Royal Meteorological Society

Balanced dynamics in the Tropics

W.T.M. Verkley [a]* and I.R. van der Velde[b]
[a] Royal Netherlands Meteorological Institute, The Netherlands
[b] Meteorology and Air Quality Section, Wageningen University, The Netherlands
*Correspondence to: W. T. M. Verkley, Royal Netherlands Meteorological Institute, PO Box 201, 3730 AE De Bilt, The Netherlands. E-mail: verkley@knmi.nl

For the shallow-water equations on an equatorial beta plane, the properties of low-frequency Rossby waves and (mixed) Rossby–gravity waves are investigated. It is shown that in the low-frequency limit the horizontal divergence of these solutions is zero and their geopotential satisfies $\varphi = f\psi$, where $f = \beta y$ is the Coriolis parameter and ψ is the stream function of the non-divergent velocity field. This type of balance is rather different from the geostrophic balance satisfied by Kelvin waves. It can be used to formulate a balanced potential vorticity equation in the single variable ψ that, while filtering out Kelvin waves and inertia–gravity waves, exactly reproduces Rossby waves and Rossby–gravity waves in the low-frequency limit.

Key Words: linear waves; equatorial beta plane; potential vorticity

Received 26 September 2008; Revised 11 September 2009; Accepted 14 September 2009; Published online in Wiley InterScience 15 January 2010

Citation: Verkley WTM, van der Velde IR. 2010. Balanced dynamics in the Tropics. *Q. J. R. Meteorol. Soc.* **136**: 41–49. DOI:10.1002/qj.530

1. Introduction

Like Matsuno (1966), we are interested in the question: 'Is there quasi-geostrophic motion even at the Equator?' The matter was still of relevance 28 years after Matsuno posed this question, as may be deduced from the fact that in Chapter 1 of the textbook by James (1994) we can read:

> *[The set of quasi-geostrophic equations] has now fallen out of favour as an equation set for modeling the atmospheric circulation since it is not uniformly valid as one approaches the equator* . . .

The question of whether there is quasi-geostrophic motion at the Equator remains of relevance today.

In his study of wave motion on the equatorial beta plane, Matsuno (1966) notices that Kelvin waves (characterized by a meridional velocity that is identically zero) are in geostrophic balance in the sense that the Coriolis force on the zonal velocity field is exactly compensated by the meridional pressure gradient force. For the Rossby waves, however, Matsuno (1966) notices that approximate geostrophic balance seems to hold but that in some cases the velocity field is somewhat curious around the Equator.

In this article we will study in more detail the properties of Rossby waves and Rossby–gravity waves. From a study of the low-frequency limit, it will be deduced that the balance that characterizes these waves differs from the geostrophic balance that characterizes Kelvin waves: the horizontal divergence is asymptotically zero and the geopotential φ is given by the Coriolis parameter f times the stream function ψ of the non-divergent velocity field. Near the Equator this entails a rather different relation between wind and pressure field than implied by geostrophic balance, and explains Matsuno's observation regarding the velocity field of Rossby waves around the Equator.

The type of balance that we find is identical with Daley's (1983) 'simplest form of the geostrophic relationship'. By incorporating this balance condition in the linearized potential vorticity equation, we obtain an equation that filters out Kelvin waves as well as inertia–gravity waves while exactly reproducing the low-frequency behaviour of Rossby waves and Rossby–gravity waves. The equation that we obtain is analogous to the linearized equivalent barotropic vorticity equation as proposed by Cressman (1958), the difference being that the full variation of $f^2 = \beta^2 y^2$ is kept in the free surface term. It is pointed out that this equation is the linearized version of a more general balanced potential vorticity equation that was discussed in the context of spherical geometry by Verkley (2009) and Schubert *et al.* (2009a).

The equatorial beta plane shallow-water model is introduced in section 2. In section 3 we review the properties of the full spectrum of linear wave solutions (i.e. Kelvin waves and non-Kelvin waves), followed by a discussion of the low-frequency limit in section 4. The low-frequency limit of non-Kelvin waves excludes inertia–gravity waves while reasonably approximating Rossby waves and (westward-moving) Rossby–gravity waves. Both types of waves are seen to satisfy the balance condition mentioned above and can be recovered from a balanced potential vorticity equation, as shown in section 5. In section 6 we discuss a few further implications, after which our conclusions follow in section 7.

2. The shallow-water equations

We consider a one-layer shallow-water model, describing the dynamics of a single hydrostatic layer of fluid with constant density ρ. The average height of the fluid is denoted by H_A; the deviation from this average height and the height of the orography are denoted by η and η_B, respectively. The fluid is assumed to move frictionlessly with a horizontal velocity $\mathbf{v}$ that is independent of height.

The system is described by the following set of equations (Pedlosky, 1987):

$$\frac{\mathrm{D}\mathbf{v}}{\mathrm{D}t} + f\mathbf{k} \times \mathbf{v} + \nabla\varphi = 0, \tag{1}$$

$$\frac{\mathrm{D}H}{\mathrm{D}t} + H\mathcal{D} = 0, \tag{2}$$

in which f is the Coriolis parameter, $\mathbf{k}$ is an upward-pointing unit vector, $\mathcal{D} = \nabla \cdot \mathbf{v}$ is the divergence of the horizontal velocity field, and the geopotential φ and total depth H are given by

$$\varphi = g\eta, \tag{3}$$

$$H = H_A + \eta - \eta_B. \tag{4}$$

Here g is the acceleration due to gravity. On the equatorial beta plane, horizontal positions are expressed in terms of the Cartesian coordinates x and y, where x is eastward and y is northward, whereas the Coriolis parameter is approximated by $f = \beta y$.

By expressing time, horizontal length and vertical height in units of the Earth's inverse angular velocity Ω^{-1}, the Earth's radius a and the fluid's average height H_A, respectively, we obtain a non-dimensional system characterized by one parameter,

$$\gamma = 4\frac{(\Omega a)^2}{gH_A}, \tag{5}$$

which is called Lamb's parameter. Written in terms of these units, the shallow-water equations on the equatorial beta plane become

$$\frac{\mathrm{D}\mathbf{v}}{\mathrm{D}t} + 2y\,\mathbf{k} \times \mathbf{v} + \nabla\varphi = 0, \tag{6}$$

$$\frac{\mathrm{D}H}{\mathrm{D}t} + H\mathcal{D} = 0, \tag{7}$$

with (3) and (4) given by

$$\varphi = \frac{4}{\gamma}\eta, \tag{8}$$

$$H = 1 + \eta - \eta_B. \tag{9}$$

Here for the Coriolis parameter the explicit expression $f = 2y$ has been substituted, noting that $\beta = (2\Omega)/a$ at the Equator. By scaling x and y in terms of the Earth's radius a, these coordinates can be identified with the longitude λ and the latitude ϕ, if the latter are expressed in radians.

The equations that we will deal with in the following are assumed to apply to small deviations from the state of rest, and the orography will be assumed to be zero. The appropriate linearized equations for the horizontal velocity and depth field are, after substituting (8) and (9),

$$\frac{\partial\mathbf{v}}{\partial t} + 2y\,\mathbf{k} \times \mathbf{v} + \frac{4}{\gamma}\nabla\eta = 0, \tag{10}$$

$$\frac{\partial\eta}{\partial t} + \mathcal{D} = 0. \tag{11}$$

These linearized equations form the basis of the discussion in the next sections.

3. Linear waves

Written out in terms of the zonal and meridional components u and v of the horizontal velocity, the linearized shallow-water equations on the equatorial beta plane are

$$\frac{\partial u}{\partial t} - 2yv + \frac{4}{\gamma}\frac{\partial\eta}{\partial x} = 0, \tag{12}$$

$$\frac{\partial v}{\partial t} + 2yu + \frac{4}{\gamma}\frac{\partial\eta}{\partial y} = 0, \tag{13}$$

$$\frac{\partial\eta}{\partial t} + \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0. \tag{14}$$

We will review in some detail the different wave solutions that are known since the work of Matsuno (1966), starting with the somewhat special class of Kelvin waves.

3.1. *Kelvin waves*

There is a subset of solutions in which the meridional velocity v is identically zero. The equations in that case simplify to

$$\frac{\partial u}{\partial t} + \frac{4}{\gamma}\frac{\partial\eta}{\partial x} = 0, \tag{15}$$

$$2yu + \frac{4}{\gamma}\frac{\partial\eta}{\partial y} = 0, \tag{16}$$

$$\frac{\partial\eta}{\partial t} + \frac{\partial u}{\partial x} = 0. \tag{17}$$

The second of these equations shows that the meridional pressure gradient force is in geostrophic balance with the zonal velocity. The zonal pressure gradient force, on the other hand, is not in geostrophic balance with the meridional velocity (which is zero) but is compensated by the zonal acceleration. We search for solutions of the form

$$u(x, y, t) = \mathrm{Re}[\hat{u}(y)\exp \mathrm{i}(mx - 2\sigma t)], \tag{18}$$

$$\eta(x, y, t) = \mathrm{Re}[\hat{\eta}(y)\exp \mathrm{i}(mx - 2\sigma t)], \tag{19}$$

where $0 \leq m < \infty, -\infty < \sigma < \infty$ and, following Longuet-Higgins (1968), the frequency is written as a multiple of 2Ω. If we substitute these expressions in (15)–(17), we find that $\hat{u}$ and $\hat{\eta}$ should satisfy

$$-2\sigma\hat{u} + \frac{4m}{\gamma}\hat{\eta} = 0, \tag{20}$$

$$2y\hat{u} + \frac{4}{\gamma}\frac{d\hat{\eta}}{dy} = 0, \tag{21}$$

$$-2\sigma\hat{\eta} + m\hat{u} = 0. \tag{22}$$

The first and the third of these equations only have a non-trivial solution if σ and m satisfy

$$\gamma\sigma^2 - m^2 = 0, \tag{23}$$

in which case $\hat{\eta}$ and $\hat{u}$ are related by

$$\hat{\eta} = \frac{m}{2\sigma}\hat{u}. \tag{24}$$

If this expression for $\hat{\eta}$ is substituted in the second of the three equations above, we obtain the following equation for $\hat{u}$:

$$\frac{d\hat{u}}{dy} = -\frac{\gamma\sigma}{m}y\hat{u}, \tag{25}$$

the solution of which is

$$\hat{u}(y) = A\exp\left(-\frac{\gamma\sigma}{2m}y^2\right) = A\exp\left(-\frac{\gamma^{1/2}y^2}{2}\right), \tag{26}$$

where A is an arbitrary amplitude and where in the second equality we have used the positive root,

$$\sigma = m/\gamma^{1/2}, \tag{27}$$

of relation (23) in order to obtain a zonal velocity field that falls off to zero far away from the Equator. Expression (26), together with (24) and the dispersion relation (27), determines the Kelvin solution of the set (12)–(14).

Taking the amplitude A to be real, we may write

$$u(x, y, t) = \hat{u}(y)\cos(mx - 2\sigma t), \tag{28}$$

$$\eta(x, y, t) = \hat{\eta}(y)\cos(mx - 2\sigma t), \tag{29}$$

where, from (24), we have that

$$\hat{\eta}(y) = \frac{m}{2\sigma}A\exp\left(-\frac{\gamma^{1/2}y^2}{2}\right). \tag{30}$$

If, following Longuet-Higgins (1968) again, we normalize the wave solutions such that

$$\int_{-\pi}^{\pi} dx \int_{-\infty}^{\infty} (u^2 + z^2)\, dy = \pi, \tag{31}$$

where $z \equiv 2\gamma^{-1/2}\eta$, we find

$$A^{-2} = \frac{2\pi^{1/2}}{\gamma^{1/4}}. \tag{32}$$

In an example of a Kelvin wave, to be shown later, this normalization will be used.

3.2. *Non-Kelvin waves*

Waves for which the meridional velocity is not identically zero will be called non-Kelvin waves, in accordance with the terminology of Ripa (1994). These solutions comprise Rossby waves, Rossby–gravity waves and inertia–gravity waves. To obtain these solutions we write

$$u(x, y, t) = \mathrm{Re}[\hat{u}(y)\exp i(mx - 2\sigma t)], \tag{33}$$

$$v(x, y, t) = \mathrm{Re}[-i\hat{v}(y)\exp i(mx - 2\sigma t)], \tag{34}$$

$$\eta(x, y, t) = \mathrm{Re}[\hat{\eta}(y)\exp i(mx - 2\sigma t)]. \tag{35}$$

Substituting these in the set (12)–(14), it follows that we should have

$$-2\sigma\hat{u} + 2y\hat{v} + \frac{4m}{\gamma}\hat{\eta} = 0, \tag{36}$$

$$-2\sigma\hat{v} + 2y\hat{u} + \frac{4}{\gamma}\frac{d\hat{\eta}}{dy} = 0, \tag{37}$$

$$-2\sigma\hat{\eta} + m\hat{u} - \frac{d\hat{v}}{dy} = 0. \tag{38}$$

The first and the third of these equations can be used to express $\hat{u}$ and $\hat{\eta}$ in terms of $\hat{v}$:

$$\hat{u} = \left(\sigma^2 - \frac{m^2}{\gamma}\right)^{-1}\left(\sigma y\hat{v} - \frac{m}{\gamma}\frac{d\hat{v}}{dy}\right), \tag{39}$$

$$\hat{\eta} = \frac{1}{2}\left(\sigma^2 - \frac{m^2}{\gamma}\right)^{-1}\left(my\hat{v} - \sigma\frac{d\hat{v}}{dy}\right). \tag{40}$$

The only condition here is that σ and m should *not* satisfy (23). By substituting these expressions in the second of the three equations above, we see that $\hat{v}$ should be a solution of

$$\frac{d^2\hat{v}}{dy^2} - \gamma y^2\hat{v} + [\gamma\sigma^2 - m^2 - \frac{m}{\sigma}]\hat{v} = 0; \tag{41}$$

this solution is given by (Gill, 1982)

$$\hat{v}(y) = A\exp\left(-\frac{\gamma^{1/2}y^2}{2}\right)H_n(\gamma^{1/4}y). \tag{42}$$

Here H_n is a Hermite polynomial and σ and m should satisfy the dispersion relation

$$\gamma\sigma^2 - m^2 - \frac{m}{\sigma} = (2n + 1)\gamma^{1/2}. \tag{43}$$

Expression (42), in combination with (39), (40) and the dispersion relation (43), determines the complete solution. We note that, formally, the frequency of the Kelvin wave is obtained from (43) by setting $n = -1$. This number is usually attached to Kelvin waves.

The dispersion relationship (43) is cubic, so that for a fixed value of γ and a given value of m there are at most three solutions. The dispersion relationship is displayed graphically in Figure 1. Besides the Kelvin wave, the diagram shows high-frequency inertia–gravity waves and low-frequency Rossby waves for $n = 1, 2, \ldots$ and the Rossby–gravity wave for $n = 0$. Following the usual practice, negative values of the frequency σ are displayed as positive

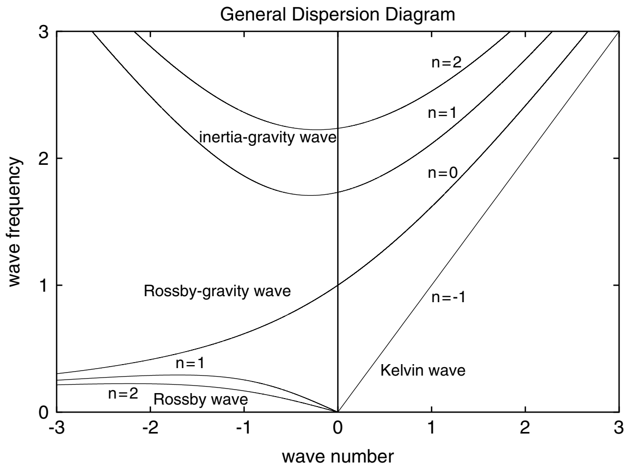

Figure 1. Dispersion relationship for $\gamma = 1$. On the horizontal axis is the wave number m; on the vertical axis the frequency σ. Negative frequencies correspond to westward-moving waves and are displayed as positive frequencies for negative wave numbers, i.e. in the left-hand part of the diagram.

values for negative wave numbers. The waves on the left-hand part of the diagram therefore denote westward-moving waves; the waves on the right-hand part of the diagram denote eastward-moving waves.

Taking the amplitude A to be real, we may write

$$u(x,y,t) = \hat{u}(y)\cos(mx - 2\sigma t), \quad (44)$$

$$v(x,y,t) = \hat{v}(y)\sin(mx - 2\sigma t), \quad (45)$$

$$\eta(x,y,t) = \hat{\eta}(y)\cos(mx - 2\sigma t), \quad (46)$$

where, using the recurrence relations for Hermite polynomials (e.g. Eqs (13.2) and (13.3) of Arfken, 1970), we may derive from (39), (40) and (42) that the fields $\hat{u}$ and $\hat{\eta}$ are given by

$$\hat{u}(y) = \frac{A}{\gamma^{1/4}}\exp\left(-\frac{\gamma^{1/2}y^2}{2}\right) \times \left[\frac{1}{2}\frac{H_{n+1}(\gamma^{1/4}y)}{(\sigma - m/\gamma^{1/2})} + n\frac{H_{n-1}(\gamma^{1/4}y)}{(\sigma + m/\gamma^{1/2})}\right], \quad (47)$$

$$\hat{\eta}(y) = \frac{1}{2}A\gamma^{1/4}\exp\left(-\frac{\gamma^{1/2}y^2}{2}\right) \times \left[\frac{1}{2}\frac{H_{n+1}(\gamma^{1/4}y)}{(\sigma - m/\gamma^{1/2})} - n\frac{H_{n-1}(\gamma^{1/4}y)}{(\sigma + m/\gamma^{1/2})}\right]. \quad (48)$$

Using expressions for the Hermite polynomials (e.g. Table 13.1 of Arfken, 1970), we may draw plots and profiles of the different waves. Using $z \equiv 2\gamma^{-1/2}\eta$, as before, we may choose the amplitude A in such a way that

$$\int_{-\pi}^{\pi} dx \int_{-\infty}^{\infty} (u^2 + v^2 + z^2)\, dy = \pi. \quad (49)$$

For non-Kelvin waves this gives

$$A^{-2} = \frac{2^n\, n!\, \pi^{1/2}}{\gamma^{1/4}} \times \left[1 + \frac{n+1}{\gamma^{1/2}(\sigma - m/\gamma^{1/2})^2} + \frac{n}{\gamma^{1/2}(\sigma + m/\gamma^{1/2})^2}\right]. \quad (50)$$

Table I. Parameters of the equatorial waves that are shown in Figures 2 and 4. The columns with η_m and $|\mathbf{v}|_m$ give the maximum values of η and $|\mathbf{v}|$ of the different waves. The lower three rows of the column refer to the low-frequency approximation.

Wave	n	σ	A	η_m	$\lvert\mathbf{v}\rvert_m$
Kelvin	−1	+1.000	+0.531	0.266	0.531
Rossby–gravity	0	+1.618	+0.395	0.194	0.395
inertia–gravity	1	+2.115	+0.323	0.196	0.391
Rossby	1	−0.254	−0.263	0.183	0.563
Rossby–gravity	0	−0.618	−0.639	0.120	0.639
inertia–gravity	1	−1.861	−0.330	0.249	0.400
Rossby	2	−0.167	−0.108	0.207	0.489
Rossby–gravity	0	−0.500	−0.531	0.161	0.531
Rossby	1	−0.250	−0.266	0.195	0.531
Rossby	2	−0.167	−0.108	0.212	0.478

To derive the latter expressions, we used the orthogonality conditions for Hermite polynomials (e.g. Eq. (13.15) of Arfken, 1970). In Figure 2 we show a few examples of equatorial waves for $\gamma = 1$ and $m = 1$, plotted at time $t = 0$. The values of the frequency σ as well as the values of A are given in Table I.

4. The low-frequency limit

In this section we will study the behaviour of the different waves in the limit that the frequency σ approaches zero. For both Kelvin waves and non-Kelvin waves it can be checked straightforwardly from (14) and (17) that

$$\hat{\mathcal{D}} = 2\sigma\hat{\eta}. \quad (51)$$

Here the divergence $\mathcal{D}$ is written, like the other fields, in normal mode form, i.e.

$$\mathcal{D} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = \mathrm{Re}[\mathrm{i}\hat{\mathcal{D}}(y)\exp\mathrm{i}(mx - 2\sigma t)]. \quad (52)$$

It implies that the divergence of low-frequency waves will have relatively small amplitudes when compared with the height field. The way in which the height field and the velocity field are related to each other will be different for the different types of waves, as we will see below.

4.1. *Kelvin waves*

It is clear from the dispersion relationship (27) that Kelvin waves with long wavelengths (for which m is small) have low frequencies, but there is no structural change in the way the velocity field is related to the height field: all Kelvin waves are characterized by geostrophic balance between the height field η and the zonal velocity field u.

4.2. *Non-Kelvin waves*

The low-frequency limit of Rossby waves, Rossby–gravity waves and inertia–gravity waves is obtained from the

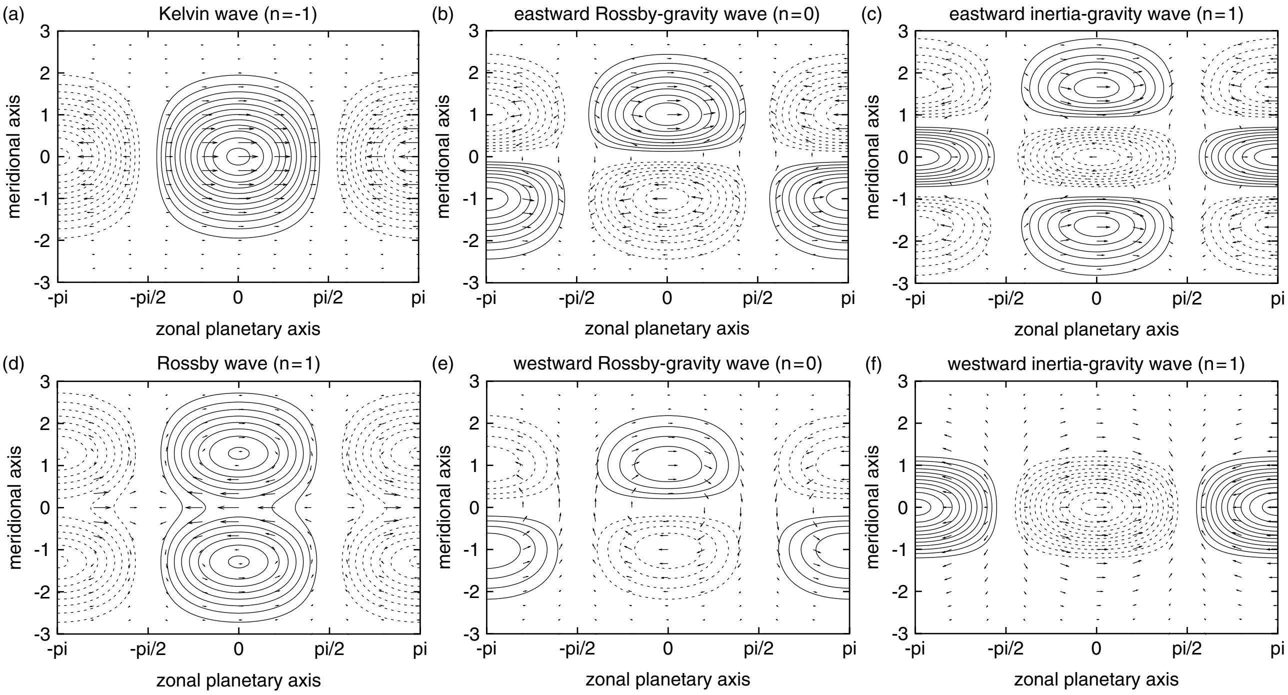

Figure 2. Examples of equatorial waves for $\gamma = 1$, $m = 1$ and plotted at time $t = 0$. The fields are normalized as in (32) and (50). The height field is displayed in terms of contours, the velocity field in terms of arrows. The contours are plotted with an interval of 0.02, the values −0.02, 0 and 0.02 being omitted. Maximum values of the height field η and the velocity field $|\mathbf{v}|$ are given in Table I. (a) (eastward-moving) Kelvin wave with $n = -1$; (b) eastward-moving Rossby–gravity wave with $n = 0$; (c) eastward-moving inertia–gravity wave with $n = 1$; (d) (westward-moving) Rossby wave with $n = 1$; (e) westward-moving Rossby–gravity wave with $n = 0$; (f) westward-moving inertia–gravity wave with $n = 1$.

dispersion relation (43) by putting the first term on the left-hand side of this equation to zero. This gives

$$-m^2 - \frac{m}{\sigma} = (2n+1)\gamma^{1/2}, \tag{53}$$

which can also be written as

$$\sigma = \frac{-m}{(2n+1)\gamma^{1/2} + m^2}. \tag{54}$$

For Rossby waves ($n = 1, 2, \dots$) this is a reasonable approximation; for Rossby–gravity waves ($n = 0$) the approximation is only reasonable if the frequencies are negative and m is not too small, i.e. for westward waves with a horizontal wavelength that is not too large. This is illustrated by Figure 3, where the solid curves are the frequencies of the original Rossby and Rossby–gravity waves (identical to those in Figure 1) and the dashed curves their approximations according to (54). We note that all inertia–gravity waves have been excluded by taking the low-frequency limit.

To investigate the structure of the associated fields in the low-frequency approximation, we consider (39), (40) and (41) in the limit that σ approaches zero. The first and second of these reduce to

$$m\hat{u} = \frac{\mathrm{d}\hat{v}}{\mathrm{d}y}, \tag{55}$$

$$\hat{\eta} = -\frac{\gamma}{2m}\, y\, \hat{v}, \tag{56}$$

whereas the third becomes

$$\frac{\mathrm{d}^2\hat{v}}{\mathrm{d}y^2} - \gamma y^2 \hat{v} + \left[-m^2 - \frac{m}{\sigma}\right]\hat{v} = 0. \tag{57}$$

The solutions of the latter equation are the same as those of (41) and are given by (42), but σ and m should satisfy the simplified dispersion relation (53). The fields $\hat{u}$ and $\hat{\eta}$ can be obtained from (42) by using (55) and (56). This leads to, making use of the recurrence relations of Hermite polynomials in the same way as before,

$$\hat{u}(y) = \frac{A\gamma^{1/4}}{m}\exp\left(-\frac{\gamma^{1/2}y^2}{2}\right) \times \left[-\frac{1}{2}H_{n+1}(\gamma^{1/4}y) + nH_{n-1}(\gamma^{1/4}y)\right], \tag{58}$$

$$\hat{\eta}(y) = \frac{A\gamma^{3/4}}{2m}\exp\left(-\frac{\gamma^{1/2}y^2}{2}\right) \times \left[-\frac{1}{2}H_{n+1}(\gamma^{1/4}y) - nH_{n-1}(\gamma^{1/4}y)\right]. \tag{59}$$

It can be checked that these expressions can also be obtained from (47) and (48) by putting σ to zero in the latter expressions. If we wish to normalize the approximated solutions, as in (49), then A should be chosen as

$$A^{-2} = \frac{2^n\, n!\, \pi^{1/2}}{\gamma^{1/4}}\left[1 + \frac{(2n+1)\gamma^{1/2}}{m^2}\right], \tag{60}$$

an expression that can be verified to be in accord with (50) by putting σ to zero. To show how well the low-frequency limit works in the case of a (westward-moving) Rossby–gravity wave and for Rossby waves, we show a few examples in Figure 4 for $\gamma = 1$ and $m = 1$. The upper panel shows the original wave solutions; the lower panel their approximations. The values of σ and A are given in Table I. We note that the original and approximated fields are remarkably similar.

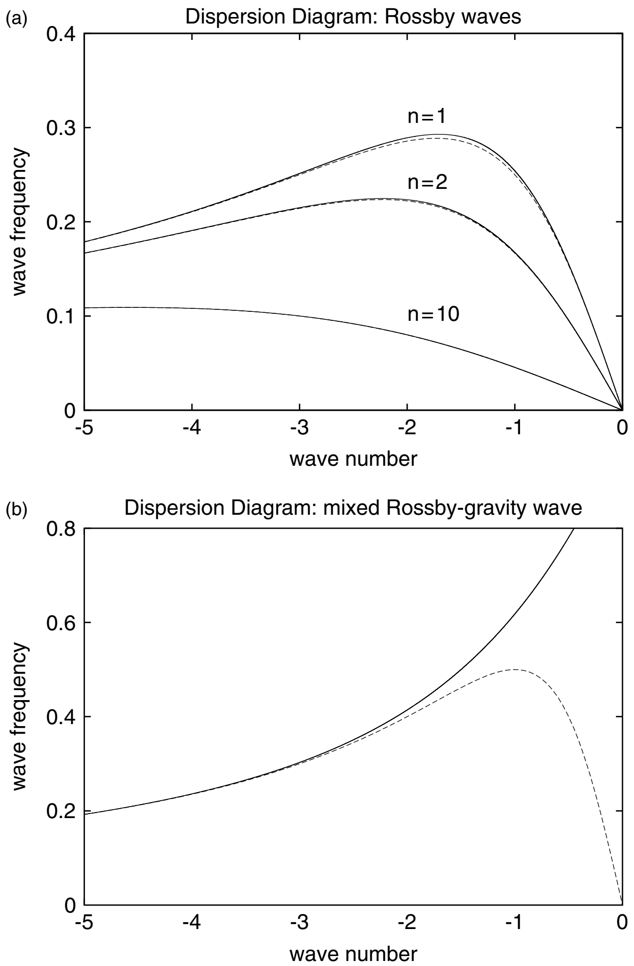

Figure 3. Comparison between the original frequencies (solid lines) as calculated from the full dispersion relationship (43) and the approximated frequencies (dashed lines) as calculated from the approximated dispersion relationship (54) for $\gamma = 1$ and $m = 1$. (a) Rossby waves with $n = 1$, $n = 2$ and $n = 10$; (b) Rossby–gravity wave with $n = 0$.

As the low-frequency approximation works least well for the Rossby–gravity wave with $n = 0$, we consider this case in somewhat more detail. For $n = 0$ the dispersion relation (43) can be factorized,† i.e. for $n = 0$ the dispersion relation (43) is equivalent to

$$\left(\sigma + \frac{m}{\gamma^{1/2}}\right)\left(\sigma^2 - \frac{m}{\gamma^{1/2}}\sigma - \frac{1}{\gamma^{1/2}}\right) = 0. \tag{61}$$

The first root, $\sigma = -m/\gamma^{1/2}$ leads to solutions that are unbounded in y, as in the second root of the Kelvin waves, and has to be abandoned. The other two roots are given by

$$\sigma = \frac{m}{2\gamma^{1/2}} \pm \frac{m}{2\gamma^{1/2}}\left(1 + \frac{4\gamma^{1/2}}{m^2}\right)^{1/2} \tag{62}$$

$$= \frac{m}{2\gamma^{1/2}} \pm \frac{1}{\gamma^{1/4}}\left(1 + \frac{m^2}{4\gamma^{1/2}}\right)^{1/2}. \tag{63}$$

The expressions above show that the root with negative frequency has $\sigma \sim -1/m$ for large $m/\gamma^{1/4}$ and $\sigma \sim -1/\gamma^{1/4}$ for small $m/\gamma^{1/4}$. This root is reproduced approximately in the low-frequency limit, which (see (54)) gives $\sigma \sim -1/m$ for large $m/\gamma^{1/4}$ and $\sigma \sim -m/\gamma^{1/2}$ for small $m/\gamma^{1/4}$. While the positive root is not reproduced at all, the negative root differs most for small values of $m/\gamma^{1/4}$, where the low-frequency approximation gives a frequency equal to the abandoned first root of the original dispersion relation, like Kelvin waves moving westward. The $\hat{v}$ fields are identical for both the unapproximated and approximated fields, given by (42) and noting that $H_0(\gamma^{1/4}y) = 1$, but this is not so for the other fields. In the unapproximated case we have, for small $m/\gamma^{1/4}$,

$$\hat{u}(y) \sim -A(\gamma^{1/4}y)\exp\left(-\frac{\gamma^{1/2}y^2}{2}\right), \tag{64}$$

$$\hat{\eta}(y) \sim \frac{A\gamma^{1/2}}{2m}(\gamma^{1/4}y)\exp\left(-\frac{\gamma^{1/2}y^2}{2}\right), \tag{65}$$

as can be verified by checking with (47) and (48) and using $H_1(\gamma^{1/4}y) = 2\gamma^{1/4}y$. The corresponding fields in the low-frequency limit are (see (58) and (59))

$$\hat{u}(y) \sim -A(\gamma^{1/4}y)\left(\frac{\gamma^{1/4}}{m}\right)\exp\left(-\frac{\gamma^{1/2}y^2}{2}\right), \tag{66}$$

$$\hat{\eta}(y) \sim \frac{A\gamma^{1/2}}{2m}(\gamma^{1/4}y)\left(\frac{\gamma^{1/4}}{m}\right)\exp\left(-\frac{\gamma^{1/2}y^2}{2}\right). \tag{67}$$

We see that in the latter fields both $\hat{u}$ and $\hat{\eta}$ are multiplied by a factor $\gamma^{1/4}/m$ compared with the former. For small values of $m/\gamma^{1/4}$ (i.e. for small values of m or large values of γ, or both) the fields in the low-frequency limit thus have relatively small meridional velocity fields. In this respect they also resemble Kelvin waves.

We will now consider in more detail the relationship between velocity and height fields for the Rossby waves and Rossby–gravity waves in the low-frequency limit. We first note that (55) implies that the divergence of the velocity field is zero. Indeed, for a wave of the general form given above we have in this case

$$\hat{\mathcal{D}}(y) = m\hat{u}(y) - \frac{d\hat{v}(y)}{dy} = 0. \tag{68}$$

The velocity field can thus be written in terms of a stream function ψ,

$$u = -\frac{\partial\psi}{\partial y}, \quad v = \frac{\partial\psi}{\partial x}, \tag{69}$$

which, when ψ is written like the other fields:

$$\psi(x, y, t) = \mathrm{Re}[\hat{\psi}(y)\exp \mathrm{i}(mx - 2\sigma t)], \tag{70}$$

implies that

$$\hat{u} = -\frac{d\hat{\psi}}{dy}, \qquad \hat{v} = -m\hat{\psi}. \tag{71}$$

Expression (56) then yields

$$\frac{4}{\gamma}\hat{\eta} = 2y\hat{\psi}. \tag{72}$$

†See Matsuno (1966), his Eq. (12). Note that Matsuno's wavenumber k and frequency ω are related to our m and σ by $k = m/\gamma^{1/4}$ and $\omega = -\sigma\gamma^{1/4}$.

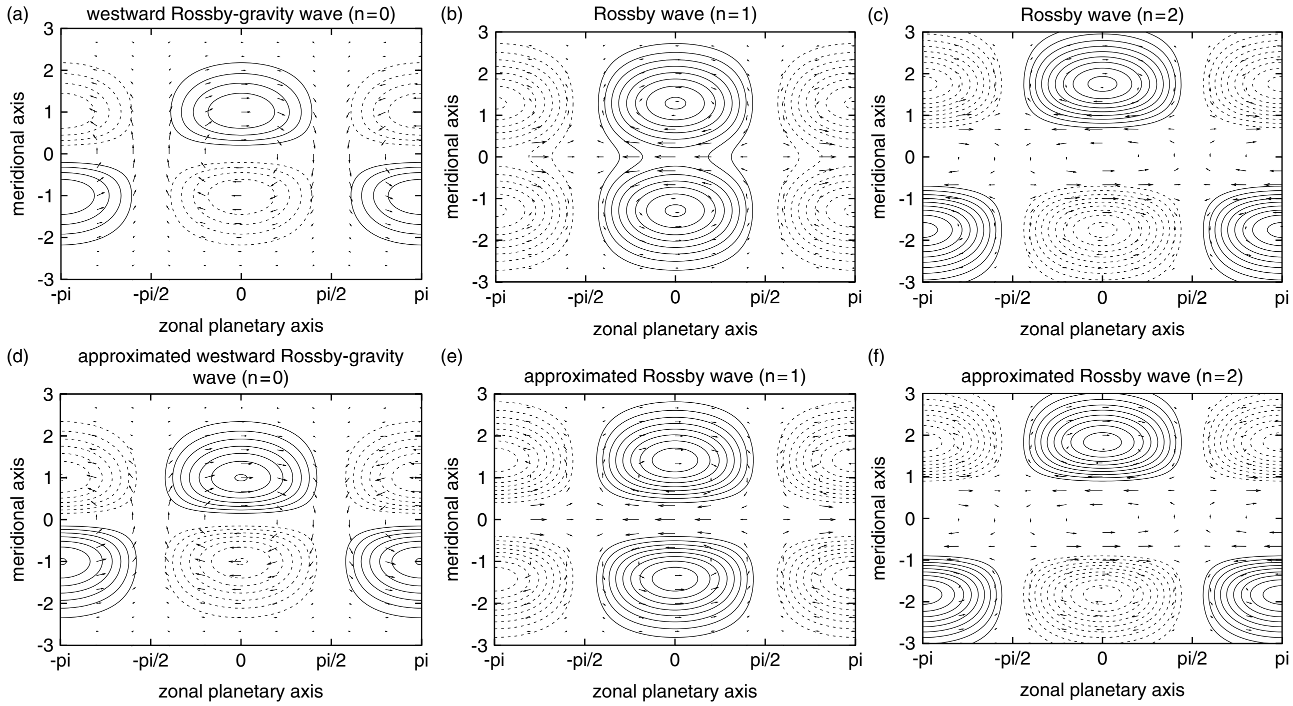

Figure 4. Height contours and velocity fields of a Rossby–gravity wave and two Rossby waves, displayed with the same conventions as in Figure 2, for $\gamma = 1$ and $m = 1$. The waves in the upper panels are solutions of the full linearized shallow-water equations; the waves in the lower panels are the corresponding approximations in the low-frequency limit. Left: Rossby–gravity wave with $n = 0$; middle: Rossby wave with $n = 1$; right: Rossby wave with $n = 2$. Maximum values of the height field η and the velocity field $|\mathbf{v}|$ are given in Table I.

In terms of the full wave fields, this would mean

$$\frac{4}{\gamma}\eta = 2y\psi, \tag{73}$$

or in dimensional fields

$$g\eta = \beta y\psi \quad \Longleftrightarrow \quad \varphi = f\psi. \tag{74}$$

Therefore, in the low-frequency limit the velocity field is asymptotically non-divergent and the geopotential is given by the Coriolis parameter multiplied by the stream function of the non-divergent velocity field.

We mention that Ripa (1994), in a parenthetical remark after his Eq. (3.3), has already alluded to this type of balance in the limit of low-frequency non-Kelvin waves. The balance is the same as what Daley (1983) calls the 'simplest form of the geostrophic relationship'. It is rather different from the geostrophic balance that characterizes Kelvin waves and explains why Matsuno (1966, p. 31), referring to a Rossby wave with $n = 2$ such as in the right panels of Figure 4, observes that the wind field is somewhat curious in the vicinity of the Equator. Given the antisymmetric height field η in that solution, the stream function of the flow would have to be symmetric according to the balance observed here, and that explains that this particular wave has closed cyclonic and anticyclonic circulation cells centred at the Equator.

5. Balanced theory

Using the balance condition observed above, it is possible to approach the low-frequency limit somewhat differently, an approach that leads to exactly the same results for Rossby waves and Rossby–gravity waves but can be generalized to the nonlinear case. To this end, we go back to the linear system (12)–(14). An equation for the relative vorticity ζ,

$$\zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}, \tag{75}$$

is obtained by taking the y derivative of the first equation, the x derivative of the second equation and then subtracting. Together with the equation for the surface elevation η, this gives the following system:

$$\frac{\partial \zeta}{\partial t} + 2v + 2y\mathcal{D} = 0, \tag{76}$$

$$\frac{\partial \eta}{\partial t} + \mathcal{D} = 0. \tag{77}$$

Eliminating the divergence, we obtain an equation for the linearized potential vorticity:

$$\frac{\partial}{\partial t}(\zeta - 2y\eta) + 2v = 0. \tag{78}$$

Using the type of balance discussed above, we might now substitute $\zeta = \nabla^2\psi$, $v = \partial\psi/\partial x$ and $\eta = (\gamma/2)y\psi$ to arrive at

$$\frac{\partial}{\partial t}(\nabla^2\psi - \gamma y^2\psi) + 2\frac{\partial \psi}{\partial x} = 0, \tag{79}$$

which is a closed, balanced, system in terms of a single variable: the stream function ψ. Note that by substituting the balance condition (73) in (77), we obtain an expression for the small residual divergence $\mathcal{D}$.

In the same way as before, we might look for normal mode solutions of (79) in the form (70). This results in the following equation:

$$\frac{d^2\hat{\psi}}{dy^2} - \gamma y^2\hat{\psi} + \left[-m^2 - \frac{m}{\sigma}\right]\hat{\psi} = 0, \quad (80)$$

which is seen to have exactly the same form as the low-frequency approximation for $\hat{v}$, i.e. (57). Because the meridional velocity field $\hat{v}$, associated with the stream function $\hat{\psi}$, is simply given by $\hat{v} = -m\hat{\psi}$, (refer to (71)), we see that if $\hat{\psi}$ satisfies (80) then $\hat{v}$ satisfies (57). This implies that we may write the solution as

$$\hat{\psi}(y) = -\frac{A}{m}\exp\left(-\frac{\gamma^{1/2}y^2}{2}\right)H_n(\gamma^{1/4}y); \quad (81)$$

this solution would then lead to exactly the same meridional velocity as given by (42). For $\hat{u}$ and $\hat{\eta}$ we would have

$$\hat{u} = -\frac{d\hat{\psi}}{dy} = \frac{1}{m}\frac{d\hat{v}}{dy}, \quad (82)$$

$$\hat{\eta} = \frac{\gamma}{2}y\hat{\psi} = -\frac{\gamma}{2m}y\hat{v}, \quad (83)$$

in accordance with (55) and (56). It means that the expressions for $\hat{u}$ and $\hat{\eta}$ in (58) and (59), as well as the normalization (60), all apply without further qualifications.

We may thus conclude that the balanced system based on the linearized potential vorticity equation (79) exactly reproduces Rossby waves and the Rossby–gravity waves in the low-frequency approximation of the linearized shallow-water system. These approximated solutions reproduce the exact solutions remarkably well, as the different figures have already shown.

6. Discussion

If we denote the linearized potential vorticity $\zeta - 2y\eta$ by r and express it, like the other fields, in normal-mode form,

$$r(x, y, t) = \text{Re}[\hat{r}(y)\exp(im - 2\sigma t], \quad (84)$$

we notice that $\hat{r} = 0$ for Kelvin waves. Thus Kelvin waves are invisible in terms of potential vorticity. As a consequence, the balanced approximation cannot be used to describe these waves, not even in the low-frequency limit. For the non-Kelvin waves, i.e. Rossby waves, Rossby–gravity waves and inertia–gravity waves, it can be checked that $\hat{r} = -\hat{v}/\sigma$. This implies that inertia–gravity waves (with large values of σ) have small potential vorticity if compared with Rossby waves and (westward-moving) Rossby–gravity waves (with small values of σ). We can thus say that the balanced system filters out Kelvin waves because they are invisible in terms of potential vorticity and inertia–gravity waves and (eastward-moving) Rossby–gravity waves because they have relatively large frequencies. Using Ripa's (1994) way of distinguishing between Kelvin waves and non-Kelvin waves, Schubert *et al.* (2009b) arrived at a similar conclusion.

We notice that the balance satisfied by Kelvin waves, low-frequency Rossby waves and low-frequency Rossby–gravity waves is geostrophic in all cases, but in a limited, partial sense. For Kelvin waves there is a balance between the Coriolis force and the *meridional* pressure gradient force, expressed in terms of dimensional fields as

$$fu + \frac{\partial\varphi}{\partial y} = 0. \quad (85)$$

Low-frequency Rossby waves and low-frequency Rossby–-gravity waves are characterized by a similar balance between the Coriolis force and the *zonal* pressure gradient force, in dimensional fields,

$$-fv + \frac{\partial\varphi}{\partial x} = 0. \quad (86)$$

Indeed, from (74) it follows that $\partial/\partial x(-f\psi + \varphi) = 0$, implying that $-f\partial\psi/\partial x + \partial\varphi/\partial x = 0$. Non-divergence of the flow gives $v = \partial\psi/\partial x$, from which (86) follows immediately. Equation (85) is exact for Kelvin waves whereas (86) is invalid. For low-frequency Rossby waves and low-frequency Rossby–gravity waves it is the other way around, i.e. (86) is exact whereas (85) is invalid or at best an approximation. Bouchut *et al.* (2005) reserve the term 'balanced' for flows that satisfy (86). Kelvin waves, which satisfy (85), are thus termed 'unbalanced'. We prefer to classify both types of waves as 'balanced', with a balance that is different for the different types of waves, and which might be characterized as 'partially geostrophic'.

We have seen that low-frequency non-Kelvin waves can be described in terms of a balanced potential vorticity equation. An advantage of such a description is that it can be generalized to finite-amplitude motions. The balanced potential vorticity equation (79) can be seen as a linearization (without orography) of a more general equation in which a potential vorticity field q, defined by

$$q = 2y + \nabla^2\psi - \gamma y^2\psi + 2y\eta_B, \quad (87)$$

is advected with the non-divergent velocity field $\mathbf{v} = \mathbf{k} \times \nabla\psi$. The non-dimensional expression above follows directly from Eq. (33) of Verkley (2009) by using that $\mu = \sin\phi \approx \phi = y$ in the equatorial beta plane. In the equatorial beta plane the differential operators are all Cartesian to first order in L/a, where L is a typical horizontal distance and a is the radius of the Earth, as shown by Verkley (1990). This equation has the same form as the equivalent barotropic vorticity equation as proposed by Cressman (1958), the difference being that the full variation of $f^2 = \beta^2y^2$ is retained instead of being approximated by a constant value f_0^2.

7. Conclusion

We have analyzed the low-frequency limit of linear wave solutions of the shallow-water equations on an equatorial beta plane. The analysis has brought to light that Rossby waves and (westward-moving) Rossby–gravity waves satisfy a form of balance that Daley (1983) calls the 'simplest form of the geostrophic relationship'. This form of balance is characterized by a non-divergent horizontal velocity field and a geopotential that satisfies $\varphi = f\psi$, where $f = \beta y$ is the Coriolis parameter and ψ is the stream function of the non-divergent velocity field.

It is shown that the linearized potential vorticity equation, in which this balance is incorporated, exactly reproduces

Rossby waves and Rossby–gravity waves in the low-frequency limit. The linear waves in the low-frequency limit are rather accurate approximations of the corresponding solutions of the full linear shallow-water system. We may thus conclude that the balanced potential vorticity equation offers an appropriate approach to slow linear non-Kelvin-wave motions, even at the Equator. Kelvin waves are excluded from this description as these waves are invisible in terms of potential vorticity. Low-frequency Kelvin waves should thus be taken into account separately.

We noticed that the balanced potential vorticity equation has the same form as the equivalent barotropic vorticity equation for a midlatitude beta plane, originally proposed by Cressman (1958), the difference being that the constant factor f_0^2 has been replaced by the variable $\beta^2 y^2$. We believe that this equation might be useful in the study of tropical dynamics, and also in the finite-amplitude case. The recent finding of Yano *et al.* (2009) that a small horizontal divergence is consistent with observational data on large-scale tropical circulations adds credit to this view. There is therefore reason to enlarge the scope of the quotation from James' (1994) textbook, cited in the Introduction, of which we now give the second part:

> *... but [the quasi-geostrophic system of equations] remains of great value in diagnosing, and gaining insight, into the dominant dynamical processes in the midlatitude and subtropical regions.*

Acknowledgements

The authors thank Dr Ronald van der A and Dr Peter Siegmund as well as Professor Hennie Kelder for sharing with us their insights into tropical dynamics. Discussions with Dr Jun-Ichi Yano and the comments of the referees were also much appreciated.

References

Arfken G. 1970. *Mathematical Methods for Physicists*. Academic Press.
Bouchut F, Le Sommer J, Zeitlin V. 2005. Breaking of balanced and unbalanced equatorial waves. *CHAOS* **15**: 013503(1)–103503(19).
Cressman GP. 1958. Barotropic divergence and very long atmospheric waves. *Mon. Weather Rev.* **86**: 293–297.
Daley R. 1983. Linear non-divergent mass–wind laws on the sphere. *Tellus* **35A**: 17–27.
Gill AE. 1982. *Atmosphere–Ocean Dynamics*. Academic Press.
James IN. 1994. *Introduction to Circulating Atmospheres*. Cambridge University Press: Cambridge, UK.
Longuet-Higgins MS. 1968. The eigenfunctions of Laplace's tidal equations over a sphere. *Philos. Trans. R. Soc. London* **A262**: 511–607.
Matsuno T. 1966. Quasi-geostrophic motions in the equatorial area. *J. Meteorol. Soc. Jpn* **44**: 25–42.
Pedlosky J. 1987. *Geophysical Fluid Dynamics*. Springer-Verlag: Berlin.
Ripa P. 1994. Horizontal wave propagation in the equatorial waveguide. *J. Fluid. Mech.* **271**: 267–284.
Schubert WH, Taft RK, Silvers LG. 2009a. Shallow water quasi-geostrophic theory on the sphere. *J. Adv. Model. Earth Syst.* **1**(2): 1–17.
Schubert WH, Silvers LG, Masarik MT, Gonzalez AO. 2009b. A filtered model of tropical wave motions. *J. Adv. Model. Earth Syst.* **1**(3): 1–11.
Verkley WTM. 1990. On the beta plane approximation. *J. Atmos. Sci.* **47**: 2453–2460.
Verkley WTM. 2009. A balanced approximation of the one-layer shallow-water equations on a sphere. *J. Atmos. Sci.* **66**: 1735–1748.
Yano JI, Mulet S, Bonazzola M. 2009. Tropical large-scale circulations: Asymptotically nondivergent? *Tellus* **61A**: 417–427.

