Non-differentiability of Payoff Functions and Non-uniqueness of Nash Equilibria

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Abstract—Given non-degenerate intervals X^i of $\mathbb{R}$ and an increasing ordered mapping $\Phi : X^1 \times \cdots \times X^N \to \mathbb{R}^N$, games in strategic form between N players with the X^i as action sets with the following three properties are studied: the set of Nash equilibria E is convex, Φ is constant on E and in each Nash equilibrium at least one payoff function is not partially differentiable w.r.t. its own action. The results are illustrated for a special class of aggregative games that include the formal transboundary pollution games with global transboundary pollution.

Keywords—Aggregative game, convex analysis, formal transboundary pollution game, non-differentiable payoff functions, uniqueness of Nash equilibria.

I. Introduction

Consider the following game in strategic form Γ_0 between two players taken from Folmer and von Mouche (2004). Each player i has action set $X^i = [0, 2]$ and payoff function

$$f^i(x_1, x_2) = \ln(x^i + 1) - \mathcal{D}^i(x^1 + x^2),$$

where

$$\mathcal{D}^1(Q) = \begin{cases} \frac{1}{3}Q \; (Q \in [0, 1]) \\ \frac{2}{3}Q^1 - \frac{1}{3} \; (Q \in [1, 2]) \end{cases},$$

$$\mathcal{D}^2(Q) = \begin{cases} \frac{1}{4}Q \; (Q \in [0, 1]) \\ 4Q - \frac{15}{4} \; (Q \in [1, 2]) \end{cases}.$$

A straightforward calculation shows that there are well-defined reaction functions given by

$$R^1(x^2) = \begin{cases} 1 - x^2 \; (0 \le x^2 \le 1/2) \\ 1/2 \; (1/2 \le x^2 \le 1) \end{cases},$$

$$R^2(x^1) = 1 - x^1.$$

This implies $\{(x, 1-x) \mid 1/2 \le x \le 1\}$ for the set of Nash equilibria. Observe the following three properties of Γ_0: (1) the sum of actions is constant (i.e. 1) in each Nash equilibrium, (2) in each Nash equilibrium no payoff function is partially differentiable w.r.t. its own action, and (3) the set of Nash equilibria is convex.

The aim of this article is to give sufficient conditions for games in strategic form that imply these properties directly. To this end a theory will be presented for a class of games $\mathcal{G}_1$ in strategic form with special attention to the subclass of so-called formal transboundary pollution games with global transboundary pollution, and in particular Γ_0. An action of a player in such a game has the real-world interpretation of the emission level of a country and the sum of emission levels across the countries is interpreted as a deposition level (see, for instance, Folmer and von Mouche, 2002). A direct result of this theory is that in a formal transboundary pollution game with global transboundary pollution each Nash equilibrium has the same deposition level.

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II. Setting and Notations

Let

$$N$$

be a positive integer, and write

$$\mathcal{N} := \{1, \ldots, N\}.$$

Fix

$$\text{non-degenerate intervals } X^i \; (i \in \mathcal{N}) \text{ of } \mathbb{R},$$

and with

$$\mathbf{X} := X^1 \times \cdots \times X^N,$$

$$\text{a mapping } \Phi = (\varphi^1, \ldots, \varphi^N) : \mathbf{X} \to \mathbb{R}^N,$$

which will be called *co-strategy mapping*.[1] It is supposed that

$$\Phi \text{ is increasing and ordered.}^2$$

Write

$$Y^i := \varphi^i(\mathbf{X}) \; (i \in \mathcal{N}), \quad \mathbf{Y} := Y^1 \times \cdots \times Y^N.$$

Sufficient for Y^i to be an interval is that φ^i is continuous.

Given $X^1, \ldots, X^N$ and Φ, let $\mathcal{G}_0$ be the class of games in strategic form with $\mathcal{N}$ as set of players, and for each player i action set X^i. The payoff function of player i will be denoted by

$$f^i : \mathbf{X} \to \mathbb{R}.$$

For the moment there are no further restrictions for the payoff functions. In the next definition a subclass $\mathcal{G}_1$ of $\mathcal{G}_0$ will be defined by assuming some specific properties for the payoff functions. For $\Gamma \in \mathcal{G}_0$ denote the set of Nash equilibria by

$$E(\Gamma)$$

[1] I thank D. Furth for suggesting this terminology.

[2] Given a positive integer n, the relations $\ge, >, \gg$ on $\mathbb{R}^n$ are defined by: $\mathbf{x} \ge \mathbf{y} : \; x_k \ge y_k \; (1 \le k \le n)$; $\mathbf{x} > \mathbf{y} : \; \mathbf{x} \ge \mathbf{y}$ and $\mathbf{x} \ne \mathbf{y}$; $\mathbf{x} \gg \mathbf{y} : x_k > y_k \; (1 \le k \le n)$. And $\le, <, \ll$ denote the dual relations of respectively $\ge, >, \gg$.

Consider a mapping $F : Z \to \mathbb{R}^n$, where $Z \subseteq \mathbb{R}^m$. In this article, F is called – *ordered* if for all $\mathbf{a}, \mathbf{b} \in Z$ it holds that $F(\mathbf{a}) \ge F(\mathbf{b})$ or that $F(\mathbf{a}) \le F(\mathbf{b})$; – *strictly ordered* if for all $\mathbf{a}, \mathbf{b} \in Z$ it holds that $F(\mathbf{a}) \gg F(\mathbf{b})$ or that $F(\mathbf{a}) = F(\mathbf{b})$ or that $F(\mathbf{a}) \ll F(\mathbf{b})$; – *increasing* if for all $\mathbf{a}, \mathbf{b} \in Z$ one has $\mathbf{a} \le \mathbf{b} \Rightarrow F(\mathbf{a}) \le F(\mathbf{b})$; – *strongly increasing* if for all $\mathbf{a}, \mathbf{b} \in Z$ one has $\mathbf{a} < \mathbf{b} \Rightarrow F(\mathbf{a}) < F(\mathbf{b})$.

and the set of interior Nash equilibria, i.e. the Nash equilibria that belong to the topological interior $\mathrm{Int}(\mathbf{X})$ of $\mathbf{X}$, by

$$E_{\mathrm{int}}(\Gamma).$$

Further notations:[3] denote $\mathbf{X}^{\hat{\imath}} := \prod_{l=1,\, l\neq i}^{N} X^l$, identify $\mathbf{X}$ with $X^i \times \mathbf{X}^{\hat{\imath}}$, and accordingly write $\mathbf{x} \in \mathbf{X}$ as $\mathbf{x} = (x^i; \mathbf{x}^{\hat{\imath}})$. The following notations for a subset I of $\mathbb{R}$ are introduced: $l(I) := \{\min(I)\}$ if $\min(I)$ exists and $l(I) := \emptyset$ if $\min(I)$ does not exist; $r(I) := \{\max(I)\}$ if $\max(I)$ exists and $r(I) := \emptyset$ if $\max(I)$ does not exist. Moreover, $\mathrm{Int}(I)$ denotes the interior of I, $I_- := \mathrm{Int}(I) \cup l(I)$ and $I_+ := \mathrm{Int}(I) \cup r(I)$. Note that $\mathrm{Int}(I) \subseteq I_- \subseteq I$ and $\mathrm{Int}(I) \subseteq I_+ \subseteq I$. Also note that for an interval I, $I \subseteq I_- \cup r(I)$ and $I \subseteq I_+ \cup l(I)$ hold.[4]

Definition 1: $\mathcal{G}_1$ is the subclass of $\Gamma \in \mathcal{G}_0$ where for each player i

I. in each point of $E \cap (X^i_+ \times \mathbf{X}^{\hat{\imath}})$ the left partial derivative $D_i^- f^i$ of f^i w.r.t. x^i exists as element of $\overline{\mathbb{R}}$ and in each point of $E \cap (X^i_- \times \mathbf{X}^{\hat{\imath}})$ the right partial derivative $D_i^+ f^i$ of f^i w.r.t. x^i exists as element of $\overline{\mathbb{R}}$.[5]

II. there exist functions $\mathcal{T}^i_+ : X^i_- \times Y^i \to \overline{\mathbb{R}}$ and $\mathcal{T}^i_- : X^i_+ \times Y^i \to \overline{\mathbb{R}}$, such that for all $\mathbf{x} \in E \cap (X^i_- \times \mathbf{X}^{\hat{\imath}})$

$$(D_i^+ f^i)(\mathbf{x}) = \mathcal{T}^i_+(x^i, \varphi^i(\mathbf{x}))$$

and for all $\mathbf{x} \in E \cap (X^i_+ \times \mathbf{X}^{\hat{\imath}})$

$$(D_i^- f^i)(\mathbf{x}) = \mathcal{T}^i_-(x^i, \varphi^i(\mathbf{x}).) \quad \diamond$$

Remarks:

1) Sufficient for Property I for i to hold, is concavity of all conditional payoff functions $f^i_{\mathbf{z}} : X^i \to \mathbb{R}$ $(\mathbf{z} \in \mathbf{X}^{\hat{\imath}})$.[6] In this case even: f^i is in each interior point of X^i left and right differentiable w.r.t. x^i . Moreover, if $\max(X^i)$ exists, then in this point the left derivative exists as element of $\mathbb{R} \cup \{-\infty\}$ and if $\min(X^i)$ exists, then in this point the right derivative exists as element of $\mathbb{R} \cup \{+\infty\}$.
2) Sufficient for Properties I and II for i to hold, is that each conditional payoff function $f^i_{\mathbf{z}} : X^i \to \mathbb{R}$ is differentiable and that there exists a function $\mathcal{T}^i : X^i \times Y \to \mathbb{R}$ such that
$$(D_i f^i)(\mathbf{x}) = \mathcal{T}^i(x^i, \varphi^i(\mathbf{x}))\ (\mathbf{x} \in \mathbf{X}).$$
Indeed, then take $\mathcal{T}^i_+ := \mathcal{T}^i \upharpoonright X^i_- \times Y^i$ and $\mathcal{T}^i_- := \mathcal{T}^i \upharpoonright X^i_+ \times Y^i$. This situation is referred to as the *differentiable case* for player i.
3) In the two formulas in II only the values of $\mathcal{T}^i_-$ ($\mathcal{T}^i_+$) on a certain subset of $X^i_+ \times Y^i$ ($X^i_- \times Y^i$) matter; thus the $\mathcal{T}^i_-$ and $\mathcal{T}^i_-$ may be not unique and payoff functions are not necessarily continuous. But below, as it will turn out, it is desirable that $\mathcal{T}^i_+$ and $\mathcal{T}^i_-$ are more broadly defined.
4) If the function φ^i is strictly increasing in x^i, and Y^i is an interval of $\mathbb{R}$, then for all $\mathbf{a} \in \mathbf{X}$
$$a^i \in X^i_- \ \Rightarrow\ \varphi^i(\mathbf{a}) \in Y^i_-,$$
$$a^i \in X^i_+ \ \Rightarrow\ \varphi^i(\mathbf{a}) \in Y^i_+.$$
To see that, for example, the first implication holds, suppose $a^i \in X^i_-$. Because X^i is a non-degenerate interval, there exists $b^i \in X^i$ with $b^i > a^i$. Because φ^i is strictly increasing in x^i, it holds that $\varphi^i(a^i; \mathbf{a}^{\hat{\imath}}) < \varphi^i(b^i; \mathbf{a}^{\hat{\imath}})$ and thus Y^i even is a non-degenerate interval. It follows that $\varphi^i(a^i; \mathbf{a}^{\hat{\imath}}) \in Y^i \setminus r(Y^i)$ and $\varphi^i(\mathbf{a}) \in Y^i_-$.
5) Sufficient for Φ to be strictly ordered is that for all $i \in \mathcal{N}$, $\varphi^i = c^i \varphi^1$ where $c^i > 0$.
An important special case of a strictly ordered strongly increasing co-strategy mapping is
$$\Xi := (Q, \ldots, Q) : \mathbf{X} \to \mathbb{R}^N$$
with component functions
$$Q(\mathbf{x}) := \sum_{l=1}^{N} T_l x^l,$$
where $T_l > 0$ $(l \in \mathcal{N})$.[7]
6) In practice, given a game in strategic form Γ, it can be easily checked whether it belongs to $\mathcal{G}_1$. Quite a lot of games in strategic form used in economic theory belong to $\mathcal{G}_1$. For instance, consider an aggregative game, i.e. a game in strategic form where each strategy set is a non-degenerate interval of $\mathbb{R}$ and where each payoff function f^i is of the form
$$f^i(x^1, \ldots, x^N) = \pi^i(x^i, x^1 + \cdots + x^N),$$
where, with $Y = X^1 + \cdots + X^N$, $\pi^i : X^i \times Y \to \mathbb{R}$. Suppose each π^i is differentiable.[8] Then $\Gamma \in \mathcal{G}_1$ (and the differentiable case holds). Indeed: take $\varphi^i(\mathbf{x}) = \sum_{l=1}^{N} x^l$ and define $\mathcal{T}^i : X^i \times Y \to \mathbb{R}$ by
$$\mathcal{T}^i(x^i, y) = D_1\pi^i(x^i, y) + D_2\pi^i(x^i, y).$$

Lemma 1: Let $\Gamma \in \mathcal{G}_1$ and $\mathbf{n} \in \mathbf{X}$.

1) If $\mathbf{n} \in E$, then for all i
$$n^i \in X^i_+ \ \Rightarrow \mathcal{T}^i_-(n^i, \varphi^i(\mathbf{n})) \geq 0,$$
$$n^i \in X^i_- \ \Rightarrow \mathcal{T}^i_+(n^i, \varphi^i(\mathbf{n})) \leq 0.$$
2) If each conditional payoff function is concave, then sufficient for $\mathbf{n}$ to be a Nash equilibrium is that for each i with $n^i \in \mathrm{Int}(X^i)$
$$\mathcal{T}^i_+(n^i, \varphi^i(\mathbf{n})) \leq 0 \leq \mathcal{T}^i_-(n^i, \varphi^i(\mathbf{n})),$$
for i with $n^i \in l(X^i)$
$$\mathcal{T}^i_+(n^i, \varphi^i(\mathbf{n})) \leq 0,$$

[3] Also often in notations like $E(\Gamma)$ the Γ-dependence will be omitted.

[4] For example, if $I =]-3, 1]$, then $\mathrm{Int}(I) =]-3, 1[$, $I_- = [-3, 1)$ and $I_+ =]-3, 1]$.

[5] $\overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty, +\infty\}$ denotes the set of extended real numbers and equip $\overline{\mathbb{R}}$ with the usual arithmetical operations.

[6] Here $f^i_{\mathbf{z}}(x^i) = f^i(x^i; \mathbf{z})$.

[7] Note that $Q(\mathbf{X}) = T_1 X^1 + \cdots + T_N X^N$ is a non-degenerate interval of $\mathbb{R}$.

[8] 'Differentiable' needs some comment, while the domain of π^i may not be open. The convention here is that a function $f : A \to \mathbb{R}$ where A is a subset of $\mathbb{R}^n$ is differentiable if it can be extended to a function on an open subset U of $\mathbb{R}^n$ containing A.

and for i with $n^i \in r(X^i)$

$$\mathcal{T}^i_-(n^i, \varphi^i(\mathbf{n})) \geq 0. \ \diamond$$

Proof.— 1. Property I implies for $\mathbf{n} \in E$ that for all i

$$n^i \in X^i_+ \Rightarrow (D^-_i f^i)(\mathbf{n}) \geq 0,$$

$$n^i \in X^i_- \Rightarrow (D^+_i f^i)(\mathbf{n}) \leq 0.$$

Next Property II implies the desired result.

2. From convex analysis one knows the following for a concave real-valued function g on a non-degenerate interval I of $\mathbb{R}$. Sufficient for $a \in I$ to be a maximiser of g is that in case $a \in \mathrm{Int}(I)$ one has $g'_+(a) \leq 0 \leq g'_-(a)$, in case $a \in l(I)$ one has $g'_+(a) \leq 0$ and in case $a \in r(I)$ one has $g'_-(a) \geq 0$. Now apply this fact together with Property II to $g = f^i_{\mathbf{n}^i}$. Q.E.D.

For $\Gamma \in \mathcal{G}_1$ and $i \in \mathcal{N}$, the following eight properties will be dealt with:

$A^i_>$. For all $\mathbf{a}, \mathbf{b} \in E$ with $\varphi^i(\mathbf{b}) > \varphi^i(\mathbf{a})$ and $b^i \in X^i_+,\ a^i \in X^i_-$,

$$\mathcal{T}^i_-(b^i, \varphi^i(\mathbf{b})) \geq \mathcal{T}^i_+(a^i, \varphi^i(\mathbf{a})) \ \Rightarrow\ b^i \leq a^i.$$

$A^i_\geq$. For all $y^i \in \varphi^i(E)$ and $b^i \in X^i_+,\ a^i \in X^i_-$,

$$\mathcal{T}^i_-(b^i, y^i) \geq \mathcal{T}^i_+(a^i, y^i) \ \Rightarrow\ b^i \leq a^i.$$

B^i_{1+}. $\mathcal{T}^i_+$ is strictly decreasing in its first variable.
B^i_{1-}. $\mathcal{T}^i_-$ is strictly decreasing in its first variable.
$B^i_1(d)$. The differentiable case for player i holds and $\mathcal{T}^i$ is strictly decreasing in its first variable.
B^i_{2+}. $\mathcal{T}^i_+$ is decreasing in its second variable.
B^i_{2-}. $\mathcal{T}^i_-$ is decreasing in its second variable.
$B^i_2(d)$. The differentiable case for player i holds and $\mathcal{T}^i$ is decreasing in its second variable.

Moreover, by omitting the superscript in a notation of the above properties, the corresponding property that holds for all i is meant. For example $A_\geq = \wedge_{i=1}^N A^i_\geq$.

Proposition 1: Let $i \in \mathcal{N}$.

1) $A^i_\geq \wedge (B^i_{2+} \vee B^i_{2-}) \ \Rightarrow\ A^i_>$.
2) $B^i_1(d) \cap B^i_2(d) \ \Rightarrow$ each $f^i_{\mathbf{z}}$ is strictly concave. $\diamond$

Proof.— 1. Suppose $\mathbf{a}, \mathbf{b} \in E$ with $\varphi^i(\mathbf{b}) > \varphi^i(\mathbf{a})$, $b^i \in X^i_+,\ a^i \in X^i_-$ and $\mathcal{T}^i_-(b^i, \varphi^i(\mathbf{b})) \geq \mathcal{T}^i_+(a^i, \varphi^i(\mathbf{a}))$.

Case $A^i_\geq \wedge B^i_{2+}$. By B^i_{2+}, $\mathcal{T}^i_+(a^i, \varphi^i(\mathbf{a})) \geq \mathcal{T}^i_+(a^i, \varphi^i(\mathbf{b}))$. Therefore $\mathcal{T}^i_-(b^i, \varphi^i(\mathbf{b})) \geq \mathcal{T}^i_+(a^i, \varphi^i(\mathbf{b}))$. Now, by $A^i_\geq$, $b^i \leq a^i$, as desired.

Case $A^i_\geq \wedge B^i_{2-}$. By B^i_{2-}, $\mathcal{T}^i_-(b^i, \varphi^i(\mathbf{b})) \leq \mathcal{T}^i_-(b^i, \varphi^i(\mathbf{a}))$. Therefore $\mathcal{T}^i_-(b^i, \varphi^i(\mathbf{a})) \geq \mathcal{T}^i_+(a^i, \varphi^i(\mathbf{a}))$. Now, by $A^i_\geq$, $b^i \leq a^i$, as desired.

2. Because $f^i_{\mathbf{z}} : X^i \to \mathbb{R}$ is differentiable, its strict concavity is equivalent with strict a strictly increasing derivative $f^{i\prime}_{\mathbf{z}}$. Let $a^i, b^i \in X^i$ with $a^i < b^i$. Because φ^i is increasing in x^i, one has $\varphi^i(a^i; \mathbf{z}) \leq \varphi^i(b^i; \mathbf{z})$. Now, because of $B^i_1(d)$ and $B^i_2(d)$

$$f^{i\prime}_{\mathbf{z}}(a^i) = (D_i f^i)(a^i; \mathbf{z}) =$$

$$\mathcal{T}^i(a^i, \varphi^i(a^i; \mathbf{z})) > \mathcal{T}^i(b^i, \varphi^i(a^i; \mathbf{z}))$$

$$\geq \mathcal{T}^i(b^i, \varphi^i(b^i; \mathbf{z}) = (D_i f^i)(b^i; \mathbf{z}) = f^{i\prime}_{\mathbf{z}}(b^i). \quad \text{Q.E.D.}$$

Note now that Property $B^i_1(d) \wedge B^i_2(d)$ implies all eight properties.

III. A UNIQUENESS RESULT

Theorem 1 below provides a uniqueness result for Nash equilibria for games in the class $\mathcal{G}_1$. A stronger version of it can be found in Folmer and von Mouche (2004).

Proposition 2: Let $\Gamma \in \mathcal{G}_1$ and $i \in \mathcal{N}$. If Γ has

1) Property $A^i_>$, then for all $\mathbf{a}, \mathbf{b} \in E$,

$$\varphi^i(\mathbf{a}) < \varphi^i(\mathbf{b}) \ \Rightarrow\ a^i \geq b^i.$$

2) Property $A^i_\geq \wedge (B^i_{2+} \vee B^i_{2-})$, then for all $\mathbf{a}, \mathbf{b} \in E$,

$$\varphi^i(\mathbf{a}) \leq \varphi^i(\mathbf{b}) \ \Rightarrow\ a^i \geq b^i. \ \diamond$$

Proof.— 1. If $a^i \in r(X^i)$ or $b^i \in l(X^i)$, then $a^i \geq b^i$ holds. Now suppose $a^i \notin r(X^i)$ and $b^i \notin l(X^i)$. Then $a^i \in X^i_-$ and $b^i \in X^i_+$. By Lemma 1(1),

$$\mathcal{T}^i_-(b^i, \varphi^i(\mathbf{b})) \geq \mathcal{T}^i_+(a^i, \varphi^i(\mathbf{a})).$$

By Property $A^i_>$, $b^i \leq a^i$.

2. If $a^i \in r(X^i)$ or $b^i \in l(X^i)$, then $a^i \geq b^i$ holds. Now suppose $a^i \notin r(X^i)$ and $b^i \notin l(X^i)$. Then $a^i \in X^i_-$ and $b^i \in X^i_+$. By Lemma 1(1),

$$\mathcal{T}^i_-(b^i, \varphi^i(\mathbf{b})) \geq \mathcal{T}^i_+(a^i, \varphi^i(\mathbf{a})).$$

- Case $A^i_\geq \wedge B^i_{2+}$. By Property B^i_{2+},

  $$\mathcal{T}^i_+(a^i, \varphi^i(\mathbf{a})) \geq \mathcal{T}^i_+(a^i, \varphi^i(\mathbf{b})).$$

  Therefore

  $$\mathcal{T}^i_-(b^i, \varphi^i(\mathbf{b})) \geq \mathcal{T}^i_+(a^i, \varphi^i(\mathbf{b})).$$

  Now, by Property $A^i_\geq$, $b^i \leq a^i$.
- Case $A^i_\geq \wedge B^i_{2-}$. By Property B^i_{2-},

  $$\mathcal{T}^i_-(b^i, \varphi^i(\mathbf{b})) \leq \mathcal{T}^i_+(b^i, \varphi^i(\mathbf{a})).$$

  Therefore

  $$\mathcal{T}^i_+(a^i, \varphi^i(\mathbf{a})) \leq \mathcal{T}^i_-(b^i, \varphi^i(\mathbf{a})).$$

  Now, by Property $A^i_\geq$, $b^i \leq a^i$.

  3. If $a^i \in r(X^i)$ or $b^i \in l(X^i)$, then $a^i \geq b^i$ holds. Now suppose $a^i \notin r(X^i)$ and $b^i \notin l(X^i)$. Then $a^i \in X^i_-$ and $b^i \in X^i_+$. By Lemma 1(1),

  $$\mathcal{T}^i_-(b^i, \varphi^i(\mathbf{b})) \geq \mathcal{T}^i_+(a^i, \varphi^i(\mathbf{a})). \ \text{Q.E.D.}$$

Theorem 1: Consider $\Gamma \in \mathcal{G}_1$ and suppose $\Phi \restriction E$ is strongly increasing. Then Property $A_\geq \wedge (B_{2+} \vee B_{2-})$ is sufficient for Γ to have at most one Nash equilibrium. In particular, it is sufficient that the differentiable case holds where each $\mathcal{T}^i$ is strictly decreasing in its first variable and decreasing in its second variable. $\diamond$

Proof.— Suppose $\mathbf{a}, \mathbf{b} \in E$. Because $\Phi \restriction E$ is ordered, one has $\Phi(\mathbf{a}) \geq \Phi(\mathbf{b})$ or $\Phi(\mathbf{a}) \leq \Phi(\mathbf{b})$. It may be assumed that $\Phi(\mathbf{a}) \leq \Phi(\mathbf{b})$, i.e. that $\varphi^i(\mathbf{a}) \leq \varphi^i(\mathbf{b})$ $(i \in \mathcal{N})$ holds. By Proposition 2(2), $a^i \geq b^i$ $(i \in \mathcal{N})$, i.e. $\mathbf{a} \geq \mathbf{b}$. Because $\Phi \restriction E$ is increasing, $\Phi(\mathbf{a}) \geq \Phi(\mathbf{a})$ holds. Thus $\Phi(\mathbf{a}) = \Phi(\mathbf{b})$ and $\mathbf{a} \geq \mathbf{b}$. Because $\Phi \restriction E$ is strongly increasing, it follows that $\mathbf{a} = \mathbf{b}$. Q.E.D.

IV. Φ IS CONSTANT ON THE SET OF NASH EQUILIBRIA

Of course under the conditions of Theorem 1, $\Phi : E \to \mathbb{R}^N$ is constant. Here is another sufficient condition:

Theorem 2: If $\Gamma \in \mathcal{G}_1$ has Property $A_>$ and $\Phi \restriction E$ is strictly ordered, then $\Phi : E \to \mathbb{R}^N$ is constant. $\diamond$

Proof.— By contradiction. So suppose $\mathbf{a}, \mathbf{b} \in E$ such that $\Phi(\mathbf{a}) \neq \Phi(\mathbf{b})$. Let $j \in \mathcal{N}$ such that $\varphi^j(\mathbf{a}) \neq \varphi^j(\mathbf{b})$. It may be supposed that the strict inequality $\varphi^j(\mathbf{a}) < \varphi^j(\mathbf{b})$ holds. Because $\Phi \restriction E$ is strictly ordered, this inequality implies

$$\Phi(\mathbf{a}) \ll \Phi(\mathbf{b}),$$

i.e. $\varphi^i(\mathbf{a}) < \varphi^i(\mathbf{b})$ $(i \in \mathcal{N})$. By Proposition 2(1), $a^i \geq b^i$ $(i \in \mathcal{N})$, i.e. $\mathbf{a} \geq \mathbf{b}$. Because $\Phi \restriction E$ is increasing, $\Phi(\mathbf{a}) \geq \Phi(\mathbf{b})$ follows, which is a contradiction. Q.E.D.

Note that by Theorem 2 and Proposition 1(1), for $\Phi : E \to \mathbb{R}^N$ to be constant it is also sufficient that Γ has Property $A_\geq \wedge (B_{2+} \vee B_{2-})$ and $\Phi \restriction E$ is strictly ordered.

Define

$$\mathcal{G}_1^\star := \{\Gamma \in \mathcal{G}_1 \mid \Phi : E(\mathbf{\Gamma}) \to \mathbb{R}^N \text{ is constant}\}.$$

For each $\Gamma \in \mathcal{G}_1^\star$ with $E(\Gamma) \neq \emptyset$, denote by

$$\mathbf{\Psi}(\Gamma) = \Psi^1(\Gamma) \times \cdots \times \Psi^N(\Gamma)$$

the constant value of $\Phi \restriction E(\mathbf{\Gamma})$. Note that $\Psi(\Gamma) \in \mathbf{Y} \subseteq \mathbb{R}^N$. In the case where $\varphi^1 = \cdots = \varphi^N$, each coefficient of $\mathbf{\Psi}(\Gamma)$ is the same and $\mathbf{\Psi}(\Gamma)$ is identified with this coefficient and denoted by $\Psi(\Gamma) \in \mathbb{R}$; then the sum of actions is constant in each Nash equilibrium. Below it will become more clear how $\mathbf{\Psi}(\Gamma)$ is related to Γ.

Lemma 2: Suppose $\Gamma \in \mathcal{G}_1^\star$, each φ^i is strictly increasing in x^i and each Y^i is an interval. Then: $\#E(\Gamma) \geq 2 \;\Rightarrow\; \mathbf{\Psi}(\Gamma) \in \mathrm{Int}(\mathbf{Y})$. $\diamond$

Proof.— Let $\mathbf{a}, \mathbf{b} \in E$ with $\mathbf{a} \neq \mathbf{b}$. By assumption, $\mathbf{\Psi} = \Phi(\mathbf{a}) = \Phi(\mathbf{b})$. Take i such that $a^i \neq b^i$. It may be supposed that $a^i < b^i$. Now $a^i \in X^i_-$ and $b^i \in X^i_+$. By Remark 5, $\Psi^i = \varphi^i(\mathbf{a}) \in Y^i_-$ and $\Psi^i = \varphi^i(\mathbf{b}) \in Y^i_+$, and therefore $\Psi^i \in \mathrm{Int}(Y^i)$. Thus $\mathbf{\Psi} \in \mathrm{Int}(\mathbf{Y})$. Q.E.D.

V. NON-DIFFERENTIABILITY

Proposition 3: Let $\Gamma \in \mathcal{G}_1^\star$, $i \in \mathcal{N}$ and suppose $\mathbf{a}, \mathbf{b} \in E(\Gamma)$ with $a^i < b^i$.

1) Suppose Property B^i_{1-} holds. Then: $a^i \in \mathrm{Int}(X^i) \;\Rightarrow\; f^i$ is not partially differentiable w.r.t. x^i in $\mathbf{a}$.
2) Suppose Property B^i_{1+} holds. Then: $b^i \in \mathrm{Int}(X^i) \;\Rightarrow\; f^i$ is not partially differentiable w.r.t. x^i in $\mathbf{b}$. $\diamond$

Proof.— 1. By contradiction. So suppose all conditions hold and f^i partially differentiable w.r.t. x^i in $\mathbf{a}$. Because $\Gamma \in \mathcal{G}_1^\star$, $\varphi^i(\mathbf{a}) = \varphi^i(\mathbf{b}) = \Psi^i$ holds. Note that $a^i \in X^i_-$ and $b^i \in X^i_+$. Because $\mathbf{a}, \mathbf{b} \in E$, it holds by Lemma 1(1) that

$$\mathcal{T}^i_+(a^i, \Psi^i) \leq 0 \leq \mathcal{T}^i_-(b^i, \Psi^i).$$

Because $\mathcal{T}^i_-$ is strictly increasing in its first variable and $a^i \in X^i_+$,

$$\mathcal{T}^i_-(b^i, \Psi^i) < \mathcal{T}^i_-(a^i, \Psi^i).$$

Therefore

$$\mathcal{T}^i_+(a^i, \Psi^i) < \mathcal{T}^i_-(a^i, \Psi^i).$$

Because f^i is partially differentiable w.r.t. x^i in $\mathbf{a}$, the equality $\mathcal{T}^i_+(a^i, \Psi^i) = \mathcal{T}^i_-(a^i, \Psi^i)$ holds, a contradiction.

2. As in 1, one obtains

$$\mathcal{T}^i_+(a^i, \Psi^i) \leq 0 \leq \mathcal{T}^i_-(b^i, \Psi^i).$$

Because $\mathcal{T}^i_+$ is strictly increasing in its first variable and $b^i \in X^i_-$,

$$\mathcal{T}^i_+(b^i, \Psi^i) < \mathcal{T}^i_+(a^i, \Psi^i).$$

Therefore

$$\mathcal{T}^i_+(b^i, \Psi^i) < \mathcal{T}^i_-(b^i, \Psi^i).$$

Because f^i is partially differentiable w.r.t. x^i in $\mathbf{b}$, the equality $\mathcal{T}^i_+(b^i, \Psi^i) = \mathcal{T}^i_-(b^i, \Psi^i)$ holds, a contradiction. Q.E.D.

Theorem 3: Consider $\Gamma \in \mathcal{G}_1^\star$ where all functions $\mathcal{T}^i_-$ and $\mathcal{T}^i_+$ are strictly decreasing in their first variable. Then $\#E_{\mathrm{int}}(\Gamma) \geq 2$ implies that for each interior Nash equilibrium $\mathbf{n}$ at least one payoff function is not partially differentiable in $\mathbf{n}$ w.r.t. its own action.[9] $\diamond$

Proof.— Fix $\mathbf{n} \in E_{\mathrm{int}}$. Let $\mathbf{a} \in E_{\mathrm{int}}$ with $\mathbf{a} \neq \mathbf{n}$. Let i be such that $a^i \neq n^i$. Now apply Proposition 3. Q.E.D.

VI. CONVEXITY OF THE SET OF NASH EQUILIBRIA

Given $\Gamma \in \mathcal{G}_1$, $i \in \mathcal{N}$ and $y^i \in Y^i$, define the subset $W^i_{y^i}$ of $\mathbb{R}$ by

$$W^i_{y^i} := \{x^i \in \mathrm{Int}(X^i) \mid \mathcal{T}^i_+(x^i, y^i) \leq 0 \leq \mathcal{T}^i_-(x^i, y^i)\} \cup$$

$$\{x^i \in r(X^i) \mid \mathcal{T}^i_-(x^i, y^i) \geq 0\} \cup \{x^i \in l(X^i) \mid \mathcal{T}^i_+(x^i, y^i) \leq 0\}.$$

Lemma 3: Let $\Gamma \in \mathcal{G}_1$, $i \in \mathcal{N}$ and $y^i \in Y^i$. Sufficient for $\mathcal{W}^i_{y^i}$ to be convex is that Property B^i_1 holds. $\diamond$

Proof.— Suppose $a^i, b^i \in W^i_{y^i}$ with $a^i < b^i$ and let $\lambda \in]0, 1[$. Then $a^i \in X^i_-$, $b^i \in X^i_+$ and for $c^i = \lambda a^i + (1 - \lambda) b^i$ one has $a^i < c^i < b^i$ and thus $c_i \in \mathrm{Int}(X^i)$. Because of $B^i_{1+} \wedge B^i_{1-}$,

$$\mathcal{T}^i_+(c^i, y^i) \leq \mathcal{T}^i_+(a^i, y^i) \leq 0 \leq \mathcal{T}^i_-(b^i, y^i) \leq \mathcal{T}^i_-(c^i, y^i),$$

and thus $c^i \in \mathcal{W}^i_{y^i}$. Q.E.D.

Theorem 4: Consider $\Gamma \in \mathcal{G}_1^\star$ where each function $\mathcal{T}^i$ is strictly decreasing in its first variable, each conditional payoff function is concave and Φ is affine.[10] Then $E(\Gamma)$ is convex. $\diamond$

Proof.— $\Phi^{<-1>}(\mathbf{\Psi}) = \{\mathbf{x} \in \mathbf{X} \mid \Phi(\mathbf{x}) = \mathbf{\Psi}\} = \{\mathbf{x} \in \mathbb{R}^N \mid A(\mathbf{x}) + \mathbf{a} = \mathbf{\Psi}\} \cap \mathbf{X}$. Because Φ is affine and $\mathbf{X}$ is convex, it follows that $\Phi^{<-1>}(\mathbf{\Psi})$ is a convex subset of $\mathbb{R}^N$. It now will be proved that

$$E = (W^1_{\Psi^1} \times \cdots \times W^N_{\Psi^N}) \cap \Phi^{<-1>}(\mathbf{\Psi}).$$

By Lemma 3 then E_{int} is convex.

[9] The own action of f^j is x^j.

[10] I.e. there exist a linear mapping $A : \mathbb{R}^N \to \mathbb{R}^N$ and $\mathbf{a} \in \mathbb{R}^N$ such that $\Phi(\mathbf{x}) = A(\mathbf{x}) + \mathbf{a}$ $(\mathbf{x} \in \mathbb{R}^N)$.

$'\supseteq'$: suppose $\mathbf{x} \in (W^1_{\Psi^1} \times \cdots \times W^N_{\Psi^N}) \cap \Phi^{<-1>}(\mathbf{\Psi})$, i.e. $\Phi(\mathbf{x}) = \mathbf{\Psi}$ and $x^i \in W^i_{\Psi^i}$ $(i \in \mathcal{N})$. Now for each i,

$$\varphi^i(\mathbf{x}) = \Psi^i,$$

$$x^i \in \mathrm{Int}(X^i) \Rightarrow \mathcal{T}^i_+(x^i, \varphi^i(\mathbf{x})) \leq 0 \leq \mathcal{T}^i_-(x^i, \varphi^i(\mathbf{x})),$$

$$x^i \in r(X^i) \Rightarrow \mathcal{T}^i_-(x^i, \varphi^i(\mathbf{x})) \geq 0,$$

$$x^i \in l(X^i) \Rightarrow \mathcal{T}^i_+(x^i, \varphi^i(\mathbf{x})) \leq 0.$$

So, by Lemma 1(2), $\mathbf{x} \in E$. Thus $\mathbf{x} \in E_{\mathrm{int}}$.

$'\subseteq'$: suppose $\mathbf{x} \in E$. Because $\Phi(\mathbf{x}) = \mathbf{\Psi}$, it holds that $\mathbf{x} \in \Phi^{<-1>}(\mathbf{\Psi})$, By Lemma 1(1),

$$x^i \in \mathrm{Int}(X^i) \Rightarrow \mathcal{T}^i_+(x^i, \Psi^i) \leq 0 \leq \mathcal{T}^i_-(x^i, \Psi^i),$$

$$x^i \in r(X^i) \Rightarrow \mathcal{T}^i_-(x^i, \varphi^i(\mathbf{x})) \geq 0,$$

$$x^i \in l(X^i) \Rightarrow \mathcal{T}^i_+(x^i, \varphi^i(\mathbf{x})) \leq 0.$$

Now $x^i \in W^i_{\Psi^i}$ $(i \in \mathcal{N})$ and thus $\mathbf{x} \in W^1_{\Psi^1} \times \cdots \times W^N_{\Psi^N}$. Q.E.D.

Theorem 4 is more or less a generalisation of a result in Szidarovszky and Yakowitz (1982) for homogeneous Cournot oligopoly games; although the result is more general, the given proof is simpler.

Theorems 2 and 4 imply:

Corollary 1: Suppose $\Gamma \in \mathcal{G}_1$, $\Phi = \Xi$, each conditional payoff function is concave, $\#E_{\mathrm{int}}(\Gamma) \geq 2$, and that Γ has Property $A_> \wedge B_{1+} \wedge B_{1-}$. Then

1) $\Phi \upharpoonright E(\Gamma)$ is constant.
2) $E(\Gamma)$ is convex.
3) For each interior Nash equilibrium $\mathbf{n}$ at least one payoff function is not partially differentiable in $\mathbf{n}$ w.r.t. its own action. ⋄

VII. Formal transboundary pollution games

Definition 2: Consider $\mathcal{G}_0$ in case $X^i = [0, m^i]$ $(i \in \mathcal{N})$ and co-strategy mapping Ξ. $\mathcal{G}_a$ denotes the subclass of $\Gamma \in \mathcal{G}_0$ where for each player i:

a. the payoff function is

$$f^i(\mathbf{x}) = \mathcal{P}^i(x^i) - \mathcal{D}^i(Q(\mathbf{x})),$$

where $\mathcal{P}^i: X^i \to \mathbb{R}$ and $\mathcal{D}^i : Q(\mathbf{X}) \to \mathbb{R}$.

b. $\mathcal{D}^i$ is continuous, convex and strictly increasing;
c. $\mathcal{P}^i$ is continuous, strictly concave and strictly increasing. ⋄

Remarks:

7. Each payoff function is continuous and each conditional payoff function is strictly concave.
8. The games in $\mathcal{G}_a$ are aggregative.
9. $\mathcal{G}_a$ is the class of formal transboundary games with global transboundary pollution (see, for instance, Folmer and von Mouche, 2002).

Proposition 4: $\mathcal{G}_a \subseteq \mathcal{G}^\star_1$. ⋄

Proof.— Let $\Gamma \in \mathcal{G}_a$. First it will be shown that $\Gamma \in \mathcal{G}_1$. Because each conditional payoff function is concave, Property I holds. And because $\mathcal{P}^i$ and $-\mathcal{D}^i$ are concave, one has (using Remark 4)

$$(D^i_- f^i)(\mathbf{x}) = (\mathcal{P}^i)'_-(x^i) - T_i(\mathcal{D}^i)'_-(Q(\mathbf{x}))\ (\mathbf{x} \in X^i_+ \times \mathbf{X}^{\hat{\imath}}),$$

$$(D^i_+ f^i)(\mathbf{x}) = (\mathcal{P}^i)'_+(x^i) - T_i(\mathcal{D}^i)'_+(Q(\mathbf{x}))\ (\mathbf{x} \in X^i_- \times \mathbf{X}^{\hat{\imath}}).$$

This implies that Property II holds if the functions $\mathcal{T}^i_+ : X^i_- \times Y^i \to \overline{\mathbb{R}}$, $\mathcal{T}^i_- : X^i_+ \times Y^i \to \overline{\mathbb{R}}$ are defined as follows:

$$\mathcal{T}^i_+(x^i, y^i) := (\mathcal{P}^i)'_+(x^i) - T_i(\mathcal{D}^i)'_+(y^i)\ (x^i \in X^i_-,\ y^i \in Y^i_-);$$

$$\mathcal{T}^i_-(x^i, y^i) := (\mathcal{P}^i)'_-(x^i) - T_i(\mathcal{D}^i)'_-(y^i)\ (x^i \in X^i_+,\ y^i \in Y^i_+);$$

$$\mathcal{T}^i_- \text{ and } \mathcal{T}^i_+ \text{ arbitrary elsewhere.}$$

Let $\Gamma \in \mathcal{G}_a$. Next it will be proved that Γ has Property $A_>$ (and then $\Gamma \in \mathcal{G}^\star_1$ by Theorem 2). Suppose $\mathbf{a}, \mathbf{b} \in E$ with $Q(\mathbf{b}) > Q(\mathbf{a})$ and $b^i \in X^i_+$ and $a^i \in X^i_-$ such that

$$\mathcal{T}^i_-(b^i, Q(\mathbf{b})) \geq \mathcal{T}^i_+(a^i, Q(\mathbf{a})).$$

Because $\mathcal{D}^i$ is convex and $Q(\mathbf{b}) > Q(\mathbf{a})$,

$$(\mathcal{D}^i)'_-(Q(\mathbf{b})) \geq (\mathcal{D}^i)'_+(Q(\mathbf{a})).$$

Now

$$(\mathcal{P}^i)'_-(b^i) = \mathcal{T}^i_-(b^i, Q(\mathbf{b})) + T_i(\mathcal{D}^i)'_-(Q(\mathbf{b}))$$

$$\geq \mathcal{T}^i_+(a^i, Q(\mathbf{a})) + T_i(\mathcal{D}^i)'_-(Q(\mathbf{b}))$$

$$= (\mathcal{P}^i)'_+(a^i) + T_i((\mathcal{D}^i)'_-(Q(\mathbf{b})) - (\mathcal{D}^i)'_+(Q(\mathbf{a}))) \geq (\mathcal{P}^i)'_+(a^i).$$

Thus $(\mathcal{P}^i)'_-(b^i) \geq (\mathcal{P}^i)'_+(a^i)$. Because $\mathcal{P}^i$ is strictly concave, $b^i \leq a^i$ follows. Q.E.D.

Theorem 5: Let $\Gamma \in \mathcal{G}_a$ and suppose $\#E(\Gamma) \geq 2$. Let

$$\mathcal{Z} := \{i \in \mathcal{N} \mid \text{ there exists } \mathbf{a}, \mathbf{b} \in E \text{ with } a^i \neq b^i\}.$$

Then

1) For $i \in \mathcal{Z}$, the function $\mathcal{D}^i$ is not differentiable in $\Psi(\Gamma)$.
2) In each Nash equilibrium each f^i $(i \in \mathcal{Z})$ is not partially differentiable w.r.t. its own action. ⋄

Proof.— 1. By contradiction. So suppose $i \in \mathcal{Z}$ and $\mathcal{D}^i$ is differentiable in Ψ. Let $\mathbf{a}, \mathbf{b} \in E$ with $a^i \neq b^i$. It may be supposed that $a^i < b^i$. One has $Q(\mathbf{a}) = Q(\mathbf{b}) = \Psi$. Because $a^i \in X^i_-$, $b^i \in X^i_+$ Lemma 1(1) gives $\mathcal{T}^i_-(b^i, \Psi) \geq 0$ and $\mathcal{T}^i(a^i, \Psi) \leq 0$. Therefore,

$$(\mathcal{P}^i)'_+(a^i) \leq T_i(\mathcal{D}^i)'_+(\Psi) \text{ and } (\mathcal{P}^i)'_-(b^i) \geq T_i(\mathcal{D}^i)'_-(\Psi).$$

Because $\mathcal{P}^i$ is strictly concave,

$$\mathcal{P}^{i\prime}_+(a^i) > \mathcal{P}^{i\prime}_-(b^i).$$

Thus $(\mathcal{D}^i)'_+(\Psi) > (\mathcal{D}^i)'_-(\Psi)$. However, this is impossible because $\mathcal{D}^i$ is differentiable in Ψ.

2. With 1. Q.E.D.

Corollary 2: Let $\Gamma \in \mathcal{G}_a$. If each $\mathcal{D}^i$ is differentiable (in $\Psi(\Gamma)$), then $\#E(\Gamma) \leq 1$. ⋄

Without the differentiability condition, Corollary 2 does not hold anymore. Indeed, the analysis of the game Γ_0 in section I shows that in that case there even may be infinitely many Nash equilibria.

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