Computing Replenishment Cycle Policy under Non-stationary Stochastic Lead Time

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Abstract

In this paper we address the general multi-period production/inventory problem with non-stationary stochastic demand and supplier lead time under service-level constraints. A replenishment cycle policy (R^n,S^n) is modeled, where R^n is the n-th replenishment cycle length and S^n is the respective order-up-to-level. Initially we extend an existing formulation for this policy in a way to incorporate a dynamic deterministic lead time with the assumption of order cross-over. Following this, we extend the model to incorporate a non-stationary stochastic lead time. Within a constraint programming framework, a dedicated constraint implementing a hybrid approach is proposed to compute replenishment cycle policy parameters.

Key words: inventory control; constraint programming; demand uncertainty; supplier lead time uncertainty

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1 Introduction

Inventory theory provides methods for managing inventories in different environments. An interesting class of production/inventory control problems is the one that considers the single location, single product case under non-stationary stochastic demand. In contrast to the production planning problem under deterministic demand (Wagner and Whitin [28]), different inventory control policies can be adopted to cope with the stochastic version.

A policy states the rules to decide when orders have to be placed and how to compute the replenishment lot-size for each order. For a discussion on inventory control policies see Silver et al. [22]. One of the well-known policies that can be adopted in inventory control is the replenishment cycle policy, (R,S). Under the non-stationary demand assumption this policy takes the dynamic form (R^n,S^n) where R^n denotes the length of the nth replenishment cycle, and S^n the order-up-to-level value for the nth replenishment.

It is a known result (Scarf [20]) that such a policy is not optimal in term of cost minimization, since non-stationary (s^n,S^n) always dominates it even when a delivery lag is considered (Kaplan [15]). However, as discussed in Tarim and Kingsman [25], (R, S) provides an effective means of dampening the planning instability. Furthermore, it is particularly appealing when items are ordered from the same supplier or require resource sharing. In such a case all items in a coordinated group can be given the same replenishment period. Periodic review also allows a reasonable prediction of the level of the workload on the staff involved and is particularly suitable for advanced planning environments. For these reasons, as stated by Silver et al. [22], (R, S) is a popular inventory policy.

Due to its combinatorial nature, (R^n,S^n) policy — even in the absence of stochastic lead time — presents a difficult problem to solve to optimality (Tarim and Kingsman [25]). Early work in the area have been carried out in Askin [2], Silver [21] and a heuristic procedure was proposed by Bookbinder and Tan [6]. Although many works in inventory control assume a penalty cost parameter for penalizing stock-outs, in all the works cited here the cost is minimized under a service level constraint, which is in practice a very popular measure, since it has been widely recognized that penalty costs, and in particular the cost of loosing customer goodwill, are usually difficult to assess (Bashyam and Fu [4]).

A common assumption, in practice very restrictive, in all these works is the absence of delivery lag. A work on stochastic lead time in continuous-time inventory models was presented in Zipkin [30]. Kaplan [15] characterized the optimal policy for a dynamic inventory problem where the time lag in delivery of an item is a discrete random variable with known distribution. Since tracking all the outstanding orders by means of dynamic programming requires a large multidimensional state vector, Kaplan assumes that orders do not cross in time and supplier lead time probabilities are independent of the size/number of outstanding orders (for details on order-crossover see Hayya et al. [12]). Under these assumptions he was able to provide a solution method for the problem and to derive the optimal policy. The first assumption is valid for systems where supplier's production system has a single-server queue structure operating under a FIFO policy. In Bashyam and Fu [4] a similar problem — operating under (s, S) policy, having a service level constraint and allowing orders to cross in time — is described and solved by means of a simulation based approach. To the best of our knowledge, there is no complete approach in the literature that addresses the (R^n,S^n) policy under stochastic supplier lead time.

In this paper, we use a "stochastic constraint programming" approach to address (R^n,S^n) policy under stochastic supplier lead time. Computing optimal policy parameters under these assumptions is a hard problem from a computational point of view. We build on the work of Eppen and Martin [9] and following a similar approach we develop a *scenario based method* [26,5] for solving (R^n,S^n) under stochastic demand and supplier lead time. Efficient methods for computing (R^n,S^n) policy parameters based on Constraint Programming were proposed in Tarim et al. [27,24]. In this paper, under the same assumptions, we develop a dedicated *constraint* that realizes a deterministic equivalent modeling of chance-constraints [8] by employing a scenario based approach [26]. A *constraint programming* (CP) [1] model is proposed and an example is given where an inventory control problem is solved to optimality under a given discrete stochastic supplier lead time with known distribution.

The paper is organized as follows. In Section 2 we provide some formal background related to the modeling techniques employed. In Section 3 we provide a formal definition for the general multi-period production/inventory problem with non-stationary stochastic demand and lead time. In Section 4 we extend Tarim and Kingsman's [25] model for the replenishment cycle policy in order to consider a dynamic deterministic supplier lead time, which assumes that orders may cross in time. In Section 5 former results are embedded in a scenario based approach to solve the problem when a stochastic supplier lead time with known probability mass function is given. In Section 6 a CP model is proposed, which incorporates former results in a dedicated constraint able to dynamically enforce the given service level constraint during search. Furthermore a demonstrative example is given in this section to clarify the approach. In Section 7 an instance is solved under deterministic and stochastic supplier lead times; solutions are then discussed. In Section 8 results are summarized and directions for future research are given.

2 Constraint Programming

A *Constraint Satisfaction Problem* (CSP) [1,7,16] is a triple $\langle V, C, D\rangle$, where V is a set of decision variables, D is a function mapping each element of V to a domain of potential values, and C is a set of constraints stating allowed combinations of values for subsets of variables in V.A *solution* to a CSP is simply a set of values of the variables such that the values are in the domains of the variables and all of the constraints are satisfied. We may also be interested in finding a feasible solution that minimizes (maximizes) the value of a given objective function over a subset of the variables. Alternatively, we can define a constraint as a mathematical function: $f : D_1 \times D_2 \times \ldots \times D_n \rightarrow \{0, 1\}$ such that $f(x_1, x_2, \ldots, x_n) = 1$ if and only if $C(x_1, x_2, \ldots, x_n)$ is satisfied. Using this functional notation, we can then define a constraint satisfaction problem (CSP) as follows (see also [1]): given n domains D_1, D_2, ..., D_n and m constraints f_1, f_2, ..., f_m find x_1, x_2, ..., x_n such that

$$f_k(x_1, x_2, \ldots, x_n) = 1, \quad 1 \leq k \leq m; \tag{1}$$

$$x_j \in D_j, \quad 1 \leq j \leq n. \tag{2}$$

The problem is only a feasibility problem, and no objective function is defined. Nevertheless, CSPs are also an important class of combinatorial optimization problems. Here the functions f_k do not necessarily have closed mathematical forms (for example, functional representations) and can be defined simply by providing the set S described above.

For key concepts in Constraint Programming (CP) such as constraint filtering algorithm, constraint propagation and arc-consistency see [1,17].

In [29] and [26] a *stochastic constraint satisfaction problem* (stochastic CSP) is defined as a 6-tuple $< V, S, D, P, C, \theta >$. V is a set of decision variables and S is a set of stochastic variables. D is a function mapping each element of V and each element of S to a domain of potential values. A decision variable in V is *assigned* a value from its domain. P is a function mapping each element of S to a probability distribution for its associated domain. C is a set of constraints. A constraint $h \in C$ that constrains at least one variable in S is a *chance-constraint.* θ_h is a threshold value in the interval $[0, 1]$, indicating the minimum satisfaction probability for chance-constraint h. Note that a chance-constraint with a threshold of 1 is equivalent to a hard constraint.

In [29] a policy based view of stochastic constraint programs is proposed. The semantics is based on a tree of decisions. Each path in a policy represents a different possible scenario (set of values for the stochastic variables), and the values assigned to decision variables in this scenario. To find satisfying policies, backtracking and forward checking algorithms, which explores the implicit AND/OR graph, are presented. Such an approach has been further investigated in [3]. An alternative semantics for stochastic constraint programs, which suggests an alternative solution method, comes from a scenario-based view [5]. In [26] the authors outline this solution method, which consists in generating a scenario-tree that incorporates all possible realizations of discrete random variables into the model explicitly. The great advantage of such an approach is that conventional constraint solvers can be used to solve stochastic CSP. Of course, there is a price to pay in this approach, as the number of scenarios grows exponentially with the number of stages and such a growth is particularly affected by random variables that contain a wide range of values in their domain.

3 Problem Definition

We consider a finite planning horizon of N periods and a demand d_t for each period $t \in \{1, \ldots, N\}$, which is a random variable with probability density function $g_t(d_t)$. We assume that the demand occurs instantaneously at the beginning of each time period. The demand we consider is non-stationary, that is it can vary from period to period, and we also assume that demands in different periods are independent.

In the following sections we will consider two different cases, respectively: a deterministic lead time of length L_t for an order placed in period $t \in \{1, ..., N\}$ and a stochastic lead time l_t with probability mass function $f_t(l_t)$ for an order placed in period $t \in \{1, ..., N\}$. Note that $\{l_t\}$ are mutually independent and each of them is also independent of the respective order quantity. A fixed delivery cost a is incurred for each order and a variable unit cost v. A linear holding cost h is incurred for each unit of product carried in stock from one period to the next. We assume that it is not possible to sell back excess items to the vendor at the end of a period and that negative orders are not allowed, so that if the actual stock exceeds the order-up-to-level for that review, this excess stock is carried forward and not returned to the supply source. However, such occurrences are regarded as rare events and accordingly the cost of carrying excess stocks and the positive effect on the service level of subsequent periods is ignored. As a service level constraint we require the probability that at the end of each and every period the net inventory will not be negative set to be at least a given value α. Our aim is to minimize the expected total cost, which is composed of ordering costs, unit costs and holding costs, over the N-period planning horizon, satisfying the service level constraints.

The actual sequence of ordering and delivery to be considered can be arbitrary as Kaplan notices in [15]. In the following we will adopt the same sequence of action he describes, since it handles all the deliveries symmetrically and allows for some delay in the arrival deliveries at the beginning of a period. The sequence is therefore as follows. At the beginning of a period, the inventory on hand after all the demands from previous periods have been realized is known. Since we are assuming complete backlogging, this quantity may be negative. Also known are orders placed in previous periods which have not been delivered yet. On the basis of this information, an ordering decision is made for the current period. All the deliveries that are to be made during a period are assumed to be made immediately after this ordering decision and hence are on hand at the beginning of the period. A further discussion that states the convenience of this sequence of events can be found in Kaplan [15]. To summarize there are three successive events at the beginning of each period. First, stock on hand and outstanding orders are determined. Second, an ordering decision is made on the basis of this information. Third, all supplier deliveries for the current period, including possibly the most recent orders, are received.

4 Dynamic Deterministic Lead Time

In this section we focus on the general multi-period production/inventory problem with stochastic demands and dynamic deterministic lead time. The reader may also refer to [11] about this topic. This problem can be formulated as finding the timing of the stock reviews and the size of the respective non-negative replenishment orders, X_t in period t, with the objective of minimizing the expected total cost $E\{TC\}$ over a finite planning horizon of N periods. Since a dynamic deterministic lead time $L_t \geq 0$ is considered in each period $t = 1, \ldots, N$, an order placed in period t will be received only at period $t + L_t$. Depending on the values assigned to L_t it may be obviously not possible to provide the required service level for some initial periods. In general we will be able to provide the required service level α starting from the period t for which the value $t + L_t$ is minimum. Let M be this period. Notice also that it will never be optimal to place any order in a period t such that $t + L_t > N$, since such an order will not be received within the given planning horizon. The problem can be formulated as a chance-constrained programming model (see Bookbinder and Tan [6]),

$$\begin{aligned} \min \;\; E\{TC\} = & \int_{d_1}\int_{d_2}\ldots\int_{d_N}\sum_{t=1}^{N}(a\delta_t + vX_t + h\cdot\max(I_t,0)) \\ & \times g_1(d_1)g_2(d_2)\ldots g_N(d_N)\mathrm{d}(d_1)\mathrm{d}(d_2)\ldots\mathrm{d}(d_N) \end{aligned} \tag{3}$$

subject to,

$$\delta_t = \begin{cases} 1, & \text{if} \quad X_t > 0 \\ 0, & \text{otherwise} \end{cases} \qquad t = 1, ..., N \qquad (4)$$

$$I_t = I_0 + \sum_{\{i|i \geq 1, L_i + i \leq t\}} X_i - \sum_{i=1}^{t} d_i \qquad t = 1, ..., N \qquad (5)$$

$$\Pr\{I_t \geq 0\} \geq \alpha \qquad t = M, ..., N \qquad (6)$$

$$I_t \in \mathbb{Z}, \quad X_t \geq 0, \quad \delta_t \in \{0, 1\} \qquad t = 1, ..., N \qquad (7)$$

where we comply with the notation used in [6],

d_t : the demand in period t, a random variable with probability density function, $g_t(d_t)$,
a : the fixed ordering cost (incurred when an order is placed),
h : the proportional stock holding cost,
v : the unit variable cost of an item,
L_t : the deterministic delivery lead time in period t, $L_t \geq 0$
δ_t : a {0,1} variable that takes the value of 1 if a replenishment occurs in period t and 0 otherwise,
I_t : the inventory level (stock on hand minus back-orders) at the end of period t,
I_0 : the initial inventory,
X_t : the size of the replenishment order placed in period t, $X_t \geq 0$, (received in period $t + L$).

Let us denote the inventory position (the total amount of stock on hand plus outstanding orders minus back-orders) at the end of period t as P_t. It directly follows that

$$P_t = I_t + \sum_{\{i|1 \leq i \leq t, L_i + i > t\}} X_i. \qquad (8)$$

where P_t is the inventory position in period t and it is assumed $P_0 = I_0$. We now reformulate the model using the inventory position,

$$\min \ E\{TC\} = \int_{d_1} \int_{d_2} \cdots \int_{d_N} \sum_{t=1}^{N} \left(a\delta_t + vX_t + h \cdot \max(P_t - \sum_{\{i|1 \leq i \leq t, L_i + i > t\}} X_i, 0) \right) \times g_1(d_1) g_2(d_2) \ldots g_N(d_N) \mathrm{d}(d_1) \mathrm{d}(d_2) \ldots \mathrm{d}(d_N) \qquad (9)$$

subject to,

$$\delta_t = \begin{cases} 1, & \text{if} \quad X_t > 0 \\ 0, & \text{otherwise} \end{cases} \qquad t = 1, ..., N \qquad (10)$$

$$P_t = I_0 + \sum_{i=1}^{t} (X_i - d_i) \qquad t = 1, ..., N \qquad (11)$$

$$\Pr\{P_t \geq \sum_{\{i|1\leq i\leq t, L_i+i>t\}} X_t\} \geq \alpha \qquad t = M, ..., N \qquad (12)$$

$$P_t \in \mathbb{Z}, \quad X_t \geq 0, \quad \delta_t \in \{0,1\} \qquad t = 1, ..., N. \qquad (13)$$

By using the expectation operator $E\{\cdot\}$, since $\{d_t\}$ are assumed to be mutually independent, we may rewrite the objective function as

$$\min \quad E\{TC\} = \sum_{t=1}^{N} \left(h \cdot E \left\{ \max(P_t - \sum_{\{i|1\leq i\leq t, L_i+i>t\}} X_i, 0) \right\} + a \cdot \delta_t + v \cdot X_t \right). \qquad (14)$$

When a stock-out occurs, all demand is back-ordered and filled as soon as an adequate supply arrives. However, the probability that net inventory will not be negative is set normally quite high by the management, so that the cost of back-orders can be ignored in the model. Moreover, Bookbinder and Tan discuss that the term $E\{\max(I_t, 0)\}$ may be approximated by $E\{I_t\}$, in view of these remarks. Therefore in our model we approximate the term $E\{\max(P_t - \sum_{\{i|1\leq i\leq t, L_i+i>t\}} X_i, 0)\}$ with the term $E\{P_t - \sum_{\{i|1\leq i\leq t, L_i+i>t\}} X_i\}$.

The general chance constrained programming formulation given above can be modified to incorporate the inventory control policy adopted. In this paper we adopt the "replenishment cycle policy", which is equivalent to Bookbinder-Tan's "static-dynamic uncertainty strategy". The replenishment cycle policy (ie, (R, S) policy) is *static* in the sense that the replenishment periods are determined once and for all at the beginning of the planning horizon, and *dynamic* as the order quantities are decided only after observing the realized demand. In what follows –based on [25], in which lead times are ignored– we formulate the replenishment cycle policy under dynamic deterministic lead times, L_t.

Consider a review schedule, which has m reviews over the N period planning horizon with orders placed at $\{T_1, T_2, \ldots, T_m\}$, where $T_i > T_{i-1}$, $T_m \leq N - L_{T_m}$. For convenience T_1 is defined as the start of the planning horizon and $T_{m+1} = N+1$ as the period immediately after the end of the planning horizon. The review schedule may be generalized to consider the case where $T_1 > 1$, if the opening stock I_0 is sufficient to cover the immediate needs at the start of the planning horizon. The associated stock reviews will take place at the beginning of periods T_i, $i = 1, \ldots, m$. In the considered dynamic review and replenishment policy clearly the orders X_i are all equal to zero except at replenishment periods $T_1, T_2, \ldots, T_m$. The inventory level I_t carried from period t to period $t+1$ is the opening stock plus any orders that have arrived up to and including period t less the total demand to date. Hence is given by

$$I_t = I_0 + \sum_{\{i|L_{T_i}+T_i \leq t\}} X_{T_i} - \sum_{k=1}^{t} d_k, \quad t = 1, \ldots, N. \tag{15}$$

Let us define

$$p(t) = \max\left\{i|\forall j, j \leq i, T_j + L_{T_j} \leq t, \quad i = 1, \ldots, m\right\}. \tag{16}$$

The inventory level I_t at the end of period t (Eq. 15) can be expressed as

$$I_t = I_0 + \sum_{i=1}^{p(t)} X_{T_i} + \sum_{\{i|i>p(t), L_{T_i}+T_i \leq t\}} X_{T_i} - \sum_{k=1}^{t} d_k, \quad t = 1, \ldots, N. \tag{17}$$

We now want to reformulate the constraints of the chance constrained model in terms of a new set of decision variables R_{T_i}, $i = 1, \ldots, m$. We define

$$P_t = R_{T_i} - \sum_{k=T_i}^{t} d_k, \quad T_i \leq t < T_{i+1}, \quad i = 1, \ldots, m \tag{18}$$

where R_{T_i} can be interpreted as an order-up-to-position which stock should be raised after placing an order at the ith review period T_i, and $R_{T_i} - \sum_{k=T_i}^{t} d_k$ is the end of period inventory position. We can now express the whole model in term of these new decision variables R_{T_i}, which are related to the inventory position in period T_i. The new problem is therefore to determine the number of reviews, m, the T_i, and the associated R_{T_i} for $i = 1, \ldots, m$.

If there is no replenishment scheduled for period t, then R_t equals the opening inventory position in period t. It follows that the variable R_t must be equal to P_{t-1} if no order is placed in period t and equal to the order-up-to-position if there is a review in period t. We can express this using the following constraints

$$R_t = P_t + d_t, \qquad t = 1, \ldots, N \tag{19}$$

$$R_t \geq P_{t-1}, \qquad t = 1, \ldots, N \tag{20}$$

$$R_t > P_{t-1} \Rightarrow \delta_t = 1, \quad t = 1, \ldots, N. \tag{21}$$

The values for the order-up-to-position variables, R_t, are then those that give the minimum expected total cost $E\{TC\}$. The desired opening stock positions, as required for the solution to the problem, will then be those values of R_t, for which $\delta_t = 1$. It is now clear that Constraints 4 and 5 can be replaced by Eq. 19, 21 and 20.

Let us now express Eq. 17 using R_{T_i} as decision variables

$$I_t = R_{T_{p(t)}} + \sum_{\{i|i>p(t),L_{T_i}+T_i\leq t\}} \left(R_{T_i} - R_{T_{i-1}} + d_{T_{i-1}} + \ldots + d_{T_i-1}\right) - \sum_{k=T_{p(t)}}^{t} d_k, \quad t = 1,\ldots,N. \tag{22}$$

As already mentioned, α is the desired minimum probability that the net inventory level in any time period will be non-negative. M is by definition the first period at which the inventory can be controlled. Keeping this in mind we require

$$\Pr\{I_t \geq 0\} \geq \alpha, \quad t = M,\ldots,N. \tag{23}$$

which implies, by substituting I_t with the right term in Eq. 22,

$$G_S\left(R_{T_{p(t)}} + \sum_{\{i|i>p(t),L_{T_i}+T_i\leq t\}} (R_{T_i} - R_{T_{i-1}})\right) \geq \alpha, \quad t = M,\ldots,N. \tag{24}$$

where $S = \sum_{k=T_{p(t)}}^{t} d_k - \sum_{\{i|i>p(t),L_{T_i}+T_i\leq t\}}(d_{T_{i-1}} + \ldots + d_{T_i-1})$ and, as given in [6], $G_{d_1+d_2+\ldots+d_t}(.)$ is the cumulative distribution function of $D(t) = d_1 + d_2 + \ldots + d_t$.

We now express the whole model in terms of the new set of variables R_i. Since we consider expectations $\tilde{P}_i$ and $\tilde{d}_i$, it follows that $R_i = \tilde{P}_i + \tilde{d}_i$ and also that the term X_t in the objective function can be expressed as $R_t - \tilde{P}_{t-1}$. We replace the service level constraint 6 using the new formulation in Eq. 24. We should note that $v\sum_{t=1}^{N}\left(R_t - \tilde{P}_{t-1}\right)$ in the objective function can be rewritten as $v\sum_{t=1}^{N}\tilde{d}_t + v\cdot P_N$, where $\sum_{t=1}^{N}\tilde{d}_t$ is obviously a constant of the problem. The resulting model is as follows,

$$E\{TC\} = v\sum_{t=1}^{N}\tilde{d}_t + \min \left[\sum_{t=1}^{N}\left(h\cdot\left(\tilde{P}_t - \sum_{\{i|1\leq i\leq t,L_i+i>t\}}(R_i - \tilde{P}_{i-1})\right) + a\cdot\delta_t\right) + v\cdot\tilde{P}_N\right] \tag{25}$$

subject to,

$$\{T_1,\ldots,T_m\} = \{t\in\{1,\ldots,N\}|\delta_t = 1\}$$
$$\text{Eq. 24}, \qquad t = M,\ldots,N$$

$$
\begin{aligned}
&R_t > \tilde{P}_{t-1} \Rightarrow \delta_t = 1, && t = 1,\dots,N && (26)\\
&R_t \geq \tilde{P}_{t-1}, && t = 1,\dots,N && (27)\\
&R_t = \tilde{P}_t + \tilde{d}_t, && t = 1,\dots,N && (28)\\
&R_t \geq 0, \quad \tilde{P}_t \geq 0, \quad \delta_t \in \{0,1\} && t = 1,\dots,N && (29)
\end{aligned}
$$

So far we treated the replenishment cycle policy formulation of the production/inventory problem under non-stationary stochastic demand, d_t, and dynamic deterministic lead time, L_t. We now recall that a deterministic equivalent formulation of this problem under the same policy, non-stationary stochastic demand, d_t, and deterministic but constant lead time, L, was proposed in [23]. According to this formulation and from the results presented here, when the lead time is deterministic and constant, it is easy to see that Eq. 24 becomes

$$G_{d_{T_{p(t)}}+d_{T_{p(t)}+1}+\ldots+d_t}(R_{T_{p(t)}}) \geq \alpha, \quad t = L+1,\dots,N. \tag{30}$$

We adopt the following change of variable: $T_{p(t)} = T_i$. Since the lead time is deterministic and constant T_i will be equal to $T_{p(t)}$ for every t such that $T_i \leq t < T_{i+1} + L$. It directly follows that

$$G_{d_{T_i}+d_{T_i+1}+\ldots+d_t}(R_{T_i}) \geq \alpha, \quad T_i \leq t < T_{i+1} + L. \tag{31}$$

By defining $k = t - L$ we can rewrite the former expression as

$$G_{d_{T_i}+d_{T_i+1}+\ldots+d_{k+L}}(R_{T_i}) \geq \alpha, \quad T_i \leq k < T_{i+1} \tag{32}$$

and therefore, since $\tilde{P}_k = R_{T_i} - \sum_{n=T_i}^{k} \tilde{d}_n$, it follows,

$$\tilde{P}_k \geq G^{-1}_{d_{T_i}+d_{T_i+1}+\ldots+d_{k+L}}(\alpha) - \sum_{n=T_i}^{k} \tilde{d}_n, \quad T_i \leq k < T_{i+1}. \tag{33}$$

G^{-1} is an "inverse function", such that $G^{-1}_{D(t)}(\alpha) = u$ means $\alpha = G_{D(t)}(u) = \Pr\{D(t) \leq u\}$. We assume that G is strictly increasing, hence G^{-1} is uniquely defined. The right-hand side of Eq. 33 can be calculated off-line and memorized in a table once the form of $g_t(\cdot)$ is selected. Let

$$\Phi[i,j] = G^{-1}_{d_i+d_{i+1}+\ldots+d_{j+L}}(\alpha) - \sum_{k=i}^{j} \tilde{d}_k. \tag{34}$$

By employing the table presented in Eq. 34, the whole model under deterministic and constant lead time, L, can be easily expressed using a CP formulation similar to the one presented in [27]. The whole model is

$$E\{TC\} =$$

$$v\sum_{t=1}^{N} \tilde{d}_t + \min \left[\sum_{t=1}^{N} \left(h \cdot \left(\tilde{P}_t - \sum_{i=t-L+1}^{t} (R_i - \tilde{P}_{i-1})\right) + a \cdot \delta_t\right) + v \cdot \tilde{P}_N\right] \tag{35}$$

subject to,

$$R_t > \tilde{P}_{t-1} \Rightarrow \delta_t = 1 \qquad t = 1, \ldots, N \tag{36}$$

$$R_t \geq \tilde{P}_{t-1} \qquad t = 1, \ldots, N \tag{37}$$

$$\tilde{P}_t \geq \Phi[\max_{j \in \{1..t\}} \{j \cdot \delta_j\}, t] \qquad t = 1, \ldots, N - L \tag{38}$$

$$R_t = \tilde{P}_t + \tilde{d}_t, \qquad t = 1, \ldots, N \tag{39}$$

$$R_t \geq 0, \quad \tilde{P}_t \geq 0, \quad \delta_t \in \{0, 1\} \qquad t = 1, \ldots, N \tag{40}$$

where elements in matrix Φ are indexed using the `element` constraint [13]. Obviously if we want to invert the cumulative distribution function in Eq. 24 as in the constant lead time case, the dimension of the table where the buffer stock levels are stored has to increase, since many decision variables take part in the computation of the stock-out probability. Instead of building this matrix, it may be therefore convenient to develop a dedicated *constraint* for the CP formulation of the model. In fact, in CP relations between decision variables can be expressed by means of dedicated constraints that may include customized algorithms to generate parameters and verify complex conditions like Eq. 24. In this constraint we simply wait for a partial assignment of decision variables $\{\delta_t\}$ and, by using Eq. 24, we dynamically generate during the search deterministic equivalent constraints in a way similar to the one presented in the example above. These deterministic constraints are enforced to guarantee the required service level under the given partial replenishment plan.

5 Non-stationary Stochastic Lead Time

We now consider the general multi-period production/inventory problem with non-stationary stochastic demand and lead time. As in Eppen and Martin [9], we consider a discrete stochastic lead time with probability mass function $f_i(\cdot)$ in each period $i = 1, \ldots, N$. This means that an order placed in period i will be received after k periods with probability $f_i(k)$. Since $f_i(k)$ is discrete we shall assume that there is a maximum lead time L for which $\sum_{k=0}^{L} f_i(k) = 1$, $i = 1, \ldots, N$. The probability of observing any lead time length $p > L$ will be always 0. Therefore the possible lead time lengths are limited to $S = \{0, \ldots, L\}$ and the probability mass function is defined on the finite set S. Depending on the probabilities assigned to each lead time length by the probability mass function, it may not be possible to provide the required service level for some initial periods. In general, reasoning in a worst case scenario, it will always be possible to provide the required service level α starting from period $L + 1$.

The chance-constrained programming model is given below,

$$\min \;\; E\{TC\} = \int_{d_1} \ldots \int_{d_N} \sum_{l_1} \ldots \sum_{l_N} \sum_{t=1}^{T} (v \cdot X_t + a \cdot \delta_t + h \cdot I_t) \tag{41}$$
$$f_1(l_1) f_2(l_2) \ldots f_N(l_N) \times g_1(d_1) g_2(d_2) \ldots g_N(d_N) d(d_1) d(d_2) \ldots d(d_N)$$

subject to,

$$I_t = I_0 + \sum_{\{i \mid i \geq 1, l_i \leq t-i\}} (X_i - d_t) \qquad t = 1, \ldots, N \tag{42}$$

$$\delta_t = \begin{cases} 1, & \text{if} \quad X_t > 0 \\ 0, & \text{otherwise} \end{cases} \qquad t = 1, \ldots, N \tag{43}$$

$$\Pr\{I_t \geq 0\} \geq \alpha \qquad t = L+1, \ldots, N \tag{44}$$

$$I_t \in \mathbb{Z}_0^+, \quad X_t \geq 0, \quad \delta_t \in \{0,1\} \qquad t = 1, \ldots, N \tag{45}$$

where

l_i : the lead time length of the order placed in period i, a discrete random variable with probability mass function $f_i(\cdot)$.

We now reformulate the model using the inventory position,

$\min \;\; E\{TC\} =$

$$\int_{d_1} \ldots \int_{d_N} \sum_{l_1} \ldots \sum_{l_N} \sum_{t=1}^{N} \left(a\delta_t + vX_t + h \cdot \left(P_t - \sum_{\{i \mid 1 \leq i \leq t, l_i > t-i\}} X_i \right) \right) \tag{46}$$
$$f_1(l_1) f_2(l_2) \ldots f_N(l_N) \times g_1(d_1) g_2(d_2) \ldots g_N(d_N) \mathrm{d}(d_1) \mathrm{d}(d_2) \ldots \mathrm{d}(d_N)$$

subject to,

$$\delta_t = \begin{cases} 1, & \text{if} \quad X_t > 0 \\ 0, & \text{otherwise} \end{cases} \qquad t = 1, ..., N \tag{47}$$

$$P_t = I_0 + \sum_{i=1}^{t} (X_i - d_i) \qquad t = 1, ..., N \tag{48}$$

$$\Pr\{P_t \geq \sum_{\{i \mid 1 \leq i \leq t, l_i > t-i\}} X_i\} \geq \alpha \qquad t = L+1, ..., N \tag{49}$$

$$P_t \in \mathbb{Z}_0^+, \quad X_t \geq 0, \quad \delta_t \in \{0,1\} \qquad t = 1, ..., N. \tag{50}$$

Let us define the cumulative distribution function $F_i(k) = \sum_{p=0}^{k} f_i(p)$, $k \geq 0$. Given the probability mass function $f_i(l_i)$ and since l_i is a discrete random variable it directly follows

$$\sum_{i=1}^{t} F_i(t-i)X_i = \sum_{i=1}^{t}\sum_{p=0}^{t-i} f_i(p) \qquad t = 1,\ldots,N. \tag{51}$$

By recalling that $\{d_t\}$ are assumed to be mutually independent, we may rewrite the objective function as

$$\min\ E\{TC\} = \sum_{t=1}^{N}\left(h \cdot E\left\{\left(P_t - \sum_{i=1}^{t}(1-F_i(t-i))X_i\right) + v \cdot X_t\right\} + a \cdot \delta_t\right) \tag{52}$$

Also in this case we want to adopt a replenishment cycle policy and we want to express the whole model in terms of the new set of variables R_i, so that order quantities have to be decided only after the demand in the former periods have been realized. The analysis developed in the former section for the replenishment condition (Eq. 43) and inventory conservation constraints (Eq. 42) still holds, since it refers to the opening-inventory-position, which by definition is not affected by the lead time length. So it is clear that these constraints can be replaced by Eq. 19, 21 and 20. Since we are considering expectations, the term X_t in the objective function can be expressed as $R_t - \tilde{P}_{t-1}$. As we did in the dynamic deterministic lead time case, we now have to express the service level constraint as a relation between the opening-inventory-positions such that the overall service level provided at the end of each period is at least α. In order to express this service level constraint we propose a scenario based approach over the discrete random variables l_i, $i = 1,\ldots,N$. Let us recall that in a scenario based approach [5,26], a scenario tree is generated which incorporates all possible realization of discrete random variables into the model explicitly. A path from the root to an extremity of the event tree represents a scenario $\omega \in \Omega$, where Ω is the set of all possible scenarios. To each scenario a given probability is associated. If S_i is the ith random variable on a path from the root to the leaf representing scenario ω and a_i is the value given to S_i in the ith stage of this scenario, then the probability of this scenario is given by $\Pr\{\omega\} = \prod_i \Pr(S_i = a_i)$. Within each scenario, we have a conventional (non-stochastic) constraint program to solve. All we have to do is replacing the stochastic variables by the values taken in the scenario and ensure that the values found for the decision variables are consistent across scenarios as certain decision variables are shared across scenarios.

In our problem we can divide random variables into two sets: the discrete random variables $\{l_i\}$ which represent lead times and the continuous random variables $\{d_i\}$ which represent demands. We deal with each set in a separate fashion, by employing a scenario based approach for the discrete random variables and a deterministic equivalent modeling approach for the continuous random variables. This is possible since, as we have already remarked, under a given scenario ω discrete random variables are treated as deterministic values. The problem is then reduced to the general multi-period production/inventory problem with dynamic deterministic lead time and stochastic demand, for which we have already presented in the former section a deterministic equivalent model that is able to represent the chance-constraints involving continuous random variables $\{d_i\}$.

Consider a review schedule Z, which has m reviews over the N period planning horizon with orders placed at $\{T_1, T_2, \dots, T_m\}$, where $T_i > T_{i-1}$, $T_m \leq N$. For convenience T_1 is defined as the start of the planning horizon and $T_{m+1} = N+1$ as the period immediately after the end of the planning horizon. The review schedule may be generalized to consider the case where $T_1 > 1$, if the opening stock I_0 is sufficient to cover the immediate needs at the start of the planning horizon. The associated stock reviews will take place at the beginning of periods T_i, $i = 1, \dots, m$. In the considered dynamic review and replenishment policy clearly the orders X_i are all equal to zero except at replenishment periods $T_1, T_2, \dots, T_m$. The inventory level I_t carried from period t to period $t+1$ is the opening stock plus any orders that have arrived up to and including period t less the total demand to date. A scenario ω_t is a possible lead time realization for all the orders placed up to period t in the given review schedule Z. Let Ω_t be the set of all the possible scenarios ω_t. The first observation we need is related to the definition of $p(t)$ (Eq. 16). We have defined $T_{p(t)}$ as the latest period before period t in the planning horizon, for which we are sure that all the former orders, including the one placed in $T_{p(t)}$ if there is any, have been delivered within period t. Under the assumption that the probability mass function $f_i(\cdot)$ is defined on a finite set S, $p(t)$ provides a bound for the scenario tree size. In fact if the possible lead time lengths in S are $0, \dots, L$, the earliest order that is delivered in period t with probability 1 under every possible scenario ω_t is the latest placed in the span $1, \dots, t-L$. Therefore since each scenario ω_t identifies the orders that have been received before or in period t, it directly follows that the number of scenarios in the tree that is needed to compute the buffer stocks for periods $t-L, \dots, t$ under any possible review schedule Z is at most 2^L, when we place $L+1$ orders in periods $t-L, \dots, t$, but it may be lower if less reviews are planned. Under a given review schedule Z and a scenario ω_t the service level constraint for a period t can be easily expressed by means of Eq. 24. It follows that the service level constraint is always a relation between at most $L+1$ decision variables P_i that represent the closing-inventory-position (or equivalently R_i which are the order-up-to-position) of the replenishment cycles covering the span $t-L, \dots, t$. Let $p_\omega(t)$ be the value of $p(t)$ under a given scenario ω_t when a review schedule Z is considered. In order to satisfy the service level constraints in our original model, we require that the overall service level under all the possible scenarios for each set of at most $L+1$ decision variables is at least α or equivalently, by using Eq. 24

$$\sum_{\omega_t \in \Omega_t} \Pr\{\omega_t\} \cdot G_S \left(R_{T_{p_\omega(t)}} + \sum_{\{i | i > p_\omega(t), (l_{T_i} | \omega_t) \leq t - T_i\}} (R_{T_i} - R_{T_{i-1}}) \right) \geq \alpha, \quad t = L+1, \ldots, N, \tag{53}$$

where $S = \sum_{k=T_{p_\omega(t)}}^{t} d_k - \sum_{\{i|i>p_\omega(t),(l_{T_i}|\omega_t)\leq t-T_i\}} (d_{T_{i-1}} + \ldots + d_{T_i - 1})$. Therefore the complete model under the replenishment cycle policy can be expressed as

$$E\{TC\} =$$

$$v \sum_{t=1}^{N} \tilde{d}_t + \min \left[\sum_{t=1}^{N} \left(h \cdot \left(\tilde{P}_t - \sum_{i=1}^{t} (1 - F_i(t-i))(R_i - \tilde{P}_{i-1}) \right) + a \cdot \delta_t \right) + v \cdot \tilde{P}_N \right] \tag{54}$$

subject to,

$$\{T_1, \ldots, T_m\} = \{t \in \{1, \ldots, N\} | \delta_t = 1\}$$

Eq. 53, $\qquad t = L+1, \ldots, N$

$$R_t > \tilde{P}_{t-1} \Rightarrow \delta_t = 1 \qquad t = 1, \ldots, N \tag{55}$$

$$R_t \geq \tilde{P}_{t-1} \qquad t = 1, \ldots, N \tag{56}$$

$$R_t = \tilde{P}_t + \tilde{d}_t \qquad t = 1, \ldots, N \tag{57}$$

$$R_t \geq 0, \quad \tilde{P}_t \geq 0, \quad \delta_t \in \{0, 1\} \qquad t = 1, \ldots, N. \tag{58}$$

6 Stochastic Lead Time: a CP Implementation

In this section we present a CP formulation for the (R^n, S^n) problem under stochastic lead time. Results from the former section will be employed in the CP formulation. In order to model the service level constraint (Eq. 53) we presented in the former section, a new constraint *serviceLevel*$(\cdot)$ will be defined. Such a constraint is needed to dynamically compute the correct buffer stock positions on the basis of the current replenishment plan, that is $\{\delta_t\}$ assignments. Without loss of generality we will consider here a different and simpler objective function. In such a function we will charge a holding cost at the end of each period based on the current inventory position, rather than the current inventory level. This will reflect the fact that we charge interests not only on the actual amount of items we have in stock, but also on outstanding orders. It should be noted that it is possible to build a CP model that considers the original objective function. We chose not to implement this function in our tool. In fact, in the research project carried out for a leading international telecommunications company that motivated this research we were explicitly required to charge holding cost on the inventory position and not on the inventory level. Doing so often make sense since companies may assess holding cost on their total invested capital and not simply on items in stock. A further and detailed justification for this can be found in [14] [2].

The CP model that incorporates our dedicated chance constraint and the objective function discussed is therefore

$$\min \quad E\{TC\} = \sum_{t=1}^{N} \left(a \cdot \delta_t + h \cdot \tilde{P}_t \right) + v \cdot \tilde{P}_N \tag{59}$$

subject to,

$$\tilde{P}_t + \tilde{d}_t - \tilde{P}_{t-1} > 0 \Rightarrow \delta_t = 1 \qquad t = 1, \dots, N \tag{60}$$

$$\delta_t = 0 \Rightarrow \tilde{P}_t + \tilde{d}_t - \tilde{P}_{t-1} = 0 \qquad t = 1, \dots, N \tag{61}$$

$$\tilde{P}_t + \tilde{d}_t - \tilde{P}_{t-1} \geq 0 \qquad t = 1, \dots, N \tag{62}$$

$$\begin{aligned} &serviceLevel(\delta_1, \dots, \delta_N, \\ &\qquad \tilde{P}_1, \dots, \tilde{P}_N, \\ &\qquad g_1(d_1), \dots, g_N(d_N), \\ &\qquad f(\cdot), \alpha) \end{aligned} \tag{63}$$

$$\tilde{P}_t \geq 0, \quad \delta_t \in \{0, 1\} \qquad t = 1, \dots, N. \tag{64}$$

It must be noted that the domain size value for the $\tilde{P}_t$ variables, exactly as in the zero lead time case, is limited and more precisely it is equal to the amount of stock required to satisfy subsequent demands till the end of the planning horizon, meeting the required service level when only a single replenishment is scheduled at the beginning of the planning horizon. In what follows we describe the signature of the new constraint we have introduced. *serviceLevel*$(\cdot)$ describes a relation between all the decision variables in the

[2] In this work the author considers a holding cost based on the inventory position rather than on-hand inventory in their order-up-to policy. He underlines how a holding cost based on inventory position provides a simple and more accurate expression for inventory holding costs in the combined manufacturing and warehouse divisions. In fact he observed that the order of a part initiates a succession of charges which are incurred throughout the lead time (direct material cost, direct labor cost and overheard cost). Certain inventory carrying costs are based on these charges – interest on investment and risk of obsolescence – and they are accrued from the time an order is placed to the manufacturing division. On the other hand other inventory carrying costs are accrued from the time the finished part is delivered to the warehouse (warehousing costs). The author suggests that a precise expression for the inventory carrying costs which reflected all these consideration would be very complex. Therefore, when interest and risk of obsolescence comprise a large portion of the total carrying cost, using a model which incurs carrying cost from the time an order is placed rather then from the time is delivered may be the correct choice

model. It also accepts as parameters the distribution of the demand in each period; the probability mass function of the lead time, which is assumed to be the same for all the periods; and the required service level. In order to enforce this constraint we consider every group of consecutive replenishment cycles that cover at least $L+1$ periods (that is the one of interest plus L former periods). Each group must have the smallest possible cardinality in term of replenishment cycle number. Obviously, to identify this group of cycles, we have to wait that a subset of consecutive δ_t variables is assigned. Then, in order to verify if the service level constraint is satisfied for the last period in this group, we check that for each replenishment cycle in the group identified at least one decision variable $\tilde{P}_t$ is assigned. If this is the case the partial policy for the span is completely defined and, by recalling that $R_t = \tilde{P}_t + \tilde{d}_t$, its feasibility can be checked by using the condition in Eq. 53. If the condition is not satisfied we backtrack. Notice that such a condition involves only the periods we identified in the group defined, this means that our constraint is able to detect infeasibility of partial assignments. A high level pseudo-code for the propagation logic of the global chance-constraint described is presented in Algorithm 1. Note that to keep the description of the algorithm simple we assume here a stochastic lead time l with probability mass function $f(l)$ in every period. The maximum lead time length is L. It should be also emphasized that, during the search, any CP solver will be able to exploit constraint propagation and detect infeasible or suboptimal assignments with respect to other constraints in the model. Furthermore many infeasible or suboptimal solutions may be pruned by using respectively dedicated *forward checking* techniques like the one described in [29] or *cost-based filtering* methods [10,24].

6.0.1 An example

We assume an initial null inventory level and a normally distributed demand with a coefficient of variation $\sigma_t/\tilde{d}_t = 0.3$ for each period $t \in \{1, \ldots, 5\}$. The expected values for the demand in each period are: {36, 28, 42, 33, 30}. The other parameters are $a = 1$, $h = 1$, $v = 0$, $\alpha = 0.95 (z_{\alpha=0.95} = 1.645)$. We consider for every period i in the planning horizon the following lead time probability mass function $f_i(t) = \{0.3, 0.2, 0.5\}$, which means that we receive an order placed in period i after $t \in \{0, \ldots, 2\}$ periods with the given probability (0 periods: 30%; 1 period: 20%; 2 periods: 50%). It is obvious that in this case we will always receive the order at most after 2 periods. In Table 1 (Fig. 1) we show the optimal solution found when our chance constraint is used to dynamically generate buffer stock levels. We now want to show that order-up-to-positions computed in this example by using condition 53 satisfy every service level constraint in the model. We assume that for the first 2 periods no service level constraint is enforced, since it is not possible to fully control the inventory in the first 2 periods. Therefore we enforce the required service level on period 3, 4 and 5, that is constraint 53 for $t = 3, \ldots, N$. Let

Algorithm 1: `propagate`

input : $\delta_1, \ldots, \delta_N$, $\tilde{P}_1, \ldots, \tilde{P}_N$, α, $d_1, \ldots, d_N$, l, L, N

begin
- $cycles \leftarrow \{\}$;
- $pointer \leftarrow 1$;
- $periods \leftarrow 0$;
- **for each period** i **in** $2, \ldots, N$ **do**
  - **if** δ_i is not assigned **then**
    - $cycles \leftarrow \{\}$;
    - $periods \leftarrow 0$;
    - $pointer = -1$;
  - **else if** δ_i **is assigned to** 1 **then**
    - **if** $pointer \neq -1$ **then**
      - $cycle \leftarrow$ a replenishment cycle over $\{pointer, ..., i-1\}$;
      - add $cycle$ to $cycles$;
    - **if** $periods \geq L$ **then**
      - checkBuffers();
    - $pointer \leftarrow i$;
    - $periods \leftarrow periods + 1$;
  - **else**
    - $periods \leftarrow periods + 1$;
- **if** $pointer \neq -1$ **then**
  - $cycle \leftarrow$ a replenishment cycle over $\{pointer, ..., N\}$;
  - add $cycle$ to $cycles$;
- **if** $periods \geq L$ **then**
  - checkBuffers();

end

Policy cost: 356					
Period (t)	1	2	3	4	5
$\tilde{d}_t$	36	28	42	33	30
R_t	125	124	129	87	55
δ_t	1	1	1	1	1
Shortage probability	–	–	5%	5%	5%

Table 1
Optimal solution.

us verify that the given order-up-to levels satisfy this condition for each of these three periods. Since we know the probability mass function $f(\cdot)$ for each period in the planning horizon we can easily compute the probability $Pr(\omega_t)$ for each scenario $\omega_t \in \Omega_t$. We have four of these scenarios for each period $t \in \{3, \ldots, N\}$, since we are placing an order in every period:

Procedure `checkBuffers`

begin
- *cycle* $\leftarrow$ the last element in *cycles*, a replenishment cycle over $\{i, \ldots, j\}$;
- **if no decision variable** $\tilde{P}_i, \ldots, \tilde{P}_j$ **is assigned then**
  - return;
- *counter* $\leftarrow 1$;
- **for each period** t **covered by** *cycle* **do**
  - $formerCycles \leftarrow cycles$;
  - remove *cycle* from $formerCycles$;
  - *coveredPeriods* $\leftarrow$ the number of periods covered by cycles in $formerCycles$;
  - *head* $\leftarrow$ first element in $formerCycles$;
  - *headLength* $\leftarrow$ periods covered by *head*;
  - **if** $counter < L$ **then**
    - **while** $coveredPeriods - headLength + counter \geq L$ **do**
      - remove *head* from $formerCycles$;
      - *head* $\leftarrow$ first element in $formerCycles$;
      - *headLength* $\leftarrow$ periods covered by *head*;
  - **else**
    - $formerCycles \leftarrow \{\}$;
  - *condition* $\leftarrow$ *true*;
  - **for each cycle** c **in** $formerCycles$ **do**
    - let $\{m, \ldots, n\}$ be the periods covered by c;
    - **if no decision variable** $\tilde{P}_m, \ldots, \tilde{P}_n$ **is assigned then**
      - *condition* $\leftarrow$ $false$;
  - **if** *condition* **then**
    - **if Eq. 53 for period** t ***in*** *cycle* **and former replenishment cycles in** $formerCycles$ **is not satisfied then**
      - backtrack();
  - *counter* $\leftarrow$ *counter* $+ 1$;

end

- S_1, $\Pr\{S_1\} = 0.15 = (0.3+0.2)0.3$; in this scenario at period t all the orders placed are received. That is the order placed in period $t-1$ is received immediately (probability 0.3), or after one period (probability 0.2), while the order placed in period t is received immediately (probability 0.3)
- S_2, $\Pr\{S_2\} = 0.35 = (0.3+0.2)(0.2+0.5)$; in this scenario at period t we don't receive the last order placed in period t. That is the order placed in period $t-1$ is received immediately (probability 0.3), or after one period (probability 0.2), while the order placed in period t is not received immediately, therefore it is received after one period (probability 0.2), or after two

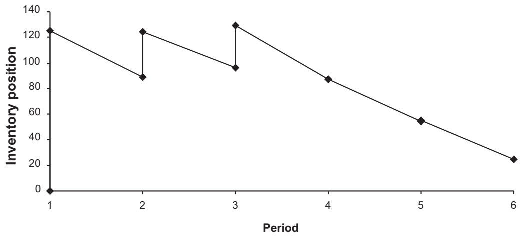

Fig. 1. Optimal policy under stochastic lead time, $f_i(t) = \{0.3, 0.2, 0.5\}$.

periods (probability 0.5)

- S_3, $\Pr\{S_3\} = 0.35 = 0.5(0.2 + 0.5)$; in this scenario at period t we don't receive the last two orders placed in periods t and $t-1$. That is the order placed in period $t-1$ is received after two periods (probability 0.5), and the order placed in period t is not received immediately, therefore it is received after one period (probability 0.2), or after two periods (probability 0.5)
- S_4, $\Pr\{S_4\} = 0.15 = 0.5 \cdot 0.3$; in this scenario at period t we don't receive the order placed in period $t-1$ and we observe order-crossover. That is the order placed in period $t-1$ is received after two periods (probability 0.5), and the order placed in period t is received immediately (probability 0.3)

In the described scenarios every possible configuration is considered. We do this without any loss in generality. In fact if some of the configurations are unrealistic (for instance if we assume that order-crossover may not take place) we just need to set the probability of the respective scenario to zero. Now it is possible to write condition 53 for each period $t \in \{3, \ldots, N\}$. Let us consider period 3:

$$\begin{aligned} &\Pr\{S_1\} \cdot G\left(\frac{129-42}{0.3\sqrt{42^2}}\right) + \Pr\{S_2\} \cdot G\left(\frac{124-(28+42)}{0.3\sqrt{28^2+42^2}}\right) + \\ &\qquad \Pr\{S_3\} \cdot G\left(\frac{125-(36+28+42)}{0.3\sqrt{36^2+28^2+42^2}}\right) + \\ &\Pr\{S_4\} \cdot G\left(\frac{125+(129-124)-(36+42)}{0.3\sqrt{36^2+42^2}}\right) = 94.60\% \cong 95\% \end{aligned} \tag{65}$$

where $G(\cdot)$ is the standard normal distribution function. This means that the combined effect of order delivery delays in our policy, all possible scenarios taken into account, gives a no stock-out probability of about 95% for period 3. Let us consider period 4:

$$\begin{aligned} &\Pr\{S_1\} \cdot G\left(\frac{87-33}{0.3\sqrt{33^2}}\right) + \Pr\{S_2\} \cdot G\left(\frac{129-(42+33)}{0.3\sqrt{42^2+33^2}}\right) + \\ &\qquad \Pr\{S_3\} \cdot G\left(\frac{124-(28+42+33)}{0.3\sqrt{28^2+42^2+33^2}}\right) + \\ &\Pr\{S_4\} \cdot G\left(\frac{124+(87-129)-(28+33)}{0.3\sqrt{28^2+33^2}}\right) = 94.89\% \cong 95\%. \end{aligned} \tag{66}$$

Period (t)	1	2	3	4	5	6	7	8
$\tilde{d}_t$	15	18	13	33	30	18	23	15

Table 2
Forecasts of period demands.

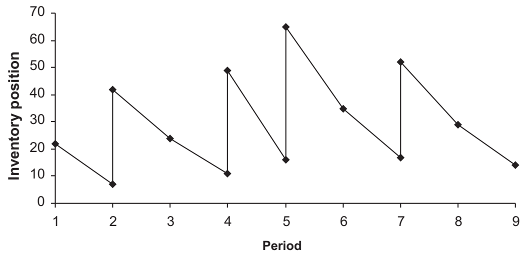

Fig. 2. Optimal policy under no lead time.

Let us consider period 5:

$$\Pr\{S_1\} \cdot G\left(\frac{55-30}{0.3\sqrt{30^2}}\right) + \Pr\{S_2\} \cdot G\left(\frac{87-(33+30)}{0.3\sqrt{33^2+30^2}}\right) + \\ \Pr\{S_3\} \cdot G\left(\frac{129-(42+33+30)}{0.3\sqrt{42^2+33^2+30^2}}\right) + \\ \Pr\{S_4\} \cdot G\left(\frac{129+(55-87)-(42+30)}{0.3\sqrt{42^2+30^2}}\right) = 94.53\% \cong 95\%. \tag{67}$$

We showed that the given solution satisfies the required service level for every period $t \in \{3, \ldots, N\}$.

7 Experiments

In this section we will solve to optimality an 8-period inventory problem under stochastic demand and lead time. Different lead time configurations are considered. The stochastic, deterministic and zero lead time cases are compared. As in the previous example we assume an initial null inventory level and a normally distributed demand with a coefficient of variation $\sigma_t/\tilde{d}_t = 0.3$ for each period $t \in \{1, \ldots, 8\}$. The expected values $\{\tilde{d}_t\}$ for the demand in each period are listed in Table 2. The other parameters are $a = 30$, $h = 1$, $v = 0$, $\alpha = 0.95(z_{\alpha=0.95} = 1.645)$. Initially we consider the problem under stochastic demand and no lead time, an efficient CP approach to find policy parameters in this case was presented in [27,24]. Obviously our approach is general and can provide solutions for this case as well, although less efficiently. The optimal solution for the instance considered is presented in Fig. 2, details about the optimal policy are reported in Table 3. We observe 5 replenishment cycles, policy parameters are: cycle lengths= $[1, 2, 1, 2, 2]$ and order-up-to-positions= $[72, 42, 49, 65, 52]$. The shortage probability is at most

$E\{TC\}$: 303	Average Inventory Level: 18.5							
Period (t)	1	2	3	4	5	6	7	8
R_t	22	42	24	49	65	35	52	29
δ_t	1	1	0	1	1	0	1	0
Shortage probability	5%	0%	5%	5%	0%	5%	0%	5%

Table 3
Optimal policy under no lead time.

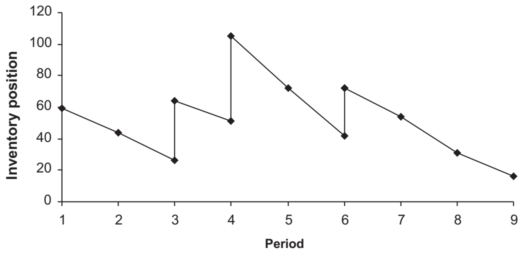

Fig. 3. Optimal policy under deterministic one period lead time.

$E\{TC\}$: 456	Average Inventory Level: 25.7							
Period (t)	1	2	3	4	5	6	7	8
R_t	59	44	64	105	72	72	54	31
δ_t	1	0	1	1	0	1	0	0
Shortage probability	–	0%	5%	5%	0%	5%	0%	5%

Table 4
Optimal policy under deterministic one period lead time, notice that the service level in the first period can obviously not be controlled.

5%, therefore the service level is met in every period. The $E\{TC\}$ is 303 and the average inventory level for the policy, computed by simulating demands and lead times according to the given probability distribution function and probability mass function respectively, is 18.5 units. Since we will consider a lead time of at most 2 periods in our examples, in order to make comparisons meaningful between different instances, for the deterministic lead time cases we computed the average inventory level over 6 periods starting from period $L + 1$, where L is the lead time length, for the stochastic lead time cases we computed again the average inventory level over 6 periods, but starting from period $\tilde{L} + 1$, where $\tilde{L}$ is the average lead time length.

We now consider the same instance, but with a deterministic lead time of one period. The optimal solution is presented in Fig. 3, details about the optimal policy are reported in Table 4. We observe now only 4 replenishment cycles, policy parameters are: cycle lengths= $[2, 1, 2, 3]$ and order-up-to-positions= $[59, 64, 105, 72]$. Again the shortage probability is at most 5% in every period, which means that the service level constraint is met. The $E\{TC\}$ is 456 and the average inventory level for the policy is 25.7 units. Therefore we observe now an expected total cost that is 50.5% higher than the zero lead time case. The replenishment plan is significantly affected by the lead time both in term of replenishment cycle lengths and order-up-to-positions. The average inventory

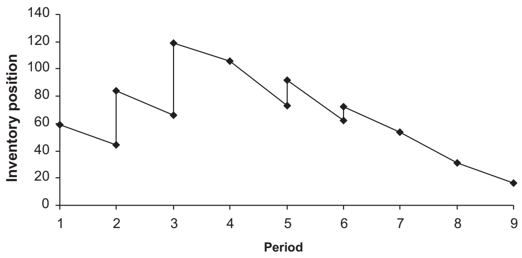

Fig. 4. Optimal policy under deterministic two periods lead time.

$E\{TC\}$: 602	Average Inventory Level: 23.2							
Period (t)	1	2	3	4	5	6	7	8
R_t	59	84	119	106	92	72	54	31
δ_t	1	1	1	0	1	1	0	0
Shortage probability	–	–	5%	5%	0%	5%	5%	5%

Table 5

Optimal policy under deterministic two periods lead time.

Lead Time	$\tilde{I}$	$E_{\tilde{I}}\{TC\}$
0	18.5	261.0
1	25.7	274.2
2	23.2	289.2

Table 6

Deterministic lead time. Average inventory levels and respective expected total cost.

level observed is higher than the one in the zero lead time case.

When a deterministic lead time of two periods is considered, as the reader may expect, we observe again higher costs and a different replenishment policy. The optimal solution is presented in Fig. 4, details about the optimal policy are reported in Table 5. The number of replenishment cycles is now again 5, policy parameters are: cycle lengths= $[1, 1, 2, 1, 3]$ and order-up-to-positions= $[59, 84, 119, 92, 72]$. The service level constraint is met in every period. The $E\{TC\}$ is 602 and the average inventory level for the policy is 23.2 units. This means that we observe a cost 98.6% and 32.0% higher than respectively the zero lead time case and the one period lead time case. The replenishment plan is again completely modified as a consequence of the lead time length. The average inventory level observed is slightly lower than in the former cases. This is due to the fact that in this replenishment plan we schedule 5 orders, while in the optimal replenishment plan under a deterministic lead time of one period only 4 orders are planned.

In Table 6 we report the expected total cost $E_{\tilde{I}}\{TC\}$ computed with respect to the average inventory level $\tilde{I}$ for the three cases presented so far.

We now concentrate on two instances where a stochastic lead time is considered and we compare results with the former cases. Firstly we analyze a stochastic

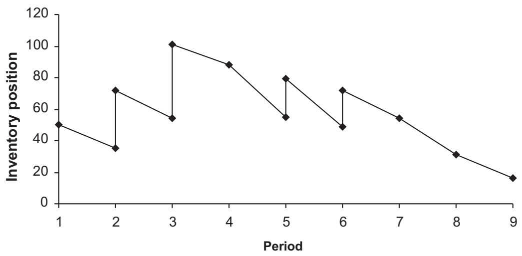

Fig. 5. Optimal policy under stochastic lead time, $f_i(t) = \{0.2(0), 0.6(1), 0.2(2)\}$.

$E\{TC\}$: 532	Average Inventory Level: 32.8							
Period (t)	1	2	3	4	5	6	7	8
R_t	50	72	101	88	79	72	54	31
δ_t	1	1	1	0	1	1	0	0
Shortage probability	–	–	5%	5%	3%	5%	5%	5%

Table 7
Optimal policy under stochastic lead time, $f_i(t) = \{0.2(0), 0.6(1), 0.2(2)\}$, in periods $\{1, 2\}$ the inventory cannot be controlled.

lead time with probability mass function $f_i(t) = \{0.2(0), 0.6(1), 0.2(2)\}$. That is an order is received immediately with probability 0.2, after one period with probability 0.6, and after two periods with probability 0.2. The optimal solution is presented in Fig. 5, details about the optimal policy are reported in Table 7. The number of replenishment cycles is again 5 as in the two period lead time case, policy parameters are: cycle lengths= $[1, 1, 2, 1, 3]$ and order-up-to-positions= $[50, 72, 101, 79, 72]$. Therefore we see that the number and the length of replenishment cycles does not change from the deterministic two period lead time case, although we observe lower order-up-to-positions as we may expect since the lead time is in average one period therefore lower than in the former case. Also the cost reflects this, in fact it is 11.6% lower than in the two period deterministic lead time case. On the other hand we observed an average inventory level of 32.8, obviously affected by the uncertainty now associated with the lead time. It should be noted that the uncertainty of the lead time plays a significant role, in fact although the average lead time is one period, the structure of the policy resembles much more the one under a two period deterministic lead time than the one under a deterministic one period lead time. Moreover the expected total cost is 16.6% higher than in this latter case.

We finally consider a different probability mass function for the lead time: $f_i(t) = \{0.5(0), 0.0(1), 0.5(2)\}$, which means that we maintain the same average lead time of one period, but we increase its variance. The optimal solution is presented in Fig. 6, details about the optimal policy are reported in Table 8. The number of replenishment cycles is still 5, policy parameters are: cycle lengths= $[1, 1, 2, 1, 3]$ and order-up-to-positions= $[50, 72, 101, 79, 72]$. Although the average lead time is still one period, order-up-to-positions are slightly higher than in the former case where the variance of the lead time was

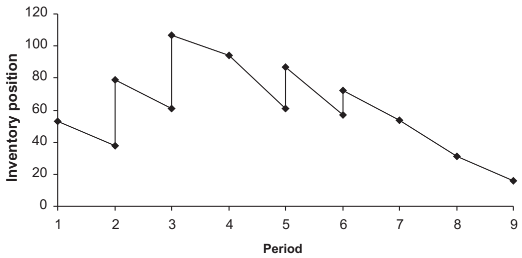

Fig. 6. Optimal policy under stochastic lead time, $f_i(t) = \{0.5(0), 0.0(1), 0.5(2)\}$.

$E\{TC\}$: 562	Average Inventory Level: 35.5							
Period (t)	1	2	3	4	5	6	7	8
R_t	53	79	107	94	87	72	54	31
δ_t	1	1	1	0	1	1	0	0
Shortage probability	–	–	5%	5%	0%	5%	5%	5%

Table 8
Optimal policy under stochastic lead time, $f_i(t) = \{0.5(0), 0.0(1), 0.5(2)\}$.

Lead Time	$\tilde{I}$	$E_{\tilde{I}}\{TC\}$
$f_i(t) = \{0.2(0), 0.6(1), 0.2(2)\}$	32.8	346.8
$f_i(t) = \{0.5(0), 0.0(1), 0.5(2)\}$	35.5	363.0

Table 9
Stochastic lead time. Average inventory levels and respective expected total cost.

lower. Also the cost reflects this, in fact it is 5.6% higher than in the former case, but still lower than the expected total cost of the two period deterministic lead time case. Moreover we observed an average inventory level of 35.5, again affected by the uncertainty associated with the lead time.

In Table 9 we report the expected total cost $E_{\tilde{I}}\{TC\}$ computed with respect to the average inventory level $\tilde{I}$ for the two cases where the lead time is stochastic.

To summarize, in our experiments we saw that supplier lead time uncertainty may significantly affect the structure of the optimal (R^n,S^n) policy. Computing optimal policy parameters constitutes a hard computational and theoretical challenge. Under different degrees of lead time uncertainty, when other input parameters for the problem remain fixed, order-up-to-positions and reorder points in the optimal policy change significantly. Realizing what the optimal decisions are for certain input parameters is a counterintuitive task. Our approach provides a systematic way to compute these optimal policy parameters.

7.1 Analyzing the cost associated with a set of optimal policy parameters

From the experiments presented interesting insights can be obtained by observing the behavior of the expected total cost and of the average inventory level for different lead time configurations. Let us firstly observe how the expected total cost changes when the lead time changes. For a deterministic lead time, as we increase its value, the cost increases significantly when the objective function considers the expected inventory position. Intuitively this is due to the fact that every replenishment cycle covering periods $i, \ldots, j$ has to cope not only with the uncertainty associated with periods $i, \ldots, j$, but also with the variability of the demand over $j+1, \ldots, j+L-1$, where L is the lead time length. In fact the order placed in period $j+1$ will be received only after L periods. When the expected inventory level is considered, the increase ratio is lower, since we only pay the cost of the uncertainty associated with the increased buffers and we do not charge holding cost on the outstanding orders. When the lead time is stochastic and the expected inventory position is considered, the optimal policy cost is affected by the expected value of the lead time and by its variability. In fact in the last two examples presented the stochastic lead time has the same expected value of one period, but in the second example the variability is obviously higher. This directly translates into a cost difference where the lead time with probability mass function $\{0.5(0), 0(1), 0.5(2)\}$ results 5.6% more costly than the one with probability mass function $\{0.2(0), 0.6(1), 0.2(2)\}$. Nevertheless in both the cases the cost observed is lower than the one observed when the lead time is deterministic and its value is two. This can be explained by the fact that the buffers required to guarantee a given service level under a deterministic two period lead time represent a worst case scenario for every instance where the lead time is stochastic and its length can be at most two periods. More formally this directly follows from Eq. 24, which determines the minimum expected inventory position required at the end of each replenishment cycle to guarantee the given service level. Although, when holding cost is charged on the expected inventory position, the behavior of the expected total cost is quite intuitive and it easily follows from the formulas presented, a dedicated reasoning must be given to explain the behavior of the average inventory level and of the expected total cost when holding cost is charged on the expected inventory level.

In the examples presented the reader may observe that a stochastic lead time distributed as follows, $\{0.2(0), 0.6(1), 0.2(2)\}$, produces an expected total cost $E\{TC\}$ lower than the one produced by a deterministic lead time of two periods. In contrast, the average inventory level $\tilde{I}$ — as well as the respective expected total cost $E_{\tilde{I}}\{TC\}$ — associated with the optimal policy computed for such a stochastic lead time is higher than the one obtained for a deterministic lead time of two periods. The reason for this is that, when we consider the expected inventory level, under a deterministic lead time we keep high buffer stocks, but we do not charge holding cost on outstanding orders, therefore the impact on the holding cost will be limited to the increase in the required buffer stocks. Under a stochastic lead time, the expected inventory level is affected by the increased buffer stocks in a similar manner, but it is also directly affected by the lead time expected value and by its variability. In fact, whenever an order has associated a short lead time, this will produce a high inventory level carried over to next periods. These scenarios may obviously affect the average inventory level of the optimal policy, while their effect on the expected inventory position is limited to the increased buffer stock levels, since the holding cost in this case is always charged also on outstanding orders. For instance a stochastic lead time distributed as follows, $\{0.5(0), 0(1), 0.5(2)\}$, produces the highest average inventory level — and expected total cost $E_{\tilde{I}}\{TC\}$ — among all the instance we considered in our set of examples. This can be explained by noticing that under a more variable lead time we will keep higher buffer stocks, and often, when the realized lead time is low, a high inventory level is accumulated and carried over to next periods before being consumed by the demand.

In conclusion we emphasize that, given a certain lead time (deterministic or stochastic), it may be relevant for certain firms to optimize the holding cost on the expected inventory position rather than on the expected inventory level. Nevertheless if we are interested in comparing the optimal policy cost for different lead time lengths and lead time probability mass functions, then we should note that the costs obtained with these two formulations do not follow the same trend, and it is necessary to compare optimal costs obtained with the specific formulation we wish to analyze. For instance if we optimize in terms of the expected inventory position ($E\{TC\}$) the instance with a deterministic lead time of two periods and the one with a stochastic lead time distributed as follows, $\{0.5(0), 0(1), 0.5(2)\}$, our model suggests that a deterministic lead time of two periods is more costly. In contrast, since both the optimal policies place the same number of orders, by analyzing the average inventory level computed for the two instances, it is easy to notice that, when the cost is computed with respect to the expected inventory level ($E_{\tilde{I}}\{TC\}$), then the stochastic lead time results more costly.

8 Conclusions

A novel approach to compute (R^n,S^n) policy parameters under stochastic lead time have been presented. We also showed how to model such a problem when a dynamic deterministic lead time is considered. The assumptions under which we developed our approach for the stochastic lead time case proved to be less restrictive than those commonly adopted in the literature for complete methods. In particular we faced the problem of order-crossover, which is a very active research topic as Riezebos show in [18] and [19]. Our approach merged well known concepts such as deterministic equivalent modeling of chance-constraints [8] and scenario based approach [26] in order to produce an effective way of solving (R^n,S^n) policy under stochastic lead time. Since we are employing CP to implement our approach we may benefit from special purpose constraint propagation techniques and cost based filtering methods that can certainly speed up the search process. Therefore in our future research we aim to develop specific filtering algorithms able to significantly speed up the search for the optimal (R^n,S^n) policy parameters under stochastic lead time.

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