectively) with rain rate ($\mathrm{mm\,h^{-1}}$) for six different consistent sets of power law relationships.

Set	$\mu_{\underline{D}_A} \times 10$	$\mu_{\underline{v}_A}$	$\mu_{\underline{V}_A} \times 10^2$	$\mu_{\underline{M}_A} \times 10^7$	$\mu_{\underline{E}_A} \times 10^7$
N_0, Λ	$4.30R^{0.210}$	$1.63R^{0.160}$	$13.9R^{0.630}$	$4.98R^{0.790}$	$10.0R^{0.950}$
N_0, v	$3.95R^{0.214}$	$1.90R^{0.143}$	$11.3R^{0.642}$	$4.46R^{0.786}$	$9.65R^{0.929}$
N_0, Z	$3.17R^{0.229}$	$2.47R^{0.086}$	$7.08R^{0.686}$	$2.87R^{0.771}$	$6.00R^{0.857}$
Λ, v	$4.07R^{0.210}$	$1.94R^{0.141}$	$12.4R^{0.630}$	$5.01R^{0.771}$	$11.1R^{0.911}$
Λ, Z	$2.79R^{0.210}$	$3.70R^{0.030}$	$5.85R^{0.630}$	$2.72R^{0.660}$	$6.34R^{0.690}$
v, Z	$3.67R^{0.258}$	$1.81R^{0.173}$	$9.07R^{0.773}$	$3.41R^{0.945}$	$7.01R^{1.12}$

Table C.3: Power law relationships of the rainfall state variables (raindrop concentration (m^{-3}), liquid rainwater content ($mg\,m^{-3}$) and radar reflectivity factor ($mm^6\,m^{-3}$)) with rain rate ($mm\,h^{-1}$) for six different consistent sets of power law relationships.

Set	$\rho_V \times 10^{-3}$	W	Z
N_0, Λ	$1.95R^{0.210}$	$88.9R^{0.840}$	$296R^{1.47}$
N_0, v	$1.89R^{0.214}$	$78.4R^{0.857}$	$237R^{1.50}$
N_0, Z	$1.85R^{0.229}$	$71.1R^{0.914}$	$200R^{1.60}$
Λ, v	$1.69R^{0.229}$	$76.8R^{0.859}$	$255R^{1.49}$
Λ, Z	$1.32R^{0.340}$	$60.1R^{0.970}$	$200R^{1.60}$
v, Z	$2.48R^{0.055}$	$82.4R^{0.827}$	$200R^{1.60}$

Table C.4: Power law relationships of the rainfall flux variables (raindrop arrival rate ($m^{-2}\,s^{-1}$), rainfall pressure (Pa) and rainfall power ($W\,m^{-2}$)) with rain rate ($mm\,h^{-1}$) for six different consistent sets of power law relationships.

Set	$\rho_A \times 10^{-3}$	$P \times 10^4$	$U \times 10^3$
N_0, Λ	$2.00R^{0.370}$	$9.94R^{1.16}$	$2.00R^{1.32}$
N_0, v	$2.46R^{0.358}$	$11.0R^{1.14}$	$2.37R^{1.29}$
N_0, Z	$3.92R^{0.314}$	$11.2R^{1.09}$	$2.35R^{1.17}$
Λ, v	$2.23R^{0.370}$	$11.2R^{1.14}$	$2.47R^{1.28}$
Λ, Z	$4.75R^{0.370}$	$12.9R^{1.03}$	$3.01R^{1.06}$
v, Z	$3.06R^{0.227}$	$10.4R^{1.17}$	$2.15R^{1.35}$

Table C.5: Power law relationships of the characteristic sizes of raindrops present in a volume of air (median-volume diameter, volume-weighted mean diameter and mean-volume diameter (all in mm)) with rain rate ($\mathrm{mm\,h^{-1}}$) for six different consistent sets of power law relationships.

Set	$D_{0,\mathrm{V}} \times 10$	$D_{m,\mathrm{V}} \times 10$	$D_{V,\mathrm{V}} \times 10$
N_0, Λ	$8.95R^{0.210}$	$9.76R^{0.210}$	$4.43R^{0.210}$
N_0, v	$8.67R^{0.214}$	$9.45R^{0.214}$	$4.30R^{0.214}$
N_0, Z	$8.46R^{0.229}$	$9.23R^{0.229}$	$4.19R^{0.229}$
Λ, v	$8.95R^{0.210}$	$9.76R^{0.210}$	$4.43R^{0.210}$
Λ, Z	$8.95R^{0.210}$	$9.76R^{0.210}$	$4.43R^{0.210}$
v, Z	$8.06R^{0.258}$	$8.78R^{0.258}$	$3.99R^{0.258}$

Table C.6: Power law relationships of the characteristic sizes of raindrops arriving at a surface (median-volume diameter, volume-weighted mean diameter and mean-volume diameter (all in mm)) with rain rate ($\mathrm{mm\,h^{-1}}$) for six different consistent sets of power law relationships.

Set	$D_{0,\mathrm{A}} \times 10$	$D_{m,\mathrm{A}} \times 10$	$D_{V,\mathrm{A}} \times 10$
N_0, Λ	$10.8R^{0.210}$	$11.6R^{0.210}$	$6.43R^{0.210}$
N_0, v	$10.3R^{0.214}$	$11.0R^{0.214}$	$6.00R^{0.214}$
N_0, Z	$9.33R^{0.229}$	$10.1R^{0.229}$	$5.13R^{0.229}$
Λ, v	$10.6R^{0.210}$	$11.4R^{0.210}$	$6.19R^{0.210}$
Λ, Z	$9.30R^{0.210}$	$10.1R^{0.210}$	$4.82R^{0.210}$
v, Z	$9.53R^{0.258}$	$10.3R^{0.258}$	$5.58R^{0.258}$

Appendix D

Spilhaus' general raindrop size distribution function revisited

This appendix is intended to reintroduce the general raindrop size distribution function proposed by Spilhaus (1948) to describe Laws and Parsons' (1943) data and to provide some additional results. Although Spilhaus' parameterization has an interesting functional form, it will be demonstrated that it is not entirely consistent and that as such it should be considered with care when applied in its original form.

Spilhaus found that Laws and Parsons' tabulated raindrop size data for all rain rates can be closely described by the (dimensionless) general function

$$\ln \overline{D} f_R(D) = -k_0^2 u^2, \tag{D.1}$$

where $\overline{D}$ is 'the median diameter dividing rain falling on a horizontal surface into equal halves by volume', $100 \times f_R(D)\mathrm{d}D$ is 'the distribution of the percentage volume of total rainfall on a horizontal surface [...] by diameter classes ($\mathrm{d}D$)', k_0^2 is a (dimensionless) factor 'constant for all rains' and

$$u = \left(\frac{D}{\overline{D}}\right)^{1/2} - 1. \tag{D.2}$$

Spilhaus found that when values of $\ln \overline{D} f_R(D)$ and u^2 are computed from Laws and Parsons' measured (smoothed) results and plotted against each other, it follows that 'the linear relationship Eq. (D.1) holds very closely for $k_0^2 = 11.5$, although there is a slight systematic deviation'.

Spilhaus motivated his choice for this particular functional form by his finding that Laws' (1941) earlier measurements of the terminal fall speed of raindrops can be closely described for diameters up to 4 mm by the theoretical relation

$$v = K D^{1/2}, \tag{D.3}$$

with $K = 1.42 \times 10^3\ \mathrm{cm}^{1/2}\,\mathrm{s}^{-1}$ (if D in cm and v in $\mathrm{cm\,s}^{-1}$), i.e. $4.49\ \mathrm{m\,s}^{-1}\,\mathrm{mm}^{-1/2}$ (if D in mm and v in $\mathrm{m\,s}^{-1}$). If this is substituted in Eq. (D.2) then u can be written as

$$u = \frac{v - \overline{v}}{\overline{v}}, \tag{D.4}$$

where $\overline{v}$, defined as $K\overline{D}^{1/2}$, is 'a median fall speed, corresponding to the median-sized drop'. Spilhaus accordingly interpreted u as 'the deviation of the fall speed of any drop from the fall speed of the median-sized drop, expressed as a ratio to the fall speed of the median-sized drop'. He claimed that since 'coalescence of drops of raindrop size depends on collision, the frequency of which in turn depends on relative speed', his general raindrop size distribution function (Eq. (D.1)), essentially corresponding to a normal distribution of the liquid rainwater content over v (as will be shown below, see Eq. (D.22)), is 'therefore not physically unreasonable'.

However, apart from being *physically* reasonable, parameterizations of raindrop size distributions should also be *mathematically* consistent. To verify the self-consistency of Spilhaus' parameterization, an explicit expression for the cumulative distribution of rain rate over drop size, defined as

$$F_R(D) = \int_0^D f_R(x)\,\mathrm{d}x, \tag{D.5}$$

is needed. Such an expression, although not provided by Spilhaus, can be derived from his general size distribution function as follows. Taking exponents on both sides of Eq. (D.1) and subsequently integrating between diameters 0 and D yields

$$F_R(D) = \frac{1}{\overline{D}} \int_0^D \exp\left[-k_0^2 u^2(x)\right] \mathrm{d}x. \tag{D.6}$$

Since $x = \overline{D}\,(u+1)^2$ (from Eq. (D.2)), $\mathrm{d}x = 2\overline{D}\,(u+1)\mathrm{d}u$. A change of variables from x to u therefore gives

$$F_R(D) = 2\int_{-1}^{u(D)} (u+1)\exp\left(-k_0^2 u^2\right) \mathrm{d}u. \tag{D.7}$$

This can also be written as

$$\begin{aligned} F_R(D) \;\; &= \;\; \frac{\sqrt{\pi}}{k_0}\left[\mathrm{erf}\left(k_0 u(D)\right) + \mathrm{erf}\left(k_0\right)\right] + \\ &\quad\; \frac{1}{k_0^2}\left[\exp\left(-k_0^2\right) - \exp\left(-k_0^2 u^2(D)\right)\right], \end{aligned} \tag{D.8}$$

where erf (x) is the error function, defined as

$$\mathrm{erf}\,(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} \mathrm{d}t, \tag{D.9}$$

with $\mathrm{erf}\,(-x) = -\,\mathrm{erf}\,(x)$ (e.g. Abramowitz and Stegun, 1972). $F_R\,(D)$ has a rather tedious functional form. A slight simplification can be achieved by noting that for typical values of k_0^2 (of the order of 10), erf (k_0) is already very close to 1 and $e^{-k_0^2}$ very close to 0. Substituting these limiting values in Eq. (D.8) yields

$$F_R(D) \approx \frac{\sqrt{\pi}}{k_0}\left[\mathrm{erf}\left(k_0 u(D)\right) + 1\right] - \frac{1}{k_0^2}\exp\left(-k_0^2 u^2(D)\right), \tag{D.10}$$

which can be shown to be a very accurate approximation.

A first check as to the consistency of Spilhaus' parameterization is provided by noting that by definition[1] $\int_0^\infty f_R(D)\mathrm{d}D = 1$, which implies $F_R(\infty) = 1$. Hence, from Eq. (D.8),

$$\frac{\sqrt{\pi}}{k_0}\left[1 + \mathrm{erf}\,(k_0)\right] + \frac{1}{k_0^2}\exp\left(-k_0^2\right) = 1, \tag{D.11}$$

an implicit constraint for k_0^2, yielding $k_0^2 \approx 12.6$. For $k_0^2 = 11.5$, the value obtained by Spilhaus, the left-hand side of Eq. (D.11) becomes 1.05, a minor violation of consistency. From Eq. (D.10) an approximation to the consistent value of k_0^2 can be obtained explicitly, yielding $k_0^2 = 4\pi$. That this is indeed a very accurate approximation, can easily be verified by substituting it in the left-hand side of Eq. (D.11). The result deviates from 1 by less than 10^{-6}.

A second check as to the consistency of Spilhaus' parameterization is provided by considering in more detail the definition of $\overline{D}$. According to Spilhaus, $\overline{D}$ is 'the median diameter dividing rain falling on a horizontal surface into equal halves by volume'. This would imply $F_R\left(\overline{D}\right) = 1/2$. From Eq. (D.2) it follows that $u\left(\overline{D}\right) = 0$. Substituting these values in Eq. (D.8) yields

$$\frac{\sqrt{\pi}}{k_0}\,\mathrm{erf}\,(k_0) + \frac{1}{k_0^2}\left[\exp\left(-k_0^2\right) - 1\right] = \frac{1}{2}, \tag{D.12}$$

another implicit constraint for k_0^2, giving $k_0^2 \approx 8.07$. For $k_0^2 = 11.5$, the left-hand side of Eq. (D.12) becomes 0.436, again a consistency violation. An explicit approximation to the consistent value of k_0^2 can again be obtained from Eq. (D.10). This yields the quadratic equation $k_0^2 - 2k_0\sqrt{\pi} + 2 = 0$, with roots $k_0 = \sqrt{\pi} \pm \sqrt{\pi - 2}$. Substitution of these roots in Eq. (D.12) shows that only $k_0 = \sqrt{\pi} + \sqrt{\pi - 2}$ satisfies the constraint and it is therefore the desired solution.

Apart from the fact that Spilhaus' value for k_0^2 is neither consistent with the constraint imposed by Eq. (D.11) nor with the constraint imposed by Eq. (D.12), the latter two are not consistent with each other either. This implies that $\overline{D}$, in contrast to what Spilhaus suggested, cannot be the median of $f_R(D)$. To what characteristic diameter does $\overline{D}$ correspond then? From Eq. (D.8) it follows that

$$F_R\left(\overline{D}\right) = \frac{\sqrt{\pi}}{k_0}\,\mathrm{erf}\,(k_0) + \frac{1}{k_0^2}\left[\exp\left(-k_0^2\right) - 1\right]. \tag{D.13}$$

Combining this with Eq. (D.11) yields

$$F_R\left(\overline{D}\right) = 1 - \frac{\sqrt{\pi}}{k_0} - \frac{1}{k_0^2}, \tag{D.14}$$

which, for $k_0^2 = 4\pi$, leads to $F_R\left(\overline{D}\right) = 1/2 - 1/(4\pi) \approx 0.420$. Starting with the approximation of Eq. (D.10) would yield numerically the same result. Hence, to render

[1]Although taking the limiting raindrop diameters as 0 and ∞ is indeed a simplification of matters, Spilhaus justified it by noting that 'the limits are taken from 0 to ∞ for simplicity, with no great error because in normal rains the number of drops approaching maximum size (about 7 mm) is negligible and the contribution of drops less than "raindrop" size (0.5 mm) is likewise negligible'.

Spilhaus' parameterization (Eq. (D.1)) consistent, $\overline{D}$ would have to correspond to the 42nd percentile of the distribution of rain rate over drop size, at least for diameter integration limits of 0 and ∞.

For many applications, it is not only the distribution of rain rate over drop size ($f_R(D)$) which is of interest, but that of liquid rainwater content ($f_W(D)$) as well. For diameter integration limits of 0 and ∞, the latter is related to the former according to

$$f_W(D) = \frac{v^{-1}(D)\, f_R(D)}{\int_0^\infty v^{-1}(D)\, f_R(D)\, \mathrm{d}D}, \tag{D.15}$$

where $v(D)$ is the relation between terminal fall speed and drop diameter. In case of Spilhaus' $v(D)$ relation (Eq. (D.3)), this becomes

$$f_W(D) = \frac{D^{-1/2} f_R(D)}{\int_0^\infty D^{-1/2} f_R(D)\, \mathrm{d}D}. \tag{D.16}$$

Upon substitution of Eq. (D.1), the integral in the denominator can be written as

$$\int_0^\infty D^{-1/2} f_R(D)\, \mathrm{d}D = \frac{1}{\overline{D}} \int_0^\infty D^{-1/2} \exp\left[-k_0^2 u^2(D)\right] \mathrm{d}D. \tag{D.17}$$

A change of variables from D to u on the right-hand side gives

$$\int_0^\infty D^{-1/2} f_R(D)\, \mathrm{d}D = 2\overline{D}^{-1/2} \int_{-1}^\infty \exp\left(-k_0^2 u^2\right) \mathrm{d}u, \tag{D.18}$$

which can also be written as

$$\int_0^\infty D^{-1/2} f_R(D)\, \mathrm{d}D = \sqrt{\frac{\pi}{k_0^2 \overline{D}}} \left[1 + \operatorname{erf}(k_0)\right]. \tag{D.19}$$

Since, as has been shown above, $\operatorname{erf}(k_0)$ is approximately 1, this integral is to a very close approximation equal to $\sqrt{4\pi / \left(k_0^2 \overline{D}\right)}$. Substituting this in Eq. (D.16) yields

$$f_W(D) \approx \sqrt{\frac{k_0^2}{4\pi}} \frac{\exp\left[-k_0^2 u^2(D)\right]}{\overline{D}\left[u(D) + 1\right]}. \tag{D.20}$$

In terms of the variable u this becomes

$$f_W(u) \approx \frac{k_0}{\sqrt{\pi}} \exp\left(-k_0^2 u^2\right), \tag{D.21}$$

an approximation found by Spilhaus as well. However, a final change of variables to v yields an even more instructive result, namely

$$f_W(v) \approx \frac{1}{\overline{v}/k_0 \sqrt{\pi}} \exp\left[-\left(\frac{v - \overline{v}}{\overline{v}/k_0}\right)^2\right]. \tag{D.22}$$

This function corresponds to the probability density function of a normal distribution with a mean of $\overline{v}$ and a standard deviation of $\overline{v}/\sqrt{2k_0^2}$, which for $k_0^2 = 4\pi$ becomes approximately $\overline{v}/5$ (e.g. Mood et al., 1974). The fact that for typical values of k_0^2 the standard deviation is much smaller than the mean guarantees that the probability mass corresponding to negative fall speeds is negligible (it is less than 10^{-6} if $k_0^2 = 4\pi$). Finally, Eq. (D.22) also resolves the problem of the definition of $\overline{D}$: it is not the median of $f_R(D)$, as Spilhaus erroneously assumed, but that of $f_W(D)$.

Appendix E

A statistical interpretation of the scaling exponent β

E.1 Introduction

The starting point for the developments in this appendix is the dimensionless form of the raindrop size distribution $N_V(D)$ [L^{-4}] derived in Chapter 3 (Eq. (3.2)), i.e.

$$N_V(D) = \frac{\rho_V}{D_C} f_{D_C}\left(\frac{D}{D_C}\right), \tag{E.1}$$

where ρ_V [L^{-3}] is the raindrop concentration and D_C [L] a characteristic raindrop diameter. In the derivation of the scaling law (Eq. (3.4)) it has been assumed that both ρ_V and D_C are uniquely related (via power law relationships) to a reference variable Ψ, itself a function of the raindrop size distribution. The net effect of this assumption is that the raindrop size distribution effectively becomes a distribution which depends on only one parameter, namely the reference variable Ψ. However, the results of Chapter 5 have indicated that not all observed variability in raindrop size distributions and power law relationships can generally be explained by one single reference variable, suggesting the extension of the scaling law formulation in such a manner that it would be able to cope with the excess variability. In this appendix, a first approach to such an extension is investigated. The problem is treated in a statistical framework. The analysis is restricted to the common case where the rain rate R plays the role of reference variable. However, the results are easily generalized to any other choice for the reference variable.

Consider rainfall related variables which are proportional to the moments of the raindrop size distribution, i.e.

$$\Omega_m = c_{\Omega_m} \int_0^\infty D^m N_V(D)\,\mathrm{d}D. \tag{E.2}$$

Substitution of Eq. (E.1) yields

$$\Omega_m = c'_{\Omega_m} \rho_V D_C^m, \tag{E.3}$$

where

$$c'_{\Omega_m} = c_{\Omega_m} \int_0^\infty x^m f_{D_C}(x)\,\mathrm{d}x \tag{E.4}$$

and $x = D/D_C$ is a dimensionless raindrop size. In particular, if ρ_V is expressed in m^{-3} and D_C in mm, the definition of the rain rate R ($\mathrm{mm\,h^{-1}}$) becomes

$$R = c'_R \rho_V D_C^{3+\gamma}, \tag{E.5}$$

with

$$c'_R = 6\pi \times 10^{-4} c \int_0^\infty x^{3+\gamma} f_{D_C}(x)\,\mathrm{d}x, \tag{E.6}$$

where c and γ are the coefficients of a power law relationship between raindrop terminal fall speed v ($\mathrm{m\,s^{-1}}$) and equivalent spherical diameter D (mm). It is seen that in general, a moment of the raindrop size distribution depends on *at least two* variables: the raindrop concentration ρ_V, a characteristic raindrop diameter D_C and possibly one or more (dimensionless) coefficients characterizing the shape of $f_{D_C}(x)$, such as its coefficients of variation, skewness or kurtosis. If, for a given climatology or a given type of rainfall, the latter are assumed to be constant (i.e. independent of the value of any rainfall related variable), only two variables remain: ρ_V and D_C.

E.2 A statistical approach

As a generalization of the methodology applied by Smith and Krajewski (1993), a simple linear regression approach is adopted to tackle the problem. Taking natural logarithms on both sides of Eqs. (E.3) and (E.5) yields

$$\ln \Omega_m = \ln c'_{\Omega_m} + \ln \rho_V + m \ln D_C \tag{E.7}$$

and

$$\ln R = \ln c'_R + \ln \rho_V + (3+\gamma) \ln D_C. \tag{E.8}$$

If $\underline{\Omega}_m$, $\underline{R}$, $\underline{\rho}_V$ and $\underline{D}_C$ are now assumed to be random variables, the slope of a simple linear regression of $\ln \underline{\Omega}_m$ on $\ln \underline{R}$ is given by

$$\widehat{\gamma}_{\Omega_m} = \frac{\mathrm{Cov}\left(\ln \underline{\Omega}_m, \ln \underline{R}\right)}{\mathrm{Var}\left(\ln \underline{R}\right)} \tag{E.9}$$

and the associated coefficient of determination, i.e. the square of the correlation coefficient between $\ln \underline{\Omega}_m$ and $\ln \underline{R}$, is given by

$$\rho^2_{\Omega_m} = \frac{\mathrm{Cov}^2\left(\ln \underline{\Omega}_m, \ln \underline{R}\right)}{\mathrm{Var}\left(\ln \underline{\Omega}_m\right)\mathrm{Var}\left(\ln \underline{R}\right)}. \tag{E.10}$$

The slope has been written as $\widehat{\gamma}_{\Omega_m}$ because it provides an estimate of the exponent of a power law relationship between $\underline{\Omega}_m$ and $\underline{R}$. Using Eqs. (E.7) and (E.8), these (co)variances can be written as

$$\begin{aligned} \mathrm{Var}\left(\ln \underline{\Omega}_m\right) &= \mathrm{Var}\left(\ln \underline{\rho}_V\right) + m^2 \mathrm{Var}\left(\ln \underline{D}_C\right) + \\ &\quad 2m\mathrm{Cov}\left(\ln \underline{\rho}_V, \ln \underline{D}_C\right), \end{aligned} \tag{E.11}$$

$$\begin{aligned}\mathrm{Var}\,(\ln \underline{R}) &= \mathrm{Var}\left(\ln \underline{\rho}_{\mathrm{V}}\right) + (3+\gamma)^2\,\mathrm{Var}\,(\ln \underline{D}_{\mathrm{C}}) + \\ &\quad 2\,(3+\gamma)\,\mathrm{Cov}\left(\ln \underline{\rho}_{\mathrm{V}}, \ln \underline{D}_{\mathrm{C}}\right)\end{aligned} \tag{E.12}$$

and

$$\begin{aligned}\mathrm{Cov}\,(\ln \underline{\Omega}_m, \ln \underline{R}) &= \mathrm{Var}\left(\ln \underline{\rho}_{\mathrm{V}}\right) + m\,(3+\gamma)\,\mathrm{Var}\,(\ln \underline{D}_{\mathrm{C}}) + \\ &\quad (m+3+\gamma)\,\mathrm{Cov}\left(\ln \underline{\rho}_{\mathrm{V}}, \ln \underline{D}_{\mathrm{C}}\right).\end{aligned} \tag{E.13}$$

Substitution of these results in Eq. (E.9) yields

$$\widehat{\gamma}_{\Omega_m} = \frac{1 + m\,(3+\gamma)\,\frac{\mathrm{Var}\left(\ln \underline{D}_{\mathrm{C}}\right)}{\mathrm{Var}\left(\ln \underline{\rho}_{\mathrm{V}}\right)} + (m+3+\gamma)\,\frac{\mathrm{Cov}\left(\ln \underline{\rho}_{\mathrm{V}}, \ln \underline{D}_{\mathrm{C}}\right)}{\mathrm{Var}\left(\ln \underline{\rho}_{\mathrm{V}}\right)}}{1 + (3+\gamma)^2\,\frac{\mathrm{Var}\left(\ln \underline{D}_{\mathrm{C}}\right)}{\mathrm{Var}\left(\ln \underline{\rho}_{\mathrm{V}}\right)} + 2\,(3+\gamma)\,\frac{\mathrm{Cov}\left(\ln \underline{\rho}_{\mathrm{V}}, \ln \underline{D}_{\mathrm{C}}\right)}{\mathrm{Var}\left(\ln \underline{\rho}_{\mathrm{V}}\right)}}, \tag{E.14}$$

which is a linear function of the order of the moment m, in accordance with the scaling law formulation (Eq. (3.21)).

Finally, this expression leads to an explicit formulation for the scaling exponent β in terms of the (co)variances between the raindrop concentration and the characteristic raindrop size according to

$$\widehat{\beta} = \frac{(3+\gamma)\,\frac{\mathrm{Var}\left(\ln \underline{D}_{\mathrm{C}}\right)}{\mathrm{Var}\left(\ln \underline{\rho}_{\mathrm{V}}\right)} + \frac{\mathrm{Cov}\left(\ln \underline{\rho}_{\mathrm{V}}, \ln \underline{D}_{\mathrm{C}}\right)}{\mathrm{Var}\left(\ln \underline{\rho}_{\mathrm{V}}\right)}}{1 + (3+\gamma)^2\,\frac{\mathrm{Var}\left(\ln \underline{D}_{\mathrm{C}}\right)}{\mathrm{Var}\left(\ln \underline{\rho}_{\mathrm{V}}\right)} + 2\,(3+\gamma)\,\frac{\mathrm{Cov}\left(\ln \underline{\rho}_{\mathrm{V}}, \ln \underline{D}_{\mathrm{C}}\right)}{\mathrm{Var}\left(\ln \underline{\rho}_{\mathrm{V}}\right)}}. \tag{E.15}$$

A comparison with Table 3.8 shows that this expression has the right limiting behavior: if $\mathrm{Var}(\ln \underline{D}_C) \gg \mathrm{Var}\left(\ln \underline{\rho}_{\mathrm{V}}\right)$ (raindrop size controlled variability) then $\widehat{\beta} = \frac{1}{3+\gamma}$; if $\mathrm{Var}\left(\ln \underline{\rho}_{\mathrm{V}}\right) \gg \mathrm{Var}(\ln \underline{D}_C)$ (raindrop concentration controlled variability, i.e. equilibrium rainfall) then $\widehat{\beta} = 0$. Hence, this expression provides a statistical interpretation of all intermediate points on the self-consistency curves in Fig. 3.2 in terms of the relative variabilities of the raindrop concentration and characteristic raindrop size.

Appendix F

An explanation for the spurious correlation between C_R and γ_R

F.1 Introduction

The problem at hand can be formulated as follows. Suppose there is available a large number of empirical values of the prefactors C_Z and exponents γ_Z of power law relationships between the radar reflectivity factor Z ($\mathrm{mm^6\,m^{-3}}$) and the rain rate R ($\mathrm{mm\,h^{-1}}$),

$$Z = C_Z R^{\gamma_Z}. \tag{F.1}$$

Although the Z–R relationship is the most common form of relationship between Z and R, often the interest lies more in obtaining R from Z (as estimated by radar) than in obtaining Z from R. Hence, an R–Z relationship of the form

$$R = C_R Z^{\gamma_R} \tag{F.2}$$

is required. If it is assumed that Z and R are uniquely related to each other, then the Z–R and R–Z relationships must be two different forms of one and the same relation. This implies that C_R and γ_R must be related to C_Z and γ_Z according to

$$C_R = C_Z^{-\gamma_R} \tag{F.3}$$

and

$$\gamma_R = \gamma_Z^{-1}. \tag{F.4}$$

The available empirical values of C_Z and γ_Z can be used in this manner to estimate the values of C_R and γ_R. The question is now how these transformations affect the dependence between C_R and γ_R, given a priori knowledge about the dependence between C_Z and γ_Z and their respective variabilities.

F.2 Case 1: Z and R are expressed in their traditional units

If Z is expressed in units of $\mathrm{mm^6\,m^{-3}}$ and R in $\mathrm{mm\,h^{-1}}$, their traditional units, then often the values of C_R and γ_R are found to be quite strongly (negatively) correlated, even though C_Z and γ_Z exhibit little or no correlation. More specifically, when plotted on semi-logarithmic paper, the values of $\log C_R$ and γ_R are generally found to lie roughly on a straight line with a negative slope, while $\log C_Z$ and γ_Z do not seem to show any systematic dependence. How can this be explained?

This is a typical example of what is called *spurious* correlation, i.e. apparent correlation between variables which in fact may be uncorrelated (e.g. Haan, 1977). In statistical terms, the problem can be posed as follows. Suppose $\underline{C}_Z$ and $\underline{\gamma}_Z$ are two independent random variables. Then what is the correlation coefficient between the random variables

$$\log \underline{C}_R = -\underline{\gamma}_R \log \underline{C}_Z \tag{F.5}$$

and

$$\underline{\gamma}_R = \underline{\gamma}_Z^{-1}? \tag{F.6}$$

By definition, the square of this correlation coefficient is given by

$$\rho^2 = \frac{\mathrm{Cov}^2\left(\log \underline{C}_R, \underline{\gamma}_R\right)}{\mathrm{Var}\left(\log \underline{C}_R\right)\mathrm{Var}\left(\underline{\gamma}_R\right)}. \tag{F.7}$$

Again by definition, the covariance between $\log \underline{C}_R$ and $\underline{\gamma}_R$ can be written in terms of expectations as

$$\begin{aligned}\mathrm{Cov}\left(\log \underline{C}_R, \underline{\gamma}_R\right) &= \mathrm{E}\left[\underline{\gamma}_R \log \underline{C}_R\right] - \mathrm{E}\left[\underline{\gamma}_R\right]\mathrm{E}\left[\log \underline{C}_R\right] \\ &= -\mathrm{E}\left[\underline{\gamma}_R^2 \log \underline{C}_Z\right] + \mathrm{E}\left[\underline{\gamma}_R\right]\mathrm{E}\left[\underline{\gamma}_R \log \underline{C}_Z\right].\end{aligned} \tag{F.8}$$

The fact that $\underline{C}_Z$ and $\underline{\gamma}_Z$ are independent implies that $\log \underline{C}_Z$ and $\underline{\gamma}_R$ are independent as well. Hence, the previous expression reduces to

$$\begin{aligned}\mathrm{Cov}\left(\log \underline{C}_R, \underline{\gamma}_R\right) &= -\mathrm{E}\left[\underline{\gamma}_R^2\right]\mathrm{E}\left[\log \underline{C}_Z\right] + \mathrm{E}^2\left[\underline{\gamma}_R\right]\mathrm{E}\left[\log \underline{C}_Z\right] \\ &= -\mathrm{E}\left[\log \underline{C}_Z\right]\mathrm{Var}\left(\underline{\gamma}_R\right),\end{aligned} \tag{F.9}$$

which is negative as long as $\mathrm{E}[\log \underline{C}_Z]$ is positive, i.e. as long as the geometric mean of $\underline{C}_Z$ exceeds one (which is typically the case for the assumed units of Z and R). This reduces the square of the correlation coefficient between $\log \underline{C}_R$ and $\underline{\gamma}_R$ to

$$\rho^2 = \frac{\mathrm{E}^2\left[\log \underline{C}_Z\right]\mathrm{Var}\left(\underline{\gamma}_R\right)}{\mathrm{Var}\left(\log \underline{C}_R\right)}. \tag{F.10}$$

By definition, the variance of $\log \underline{C}_R$ is

$$\mathrm{Var}\left(\log \underline{C}_R\right) = \mathrm{E}\left[\log^2 \underline{C}_R\right] - \mathrm{E}^2\left[\log \underline{C}_R\right]. \tag{F.11}$$

Hence

$$\begin{aligned}\mathrm{Var}\left(\log \underline{C}_R\right) &= \mathrm{E}\left[\underline{\gamma}_R^2 \log^2 \underline{C}_Z\right] - \mathrm{E}^2\left[\underline{\gamma}_R \log \underline{C}_Z\right] \\ &= \mathrm{E}\left[\underline{\gamma}_R^2\right] \mathrm{E}\left[\log^2 \underline{C}_Z\right] - \mathrm{E}^2\left[\underline{\gamma}_R\right] \mathrm{E}^2\left[\log \underline{C}_Z\right] \\ &= \mathrm{Var}\left(\underline{\gamma}_R\right) \mathrm{Var}\left(\log \underline{C}_Z\right) + \mathrm{E}^2\left[\underline{\gamma}_R\right] \mathrm{Var}\left(\log \underline{C}_Z\right) + \\ &\quad\; \mathrm{E}^2\left[\log \underline{C}_Z\right] \mathrm{Var}\left(\underline{\gamma}_R\right). \end{aligned} \tag{F.12}$$

This finally implies for the square of the correlation coefficient between $\log \underline{C}_R$ and $\underline{\gamma}_R$

$$\begin{aligned}\rho^2 &= \frac{\mathrm{E}^2\left[\log \underline{C}_Z\right] \mathrm{Var}\left(\underline{\gamma}_R\right)}{\mathrm{E}^2\left[\log \underline{C}_Z\right] \mathrm{Var}\left(\underline{\gamma}_R\right) + \mathrm{E}^2\left[\underline{\gamma}_R\right] \mathrm{Var}\left(\log \underline{C}_Z\right) + \mathrm{Var}\left(\underline{\gamma}_R\right) \mathrm{Var}\left(\log \underline{C}_Z\right)} \\ &= \frac{\mathrm{CV}^2\left(\underline{\gamma}_Z^{-1}\right)}{\mathrm{CV}^2\left(\underline{\gamma}_Z^{-1}\right) + \mathrm{CV}^2\left(\log \underline{C}_Z\right) + \mathrm{CV}^2\left(\underline{\gamma}_Z^{-1}\right) \mathrm{CV}^2\left(\log \underline{C}_Z\right)}, \end{aligned} \tag{F.13}$$

the desired expression for the square of the correlation coefficient between $\log \underline{C}_R$ and $\underline{\gamma}_R$ in terms of the coefficients of variation (the ratios of the standard deviations to the means) of $\log \underline{C}_Z$ and $\underline{\gamma}_Z^{-1}$. Eq. (F.13) shows that independent fluctuations in $\log \underline{C}_Z$ and $\underline{\gamma}_Z$ alone are enough to produce (spurious) correlations between $\log \underline{C}_R$ and $\underline{\gamma}_R$. Moreover, it shows that if $\mathrm{CV}\left(\underline{\gamma}_Z^{-1}\right) \gg \mathrm{CV}(\log \underline{C}_Z)$ ρ^2 will tend to one (with the sign of ρ equal to that of $-\mathrm{E}[\log \underline{C}_Z]$, see Eq. (F.9)), whereas if $\mathrm{CV}(\log \underline{C}_Z) \gg \mathrm{CV}\left(\underline{\gamma}_Z^{-1}\right)$ ρ^2 will tend to zero. Since the last term in the denominator is of fourth order and the others are all of second order, it can be neglected if both coefficients of variation are small enough. Two examples will serve to illustrate the usefulness of this expression.

For 28 rainfall events collected in 1968 and 1969 in De Bilt, The Netherlands (Wessels, 1972), $\log C_Z$ and γ_Z are found to be virtually uncorrelated, with a sample correlation coefficient r of only 0.0867. The sample geometric mean of C_Z is 246, well above one, and therefore the correlation between $\log C_R$ and γ_R will be negative, as can be seen from Eq. (F.9). The sample coefficients of variation of $\log C_Z$ and γ_Z^{-1} are 0.0785 and 0.1265, respectively. Substituting these values in Eq. (F.13) yields a correlation coefficient of -0.85, which is quite close to the actual sample correlation coefficient between $\log C_R$ and γ_R (-0.82).

The 69 Z–R relationships reported by Battan (1973) exhibit a slight negative correlation between $\log C_Z$ and γ_Z, their sample correlation coefficient is -0.2146. Hence, Eq. (F.13), being based on the assumption that $\log C_Z$ and γ_Z are independent, is expected to yield less satisfactory results in this case. The sample geometric mean of C_Z is 238, again implying a negative correlation between $\log C_R$ and γ_R. The sample coefficients of variation of $\log C_Z$ and γ_Z^{-1} are now 0.1140 and 0.1213, respectively, resulting in a predicted correlation coefficient of -0.73. This is still reasonably close to the actual sample correlation coefficient between $\log C_R$ and γ_R (-0.78). The conclusion is that Eq. (F.13) provides relatively robust estimates of the spurious correlation between $\log C_R$ and γ_R.

F.3 Case 2: Z and R are expressed in SI-units

Suppose now the radar reflectivity factor and the rain rate are both expressed in SI-units. To distinguish them from their traditional counterparts, they will be called Z_{SI} and R_{SI}, respectively. Hence, Z_{SI} has units of m^3 and R_{SI} has units of $\mathrm{m\,s^{-1}}$. As the units of the prefactor of a power law radar reflectivity – rain rate relationship depend both on those of radar reflectivity and rain rate and on the value of the exponent, it will be clear that changing their units will affect the correlation between prefactor and exponent. This problem will first be treated for the case of Z_{SI}–R_{SI} relationships, then for that of R_{SI}–Z_{SI} relationships.

Z_{SI} (m^3) and R_{SI} ($\mathrm{m\,s^{-1}}$) are numerically related to Z ($\mathrm{mm^6\,m^{-3}}$) and R ($\mathrm{mm\,h^{-1}}$) according to

$$\begin{cases} Z = 10^{18} Z_{\mathrm{SI}} \\ R = 3.6 \times 10^6 R_{\mathrm{SI}} \end{cases}. \tag{F.14}$$

For convenience in notation, the proportionality factors will be called p_Z and p_R in the sequel. This yields for the power law relationship between Z_{SI} and R_{SI}

$$Z_{\mathrm{SI}} = C_{Z_{\mathrm{SI}}} R_{\mathrm{SI}}^{\gamma_Z}, \tag{F.15}$$

where

$$C_{Z_{\mathrm{SI}}} = C_Z \left(p_R\right)^{\gamma_Z} p_Z^{-1}, \tag{F.16}$$

or

$$\log C_{Z_{\mathrm{SI}}} = \log C_Z + \gamma_Z \log p_R - \log p_Z. \tag{F.17}$$

Hence, only the prefactor is affected by a change of units. Because it is dimensionless, the exponent of the power law remains the same. If $\underline{C}_{Z_{\mathrm{SI}}}$, $\underline{C}_Z$ and $\underline{\gamma}_Z$ are three random variables, then the covariance between $\log \underline{C}_{Z_{\mathrm{SI}}}$ and $\underline{\gamma}_Z$ can be written as

$$\begin{aligned} \mathrm{Cov}\left(\log \underline{C}_{Z_{\mathrm{SI}}}, \underline{\gamma}_Z\right) &= \mathrm{Cov}\left(\log \underline{C}_Z + \underline{\gamma}_Z \log p_R - \log p_Z, \underline{\gamma}_Z\right) \\ &= \mathrm{Cov}\left(\log \underline{C}_Z, \underline{\gamma}_Z\right) + \log p_R \mathrm{Var}\left(\underline{\gamma}_Z\right). \end{aligned} \tag{F.18}$$

If $\underline{C}_Z$ and $\underline{\gamma}_Z$ are uncorrelated (not necessarily independent) then this reduces to

$$\mathrm{Cov}\left(\log \underline{C}_{Z_{\mathrm{SI}}}, \underline{\gamma}_Z\right) = \log p_R \mathrm{Var}\left(\underline{\gamma}_Z\right), \tag{F.19}$$

which is positive as long as $\log p_R$ is positive, i.e. as long as p_R exceeds one (as is the case here). The variance of $\log \underline{C}_{Z_{\mathrm{SI}}}$ is

$$\begin{aligned} \mathrm{Var}\left(\log \underline{C}_{Z_{\mathrm{SI}}}\right) &= \mathrm{Var}\left(\log \underline{C}_Z + \underline{\gamma}_Z \log p_R - \log p_Z\right) \\ &= \mathrm{Var}\left(\log \underline{C}_Z\right) + \log^2 p_R \mathrm{Var}\left(\underline{\gamma}_Z\right) + 2 \log p_R \mathrm{Cov}\left(\log \underline{C}_Z, \underline{\gamma}_Z\right). \end{aligned} \tag{F.20}$$

If $\underline{C}_Z$ and $\underline{\gamma}_Z$ are uncorrelated then this reduces to

$$\mathrm{Var}\left(\log \underline{C}_{Z_{\mathrm{SI}}}\right) = \mathrm{Var}\left(\log \underline{C}_Z\right) + \log^2 p_R \mathrm{Var}\left(\underline{\gamma}_Z\right). \tag{F.21}$$

This finally implies for the square of the correlation coefficient between $\log \underline{C}_{Z_{\mathrm{SI}}}$ and $\underline{\gamma}_Z$ (if $\underline{C}_Z$ and $\underline{\gamma}_Z$ are uncorrelated)

$$\begin{aligned}\rho^2 &= \frac{\mathrm{Cov}^2\left(\log \underline{C}_{Z_{\mathrm{SI}}}, \underline{\gamma}_Z\right)}{\mathrm{Var}\left(\log \underline{C}_{Z_{\mathrm{SI}}}\right) \mathrm{Var}\left(\underline{\gamma}_Z\right)} \\ &= \frac{\log^2 p_R \mathrm{Var}\left(\underline{\gamma}_Z\right)}{\log^2 p_R \mathrm{Var}\left(\underline{\gamma}_Z\right) + \mathrm{Var}\left(\log \underline{C}_Z\right)}. \end{aligned} \tag{F.22}$$

This expression shows that if $\log^2 p_R \mathrm{Var}\left(\underline{\gamma}_Z\right) \gg \mathrm{Var}(\log \underline{C}_Z)$ then ρ^2 tends to one, whereas if $\mathrm{Var}(\log \underline{C}_Z) \gg \log^2 p_R \mathrm{Var}\left(\underline{\gamma}_Z\right)$ then ρ^2 tends to zero. Since in the case treated here p_R equals 3.6×10^6, ρ^2 will almost surely be close to one. As p_R exceeds one, this indicates a strong positive correlation between $\log \underline{C}_{Z_{\mathrm{SI}}}$ and $\underline{\gamma}_Z$. Indeed, for the Dutch data reported in the first section of this appendix Eq. (F.22) predicts $\rho = +0.99$, for Battan's Z–R relationships $\rho = +0.98$. Both these values deviate less than 0.01 from their actual sample values, which confirms the validity of Eq. (F.22).

In accordance with what has been shown in the first section of this appendix for R–Z relationships, for R_{SI}–Z_{SI} relationships the problem is to find an appropriate expression for the correlation coefficient between the random variables

$$\log \underline{C}_{R_{\mathrm{SI}}} = -\underline{\gamma}_R \log \underline{C}_{Z_{\mathrm{SI}}} \tag{F.23}$$

and

$$\underline{\gamma}_R = \underline{\gamma}_Z^{-1}. \tag{F.24}$$

Substitution of Eq. (F.17) into the expression for $\log \underline{C}_{R_{\mathrm{SI}}}$ yields

$$\begin{aligned}\log \underline{C}_{R_{\mathrm{SI}}} &= -\underline{\gamma}_R \left(\log \underline{C}_Z + \underline{\gamma}_Z \log p_R - \log p_Z\right) \\ &= -\underline{\gamma}_R \log \underline{C}_Z + \underline{\gamma}_R \log p_Z - \log p_R & \text{(F.25)} \\ &= \log \underline{C}_R + \underline{\gamma}_R \log p_Z - \log p_R. & \text{(F.26)}\end{aligned}$$

Hence, the covariance of $\log \underline{C}_{R_{\mathrm{SI}}}$ and $\underline{\gamma}_R$ is

$$\mathrm{Cov}\left(\log \underline{C}_{R_{\mathrm{SI}}}, \underline{\gamma}_R\right) = \mathrm{Cov}\left(\log \underline{C}_R, \underline{\gamma}_R\right) + \log p_Z \mathrm{Var}\left(\underline{\gamma}_R\right). \tag{F.27}$$

Substitution of Eq. (F.9) into this expression yields

$$\mathrm{Cov}\left(\log \underline{C}_{R_{\mathrm{SI}}}, \underline{\gamma}_R\right) = \left\{\log p_Z - \mathrm{E}\left[\log \underline{C}_Z\right]\right\} \mathrm{Var}\left(\underline{\gamma}_R\right), \tag{F.28}$$

which is positive as long as $\log p_Z - \mathrm{E}[\log \underline{C}_Z]$ is positive, i.e. as long as p_Z exceeds the geometric mean of $\underline{C}_Z$ (which is largely the case here). The variance of $\log \underline{C}_{R_{\mathrm{SI}}}$ is

$$\mathrm{Var}\left(\log \underline{C}_{R_{\mathrm{SI}}}\right) = \mathrm{Var}\left(\log \underline{C}_R\right) + \log^2 p_Z \mathrm{Var}\left(\underline{\gamma}_R\right) + 2 \log p_Z \mathrm{Cov}\left(\log \underline{C}_R, \underline{\gamma}_R\right). \tag{F.29}$$

Substituting Eqs. (F.12) and (F.9) into this expression yields

$$\begin{aligned}
\mathrm{Var}\left(\log \underline{C}_{R_{\mathrm{SI}}}\right) &= \mathrm{Var}\left(\underline{\gamma}_R\right)\mathrm{Var}\left(\log \underline{C}_Z\right) + \mathrm{E}^2\left[\underline{\gamma}_R\right]\mathrm{Var}\left(\log \underline{C}_Z\right) + \\
&\quad \mathrm{E}^2\left[\log \underline{C}_Z\right]\mathrm{Var}\left(\underline{\gamma}_R\right) + \log^2 p_Z \mathrm{Var}\left(\underline{\gamma}_R\right) - \\
&\quad 2\log p_Z \mathrm{E}\left[\log \underline{C}_Z\right]\mathrm{Var}\left(\underline{\gamma}_R\right) \\
&= \mathrm{Var}\left(\underline{\gamma}_R\right)\mathrm{Var}\left(\log \underline{C}_Z\right) + \mathrm{E}^2\left[\underline{\gamma}_R\right]\mathrm{Var}\left(\log \underline{C}_Z\right) + \\
&\quad \left\{\log p_Z - \mathrm{E}\left[\log \underline{C}_Z\right]\right\}^2 \mathrm{Var}\left(\underline{\gamma}_R\right). \qquad (\mathrm{F.30})
\end{aligned}$$

This finally implies for the square of the correlation coefficient between $\log \underline{C}_{R_{\mathrm{SI}}}$ and $\underline{\gamma}_R$

$$\begin{aligned}
\rho^2 &= \frac{\mathrm{Cov}^2\left(\log \underline{C}_{R_{\mathrm{SI}}}, \underline{\gamma}_R\right)}{\mathrm{Var}\left(\log \underline{C}_{R_{\mathrm{SI}}}\right)\mathrm{Var}\left(\underline{\gamma}_R\right)} \\
&= \frac{\left\{\log p_Z - \mathrm{E}\left[\log \underline{C}_Z\right]\right\}^2 \mathrm{Var}\left(\underline{\gamma}_R\right)}{\left\{\log p_Z - \mathrm{E}\left[\log \underline{C}_Z\right]\right\}^2 \mathrm{Var}\left(\underline{\gamma}_R\right) + \mathrm{E}^2\left[\underline{\gamma}_R\right]\mathrm{Var}\left(\log \underline{C}_Z\right) + \mathrm{Var}\left(\underline{\gamma}_R\right)\mathrm{Var}\left(\log \underline{C}_Z\right)} \\
&= \frac{\left\{\frac{\log p_Z}{\mathrm{E}\left[\log \underline{C}_Z\right]} - 1\right\}^2 \mathrm{CV}^2\left(\underline{\gamma}_Z^{-1}\right)}{\left\{\frac{\log p_Z}{\mathrm{E}\left[\log \underline{C}_Z\right]} - 1\right\}^2 \mathrm{CV}^2\left(\underline{\gamma}_Z^{-1}\right) + \mathrm{CV}^2\left(\log \underline{C}_Z\right) + \mathrm{CV}^2\left(\underline{\gamma}_Z^{-1}\right)\mathrm{CV}^2\left(\log \underline{C}_Z\right)},
\end{aligned}$$

(F.31)

which reduces to Eq. (F.13) if $p_Z = 1$, as it should. This expression shows that if $\log p_Z \gg \mathrm{E}[\log \underline{C}_Z]$, i.e. if p_Z largely exceeds the geometric mean of $\underline{C}_Z$ (which is the case here), then ρ^2 will tend to one (unless $\mathrm{CV}^2\left(\log \underline{C}_Z\right) \gg \mathrm{CV}^2\left(\underline{\gamma}_Z^{-1}\right)$). Indeed, both for the Dutch data reported in the first section of this appendix and for Battan's 69 Z–R relationships Eq. (F.31) predicts $\rho = +0.99$. Again, these values deviate less than 0.01 from their actual sample values, thus confirming the validity of Eq. (F.31).

In summary, the derivations in this appendix lead to the following conclusions: (1) the empirically established strong negative correlation between $\log C_R$ and γ_R when Z and R are expressed in their traditional units ($\mathrm{mm}^6\,\mathrm{m}^{-3}$ and $\mathrm{mm\,h}^{-1}$) can be explained theoretically as a spurious correlation; (2) both the strong negative correlation between $\log C_R$ and γ_R and the quasi-independence of $\log C_Z$ and γ_Z should not be construed as to have any physical meaning. These dependencies are completely determined by the units employed for Z and R. A simple change of units (to SI-units for instance) may alter them radically.

Appendix G

An explanation for the dependence between N_0 and μ[1]

G.1 Introduction

Assuming the parameter N_0 to be independent of rain rate, Ulbrich (1983) has employed the prefactors and exponents of the 69 power law Z–R relationships quoted by Battan (1973) and those of 11 other relationships reported in the literature to estimate the parameters N_0 and μ of the gamma raindrop size distribution. A plot of the values thus obtained on semi-logarithmic paper has revealed that the data points lie more or less on a straight line. A linear least squares regression of $\ln N_0$ (with N_0 expressed in units of $\mathrm{cm}^{-(1+\mu)}\,\mathrm{m}^{-3}$) on μ (–) has yielded an expression of the form

$$N_0 = 6 \times 10^4 \exp(3.2\mu)\,, \qquad \text{(G.1)}$$

with a linear correlation coefficient between $\ln N_0$ and μ exceeding 0.98. Ulbrich argues that 'this very high correlation is not surprising in view of the dependence of N_0 on μ implied theoretically'. Indeed, when plotted on semi-logarithmic paper for given values of the prefactor C_Z of the Z–R relationship, the theoretical N_0–μ relationship implied by Ulbrich's approach yields approximately straight lines for the range of values of μ encountered experimentally ($-4 \leq \mu \leq 10$) (Fig. 6.11(b) and Fig. 6.12, p. 194–195). Ulbrich (1983) concludes that Eq. (G.1) has 'both theoretical and empirical justification'.

However, several authors have questioned the validity of this and similar equations. Using their own experimental raindrop size distributions, Feingold and Levin (1986) find that the correlation between $\ln N_0$ and μ decreases if N_0 is expressed in units of $\mathrm{mm}^{-(1+\mu)}\,\mathrm{m}^{-3}$ instead of $\mathrm{cm}^{-(1+\mu)}\,\mathrm{m}^{-3}$. They argue that the reason for this is 'the fact that the units of N_0 depend on the value of μ itself'. This sensitivity of the N_0–μ relationship on the units of N_0 has been confirmed by Chandrasekar and Bringi (1987) through extensive simulation experiments. These investigators conclude that 'the mean N_0–μ relationship derived by Ulbrich (1983) is due to the nature of the

[1] Adapted version of Uijlenhoet, R. (1999). An explanation for the spurious correlation between the parameters N_0 and μ of the gamma raindrop size distribution. *J. Appl. Meteor.* (submitted).

quantities involved' and that as a result 'the three-parameter gamma raindrop size distribution cannot be reduced to a two parameter form'.

Despite the apparently well-founded criticism which this approach has yielded, the relationship between N_0 and μ has remained widely used (e.g. Ulbrich and Atlas, 1998). It therefore seems appropriate to investigate this relationship a little more in detail. The two major aspects of the problem are: (1) the mathematical origin of the approximately linear *theoretical* relationship between $\ln N_0$ and μ for a given value of C_Z if N_0 is independent of rain rate; (2) the influence of the units of N_0 on the correlation between *empirical* values of $\ln N_0$ and μ.

G.2 An approximate linear relationship between $\ln N_0$ and μ

The starting point of the derivation here is Eq. (6.13) (p. 192), which gives the value of N_0 ($\mathrm{mm}^{-(1+\mu)}\,\mathrm{m}^{-3}$) (independent of rain rate) as a function of μ (–) for a given value of C_Z ($\mathrm{mm}^6\,\mathrm{m}^{-3}\,(\mathrm{mm\,h}^{-1})^{-\gamma_Z}$)

$$N_0 = \frac{10^4}{6\pi c}\frac{1}{\Gamma(4+\gamma+\mu)}\left[\frac{10^4}{6\pi c}\frac{\Gamma(7+\mu)}{\Gamma(4+\gamma+\mu)}\right]^{(4+\gamma+\mu)/(3-\gamma)} C_Z^{-(4+\gamma+\mu)/(3-\gamma)}. \tag{G.2}$$

Taking the natural logarithm on both sides of this equation gives

$$\begin{aligned}\ln N_0 \;=\;& \frac{7+\mu}{3-\gamma}\ln\left(\frac{10^4}{6\pi c}\right) - \frac{4+\gamma+\mu}{3-\gamma}\ln C_Z + \\ & \frac{4+\gamma+\mu}{3-\gamma}\ln\Gamma(7+\mu) - \frac{7+\mu}{3-\gamma}\ln\Gamma(4+\gamma+\mu)\,.\end{aligned} \tag{G.3}$$

The first two terms of this expression are linear in μ, the last two not. However, for moderate values of μ (the interest here lies typically in the range $-4 \leq \mu \leq 10$) $\ln\Gamma(7+\mu)$ and $\ln\Gamma(4+\gamma+\mu)$ may be approximated by their respective Taylor series expansions about $\mu = 0$, i.e.

$$\ln\Gamma(7+\mu) = \ln\Gamma(7) + \mu\Psi(7) + \mathrm{O}\left(\mu^2\right) \tag{G.4}$$

and

$$\ln\Gamma(4+\gamma+\mu) = \ln\Gamma(4+\gamma) + \mu\Psi(4+\gamma) + \mathrm{O}\left(\mu^2\right), \tag{G.5}$$

where $\Psi(x)$ is the psi (or digamma) function, defined as

$$\Psi(x) = \frac{\mathrm{d}\ln\Gamma(x)}{\mathrm{d}x} = \frac{\Gamma'(x)}{\Gamma(x)} \tag{G.6}$$

(Abramowitz and Stegun, 1972). Substituting these expansions in Eq. (G.3), collecting terms of the same order in μ and retaining only those of zeroth and first order yields the approximation

$$\begin{aligned}\ln N_0 \approx\ & \frac{7}{3-\gamma}\ln\left[\frac{10^4}{6\pi c}\frac{1}{\Gamma(4+\gamma)}\right]+\frac{4+\gamma}{3-\gamma}\ln\left[\frac{\Gamma(7)}{C_Z}\right]+\\ & \left\{\frac{1}{3-\gamma}\ln\left[\frac{10^4}{6\pi c}\frac{\Gamma(7)}{C_Z\Gamma(4+\gamma)}\right]-\frac{7}{3-\gamma}\Psi(4+\gamma)+\frac{4+\gamma}{3-\gamma}\Psi(7)\right\}\mu.\end{aligned} \tag{G.7}$$

This can be written alternatively as

$$N_0 \approx C_{N_0}\exp\left(\gamma_{N_0}\mu\right), \tag{G.8}$$

with

$$C_{N_0}=\left[\frac{10^4}{6\pi c\Gamma(4+\gamma)}\right]^{7/(3-\gamma)}\left[\frac{\Gamma(7)}{C_Z}\right]^{(4+\gamma)/(3-\gamma)} \tag{G.9}$$

and

$$\gamma_{N_0}=\frac{1}{3-\gamma}\ln\left[\frac{10^4}{6\pi c C_Z}\frac{\Gamma(7)}{\Gamma(4+\gamma)}\right]-\frac{7}{3-\gamma}\Psi(4+\gamma)+\frac{4+\gamma}{3-\gamma}\Psi(7). \tag{G.10}$$

If $c = 3.778\ \mathrm{m\,s^{-1}\,mm^{-\gamma}}$ and $\gamma = 0.67$ (Atlas and Ulbrich, 1977) then numerically these expressions reduce to

$$C_{N_0} = 4.62\times 10^8 C_Z^{-2.00} \tag{G.11}$$

and

$$\gamma_{N_0} = 3.25 - 0.429\ln C_Z, \tag{G.12}$$

where C_{N_0} is expressed in units of $\mathrm{mm}^{-(1+\mu)}\,\mathrm{m}^{-3}$ and C_Z in $\mathrm{mm}^6\,\mathrm{m}^{-3}\,(\mathrm{mm\,h}^{-1})^{-\gamma_Z}$. If, in accordance with Eq. (G.1), N_0 is expressed in $\mathrm{cm}^{-(1+\mu)}\,\mathrm{m}^{-3}$ then Eq. (G.8) should be multiplied with $10^{1+\mu}$. It then retains its exponential form, but with C_{N_0} increased by a factor 10 and γ_{N_0} increased with $\ln 10$, i.e.

$$C_{N_0} = 4.62\times 10^9 C_Z^{-2.00} \tag{G.13}$$

and

$$\gamma_{N_0} = 5.55 - 0.429\ln C_Z. \tag{G.14}$$

For the typical mean value of $C_Z = 250$, these equations yield $C_{N_0} = 7.39\times 10^4$ and $\gamma_{N_0} = 3.18$, quite close to Ulbrich's (1983) empirically determined values of 6×10^4 and 3.2. If $C_Z = 277$ then Eq. (G.8) fits his regression line almost perfectly ($C_{N_0} = 6.02\times 10^4$ and $\gamma_{N_0} = 3.14$). This happens to be exactly the (arithmetic) mean value of the prefactors of all 69 Z–R relationships quoted by Battan (1973). Another point which follows from these equations is that the sensitivity of the intercept $\ln C_{N_0}$ and the slope γ_{N_0} of the $\ln N_0$–μ relation to changes in C_Z has decreased after the change of units of N_0 from $\mathrm{mm}^{-(1+\mu)}\,\mathrm{m}^{-3}$ to $\mathrm{cm}^{-(1+\mu)}\,\mathrm{m}^{-3}$. As a matter of fact, this sensitivity can be made to disappear almost entirely when N_0 is expressed in SI-units ($\mathrm{m}^{-(4+\mu)}$).

G.3 The correlation between empirical values of $\ln N_0$ and μ

The second aspect of N_0–μ relations which will be dealt with here is the influence of the units of N_0 on the (spurious) correlation between empirical values of $\ln N_0$ and μ. For example, a practical question would be how the intercept $\ln C_{N_0}$, the slope γ_{N_0} and the associated correlation coefficient of the linear relationship between $\ln N_0$ and μ established via least squares regression by Ulbrich (1983) (Eq. (G.1)) would be affected if the units of N_0 would be changed from $\mathrm{cm}^{-(1+\mu)}\,\mathrm{m}^{-3}$ to $\mathrm{mm}^{-(1+\mu)}\,\mathrm{m}^{-3}$.

Consider the relation

$$\ln N_0 = \ln C_{N_0} + \gamma_{N_0}\mu, \tag{G.15}$$

with N_0 expressed in arbitrary units (within the constraints posed by its dimensions $\mathrm{L}^{-(4+\mu)}$). If this functional relation is the result of a linear least squares regression analysis, then $\ln C_{N_0}$ and γ_{N_0} can be interpreted as the intercept and the slope of the regression line, i.e.

$$\ln C_{N_0} = \mathrm{E}\left[\ln \underline{N}_0\right] - \gamma_{N_0}\mathrm{E}\left[\underline{\mu}\right] \tag{G.16}$$

and

$$\gamma_{N_0} = \frac{\mathrm{Cov}\left(\ln \underline{N}_0, \underline{\mu}\right)}{\mathrm{Var}\left(\underline{\mu}\right)}, \tag{G.17}$$

where $\ln \underline{N}_0$ and $\underline{\mu}$ are random variables. Similarly, the square of the correlation coefficient between $\ln \underline{N}_0$ and $\underline{\mu}$ is

$$\rho_{N_0}^2 = \frac{\mathrm{Cov}^2\left(\ln \underline{N}_0, \underline{\mu}\right)}{\mathrm{Var}\left(\ln \underline{N}_0\right)\mathrm{Var}\left(\underline{\mu}\right)}. \tag{G.18}$$

Now a new random variable $\underline{N}_0'$ is defined, which is related to $\underline{N}_0$ according to

$$\underline{N}_0' = s^{1+\underline{\mu}}\underline{N}_0, \tag{G.19}$$

where s denotes the change in scale of the raindrop diameter, i.e. $D' = s^{-1}D$. For instance, if D were originally expressed in units of mm and now changed to D' expressed in cm, then $s = 10$. This implies for the natural logarithm of $\underline{N}_0'$

$$\ln \underline{N}_0' = \ln \underline{N}_0 + \left(1 + \underline{\mu}\right)\ln s. \tag{G.20}$$

The covariance of $\ln \underline{N}_0'$ and $\underline{\mu}$ is therefore given by

$$\begin{aligned}\mathrm{Cov}\left(\ln \underline{N}_0', \underline{\mu}\right) &= \mathrm{Cov}\left[\ln \underline{N}_0 + \left(1 + \underline{\mu}\right)\ln s, \underline{\mu}\right] \\ &= \mathrm{Cov}\left(\ln \underline{N}_0, \underline{\mu}\right) + \ln s \mathrm{Var}\left(\underline{\mu}\right)\end{aligned} \tag{G.21}$$

and the variance of $\ln \underline{N}_0'$ by

$$\mathrm{Var}\left(\ln \underline{N}_0'\right) = \mathrm{Var}\left(\ln \underline{N}_0\right) + \ln^2 s \mathrm{Var}\left(\underline{\mu}\right) + 2\ln s \mathrm{Cov}\left(\ln \underline{N}_0, \underline{\mu}\right). \tag{G.22}$$

This implies for the slope of the regression line between $\ln \underline{N}'_0$ and $\underline{\mu}$

$$\begin{aligned}\gamma_{N'_0} &= \frac{\mathrm{Cov}\left(\ln \underline{N}'_0, \underline{\mu}\right)}{\mathrm{Var}\left(\underline{\mu}\right)} \\ &= \frac{\mathrm{Cov}\left(\ln \underline{N}_0, \underline{\mu}\right) + \ln s \mathrm{Var}\left(\underline{\mu}\right)}{\mathrm{Var}\left(\underline{\mu}\right)} \\ &= \gamma_{N_0} + \ln s, \end{aligned} \tag{G.23}$$

for its intercept

$$\begin{aligned}\ln C_{N'_0} &= \mathrm{E}\left[\ln \underline{N}'_0\right] - \gamma_{N'_0} \mathrm{E}\left[\underline{\mu}\right] \\ &= \mathrm{E}\left[\ln \underline{N}_0\right] + \left(1 + \mathrm{E}\left[\underline{\mu}\right]\right) \ln s - \left(\gamma_{N_0} + \ln s\right) \mathrm{E}\left[\underline{\mu}\right] \\ &= \mathrm{E}\left[\ln \underline{N}_0\right] - \gamma_{N_0} \mathrm{E}\left[\underline{\mu}\right] + \ln s \\ &= \ln C_{N_0} + \ln s \end{aligned} \tag{G.24}$$

(or $C_{N'_0} = sC_{N_0}$) and for the corresponding square of the correlation coefficient

$$\begin{aligned}\rho^2_{N'_0} &= \frac{\mathrm{Cov}^2\left(\ln \underline{N}'_0, \underline{\mu}\right)}{\mathrm{Var}\left(\ln \underline{N}'_0\right) \mathrm{Var}\left(\underline{\mu}\right)} \\ &= \gamma^2_{N'_0} \frac{\mathrm{Var}\left(\underline{\mu}\right)}{\mathrm{Var}\left(\ln \underline{N}'_0\right)} \\ &= \frac{\rho^2_{N_0}\left(\gamma_{N_0} + \ln s\right)^2}{\rho^2_{N_0}\left(\gamma_{N_0} + \ln s\right)^2 + \left(1 - \rho^2_{N_0}\right)\gamma^2_{N_0}}. \end{aligned} \tag{G.25}$$

Obviously, for $s = 1$ $\gamma_{N'_0}$ reduces to γ_{N_0}, $C_{N'_0}$ to C_{N_0} and $\rho^2_{N'_0}$ to $\rho^2_{N_0}$. The correlation between $\ln \underline{N}'_0$ and $\underline{\mu}$ remains positive as long as the slope of the regression line ($\gamma_{N'_0}$) remains positive, i.e. as long as $\gamma_{N_0} + \ln s > 0$. It becomes zero when $s = \exp\left(-\gamma_{N_0}\right)$. For $s = 1$, $\rho^2_{N'_0}$ obviously reduces to $\rho^2_{N_0}$. The parameter values of Ulbrich's (1983) regression line are $C_{N_0} = 6 \times 10^4$, $\gamma_{N_0} = 3.2$ and $\rho^2_{N_0} = 0.96$ (with N_0 expressed in $\mathrm{cm}^{-(1+\mu)}\,\mathrm{m}^{-3}$). Hence, if the units of N_0 were to be changed to $\mathrm{mm}^{-(1+\mu)}\,\mathrm{m}^{-3}$ (which corresponds to $s = 0.1$) then C_{N_0} would become 6×10^3, γ_{N_0} would be reduced to 0.9 and $\rho^2_{N_0}$ would be reduced to 0.66 (corresponding to a correlation coefficient of approximately 0.8).

In short, the parameter N_0, with units which depend on the value of the parameter μ, is not a very suitable concentration parameter in the gamma raindrop size distribution. Alternative parameters, such as the raindrop concentration (Chandrasekar and Bringi, 1987) or the recently proposed parameters N_L (Illingworth and Blackman, 1999; Illingworth and Johnson, 1999) and N^*_0 (Dou et al., 1999; Testud et al., 1999), all parameters with units independent of the value of μ, are preferable.

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