

# Hyperbolic Automorphisms and Holomorphic Motions in $\mathbb{C}^2$

GREGERY T. BUZZARD & KAUSHAL VERMA

## 1. Introduction

Holomorphic motions have been an important tool in the study of complex dynamics in one variable. In this paper we provide one approach to using holomorphic motions in the study of complex dynamics in two variables. To introduce these ideas more fully, let  $\Delta_r$  be the disk of radius  $r$  and center 0 in the plane, let  $\mathbb{P}^1$  be the Riemann sphere, and recall that a holomorphic motion of a set  $E \subset \mathbb{P}^1$  is a function  $\alpha: \Delta_r \times E \rightarrow \mathbb{P}^1$  such that  $\alpha(0, z) = z$  for each  $z \in E$ ,  $\alpha(\lambda, \cdot): E \rightarrow \mathbb{P}^1$  is injective for each fixed  $\lambda \in \Delta_r$ , and  $\alpha(\cdot, z): \Delta_r \rightarrow \mathbb{P}^1$  is holomorphic for each fixed  $z \in E$ . For future reference, we note that this definition (as well as most results about holomorphic motions) applies equally well when the parameter  $\lambda$  is allowed to vary in the complex polydisk:  $\lambda \in \Delta_r^n$ .

One of the first uses of holomorphic motions in the study of complex dynamics was in [MSS], where holomorphic motions were used to prove the density of structurally stable maps within the family of polynomial maps of  $\mathbb{C}$  of degree  $d$ . In general, a map  $f: M \rightarrow M$ ,  $M$  a manifold, is structurally stable within a family  $\mathcal{F}$  of maps if there is some neighborhood of  $f$ , say  $\mathcal{U} \subset \mathcal{F}$ , such that any map in  $\mathcal{U}$  is conjugate to  $f$  via a homeomorphism of  $M$ . Mañé-Sad-Sullivan [MSS] obtained structural stability for polynomial maps by showing that (subject to certain restrictions) the holomorphic motion defined naturally on the Julia set of a polynomial map extends to give a conjugacy on all of  $\mathbb{C}$  to nearby polynomial maps. More precisely, they did this by starting with the canonical holomorphic motion defined on hyperbolic periodic points and on periodic points satisfying a critical orbit relation; by their  $\lambda$ -lemma, this holomorphic motion extends uniquely to a holomorphic motion of the closure of the periodic points. The authors then constructed (by hand) certain holomorphic motions which give partial conjugacies and which extend by iteration to give a holomorphic motion of a dense set of the plane, which again extends uniquely to give a topological conjugacy on the whole sphere.

Shortly after this work, Bers and Royden [BeR] used the notion of a harmonic Beltrami coefficient (defined in Section 6) to show that, given any holomorphic motion of a set  $E$ , there is a canonical extension of this motion to a holomorphic

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motion of the sphere, although with a restriction to  $\lambda \in \Delta_{r/3}$ . The characterization of this extension is that in any component  $S$  of the complement of  $\bar{E}$ , the Beltrami coefficient  $(\partial\alpha/\partial\bar{z})/(\partial\alpha/\partial z)$  is a harmonic Beltrami coefficient. Using this result, McMullen and Sullivan [McS] proved the density of structurally stable maps within the family of rational maps of  $\mathbb{P}^1$  of degree  $d$  as follows. As before, given a family  $f_\lambda$  ( $\lambda \in \Delta$ ) with certain regularity properties, there is a canonical holomorphic motion on the closure of the set consisting of periodic points and orbits of critical points. By the Bers–Royden result, this motion extends canonically to a motion  $\alpha_\lambda$  of the entire sphere. Then  $f_\lambda^{-1} \circ \alpha_\lambda \circ f_0(z)$  defines a second holomorphic motion which agrees with the original motion on the periodic points and critical orbits and which also has a harmonic Beltrami coefficient. By the uniqueness of the Bers–Royden extension, this second holomorphic motion agrees with the first, and hence  $\alpha_\lambda$  is a global topological conjugacy.

Turning to higher dimensions, one natural family of maps with interesting dynamics in  $\mathbb{C}^2$  is the family of (generalized) Hénon maps: compositions of holomorphic diffeomorphisms of the form  $f(z, w) = (w, p(w) - az)$ , where  $p$  is a polynomial of degree  $d \geq 2$  and  $a \neq 0$ . We note here that, for questions of structural stability, we will restrict ourselves to families of maps all having the same degree. This corresponds, for example, to considering structural stability of quadratic polynomials in one variable. With this restriction, the topology on Hénon maps can be specified either in terms of the coefficients of the defining maps or in terms of the compact-open topology applied to the map and its inverse. Section 2 provides a more detailed account of Hénon maps and hyperbolicity. For further references, see the bibliography in [BuS].

There is an immediate generalization of holomorphic motions to two dimensions: simply allow each point  $z \in E$  to vary holomorphically within  $\mathbb{C}^2$ . In fact, by work of Jonsson [J], given a family  $f_\lambda$  of hyperbolic Hénon maps, the set  $J_\lambda$  (which is the closure of the set of saddle periodic points of  $f_\lambda$ ) varies as a holomorphic motion in this sense. However, this generalization fails to have many of the important properties of 1-variable holomorphic motions; in particular, given this kind of holomorphic motion on a set  $E$ , there is in general no unique extension to  $\bar{E}$  and no canonical extension in the sense of Bers and Royden.

Our approach in this paper is to use the technique of McMullen and Sullivan to construct holomorphic motions on dynamically defined 1-dimensional subsets of  $\mathbb{C}^2$  and then show that these maps define homeomorphisms on the union of these 1-dimensional subsets. To be more precise, let  $f$  be a hyperbolic Hénon map, let  $J^+$  (resp.  $J^-$ ) be the boundary of the set of points with bounded forward (resp. backward) orbit, and let  $J = J^+ \cap J^-$ . Then  $J^+$  and  $J^-$  are laminated by Riemann surfaces; each of these Riemann surfaces is conformally equivalent to the plane and is the stable or unstable manifold of a point in  $J$ . Given a 1-parameter family  $f_\lambda$  of such maps, the points of intersection between  $J_\lambda^-$  and  $J_\lambda^+$  define a holomorphic motion in each leaf, which extends canonically to the entire leaf by the Bers–Royden theorem. As in [McS], this defines a conjugacy between  $f_0$  on a leaf of  $J_0^+$  and  $f_\lambda$  on a leaf of  $J_\lambda^+$ . However, since each leaf of  $J_0^+$  is dense in

$J_0^+$ , it is not clear that the resulting conjugacy gives a homeomorphism of  $J_0^+$  to  $J_\lambda^+$ . To establish that this map is a homeomorphism, we use the notion of an affine structure (see [G1; G2; BS5]) to provide a coherent framework for discussing holomorphic motions on the leaves of the lamination. We show that the affine structure of  $J_\lambda^+$  varies holomorphically with  $\lambda$  and that, suitably normalized, the global parametrizing functions for the leaves of  $J_\lambda^+$  converge locally uniformly when approaching a limit leaf. With this, the continuity of the conjugacy follows essentially from the uniqueness of the Bers–Royden extension.

The first main result of this paper is the following theorem, which is an analog of the results of [MSS] and [McS], and which states that a hyperbolic Hénon map restricted to  $J^+ \cup J^-$  is conjugate to nearby Hénon maps via a holomorphic motion of each leaf of  $J^+ \cup J^-$ .

**THEOREM 1.1.** *Let  $f_\lambda$  be a 1-parameter family of hyperbolic Hénon maps depending holomorphically on  $\lambda \in \Delta^n$ . Then there exists  $r > 0$  and a map*

$$\Psi : \Delta_r^n \times (J_0^+ \cup J_0^-) \rightarrow J_\lambda^+ \cup J_\lambda^-$$

such that, defining  $\Psi_\lambda(p) = \Psi(\lambda, p)$ , we have:

- (1)  $\Psi_0(p) = p$ ;
- (2)  $\Psi_\lambda$  is a homeomorphism for each fixed  $\lambda$ ;
- (3)  $\Psi_\lambda(p)$  is holomorphic in  $\lambda$  for each fixed  $p \in J_0^+ \cup J_0^-$ ;
- (4)  $\Psi_\lambda$  maps each leaf of  $J_0^-$  ( $J_0^+$ ) to a leaf of  $J_\lambda^-$  ( $J_\lambda^+$ ); and
- (5)  $\Psi_\lambda f_0 = f_\lambda \Psi_\lambda$  on  $J_0^+ \cup J_0^-$ .

The first three properties are direct analogs of holomorphic motions in one variable, while the fourth property shows that the map respects the dynamically defined stable and unstable laminations.

In the study of the dynamics of polynomials in the plane, the polynomials with connected Julia set play a special role. In [BS4], Bedford and Smillie defined the notion of an unstably connected Hénon map, which is an analog of a polynomial with a connected Julia set in one variable. They also showed that, given a hyperbolic Hénon map that is unstably connected, the lamination of  $J^+$  extends to a lamination of  $J^+ \cup U^+$ , where  $U^+$  is the set of points with unbounded forward orbits. With this additional structure, we obtain a conjugacy as before on  $J^+ \cup U^+$ .

**THEOREM 1.2.** *In addition to the hypotheses of Theorem 1.1, assume that  $f_0$  is unstably connected. Then the conclusions of that theorem remain valid when  $J_0^+$  and  $J_\lambda^+$  are replaced by  $J_0^+ \cup U_0^+$  and  $J_\lambda^+ \cup U_\lambda^+$ , respectively.*

In particular, when  $f_0$  is hyperbolic and unstably connected, this gives a canonical conjugacy between  $f_0$  and  $f_\lambda$  on all of  $\mathbb{C}^2$  except for the basins of any attracting periodic points.

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### 2. Preliminaries

We recall some standard terminology and some known results, which are discussed more fully in [BS1; BS2; BS5]. Friedland and Milnor [FrM] divided the polynomial automorphisms of  $\mathbb{C}^2$  into two classes: elementary (which have relatively simple dynamics) and nonelementary. For brevity, we will use the term ‘‘Hénon map’’ to describe a nonelementary polynomial automorphism of  $\mathbb{C}^2$ . Such maps can be characterized by having dynamical degree  $d \geq 2$ , where the dynamical degree of a polynomial automorphism of  $\mathbb{C}^2$  is defined as in [BS2] by

$$d = \lim_{n \rightarrow \infty} (\deg f^n)^{1/n};$$

here  $\deg f^n$  denotes the maximum of the degrees of the two (polynomial) components of  $f^n$ .

Given a Hénon map  $f$ , we let  $K^+/K^-$  denote the set of points in  $\mathbb{C}^2$  with bounded forward/backward orbits under  $f$ , and we let  $J^\pm = \partial K^\pm$  and  $J = J^+ \cap J^-$ . Since  $\det DF$  is constant on  $\mathbb{C}^2$ , we may replace  $f$  by  $f^{-1}$  if necessary to obtain  $|\det Df| \leq 1$ . From [BS1] and [BS2] it follows that, if  $f$  is hyperbolic when restricted to  $J$ , then  $f$  is Axiom A; in this case, the nonwandering set consists of the basic set  $J$  plus a finite set  $S$  of periodic sinks. The stable set of  $J$ ,  $W^s(J)$ , is  $J^+ = \partial K^+$ , and the interior of  $K^+$  consists of the basins of the sinks. The unstable set of  $J$ ,  $W^u(J)$ , is  $J^- \setminus S$ , and the interior of  $K^-$  is empty. The sets  $W^{s/u}(J)$  have dynamically defined Riemann surface laminations  $\mathcal{W}^{s/u}$ , whose leaves consist of stable/unstable manifolds of points in  $J$ . Each leaf of either lamination is conformally equivalent to  $\mathbb{C}$ . Also,  $J$  has local product structure, which means that there exist positive  $\delta$  and  $\varepsilon$  such that, if  $x, y \in J$  with  $\|x - y\| < \delta$ , then  $W_\varepsilon^s(x)$  and  $W_\varepsilon^u(y)$  intersect in a unique point that is contained in  $J$ . Here  $W_\varepsilon^s(x)$  is the local stable manifold of  $x$  defined as  $\{p : \|f^n(x) - f^n(p)\| < \varepsilon \forall n \geq 0\}$ , with an analogous definition for the local unstable manifold. As usual, we will use  $W^s(p)$  and  $W^u(p)$  for the stable and unstable manifolds of a point  $p$ .

Note that, if  $f_\lambda$  is a 1-parameter family of Hénon maps depending holomorphically on  $\lambda \in \Delta$  and if  $f_0$  is hyperbolic, then  $f_\lambda$  is also hyperbolic for all  $\lambda$  in some neighborhood of 0. Also, by [BS1],  $f_0$  is  $\Omega$ -stable, meaning that there is a 1-parameter family of homeomorphisms  $\psi_\lambda : J_0 \rightarrow J_\lambda$  conjugating  $f_0|_{J_0}$  to  $f_\lambda|_{J_\lambda}$ . In fact, by [J], for each  $p \in J_0$  the map  $\lambda \mapsto \psi_\lambda(p)$  is holomorphic in  $\lambda$ . Hence there is a natural holomorphic motion defined on  $J_0$ . Moreover, by restricting the domain of  $\lambda$  and possibly shrinking  $\delta$  and  $\varepsilon$ , we may assume that the  $\delta$  and  $\varepsilon$  chosen for the local product structure on  $J_0$  apply equally to  $J_\lambda$  for each  $\lambda$ . For the remainder of the paper, we let  $\delta_0$  and  $\varepsilon_0$  represent such a choice of  $\delta$  and  $\varepsilon$ .

### 3. Unstable Connectivity and Critical Points

For Theorem 1.2, we need also the notion of an unstably connected Hénon map. Let  $U^+ = \mathbb{C}^2 \setminus K^+$  be the set of points with unbounded forward orbit. Bedford and Smillie [BS4] defined a Hénon map to be unstably connected with respect to a

saddle point  $p$  if some component of  $W^u(p) \cap U^+$  is simply connected. By [BS4, Thm. 0.1], this is equivalent to the condition that, for *any* saddle periodic point  $p$ , *each* component of  $W^u(p) \cap U^+$  is simply connected, and in this case they say that  $f$  is *unstably connected*. By [BS4, Thm. 0.2], the assumption  $|\det Df| \leq 1$  implies that  $f$  is unstably connected if and only if  $J$  is connected. As mentioned earlier, if  $f$  is hyperbolic then  $f$  is  $\Omega$ -stable, so if  $f$  is hyperbolic with connected  $J$  then all nearby Hénon maps are also hyperbolic with connected  $J$ . Summarizing this argument, we have the following.

**PROPOSITION 3.1.** *Let  $f$  be a Hénon map of dynamical degree  $d$ , with  $|\det Df| \leq 1$ , and suppose that  $f$  is hyperbolic and unstably connected. Then there is a neighborhood  $\mathcal{U}$  of  $f$  in the space of Hénon maps of degree  $d$  such that each  $g \in \mathcal{U}$  is hyperbolic and unstably connected.*

As observed in [H] (see also [HO] and [BS1]), there is a plurisubharmonic function  $G^+$  on  $\mathbb{C}^2$  defined by

$$G^+(p) = \lim_{n \rightarrow \infty} \frac{1}{d^n} \log^+ \|f^n(p)\|,$$

and this function is pluriharmonic on  $U^+$  and satisfies  $G^+ \circ f(p) = d \cdot G^+(p)$  and  $G^+(x, y) = \log^+ |y| + O(1)$  for  $(x, y) \in V_R^+ = \{|y| > R, |x| < |y|\}$ ,  $R$  large. There is an analogous definition of  $G^-$  with  $f^{-n}$  in place of  $f^n$ . Since  $G^+$  is pluriharmonic on  $U^+$ , it is locally the real part of a holomorphic function. In fact, in [HO, Prop. 5.4] it is shown that  $G^+ = \text{Re } \log(\phi^+)$  in  $V^+$ , where  $\phi^+(x, y) = y + O(1)$ . Hence the level sets of  $\phi^+$  define a nondegenerate holomorphic foliation  $\mathcal{G}^+$  defined in  $V^+$ . Since  $U^+$  is the union of all backward images of  $V^+$  under  $f$ , and since  $f$  is a diffeomorphism, this foliation pulls back to give a holomorphic foliation  $\mathcal{G}^+$  on  $U^+$ .

Finally, let  $\mathcal{W}^s$  denote the lamination of  $J^+$  by stable manifolds of  $J$ . We restate here a proposition due to Bedford and Smillie to the effect that, if  $f$  is hyperbolic and unstably connected, then the foliation  $\mathcal{G}^+$  and the lamination  $\mathcal{W}^s$  fit together to form a lamination of  $J^+ \cup U^+$ .

**PROPOSITION 3.2** [BS4, Prop. 2.7]. *If  $f$  is hyperbolic and unstably connected, then there is a locally trivial lamination of  $J^+ \cup U^+$  whose leaves are the leaves of  $\mathcal{W}^s$  and  $\mathcal{G}^+$ .*

For polynomials of one complex variable, there is a close connection between connectivity of the Julia set and the behavior of critical points. In two variables, Bedford and Smillie [BS3] defined the set of *unstable critical points* of a Hénon map to be the union over points  $p \in J$  of the set of critical points of the Green function  $G^+$  restricted to  $W^u(p)$  (actually the union over all  $p$  for which the unstable manifold exists, which is a set of full  $\mu$ -measure, where  $\mu$  is the unique measure of maximal entropy). They showed also that such a critical point is exactly a point of tangency between an unstable manifold of a point in  $J$  and a leaf of the foliation  $\mathcal{G}^+$ .

In case  $f$  is hyperbolic and unstably connected, there are no tangencies between the leaves of the unstable set  $W^u(J)$  and the foliation  $\mathcal{G}^+$ ; equivalently, for each  $p \in J$ , the set  $W^u(p) \cap U^+$  contains no unstable critical points. This fact was used in the proof of [BS5, Cor. A2] but was not stated explicitly. Rather, Bedford and Smillie showed in [BS4, Thm. 7.3] that  $f$  is unstably connected if and only if, for  $\mu$  almost every point  $p$ ,  $W^u(p) \cap U^+$  contains no unstable critical points. For completeness, we provide here a proof of the stronger result when  $f$  is hyperbolic and unstably connected.

**PROPOSITION 3.3.** *Let  $f$  be hyperbolic. Then  $f$  is unstably connected if and only if, for each point  $p \in J$ ,  $W^u(p) \cap U^+$  has no unstable critical points if and only if, for each point  $p \in J$ ,  $W^u(p)$  is nowhere tangent to the leaves of the foliation  $\mathcal{G}^+$ .*

*Proof.* From [BS4, Thm. 7.3],  $f$  is unstably connected if and only if, for  $\mu$  almost every point  $p$ ,  $W^u(p) \cap U^+$  contains no unstable critical points; by [BS3, Prop. B.1], an unstable critical point in  $W^u(p) \cap U^+$  is exactly a tangency between  $W^u(p)$  and a leaf of the foliation  $\mathcal{G}^+$ . Thus, we need prove only that if  $f$  is unstably connected then, for each point  $p \in J$ ,  $W^u(p) \cap U^+$  has no unstable critical points.

That  $f$  is hyperbolic implies that  $W^u(p)$  exists for each  $p \in J$  and that the unstable set  $W^u(J)$  is a locally trivial lamination of  $J^-$ . Suppose there exists  $p \in J$  such that  $W^u(p)$  is tangent to a leaf of  $\mathcal{G}^+$ . Making a local biholomorphic change of coordinates in a neighborhood of the point of tangency, we may assume that the point of tangency is the origin in  $(z, w)$  coordinates, that  $\mathcal{G}^+$  has leaves that are complex lines parallel to the  $z$ -axis, and that  $W^u(p)$  is locally the graph of a holomorphic function  $z \mapsto z^k h(z)$  with  $h(0) \neq 0$  and  $k \geq 2$ . For any piece of a leaf of  $W^u(q)$  sufficiently near this graph, the derivative of the corresponding graph for  $W^u(q)$  will have a zero near the origin; hence there will be a tangency between  $W^u(q)$  and  $\mathcal{G}^+$ . Since each leaf of  $W^u(J)$  is dense in  $J^-$  [BS2] and since these leaves form a locally trivial lamination, we see that there is a tangency between  $W^u(p)$  and  $\mathcal{G}^+$  for each  $p \in J$ .

Thus, if  $f$  is hyperbolic then a tangency between  $W^u(p)$  and  $\mathcal{G}^+$  for one  $p \in J$  implies a tangency between  $W^u(q)$  and  $\mathcal{G}^+$  for all  $q \in J$ , and hence for a set of full  $\mu$  measure; therefore, as already noted,  $f$  is not unstably connected. Taking the contrapositive, if  $f$  is unstably connected then, for each  $p \in J$ , there is no tangency between  $W^u(p)$  and  $\mathcal{G}^+$  and hence there are no unstable critical points on  $W^u(p) \cap U^+$ . As noted previously, this completes the proof.  $\square$

#### 4. Holomorphic Families of Laminations

In this section we discuss some uniformization properties of Riemann surface laminations and of holomorphic families of such laminations. Roughly, the main result is that, given a holomorphic family of Riemann surface laminations in which each leaf is conformally equivalent to the complex plane and given two holomorphic transversals to these laminations, there is a natural way of parametrizing a given

leaf by the plane so that the parametrization of this leaf varies holomorphically with the family and so that the points of intersection of this leaf with the two transversals are the images of 0 and 1 under the parametrization. Moreover, locally, this parametrization can be done in such a way that the parametrization converges locally uniformly when approaching a limit leaf. Precise definitions and results are given shortly.

We first recall the definition of a Riemann surface lamination of a topological space  $X$ , following [BS4] (see also [C; G1; G2]). A *chart* consists of an open set  $U_j \subset X$ , a topological space  $Y_j$ , and a map  $\rho_j: U_j \rightarrow \mathbb{C} \times Y_j$  that is a homeomorphism onto its image. An *atlas* consists of a collection of charts that covers  $X$ . For fixed  $y \in Y_j$ , the set of points  $\rho_j^{-1}(\mathbb{C} \times \{y\})$  is called a *plaque*. For coordinate charts  $(\rho_i, U_i, Y_i)$  and  $(\rho_j, U_j, Y_j)$  with  $U_i \cap U_j \neq \emptyset$ , the *transition function* is the homeomorphism from  $\rho_j(U_i \cap U_j)$  to  $\rho_i(U_i \cap U_j)$  defined by  $\rho_{ij} = \rho_i \circ \rho_j^{-1}$ . A *Riemann surface lamination*,  $\mathcal{L}$ , of a topological space  $X$  is determined by an atlas of charts that satisfy the following consistency condition: the transition functions may be written in the form  $\rho_{ij} = (g(z, y), h(y))$ , where the function  $z \mapsto g(z, y)$  is holomorphic for fixed  $y \in Y_j$ . The condition on the transition functions gives a consistency between the plaques defined in  $U_j$  and those in  $U_i$ . Thus, plaques fit together to make global manifolds called *leaves* of the lamination, and each leaf has the structure of a Riemann surface.

In the current setting, we are interested in the Riemann surface laminations of  $J^+$  and  $J^-$  given by stable and unstable manifolds and in the lamination of  $U^+$  given by the foliation  $\mathcal{G}^+$ . Since these leaves have a natural holomorphic structure induced from  $\mathbb{C}^2$ , we will require additionally that each map  $\rho_j$  be holomorphic on each plaque. With this additional requirement, we can view a lamination of  $X$  as locally the “graph” of a holomorphic motion, as follows. At a point  $p \in X$ , let  $\mathbf{v}$  be a vector in  $\mathbb{C}^2$  such that  $T = \mathbb{C}\mathbf{v}$  is a complex line transverse to the plaque through  $p$ . After a biholomorphic change of coordinates, we may assume that  $p$  is the origin and that  $\mathbf{v} = (0, 1)$ . Let  $V$  be a small neighborhood of  $p$ , and let  $E$  be the set of points in  $V$  that lie on  $T$ . Then the plaques in  $\mathcal{L}$  near the origin define a holomorphic motion with parameter  $z$ . In other words, there is a function  $\alpha(z, w)$  defined for  $(z, w) \in \Delta_\varepsilon \times E$  that is holomorphic in  $z$  for each fixed  $w \in E$  such that  $\alpha(0, w) = w$ ,  $\alpha(z, \cdot)$  is injective for each  $z$ , and a plaque of  $\mathcal{L}$  through the point  $(0, w)$  is given by the set of points  $(z, \alpha(z, w))$ ,  $z \in \Delta_\varepsilon$ . Moreover, there is a coherence property corresponding to the consistency requirement on the transition functions given previously. In the current setting, the map  $H(z, w) = (z, \alpha(z, w))$  is a homeomorphism from  $\Delta_\varepsilon \times E$  to an open set  $U \subset X$  that is holomorphic for each fixed  $w \in E$ . Given a second point  $\hat{p}$  and given  $\hat{H}: \Delta_\varepsilon \times \hat{E} \rightarrow \hat{U}$  with  $U \cap \hat{U} \neq \emptyset$ , we have a transition function  $H^{-1} \circ \hat{H}$ , which can be written in the form  $H^{-1} \circ \hat{H}(z, w) = (g(z, w), h(w))$ , where the map  $z \mapsto g(z, w)$  is holomorphic for fixed  $w$ .

A holomorphic family of laminations is a generalization in which each plaque varies holomorphically with some parameter  $\lambda \in \Delta_\varepsilon^n$ . For this purpose, we will restrict ourselves to families of laminations of sets in  $\mathbb{C}^2$ , and we will adopt the holomorphic motion view of laminations. So we say that  $\mathcal{L}_\lambda$  is a holomorphic

family of laminations depending on the parameter  $\lambda \in \Delta_r^n$  if, for each fixed  $\lambda$ ,  $\mathcal{L}_\lambda$  is a lamination of a set  $X_\lambda$  in  $\mathbb{C}^2$  such that each plaque is a Riemann surface as before and such that each plaque depends holomorphically on  $\lambda$  in the following sense. Again, for each point  $p \in X_{\lambda_0}$  there is a local biholomorphic change of coordinates such that the image of  $p$  is the origin and  $\mathbf{v} = (0, 1)$  is transverse to the plaque of  $\mathcal{L}_{\lambda_0}$  through the origin. Let  $E$  be the intersection of  $T = \mathbb{C}\mathbf{v}$  and a small neighborhood of  $p$  in  $X_{\lambda_0}$ . Then we require  $\varepsilon > 0$  and the existence of a function  $\alpha(z, w, \lambda)$  defined on  $\Delta_\varepsilon \times E \times \Delta_\varepsilon^n(\lambda_0)$  that is holomorphic in  $(z, \lambda)$  for each fixed  $w$  and such that (a)  $\alpha(0, w, \lambda_0) = w$ ,  $\alpha(z, \cdot, \lambda)$  is injective for each fixed  $(z, \lambda)$ , (b) the point  $(0, \alpha(0, w, \lambda))$  is contained in  $X_\lambda$  for each  $\lambda \in \Delta_\varepsilon^n$ , and (c) for each  $\lambda \in \Delta_\varepsilon^n$ , the plaque of  $\mathcal{L}_\lambda$  through  $(0, \alpha(0, w, \lambda))$  is given by the set of points  $(z, \alpha(z, w, \lambda))$ ,  $z \in \Delta_\varepsilon$ . That is,  $\alpha$  is a holomorphic motion of points  $w \in E$  with parameters  $(z, \lambda) \in \Delta_\varepsilon \times \Delta_\varepsilon^n(\lambda_0)$ .

We will need a coherence condition on families of laminations also. We can view the family  $\mathcal{L}_\lambda$  as sitting in  $\mathbb{C}^2 \times \Delta_r^n$ . Given a point  $p \in X_{\lambda_0}$  and local change of coordinates as before, we require that the map  $H(z, w, \lambda) = (z, \alpha(z, w, \lambda), \lambda)$  be a homeomorphism from  $\Delta_\varepsilon \times E \times \Delta_\varepsilon^n(\lambda_0)$  to an open set  $U$  in  $\bigcup_\lambda (X_\lambda \times \{\lambda\})$ . Moreover, given a second point  $\hat{p} \in X_{\hat{\lambda}_0}$  with  $\hat{H}: \Delta_\varepsilon \times \hat{E} \times \Delta_\varepsilon^n(\hat{\lambda}_0) \rightarrow \hat{U}$ , we require that the transition function  $H^{-1} \circ \hat{H}$  can be written in the form  $H^{-1} \circ \hat{H}(z, w, \lambda) = (g(z, w, \lambda), h(w), \lambda)$ , where, for fixed  $w$ , the map  $(z, \lambda) \mapsto g(z, w, \lambda)$  is holomorphic in  $z$  and  $\lambda$ .

Observe that the set  $\{(z, \alpha(z, w, \lambda), \lambda) : z \in \Delta_\varepsilon, \lambda \in \Delta_\varepsilon^n(\lambda_0)\}$  is an  $(n + 1)$ -dimensional holomorphic submanifold of  $\mathbb{C}^{n+2}$ . Hence the plaque of  $\mathcal{L}_{\lambda_0}$  through  $p$  can be said to vary holomorphically with  $\lambda$  by viewing it as a slice of this submanifold. We call this submanifold a *family* of plaques associated with  $p$ . Each plaque in this family is associated with a unique leaf in the corresponding lamination  $\mathcal{L}_\lambda$ , so we may speak also of the family of leaves associated with  $p$ . We will see shortly that, in the cases of interest for Hénon maps, the family of leaves through  $p$  is biholomorphic to  $\mathbb{C} \times \Delta_r^n$ .

The following is an immediate consequence of the implicit function theorem and the definitions already given. It says essentially that a point of transverse intersection between a holomorphic family of curves and a holomorphic family of plaques associated with a point varies holomorphically with the parameter.

LEMMA 4.1. *Let  $\mathcal{L}_\lambda$  be a holomorphic family of laminations, let  $P_\lambda$  be the family of plaques associated with a point  $p \in L_0$ , and let  $F: \Delta \times \Delta^n \rightarrow \mathbb{C}^2$  be holomorphic such that  $F(0, 0) = p$  and such that, for each fixed  $\lambda$ ,  $F(\cdot, \lambda)$  is an injective immersion that is transverse to  $P_\lambda$ . Then there exist an  $\varepsilon > 0$  and a holomorphic function  $p: \Delta_\varepsilon^n \rightarrow \mathbb{C}^2$  such that  $p(0) = p_0$  and  $p(\lambda) \in P_\lambda \cap F(\Delta, \lambda)$  for all  $\lambda \in \Delta_\varepsilon^n$ .*

Note that if the point  $p(\lambda)$  does not escape out the boundary of the image of  $F$  or the boundary of a plaque  $P_\lambda$ , then by the monodromy theorem  $p(\lambda)$  may be analytically continued to all of  $\Delta^n$ .



### 5. Stable Manifolds and Affine Structures

Let  $f_\lambda$  be a 1-parameter family of hyperbolic Hénon maps and recall from Section 2 that there is a homeomorphism  $\psi_\lambda$  from  $J_0$  to  $J_\lambda$  which is holomorphic in  $\lambda$  and which conjugates  $f_0|_{J_0}$  to  $f_\lambda|_{J_\lambda}$ . Given a point  $p_0$  in  $J_0$ , let  $p_\lambda$  be its image under  $\psi_\lambda$ , and let  $W^{s/u}(p_\lambda)$  be the corresponding stable and unstable manifolds. In this section we show that the stable (and unstable) manifolds of  $p_\lambda$  can be parametrized by  $\mathbb{C}$  in a way that depends holomorphically on  $\lambda$  and so that the parametrizations of nearby leaves converge locally uniformly to the parametrization of the family of leaves through  $p_\lambda$ .

Let  $S_\lambda$  denote the set of sink orbits for  $f_\lambda$ , and let  $\mathcal{W}_\lambda^u$  denote the lamination of  $J_\lambda^- \setminus S_\lambda$ . Given  $p \in J_\lambda^-$ , write  $L_\lambda(p)$  for the leaf of the lamination  $\mathcal{W}_\lambda^u$  containing  $p$ .

As in [G1; G2; BS5], we define an affine structure on a holomorphic curve  $L$  to be an atlas consisting of holomorphic diffeomorphisms  $\chi_j$  from open sets  $U_j$  of  $L$  to open sets of  $\mathbb{C}$  such that the  $U_j$  cover  $L$  and the  $\chi_j \circ \chi_k^{-1}$  are restrictions of affine diffeomorphisms of  $\mathbb{C}$  to their domains of definition. For three distinct points  $x, y, z$  in  $\mathbb{C}$ , the ratio  $(x - y)/(x - z)$  is invariant under the group of affine diffeomorphisms of  $\mathbb{C}$ . If  $x, y, z$  are distinct nearby points of  $U_j$ , then the ratio  $(\chi_j(x) - \chi_j(y))/(\chi_j(x) - \chi_j(z))$  depends only on the points  $x, y, z$ , not on the particular coordinate chart  $\chi_j$  whose domain contains  $x, y, z$ . Hence we may denote this function by  $(x - y)/(x - z)$ , which is holomorphic in  $x, y, z$  and which in fact is holomorphic as a map into  $\mathbb{P}^1$  whenever  $x, y, z$  are not all equal. An affine structure on a simply connected Riemann surface is said to be *complete* if it is isomorphic to  $\mathbb{C}$  with its canonical affine structure.

If  $f_0$  is hyperbolic, then for each  $p_0 \in J_0$  there is an injective holomorphic map from  $\mathbb{C}$  to the unstable manifold of  $p_0$ , and this map defines a complete affine structure on this unstable manifold. Moreover, the iterates of  $f_0$  respect this affine structure in the sense that the pull-back or push-forward of the affine structure from one leaf to another agrees with the original affine structure on the new leaf.

Let  $\varepsilon = \varepsilon_0$  be as chosen for the local product structure in Section 2, and fix  $x_0 \in J_0^-$ . Choose disjoint transversals  $T_1, T_2$  to the local unstable manifold  $W_\varepsilon^u(x_0)$ , and let  $T_3$  be any other transversal to this local unstable manifold. For  $x \in J_0^-$  near  $x_0$ , there are three points  $p_j(x) = T_j \cap W_\varepsilon^u(x)$ ,  $j = 1, 2, 3$ , and  $p_1, p_2$  are distinct. The ratio  $(p_1 - p_3)/(p_1 - p_2)$  is well-defined, independently of any particular choice of complex affine coordinate on  $W^u(x)$ . To say that the affine structure is *continuous* is to say that this ratio varies continuously with  $x$ , and [BS5, Prop. 5.1] implies that the affine structure on  $\mathcal{W}^u$  is continuous. In fact, the following theorem of Ghys implies a stronger continuity property.

**THEOREM 5.1** [G2]. *Let  $\mathcal{L}$  be a Riemann surface lamination of a subset  $X$  of a complex manifold such that each leaf of  $\mathcal{L}$  is parabolic (conformally equivalent to the plane). Then the affine structure on leaves is continuous in the following sense. Let  $U$  be a chart of  $\mathcal{L}$  and, for each  $i \geq 0$ , let  $x_i, y_i, z_i$  be a triple of distinct*

points in  $U$  that, for each fixed  $i$ , are all three contained in the same plaque of  $\mathcal{L}$ . Suppose also that  $(x_i, y_i, z_i)$  converges to distinct points  $(x_\infty, y_\infty, z_\infty)$  in  $U$ . Then the ratio  $(x_i - y_i)/(x_i - z_i)$  converges to  $(x_\infty - y_\infty)/(x_\infty - z_\infty)$ .

In [G2], the laminated space is assumed to be compact. However, the compactness is used only to deduce that the conformal type of each leaf is independent of the Riemannian metric on the space. In the current setting, each leaf is parabolic using the standard metric on  $\mathbb{C}^2$ , so we may dispense with compactness.

We use the continuity of the affine structure to construct holomorphic parametrizations of leaves that converge locally uniformly when approaching a limit leaf. The essential idea is to choose a limit leaf along with two transversals to this leaf. Nearby leaves will also intersect these transversals, and we can choose the parametrization of leaves by the plane so that the images of 0 and 1 lie on these transversals. The continuity of the affine structure gives the local uniform convergence almost immediately. Note that we take a very myopic view when parametrizing leaves. In practice, one leaf will come back and accumulate on itself everywhere. For purposes of the parametrization, we work locally and regard each plaque as part of a separate leaf with its own parametrization. Thus one leaf may have many different parametrizations, any two of which differ by an affine transformation.

For the following proposition, let  $\mathcal{L}$  be a lamination of a closed subset  $X$  of  $\mathbb{C}^2$  such that each leaf  $L$  of  $\mathcal{L}$  is parabolic. Also, let  $U$  be a chart of  $\mathcal{L}$ , and let  $I = \mathbb{Z}^+ \cup \{\infty\}$ .

**PROPOSITION 5.2.** *Let  $x_i, y_i \in U$  for  $i \in I$  with  $x_i \rightarrow x_\infty$  and  $y_i \rightarrow y_\infty$  ( $x_\infty \neq y_\infty$ ) and such that, for each  $i \in I$ ,  $x_i$  and  $y_i$  are contained in the same leaf  $L_i$  of  $\mathcal{L}$  and in the same plaque within  $U$ . Let  $\phi_i: \mathbb{C} \rightarrow L_i$  be injective holomorphic for  $i \in I$  with  $\phi_i^{-1}(x_i) \rightarrow \phi_\infty^{-1}(x_\infty)$  and  $\phi_i^{-1}(y_i) \rightarrow \phi_\infty^{-1}(y_\infty)$ . Then  $\phi_i \rightarrow \phi_\infty$  uniformly on each compact subset of  $\mathbb{C}$ .*

*Proof.* Let  $P_i$  be the plaque of  $U$  containing  $x_i, y_i$ . We will show first that  $\phi_i$  converges to  $\phi_\infty$  uniformly on each compact subset of  $\phi_\infty^{-1}(P_\infty) \subset \mathbb{C}$ . By assumption on  $U$ , there exists a biholomorphic change of coordinates such that  $P_\infty$  is an open set in the  $z$ -axis ( $\mathbb{C} \times \{0\}$ ). By restricting to sufficiently large  $i$ , we may assume that the projection  $\pi_i: P_i \rightarrow (\mathbb{C} \times \{0\})$  is injective holomorphic for each  $i$  (and that  $\pi_\infty = \text{Id}$ ). Moreover,  $\pi_i^{-1}\pi_\infty$  converges to the identity uniformly on compact subsets of  $P_\infty$  as  $i \rightarrow \infty$  (e.g., by the  $\lambda$ -lemma of [MSS]).

Let  $\gamma \in \pi_\infty(P_\infty)$  be a simple closed curve with  $x_\infty, y_\infty \notin \pi_\infty^{-1}(\gamma)$ , and let  $N_\gamma = \cup_{i \in I} \pi_i^{-1}(\gamma)$ . Then  $N_\gamma$  is compact and  $x_i, y_i \notin N_\gamma$  for  $i$  large. Define  $R_i(p)$  on  $P_i$  ( $i \in I$ ) by

$$R_i(p) = \frac{\phi_i^{-1}(x_i) - \phi_i^{-1}(p)}{\phi_i^{-1}(x_i) - \phi_i^{-1}(y_i)}.$$

Since  $x_\infty \neq y_\infty$  and since the preimages of  $x_i$  and  $y_i$  converge (respectively) to the preimages of  $x_\infty$  and  $y_\infty$ , we see that, for large  $i$ ,  $R_i$  is well-defined and holomorphic on  $P_i$ . Moreover,  $R_i(p)$  is precisely the ratio function applied to the triple

$(x_i, y_i, p)$ . Viewing  $R_i(p) = R(i, p)$  as a function on the compact set  $N_\gamma$ , the theorem of Ghys implies that  $R$  is continuous on  $N_\gamma$  and hence uniformly continuous. In particular,  $\phi_i^{-1} \circ \pi_i^{-1} \rightarrow \phi_\infty^{-1} \circ \pi_\infty^{-1}$  uniformly on  $\gamma$ , hence on the interior of  $\gamma$  by Cauchy's formula, hence on each compact subset of  $P_\infty$ .

Thus  $(\pi_i \circ \phi_i)^{-1} \rightarrow (\pi_\infty \circ \phi_\infty)^{-1}$  uniformly on compact subsets of  $\pi_\infty(P_\infty)$ . Since  $\pi_\infty \circ \phi_\infty$  is injective holomorphic, this implies that  $\pi_i \circ \phi_i$  converges to  $\pi_\infty \circ \phi_\infty$  uniformly on compact subsets of  $\phi_\infty^{-1}(P_\infty)$  (e.g., by the integral formula for the inverse of a holomorphic map). Since  $\pi_i^{-1} \circ \pi_\infty$  converges to the identity uniformly on compact subsets of  $P_\infty$ , this implies that  $\phi_i$  converges to  $\phi_\infty$  uniformly on compact subsets of  $\phi_\infty^{-1}(P_\infty)$ .

To complete the proof, let  $K \subset \mathbb{C}$  be compact, and cover  $\phi_\infty(K)$  by finitely many plaques  $P_{\infty,1}, \dots, P_{\infty,m}$  with  $P_{\infty,j} \cap P_{\infty,j+1} \neq \emptyset$  for  $j = 1, \dots, m - 1$  and  $P_{\infty,1} = P_\infty$ . The preceding construction implies that  $\phi_i$  converges to  $\phi_\infty$  uniformly on compact subsets of  $\phi_\infty^{-1}(P_{\infty,1})$ . Since  $P_{\infty,1}$  and  $P_{\infty,2}$  are open and have nonempty intersection, we can apply the same argument to two new sequences of points with limits in their intersection to conclude that  $\phi_i$  converges to  $\phi_\infty$  uniformly on compact subsets of  $\phi_\infty^{-1}(P_{\infty,2})$ . By induction, we obtain uniform convergence on all of  $K$ . □

In dealing with families of Hénon maps, we will need a parametrized version of the foregoing result. We begin with a definition.

**DEFINITION 5.3.** Let  $\mathcal{L}_\lambda$  ( $\lambda \in \Delta^n$ ) be a holomorphic family of laminations. We say that  $\mathcal{L}_\lambda$  is *leafwise trivial* if, for each leaf  $L_{\lambda,0}$ , there exists some  $\varepsilon > 0$  such that the set  $Z := \{(\lambda, p) : \lambda \in \Delta_\varepsilon^n(\lambda_0), p \in L_\lambda\}$  is biholomorphic to  $\Delta_\varepsilon^n \times \mathbb{C}$ .

As an example of how a holomorphic family of leaves could fail to be trivial in this sense, consider a  $\mathbb{P}^1$  bundle over  $\Delta^n$ ; then remove a section over  $\Delta^n$  that is not holomorphic. Then each leaf is biholomorphic to  $\mathbb{C}$ , but the bundle is not biholomorphic to  $\Delta^n \times \mathbb{C}$ .

In the following theorem,  $I = \mathbb{Z}^+ \cup \{\infty\}$ , as before.

**THEOREM 5.4.** Let  $\mathcal{L}_\lambda$  ( $\lambda \in \Delta^n$ ) be a leafwise trivial holomorphic family of laminations. Let  $x_i(\lambda)$  and  $y_i(\lambda)$  ( $i \in I$ ) be holomorphic in  $\lambda$  with  $x_i(\lambda) \neq y_i(\lambda)$  for each  $i$  and  $\lambda$  and such that, for all  $\lambda$ ,  $y_i(\lambda)$  is contained in the plaque through  $x_i(\lambda)$ . Suppose also that  $x_i(\lambda)$  converges to  $x_\infty(\lambda)$  and  $y_i(\lambda)$  converges to  $y_\infty(\lambda)$  uniformly on compact subsets of  $\Delta^n$  as  $i \rightarrow \infty$ . Let  $L_{i,\lambda}$  be the leaf through  $x_i(\lambda)$ , and let  $\phi_{i,\lambda}: \mathbb{C} \rightarrow L_{i,\lambda}$  be injective holomorphic with  $\phi_{i,\lambda}(0) = x_i(\lambda)$  and  $\phi_{i,\lambda}(1) = y_i(\lambda)$ .

Then  $\phi_i(\lambda, z) = \phi_{i,\lambda}(z)$  is holomorphic in  $(\lambda, z)$ , and  $\phi_i$  converges to  $\phi_\infty$  uniformly on compact subsets of  $(z, \lambda) \in \mathbb{C} \times \Delta^n$ .

*Proof.* Since  $\mathcal{L}_\lambda$  is leafwise trivial, it is a locally trivial fibration over  $\Delta^n$  and hence is biholomorphic to  $\Delta^n \times \mathbb{C}$  by [W, Lemma 4.4]. Consequently, there exist injective holomorphic maps  $\Phi_{i,\lambda}: \mathbb{C} \rightarrow L_{i,\lambda}$  such that  $\Phi_{i,\lambda}(z)$  is holomorphic in  $(\lambda, z) \in \Delta^n \times \mathbb{C}$ .

Since  $x_i(\lambda)$  and  $y_i(\lambda)$  are holomorphic in  $\lambda$ , we see that  $X_i(\lambda) := \Phi_{i,\lambda}^{-1}(x_i(\lambda))$  and  $Y_i(\lambda) := \Phi_{i,\lambda}^{-1}(y_i(\lambda))$  are holomorphic from  $\Delta^n$  to  $\mathbb{C}$ ; by the injectivity of  $\Phi_{i,\lambda}$ , we then have  $X_i(\lambda) \neq Y_i(\lambda)$ . Since injective maps from the plane to itself are unique up to affine map, we see that  $\phi_{i,\lambda}(z) = \Phi_{i,\lambda}(X_i(\lambda) + z(Y_i(\lambda) - X_i(\lambda)))$  is holomorphic in  $(\lambda, z)$ , as desired.

Finally, the uniform convergence of  $\phi_i$  to  $\phi_\infty$  follows almost exactly as in the proof of Proposition 5.2, using the function  $R_{i,\lambda}$  given by the formula for  $R_i$  with  $\phi_{i,\lambda}^{-1}$  in place of  $\phi_i^{-1}$ . □

Next, we show that the leaves of the dynamical laminations generated by a hyperbolic Hénon map are leafwise trivial holomorphic families of laminations.

**THEOREM 5.5.** *Let  $f_\lambda$  be a family of hyperbolic Hénon maps depending holomorphically on  $\lambda \in \Delta^n$ , and let  $\mathcal{W}_\lambda^u$  be the lamination of  $J_\lambda^-$  whose leaves are the unstable manifolds of  $J_\lambda$ . Then  $\mathcal{W}_\lambda^u$  is a leafwise trivial holomorphic family of laminations; likewise,  $\mathcal{W}_\lambda^s$  is a leafwise trivial holomorphic family of laminations.*

*Moreover, if each  $f_\lambda$  is unstably connected and if  $\mathcal{L}_\lambda = \mathcal{W}_\lambda^s \cup \mathcal{G}_\lambda^+$ , then  $\mathcal{L}_\lambda$  also is a leafwise trivial holomorphic family of laminations.*

*Proof.* The proof of the (un)stable manifold theorem for hyperbolic sets as in [SFLC, Chap. 6] relies on a contraction mapping argument applied to a Banach space of bounded sections over  $J_\lambda$ . Starting with initial approximations to the unstable manifolds that vary holomorphically with  $\lambda$ , the uniform convergence obtained from the contraction implies that the unstable manifolds for  $J_\lambda$  will vary holomorphically with  $\lambda$  in the sense that the family of leaves associated with a point varies holomorphically with  $\lambda$ . Thus,  $\mathcal{L}_\lambda$  is a holomorphic family of laminations.

For the leafwise triviality, [BS1, Thm. 5.4] implies that, for  $x_{\lambda_0} \in J_{\lambda_0}$ , we can exhaust  $W^u(x_{\lambda_0})$  by an increasing union of disks. Since the family of leaves  $L_\lambda$  associated with  $x_{\lambda_0}$  varies holomorphically with  $\lambda$ , the same argument implies that there exist an  $\varepsilon > 0$  and injective holomorphic maps  $H_j: \Delta \times \Delta_\varepsilon^n(\lambda_0) \rightarrow Z$ , where  $Z$  is the manifold of leaves associated with  $x_{\lambda_0}$  as in Definition 5.3, such that the image of  $H_j$  is contained in the image of  $H_{j+1}$  and such that the union of their images is all of  $Z$ . Because each leaf is conformally equivalent to  $\mathbb{C}$ , [FS] implies that  $Z$  is biholomorphic to  $\mathbb{C} \times \Delta_\varepsilon^n$ , so  $\mathcal{L}_\lambda$  is leafwise trivial.

Finally, suppose  $f_\lambda$  is unstably connected for all  $\lambda$ . The function  $G_\lambda^+(p)$  is pluriharmonic in  $(\lambda, p)$  by [BS1, Prop. 3.3] and hence is locally the real part of a function  $\Psi$  that is holomorphic in  $(\lambda, p)$ . Then the plaques of  $\mathcal{G}_\lambda^+$  are precisely the level sets of  $\Psi(\lambda, \cdot)$ ; hence these plaques vary holomorphically in  $\lambda$ , so  $\mathcal{L}_\lambda$  is a holomorphic family of laminations. The fact that  $\mathcal{L}_\lambda$  is leafwise trivial in this case follows as before, using the ideas in the proof of [HO, Thm. 7.2] to produce the increasing sequence of biholomorphic images of bi-disks. □

Collecting the results of this section, we obtain the following result, which allows us to parametrize leaves of  $\mathcal{W}_\lambda^u$  and  $\mathcal{W}_\lambda^s$  holomorphically in  $\lambda$  so that the parametrizations converge locally uniformly when approaching a limit leaf. For this proposition, let  $\varepsilon = \varepsilon_0$  be as chosen for local product structure. Moreover, if

necessary we may shrink this  $\varepsilon$  so that, at each point of  $J_\lambda$ , the bi-disk of size  $2\varepsilon$  with axes parallel to the stable and unstable directions at this point defines a chart for the stable and unstable laminations.

**THEOREM 5.6.** *Let  $f_\lambda$  be a family of hyperbolic Hénon maps depending holomorphically on  $\lambda \in \overline{\Delta}^n$ . Let  $p \in J_0$  and  $q \in J_0 \cap W_\varepsilon^s(p)$  with  $q \neq p$ , and let  $p_\lambda = \psi_\lambda(p)$  and  $q_\lambda = \psi_\lambda(q)$ . Then there exists a  $\phi_\lambda: \mathbb{C} \rightarrow \mathbb{C}^2$ , injective for each fixed  $\lambda$  and holomorphic in  $(z, \lambda) \in \mathbb{C} \times \overline{\Delta}^n$ , such that  $\phi_\lambda(\mathbb{C}) = W^s(p_\lambda)$ ,  $\phi_\lambda(0) = p_\lambda$ , and  $\phi_\lambda(1) = q_\lambda$ . Moreover, if  $p^j \in J_0$  with  $p^j \rightarrow p$  and if  $q^j \in J_0 \cap W_\varepsilon^s(p^j)$  with  $q^j \rightarrow q$  and if  $\phi^j$  is the corresponding parametrization for each  $j$ , then  $\phi_\lambda^j$  converges to  $\phi_\lambda$  uniformly on compact subsets of  $\mathbb{C} \times \overline{\Delta}^n$ . There is an analogous result for  $W^u(p_\lambda)$ .*

*Proof.* By Theorem 5.5,  $\mathcal{W}_\lambda^u$  is a leafwise trivial family of laminations. As a result, Theorem 5.4 applies to give  $\phi_\lambda$  with the stated properties and shows that if  $p_\lambda^j$  and  $q_\lambda^j$  converge uniformly on compacts to  $p_\lambda$  and  $q_\lambda$ , respectively, then  $\phi_\lambda^j$  converges uniformly on compacts to  $\phi_\lambda$ . Hence it suffices to show the uniform convergence of  $p_\lambda^j$  and  $q_\lambda^j$  to  $p_\lambda$  and  $q_\lambda$ .

To do this, define holomorphic maps  $h_j(\lambda) = p_\lambda^j$  and  $h(\lambda) = p_\lambda$ , where  $p_\lambda^j = \psi_\lambda(p^j)$ . Note that, since we have restricted to  $\lambda$  in the closed polydisk  $\overline{\Delta}^n$ , the filtration argument in [BS1] implies that there exists some  $R > 0$  such that  $J_\lambda$  is contained in  $\Delta_R^2$  independently of  $\lambda$ . In particular,  $h_j$  is uniformly bounded by  $R$ , independently of  $\lambda$  and  $j$ . Note also that, for each fixed  $\lambda$ ,  $\psi_\lambda$  is a homeomorphism; since  $p^j \rightarrow p$ , we have  $\psi_\lambda(p^j) \rightarrow \psi_\lambda(p)$  for each fixed  $\lambda$ . Hence  $\{h_j\}_j$  is a uniformly bounded sequence of holomorphic maps that converges pointwise to  $h$ . Because the sequence is uniformly bounded it is also equicontinuous, and this plus pointwise convergence implies uniform convergence. Thus  $p_\lambda^j$  converges uniformly on compacts to  $p_\lambda$  and likewise for  $q_\lambda^j$ , which (as noted previously) implies the convergence of  $\phi_\lambda^j$  to  $\phi_\lambda$ . □

We need an analogous parametrization for leaves of  $\mathcal{G}_\lambda^+$  in the unstably connected case. Since  $\psi_\lambda$  is not defined outside  $J_0$ , we will have to work a bit harder. Our next theorem will allow us to extend  $\psi_\lambda$  to  $U_0^+ \cap J_0^-$ .

**THEOREM 5.7.** *Let  $f_\lambda$  be a family of unstably connected hyperbolic Hénon maps depending holomorphically on  $\lambda \in \overline{\Delta}^n$ . Let  $p \in (J_0^+ \cup U_0^+) \cap J_0^-$ . Let  $L_\lambda^+$  be the family of leaves of  $\mathcal{G}_\lambda^+ \cup \mathcal{W}_\lambda^s$  through  $p$ , and let  $L_\lambda^-$  be the family of leaves of  $\mathcal{W}_\lambda^u$  through  $p$ . Then there exists a unique map  $\lambda \rightarrow p_\lambda \in \mathbb{C}^2$ , bounded and holomorphic in  $\lambda \in \overline{\Delta}^n$ , such that  $p_0 = p$  and  $p_\lambda \in L_\lambda^- \cap L_\lambda^+$  for each  $\lambda$ .*

*Moreover, if  $p^j \in (J_0^+ \cup U_0^+) \cap J_0^-$  and  $p^j \rightarrow p$ , then  $p_\lambda^j$  converges to  $p_\lambda$  uniformly on  $\overline{\Delta}^n$ .*

*Proof.* We first construct  $p_\lambda$ . For this purpose, if  $p \in J_0$  then  $p_\lambda = \psi_\lambda(p)$  satisfies the conclusions; hence we assume  $p \in U_0^+$ . Choose a chart containing  $p$  for the family of laminations  $\mathcal{G}_\lambda^+$ ,  $\lambda \in \Delta_\varepsilon^n$ , and let  $P_\lambda^+$  be the family of plaques through  $p$ . Likewise, let  $P_\lambda^-$  be the family of plaques of  $\mathcal{W}_\lambda^u$  through  $p$ . Since  $f_0$  is hyperbolic

and unstably connected, Lemma 4.1 implies that  $p_\lambda$  is defined uniquely for  $\lambda$  near 0 as the intersection of  $P_\lambda^+$  and  $P_\lambda^-$ .

Note that, by definition of the lamination  $\mathcal{G}_\lambda^+$ , the function  $G_\lambda^+(p_\lambda)$  is constant. Note also that, since  $\lambda$  is restricted to the closed polydisk in the hypothesis of the lemma, it follows from [BS1] that there exists an  $R > 0$  independent of  $\lambda$  such that  $J_\lambda^-$  is contained in  $\Delta_R^2 \cap V_R^+$  and such that, for a given constant  $C$ , the intersection of  $\Delta_R^2 \cap V_R^+$  with the level set  $\{G_\lambda^+(x, y) = C\}$  is contained in  $\{|y| < R'\}$  for some  $R' > 0$  independent of  $\lambda$ . Hence, replacing  $R$  by the max of  $R$  and  $R'$ , we have that  $p_\lambda$  is contained in  $\Delta_R^2$ , and this will remain true if we continue  $p_\lambda$  within the intersection of  $J_\lambda^-$  and the same level set of  $G_\lambda^+$ .

We now continue  $p_\lambda$  throughout  $\overline{\Delta^n}$ . Suppose that  $\gamma$  is any closed curve from  $[0, 1]$  to  $\overline{\Delta^n}$  and suppose that  $p_\lambda$  is defined and holomorphic at each point  $\lambda \in \gamma((0, 1))$ . Since  $p_\lambda$  is uniformly bounded, we can take a sequence  $t_j \in [0, 1]$  with  $t_j$  increasing to 1 and such that, for  $\lambda_j = \gamma(t_j)$ , the points  $p_{\lambda_j}$  converge to some point  $q$ . Let  $\lambda_0 = \gamma(1)$ . Since  $p_\lambda \in J_\lambda^-$  for all  $\lambda$  and since the union over  $\lambda \in \overline{\Delta^n}$  of  $J_\lambda^- \times \{\lambda\}$  is closed as a subset of  $\mathbb{C}^2 \times \overline{\Delta^n}$ , we have  $q \in J_{\lambda_0}^-$ . Also, since  $G_\lambda^+(p_\lambda)$  is a constant  $C > 0$ , we have  $G_\lambda^+(q) = C$  and hence  $q \in U_{\lambda_0}^+$ . In particular,  $q$  is the point of intersection of plaques of the corresponding laminations and hence has an extension  $q_\lambda$  as above for  $\lambda$  in some neighborhood of  $\lambda_0$ .

Note that if  $q_\lambda = p_\lambda$  at some point  $\lambda$  in their set of common definition, then the local unique extension in terms of intersecting plaques implies that they agree on an open set and hence everywhere they are both defined. Thus  $q_\lambda$  will be a continuation of  $p_\lambda$  once we show that they agree at one point.

In a neighborhood of  $q_{\lambda_0}$ , let  $\Psi_\lambda(x, y)$  be holomorphic in  $(\lambda, x, y)$  with  $\text{Re } \Psi_\lambda(x, y) = G_\lambda^+(x, y)$ . Then the level sets of  $\Psi_\lambda$  define the lamination  $\mathcal{G}_\lambda^+$ , so  $\Psi_{\lambda_j}(p_{\lambda_j})$  is a constant  $C$  independent of  $j$  and hence is equal to  $\Psi_{\lambda_0}(q)$ . In a neighborhood of  $q$  and for  $\lambda$  near  $\lambda_0$ , there is a fixed complex line independent of  $\lambda$  through  $q$  such that the projection of the level set  $\{\Psi_\lambda = C\}$  to this line is injective holomorphic. Moreover, the points of intersection of  $J_\lambda^-$  with this level set define a holomorphic motion via projection to this complex line. Because  $J_\lambda^-$  intersects the set  $\{\Psi_\lambda = C\}$  transversally for all  $\lambda$  near  $\lambda_0$ , we can choose a small neighborhood  $Y$  of  $q$  and then restrict  $\lambda$  to a sufficiently small neighborhood of  $\lambda_0$  such that each point in  $Y$  that is a point of intersection between  $\{\Psi_\lambda = C\}$  and  $J_\lambda^-$  has a continuation as such a point of intersection for all  $\lambda$  in this small neighborhood.

For  $j$  sufficiently large,  $p_{\lambda_j}$  is such a point of intersection, and the continuation of  $p_{\lambda_j}$  must agree with the extension of  $p_{\lambda_k}$  because  $p_\lambda$  is defined as a point of intersection. Hence  $p_\lambda$  has an extension to  $\lambda$  in a neighborhood of  $\lambda_0$ . Then  $p_\lambda$  and  $q_\lambda$  both project to the complex line chosen previously, and their images are points in the holomorphic motion. Corollary 2 of [BeR] implies that, given  $r > 0$  small, these points of the holomorphic motion are constrained to lie in a small neighborhood of  $q$  for  $\|\lambda - \lambda_0\| \leq r$ . From the injectivity of a holomorphic motion and the compactness of this parameter range, these two points must be either identical for all such  $\lambda$  or distinct with a positive lower bound on their closest approach. Since  $p_{\lambda_j}$  converges to  $q$  by hypothesis, the two images must be identical.

Hence  $q_\lambda$  agrees with  $p_\lambda$  for some  $\lambda$  where both are defined. As noted before, this implies that they agree on an open set and hence  $q_\lambda$  is a continuation of  $p_\lambda$ . By the monodromy theorem,  $p_\lambda$  extends to all of  $\overline{\Delta}^n$ .

Suppose now that  $p^j$  converges to  $p$  as in the statement of the theorem. We wish to show that  $p_\lambda^j$  converges uniformly on  $\overline{\Delta}^n$  to  $p_\lambda$ . However, since the  $p_\lambda^j$  are uniformly bounded, the argument in the proof of Theorem 5.6 implies that we need show only that  $p_\lambda^j$  converges to  $p_\lambda$  for each fixed  $\lambda$ .

Let  $P_\lambda^+$  and  $P_\lambda^-$  be the family of plaques through  $p$  for  $\lambda$  in some small neighborhood of 0. The family of plaques  $\{P_\lambda^+\}_\lambda$  form a holomorphic manifold  $M^+$  of dimension  $n + 1$  in  $\mathbb{C}^2 \times \Delta_\varepsilon^n$ , so there is an open set in this ambient space and a bounded holomorphic function  $H^+$  defined on this open set such that  $M^+$  is precisely the zero set of  $H^+$  (and likewise for  $H^-$  and  $M^-$ ).

For  $j$  sufficiently large and  $\lambda$  in some small polydisk  $D^n$  independent of  $j$ , the point  $p_\lambda^j$  is contained in the set where  $H^\pm$  are defined;  $p_\lambda^j$  is defined as the point of intersection of two leaves of the stable and unstable laminations and so we see that, for fixed  $j$ ,  $H^\pm(p_\lambda^j)$  is either 0 for all  $\lambda$  near 0 or never 0. Moreover, since  $H^\pm$  is bounded, the set of functions  $h_j^\pm(\lambda) = H^\pm(p_\lambda^j)$  is a normal family. Given any subsequence of  $h_j^\pm$ , we can now extract a locally uniformly convergent subsequence; since  $p^j = p_0^j$  converges to  $p = p^0$ , the limit function must have a zero at  $\lambda = 0$  and hence must be identically 0 by Hurwitz's theorem. Since this is true for any initial subsequence, it follows that  $h_j^\pm$  converges to 0 pointwise as  $j \rightarrow \infty$  for each  $\lambda \in D^n$ . Since the  $h_j^\pm$  are uniformly bounded, we have as before that the convergence to 0 is uniform on compact sets. From the definition of  $h_j^\pm$  in terms of  $H^\pm$ , this implies that  $p_\lambda^j$  converges to  $p_\lambda$  uniformly for  $\lambda$  in compact subsets of  $D^n$ .

Finally, recall that the points  $p_\lambda^j$  are uniformly bounded and so form a normal family. Given any subsequence and any further locally uniformly convergent subsequence, the preceding argument implies that the limit function agrees with  $p_\lambda$  on some neighborhood of 0 and hence everywhere. Since this is true for any initial subsequence, the functions  $p_\lambda^j$  must converge pointwise to  $p_\lambda$  on all of  $\overline{\Delta}^n$ , and since they are uniformly bounded, we see that the convergence is uniform on this compact set. □

**COROLLARY 5.8.** *Let  $f_\lambda$  be as in the previous theorem. Then the map  $\psi_\lambda: J_0 \rightarrow J_\lambda$  extends to a map  $\psi_\lambda: (J_0^+ \cup U_0^+) \cap J_0^-$  such that  $\psi_0$  is the identity,  $\psi_\lambda$  is a homeomorphism for each fixed  $\lambda$ , and  $\psi_\lambda(p)$  is holomorphic in  $\lambda$  for each fixed  $p$ .*

*Proof.* The theorem implies that, given  $p \in (J_0^+ \cup U_0^+) \cap J_0^-$ , we can define  $\psi_\lambda(p) = p_\lambda$ , and also that this extension is continuous and holomorphic in  $\lambda$ . Moreover, for any fixed  $\lambda_0$ , we can apply the theorem to obtain  $\psi_{\lambda_0, \lambda}$  taking  $(J_{\lambda_0}^+ \cup U_{\lambda_0}^+) \cap J_{\lambda_0}^-$  to  $(J_\lambda^+ \cup U_\lambda^+) \cap J_\lambda^-$ . The uniqueness part of the theorem implies that  $\psi_\lambda^{-1} = \psi_{\lambda, \lambda_0}$ . Thus,  $\psi_\lambda$  is injective with continuous inverse, as desired. □

We are now ready to give a version of Theorem 5.6 in the unstably connected case. The proof is the same as the proof of Theorem 5.6, using the corollary to obtain the homeomorphism  $\psi_\lambda$ .

**THEOREM 5.9.** *Let  $f_\lambda$  be a family of hyperbolic and unstably connected Hénon maps depending holomorphically on  $\lambda \in \overline{\Delta}^n$ . Let  $A_0 = (J_0^+ \cup U_0^+) \cap J_0^-$  and let  $p \in A_0$  and  $q \in A_0$  be in the same plaque of  $\mathcal{W}^s \cup \mathcal{G}^+$  as  $p$  ( $p \neq q$ ). Let  $p_\lambda$  and  $q_\lambda$  be the points defined in the previous theorem. Then there exists a  $\phi_\lambda: \mathbb{C} \rightarrow \mathbb{C}^2$ , injective for each fixed  $\lambda$  and holomorphic in  $(z, \lambda) \in \mathbb{C} \times \Delta^n$ , such that  $\phi_\lambda(\mathbb{C})$  equals the leaf of  $\mathcal{W}_\lambda^s \cup \mathcal{G}_\lambda^+$  through  $p_\lambda$  with  $\phi_\lambda(0) = p_\lambda$  and  $\phi_\lambda(1) = q_\lambda$ . Moreover, if  $p^j \in A_0$  with  $p^j \rightarrow p$  and  $q^j \in A_0$  in the same plaque as  $p^j$  with  $q^j \rightarrow q$  and if  $\phi^j$  is the corresponding parametrization for each  $j$ , then  $\phi_\lambda^j$  converges to  $\phi_\lambda$  uniformly on compact subsets of  $\mathbb{C} \times \Delta^n$ . There is an analogous result for leaves of  $\mathcal{W}_\lambda^u$ .*

### 6. Holomorphic Motions

We recall the following theorem, due to Bers and Royden, on the canonical extension of a holomorphic motion of a set  $E \subset \mathbb{P}^1$  to a holomorphic motion on  $\mathbb{P}^1$ . For more background, see [BeR].

**THEOREM 6.1** [BeR]. *Let  $\tau: \Delta \times E \rightarrow \mathbb{P}^1$  be a holomorphic motion. Then  $\tau$  restricted to  $\Delta_{1/3} \times E$  has a canonical extension to a holomorphic motion  $\tau: \Delta_{1/3} \times \mathbb{P}^1 \rightarrow \mathbb{P}^1$ . This extension is characterized by the following property. Let  $\mu(\lambda, z)$  be the Beltrami coefficient of  $z \mapsto \tau(\lambda, z)$  and let  $S$  be any component of  $\mathbb{P}^1 \setminus \hat{E}$ , where  $\hat{E}$  is the closure of  $E$  in  $\mathbb{P}^1$ . Then*

$$\mu(\lambda, z) = \rho_S(z)^{-2} \overline{\psi(\lambda, z)} \tag{6.1}$$

for  $z \in S$  and  $\lambda \in \Delta_{1/3}$ , where  $\rho_S(z)|dz|$  is the hyperbolic metric in  $S$  and where the function  $\psi(\lambda, z)$  is holomorphic in  $z \in S$  and antiholomorphic in  $\lambda \in \Delta_{1/3}$ .

This theorem is true also if the disk is replaced by the ball in  $\mathbb{C}^n$ ; see [Su] or [Mi].

A Beltrami coefficient of the form in (6.1) is said to be a *harmonic* Beltrami coefficient. The hyperbolic metric is also known as the Poincaré metric and the infinitesimal Kobayashi metric.

The parametrization of leaves given in the previous section gives us a way to speak of a holomorphic motion on leaves.

**DEFINITION 6.2.** Let  $\phi: \Delta^n \times \mathbb{C} \rightarrow \mathbb{C}^2$  be holomorphic and suppose that  $\phi_\lambda = \phi(\lambda, \cdot)$  is injective for each fixed  $\lambda \in \Delta^n$ . Let  $E_0 \subset \phi(0, \mathbb{C})$ . Then  $\tau: \Delta^n \times E_0 \rightarrow \mathbb{C}^2$  is a *holomorphic motion* of  $E_0$  on the family of leaves defined by  $\phi$  if  $\tau_\lambda(E_0) = \tau(\lambda, E_0)$  is contained in the leaf  $\phi(\lambda, \mathbb{C})$  for each  $\lambda$  and if  $\phi_\lambda^{-1} \tau_\lambda \phi_0$  is a standard holomorphic motion in  $\mathbb{C}$  of the set  $\phi_0^{-1}(E_0)$ .

In particular, given a holomorphic motion on leaves, we can pull it back to a holomorphic motion in the plane, then apply the Bers–Royden extension, and push forward to obtain an extended holomorphic motion on leaves. We will call this extension the Bers–Royden extension also.



We record here also a notion for the convergence of holomorphic motions on leaves when approaching a limit leaf. Let  $I = \mathbb{Z}^+ \cup \{\infty\}$ . In the following definition, the Hausdorff metric on sets in the plane is defined with respect to the spherical metric, denoted here by  $d_s$ , on the Riemann sphere.

NOTATION. With  $\phi$  and  $\tau$  as in the previous definition, let  $\phi_*[\tau_\lambda]$  denote the map  $\phi_\lambda^{-1}\tau_\lambda\phi_0$  defined on  $\phi_0^{-1}(E_0)$ .

DEFINITION 6.3. For each  $i \in I$ , let  $\phi^i : \Delta^n \times \mathbb{C} \rightarrow \mathbb{C}^2$  be holomorphic with  $\phi_\lambda^i = \phi^i(\lambda, \cdot)$  injective for each fixed  $\lambda$ , and suppose that  $\phi^i$  converges to  $\phi^\infty$  uniformly on compact sets. Let  $E^i \subset \phi^i(0, \mathbb{C})$  for each  $i \in I$ , and let  $\tau^i : \Delta^n \times E^i$  be a holomorphic motion on the leaves defined by  $\phi^i$ . Then  $\tau^i$  converges uniformly to  $\tau^\infty$  means that the sets  $A^i = (\phi_0^i)^{-1}(E^i)$  converge to  $A^\infty$  in the Hausdorff metric and that the corresponding holomorphic motions in the plane converge uniformly on compacts: for each  $\varepsilon > 0$  there exist  $\delta > 0$  and  $N > 0$  such that, if  $i > N$  and  $\|\lambda_1 - \lambda_2\| + d_s(z_1, z_2) < \delta$  ( $z_1 \in A^i, z_2 \in A^\infty$ ), then

$$d_s(\phi_*^i[\tau_{\lambda_1}^i](z_1), \phi_*^\infty[\tau_{\lambda_2}^\infty](z_2)) < \varepsilon.$$

The uniqueness of the Bers–Royden extension allows us to conclude that, given a sequence of holomorphic motions on leaves converging as just described, the extensions also converge in this sense.

PROPOSITION 6.4. Let  $\phi^i$  and  $\tau^i$  be as in Definition 6.3, and let  $\hat{\tau}^i$  denote the Bers–Royden extension of  $\tau^i$ . Then  $\hat{\tau}^i$  converges uniformly to  $\hat{\tau}^\infty$ .

Proof. That  $A^0$  converges to  $A^\infty$  in the Hausdorff metric implies, for a given compact  $K \subset \mathbb{C} \setminus A^\infty$ , that  $K$  is also contained in the complement of  $A^i$  for large  $i$  and that the hyperbolic metric of the component of the complement of  $A^i$  containing  $K$  converges uniformly on  $K$  to the hyperbolic metric of the complement of  $A^\infty$ . Moreover, since each  $\phi_*^i[\hat{\tau}_\lambda^i]$  has a harmonic Beltrami coefficient, say  $\mu_i(\lambda, z) = \rho_i(z)^{-2}\overline{\psi_i(\lambda, z)}$ , and since  $\|\mu_i(\lambda, z)\|\rho_i(z)^2$  is uniformly bounded for  $\lambda \in \overline{\Delta^n}$  and  $z \in K$ , we see that the family  $\{\psi_i\}$  is a normal family.

Hence there exists a subsequence of  $\psi^i$  converging uniformly on each compact subset of  $\Delta^n \times (\mathbb{C} \setminus A^\infty)$  to  $\psi(\lambda, z)$ . Moreover, from [BeR, Thm. 1], we have for each  $i$  that

$$\|\mu_i(\lambda, z)\|_\infty < \|\lambda\|.$$

Hence this estimate holds also for  $\mu(\lambda, z) = \rho_\infty(z)^{-2}\overline{\psi(\lambda, z)}$ , and the subsequence of holomorphic motions corresponding to the chosen subsequence of  $\psi^i$  converges uniformly to a holomorphic motion with the harmonic Beltrami coefficient  $\mu$ . But this limit motion must agree with  $\phi_*^\infty[\hat{\tau}_\lambda^\infty]$  on  $A^\infty$ , and since this latter motion also has a harmonic Beltrami coefficient, the uniqueness of the Bers–Royden extension implies that the limit motion must equal  $\phi_*^\infty[\hat{\tau}_\lambda^\infty]$ . Since any subsequence must have the same limit, we obtain pointwise convergence, and [BeR, Cor. 2] implies equicontinuity of the sequence and hence uniform convergence, as in the preceding definition. □

We prove next that the natural motion of  $J_0$  given by  $\psi_\lambda$  is a holomorphic motion on leaves and that the motion on a sequence of leaves approaching a limit leaf converge to the motion on the limit leaf.

**THEOREM 6.5.** *Let  $f_\lambda$  be a family of hyperbolic Hénon maps depending holomorphically on  $\lambda \in \overline{\Delta}^n$ . Let  $\mathcal{L}_\lambda$  be either of the laminations  $\mathcal{W}_\lambda^u$  or  $\mathcal{W}_\lambda^s$ . Let  $p \in J_0$  and  $p_\lambda = \psi_\lambda(p)$ , and let  $L_\lambda = L_\lambda(p)$  be the leaf of  $\mathcal{L}_\lambda$  through  $p_\lambda$ . Let  $E_0 = L_0 \cap J_0$ . Then  $\psi(\lambda, \cdot) = \psi_\lambda(\cdot)$  is a holomorphic motion of  $E_0$  on the family of leaves  $\{L_\lambda\}$ .*

*Moreover, if  $p^j \in J_0$  converges to  $p \in J_0$  and  $L_\lambda^j = L_\lambda(p^j)$  is the leaf through  $p_\lambda^j$ , then the holomorphic motion of  $E_0^j = L_0(p^j) \cap J_0$  on the family of leaves  $\{L_\lambda^j\}$  converges uniformly to the holomorphic motion of  $E_0$  on the family of leaves  $\{L_\lambda\}$ .*

*Finally, the Bers–Royden extensions of the motions of  $E_0^j$  converge uniformly to the Bers–Royden extensions of the motion of  $E_0$ .*

*Proof.* Since  $\psi_\lambda$  is a homeomorphism of  $J_0$  to  $J_\lambda$  that conjugates  $f_0$  to  $f_\lambda$ , it follows that  $\psi_\lambda$  maps  $L_0 \cap J_0$  onto  $L_\lambda \cap J_\lambda$ . Hence  $\psi_\lambda(E_0)$  is contained in  $L_\lambda$ . Moreover, Theorem 5.6 implies that there exist holomorphically varying parametrizations  $\phi_\lambda: \mathbb{C} \rightarrow L_\lambda$ . Since  $\psi_\lambda(q)$  is holomorphic in  $\lambda$  for each fixed  $q \in J_0$ , we see that  $\phi_*[\psi_\lambda]$  is a holomorphic motion in the plane; hence  $\psi_\lambda$  is a holomorphic motion on the family of leaves through  $p$ .

For the convergence result, assume without loss of generality that  $\mathcal{L}_\lambda$  is the unstable lamination. For the remainder of this proof, let  $\delta = \delta_0$  and  $\varepsilon = \varepsilon_0$  be the constants chosen earlier from the definition of local product structure: if  $a_\lambda, b_\lambda \in J_\lambda$  with  $\|a - b\| < \delta$ , then  $W_\varepsilon^s(a_\lambda)$  and  $W_\varepsilon^u(b_\lambda)$  intersect in a unique point contained in  $J_\lambda$ .

Theorem 5.6 implies that there exist functions  $\phi_\lambda^j: \mathbb{C} \rightarrow W^u(p_\lambda^j)$  which are holomorphic in  $z \in \mathbb{C}$  and in  $\lambda \in \overline{\Delta}^n$  and bijective for each fixed  $\lambda$  and which converge locally uniformly to the map  $\phi_\lambda^\infty$  parametrizing  $W^u(p_\lambda)$ .

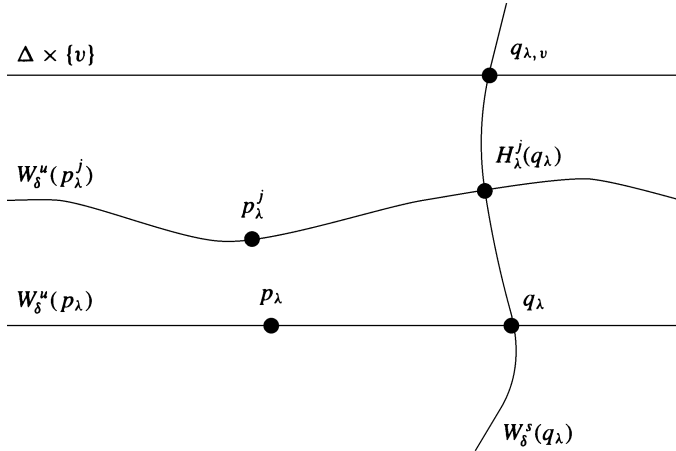
With these parametrizations, the first part of this proof implies that the holomorphic motion on the family of leaves through  $p^j$  is defined on the set  $A^j = (\phi_0^j)^{-1}(J_0 \cap W^u(p^j))$  and is given by the pull-back  $\tau_\lambda^j = \phi_*^j[\psi_\lambda]$ . The set  $A^\infty$  and  $\tau_\lambda^\infty$  are defined similarly, using  $p$  and  $\phi$ .

Choose  $R > 0$  and let  $K = \overline{\Delta}_R \subset \mathbb{C}$ . Since we are using the spherical metric to define the Hausdorff metric, the proposition to be established once we show that  $A^j \cap K$  converges to  $A^\infty \cap K$  in the Hausdorff metric and that  $\tau_\lambda^j = \phi_*^j[\psi_\lambda]$  converges uniformly on  $(z, \lambda) \in (K \cap E^j) \times \overline{\Delta}^n$  to  $\tau_\lambda^\infty$ .

Since  $\phi_\lambda^\infty(K)$  is contained in  $W^u(p_\lambda)$ , it follows that, for large  $n$ ,  $f_\lambda^{-n}(\phi_\lambda^\infty(K))$  is contained in  $W_\delta^u(f_\lambda^{-n}(p_\lambda))$ . Hence, for large  $j$ ,  $f_\lambda^{-n}(\phi_\lambda^j(K))$  is also within  $\delta$  of  $f_\lambda^{-n}(p_\lambda)$ . It suffices to prove the convergence result near  $f_\lambda^{-n}(p_\lambda)$  and then apply  $f_\lambda^n$ ; for clarity, we drop the  $f_\lambda^{-n}$  for the remainder of the proof.

Choose distinct points  $a$  and  $b$  in  $W_\varepsilon^u(p) \cap J_0$  so that each of  $a_\lambda$  and  $b_\lambda$  is of distance no more than  $\delta/2$  from  $p_\lambda$  for any  $\lambda$ . Then, for large  $j$ ,  $W_\varepsilon^u(p_\lambda^j)$  and  $W_\varepsilon^s(a_\lambda)$  intersect in a unique point of  $J_\lambda$ , and likewise for  $b_\lambda$ .

Using a local biholomorphic change of variables from a neighborhood of  $W_\delta^u(p_\lambda)$  to the unit bidisk  $\{|u| < 1, |v| < 1\}$  (with the change of variables depending holomorphically on  $\lambda$ ), we may assume that  $W_\delta^u(p_\lambda)$  is  $\Delta \times \{0\}$  and that  $W_\delta^s(a_\lambda)$  and  $W_\delta^s(b_\lambda)$  are  $\{0\} \times \Delta$  and  $\{1/2\} \times \Delta$ , respectively. Then, for each  $q \in J_0 \cap W_\delta^u(p_0)$  and for given values of  $\lambda \in \Delta^n$  and  $v \in \Delta$ , we associate the point given by taking the intersection of  $W_\delta^s(q_\lambda)$  with  $\Delta \times \{v\}$  and then projecting to the  $u$ -coordinate. This defines a holomorphic motion of the point  $q$  with parameters  $\lambda$  and  $v$ .



**Figure 1** The holomorphic motion  $q(\lambda, v)$  and the projection  $H_\lambda^j$

We can view this holomorphic motion as a lamination with leaves defined by  $\{q_{\lambda,v} : v \in \Delta\}$  as in Figure 1, and the holonomy map associated with the leaves of this lamination gives a projection  $H_\lambda^j$  from  $W_\delta^u(p_\lambda) \cap J_\lambda$  to  $W_\delta^u(p_\lambda^j) \cap J_\lambda$ . As  $j$  tends to  $\infty$ , the  $v$ -coordinate of  $W_\delta^u(p_\lambda^j)$  converges uniformly to 0. Hence the estimate in [BeR, Cor. 2] implies that  $H_\lambda^j$  (and  $(H_\lambda^j)^{-1}$ ) converges to the identity uniformly in  $q$  and  $\lambda$ . In particular, this establishes the convergence of  $E^j \cap K$  to  $E^\infty \cap K$  in the Hausdorff metric. Moreover, given  $q \in J_0 \cap W^u(p_0^j)$ , we have  $\psi_\lambda(q) = H_\lambda^j \psi_\lambda(H_0^j)^{-1}(q)$ . Hence

$$\tau_\lambda^j = (\phi_\lambda^j)^{-1} H_\lambda^j \psi_\lambda [(\phi_0^j)^{-1} H_0^j]^{-1},$$

where  $\psi_\lambda$  is restricted to  $W_\delta^u(p_\lambda)$ . The right-hand side converges uniformly to

$$(\phi_\lambda^\infty)^{-1} \psi_\lambda (\phi_0^\infty)^{-1} = \tau_\lambda^\infty,$$

as desired.

Finally, the convergence of the Bers–Royden extensions follows from Proposition 6.4. □

We next prove an analogous result in the unstably connected case.

**PROPOSITION 6.6.** *Let  $f_\lambda$  be as in Proposition 6.5 and assume also that each  $f_\lambda$  is unstably connected. Let  $\mathcal{L}_\lambda$  be the lamination  $\mathcal{W}_\lambda^s \cup \mathcal{G}_\lambda^+$ . Let  $p \in (J_0^+ \cup U_0^+) \cap J_0^-$  and  $p_\lambda = \psi_\lambda(p)$ , and let  $L_\lambda = L_\lambda(p)$  be the leaf of  $\mathcal{L}_\lambda$  through  $p_\lambda$ . Let  $E_0 = L_0 \cap J_0^-$ . Then  $\psi(\lambda, \cdot) = \psi_\lambda(\cdot)$  is a holomorphic motion of  $E_0$  on the family of leaves  $\{L_\lambda\}$ .*

*Moreover, if  $p^j \in (J_0^+ \cup U_0^+) \cap J_0^-$  converges to  $p$  in the same set, then the holomorphic motion of  $E_0^j = L_0(p^j) \cap J_0^-$  on the family of leaves  $\{L_\lambda(p^j)\}$  converges uniformly to the holomorphic motion of  $E_0$  on the family of leaves  $\{L_\lambda\}$ , and the Bers–Royden extensions of the motions of  $E_0^j$  converge uniformly to the Bers–Royden extension of the motion of  $E_0$ .*

*Proof.* Since  $f_\lambda$  is unstably connected, we can use Corollary 5.8 to obtain the homeomorphism  $\psi_\lambda$  and then use Theorem 5.9 in place of Theorem 5.6 in the proof of the previous theorem to obtain the holomorphic motion of  $E_0$ .

For the convergence result, if  $p \in J_0$  then the proof is the same as that of the previous theorem, so we assume that  $p \in U_0^+ \cap J_0^-$ . In this case, proof of the previous theorem still applies except for the existence of  $\delta$  and  $\varepsilon$ . However, instead of applying  $f^{-n}$  for some large  $n$ , we now apply  $f^n$ . Since leaves of the lamination of  $U_0^+$  are super-stable manifolds (as shown in [BS3]), it follows that, for large  $n$  and  $j$ ,  $f_\lambda^n(\phi_\lambda^j(K))$  is again contained in a small neighborhood of  $f_\lambda^n(p_\lambda)$ ; the discussion of  $G^+$  after Proposition 3.1 implies that these images of  $K$  will be nearly horizontal disks. A simple calculation implies that the local unstable manifolds of points in  $J_\lambda^-$  near  $p_\lambda$  are nearly vertical disks. Hence again there are unique points of intersection between local stable and unstable leaves, so the remainder of the proof of the previous theorem applies without change. □

### 7. Proof of Main Theorems

*Proof of Theorem 1.1.* Choose  $p_0 \in J_0$  and let  $p_\lambda = \psi_\lambda(p_0)$ . We will first construct the map  $\Psi_\lambda$  on the set  $W^u(p_0)$ . Toward this end, let  $\phi_\lambda : \mathbb{C} \rightarrow W^u(p_\lambda)$  be a parametrization obtained by Theorem 5.6; that is,  $\phi_\lambda$  is holomorphic in  $(\lambda, z)$ , with  $\phi_\lambda(0) = p_\lambda$  and  $\phi_\lambda(1) = \psi_\lambda(q_0)$  for some  $q_0 \in W^u(p_0) \setminus \{p_0\}$ . Let  $E_0 = J_0 \cap W^u(p_0)$ , and define a holomorphic motion of  $A_0 = \phi_0^{-1}(E_0)$  by

$$\alpha_\lambda = \phi_\lambda^{-1} \psi_\lambda \phi_0 = \phi_*[\psi_\lambda].$$

By the theorem of Bers and Royden,  $\alpha$  extends canonically to a holomorphic motion  $\hat{\alpha}_\lambda$  of  $\mathbb{C}$  with a harmonic Beltrami coefficient.

We define  $\Psi_\lambda : W^u(p_0) \rightarrow W^u(p_\lambda)$  by  $\Psi_\lambda = \phi_\lambda \hat{\alpha}_\lambda \phi_0^{-1}$ . Note that on  $E_0$ ,  $\Psi_\lambda = \psi_\lambda$ . Moreover,  $\Psi_\lambda$  is independent of the choice of  $\phi_\lambda$ . To see this, suppose that  $\gamma : \Delta^n \times \mathbb{C} \rightarrow W^u(p_\lambda)$  is holomorphic in  $(\lambda, z)$ , and let  $B_0$  and  $\beta_\lambda$  be the analogs of  $A_0$  and  $\alpha_\lambda$  with  $\gamma$  in place of  $\phi$ . Then  $\phi_\lambda^{-1} \gamma_\lambda : \mathbb{C} \rightarrow \mathbb{C}$  is affine linear and holomorphic in  $\lambda$ , say  $\phi_\lambda^{-1} \gamma_\lambda(z) = Q_\lambda(z)$  or  $\gamma_\lambda(z) = \phi_\lambda Q_\lambda(z)$ . Hence

$$\beta_\lambda(z) = Q_\lambda^{-1} \phi_\lambda^{-1} \psi_\lambda \phi_0 Q_0(z) = Q_\lambda^{-1} \alpha_\lambda Q_0^{-1}(z).$$

Since  $Q_\lambda$  is affine linear, the canonical extension of  $\beta_\lambda$  is  $\hat{\beta}_\lambda = Q_\lambda^{-1} \hat{\alpha}_\lambda Q_0(z)$ . Using this with the expression for  $\gamma_\lambda$  just given and canceling terms, we obtain

$\gamma_\lambda \hat{\beta}_\lambda \gamma_0^{-1}(p) = \Psi_\lambda(p)$  for each  $p \in W^u(p_0)$ . That is,  $\Psi_\lambda$  is independent of the choice of parametrization.

Hence we may apply the foregoing construction to each  $p_0 \in J_0$  to obtain  $\Psi_\lambda: J_0^- \setminus S_0 \rightarrow J_\lambda^- \setminus S_\lambda$  satisfying properties (1), (3), and (4) of the theorem, where  $S_\lambda$  is the set of sink orbits for  $f_\lambda$ . The same construction applies to give  $\Psi_\lambda$  on  $J_0^+$ , and we can define  $\Psi_\lambda$  on  $S_0$  by using the implicit function theorem to follow the sink orbits.

As in [McS], we can use the uniqueness of the Bers–Royden extension to show that  $\Psi_\lambda$  conjugates  $f_0$  on  $W^u(p_0)$  to  $f_\lambda$  on  $W^u(p_\lambda)$ . To do this, let  $\hat{\alpha}_\lambda$  be the holomorphic motion of  $\mathbb{C}$  induced as before by  $\psi_\lambda$  on  $W^u(p_0)$  and let  $\hat{\beta}_\lambda$  be the motion induced by  $\psi_\lambda$  on  $W^u(f_0(p_0))$ , where  $W^u(f_\lambda(p_\lambda))$  is parametrized by  $\gamma_\lambda$ . (Note that  $\beta$  and  $\gamma$  are different from the maps of the same name in the preceding section.) We obtain the following diagram, with the left and right portions commuting as indicated.

$$\begin{array}{ccccccc}
 \mathbb{C} & \xrightarrow{\phi_0} & W^u(p_0) & \xrightarrow{f_0} & W^u(f_0(p_0)) & \xleftarrow{\gamma_0} & \mathbb{C} \\
 \downarrow \hat{\alpha}_\lambda \quad \text{///} & & \downarrow \psi_\lambda \rightsquigarrow \Psi_\lambda & & \downarrow \psi_\lambda \rightsquigarrow \Psi_\lambda \quad \text{///} & & \downarrow \hat{\beta}_\lambda \\
 \mathbb{C} & \xrightarrow{\phi_\lambda} & W^u(p_\lambda) & \xrightarrow{f_\lambda} & W^u(f_\lambda(p_\lambda)) & \xleftarrow{\gamma_\lambda} & \mathbb{C}
 \end{array}$$

Note that  $\gamma_\lambda^{-1} f_\lambda \phi_\lambda$  is a biholomorphic map of  $\mathbb{C}$  to itself and hence is equal to some affine linear map  $Q_\lambda$  depending holomorphically on  $\lambda$ . As a result,  $Q_\lambda \hat{\alpha}_\lambda Q_0^{-1}$  is a holomorphic motion of  $\mathbb{C}$ . Moreover, since  $Q_\lambda$  is an affine linear map, the Beltrami coefficient of this new holomorphic motion is simply a constant times the Beltrami coefficient of  $\hat{\alpha}_\lambda$ , so the new holomorphic motion has a harmonic Beltrami coefficient.

Furthermore,  $\psi_\lambda = f_\lambda \psi_\lambda f_0^{-1}$  on  $J_0$  implies that  $\psi_\lambda = f_\lambda \phi_\lambda \alpha_\lambda \phi_0^{-1} f_0^{-1}$  on  $W^u(f_0(p_0)) \cap J_0$ , and hence  $Q_\lambda \alpha_\lambda Q_0^{-1} = \gamma_\lambda^{-1} \psi_\lambda \gamma_0$  on the same set. But also  $\beta_\lambda = \gamma_\lambda^{-1} \psi_\lambda \gamma_0$  by construction and so, by the uniqueness of the extension of this motion to a motion with harmonic Beltrami coefficient, we see that  $\hat{\beta}_\lambda = Q_\lambda \hat{\alpha}_\lambda Q_0^{-1}$ .

Since  $\hat{\alpha}_\lambda = \phi_\lambda^{-1} \Psi_\lambda \phi_0$  and  $\hat{\beta}_\lambda = \gamma_\lambda^{-1} \Psi_\lambda \gamma_0$ , we have

$$\gamma_\lambda^{-1} \Psi_\lambda \gamma_0 = Q_\lambda \phi_\lambda^{-1} \Psi_\lambda \phi_0 Q_0;$$

using  $Q_\lambda = \gamma_\lambda^{-1} f_\lambda \phi_\lambda$  and canceling common factors, we obtain  $\Psi_\lambda = f_\lambda \Psi_\lambda f_0^{-1}$ .

The argument just given applies to any  $p_0 \in J_0$ , so  $f_\lambda = \Psi_\lambda f_0 \Psi_\lambda^{-1}$  on  $J_0^- \setminus S_0$ . Finally, the extension of  $\Psi_\lambda$  to  $S_0$  using the implicit function theorem respects the dynamics on the sink orbits and therefore  $\Psi_\lambda$  conjugates  $f_0$  to  $f_\lambda$  on all of  $J_0^-$ . Applying this to  $f_\lambda^{-1}$  gives  $\Psi_\lambda$  on  $J_0^- \cup J_0^+$  satisfying properties (1), (3), (4), and (5) of the theorem.

Note that  $\Psi_\lambda$  is bijective, since it is bijective on each leaf and since there is a 1-to-1 correspondence between leaves. We need to check that  $\Psi_\lambda$  is continuous with continuous inverse, but it suffices to show that it is continuous and proper (as a map from a subset of  $\mathbb{C}^2$  into  $\mathbb{C}^2$ ) since then we can use a 1-point compactification to get a continuous 1-to-1 map on a compact set, which automatically has a continuous inverse.

To show continuity, let  $q^j$  be a sequence of points in  $J_0^-$  converging to a point  $q^\infty$  in  $J_0^-$ , and suppose first that  $q^\infty$  is not a sink. We want to show that  $\Psi_\lambda(q^j)$  converges to  $\Psi_\lambda(q^\infty)$ . Let  $p^\infty \in J_0$  so that  $q^\infty$  is in the unstable manifold of  $p^\infty$  for  $f_0$ , and likewise let  $p^j \in J_0$  so that  $q^j$  is in the unstable manifold of  $p^j$ . Dropping to a subsequence if necessary, we may assume that  $p^j$  converges to  $p^\infty$ .

Theorem 6.5 implies that the holomorphic motion of  $W^u(p^j) \cap J_0$  converges uniformly to the holomorphic motion of  $W^u(p^\infty) \cap J_0$  and also that the Bers–Royden extensions of the former motions converge to the Bers–Royden extension of the latter. Since  $\Psi_\lambda$  is precisely the Bers–Royden extension of these motions, it follows at once that  $\Psi_\lambda(q^j)$  converges to  $\Psi_\lambda(q^\infty)$ .

We claim next that  $p_0 \in J_0^-$  is in the basin of attraction of a sink orbit if and only if  $\Psi_\lambda(p_0)$  is in the basin of attraction of a sink orbit for each  $\lambda$ . First,  $p_0 \in J_0^-$  is in  $J_0^+$  precisely when  $\Psi_\lambda(p_0) \in J_\lambda^+$ , since  $\Psi$  is injective and is a homeomorphism of  $J_0$  to  $J_\lambda$ . Since  $J_0^+$  is the boundary of all basins of attraction of sink orbits of  $f_0$ , we may assume either that  $p_0$  is in the basin of a sink or that  $p_0$  is in the set of points with unbounded forward orbit. We can then write  $\Delta^n$  as the disjoint union of the set  $A$  of  $\lambda$  such that  $\Psi_\lambda(p_0)$  is in the basin of some sink and the set  $B$  of  $\lambda$  such that  $\Psi_\lambda(p_0)$  has unbounded forward orbit. Note that, if  $\Psi_\lambda(p_0)$  is attracted to a sink of  $f_\lambda$ , then some small closed neighborhood is attracted to this sink; for all sufficiently small perturbations of  $f_\lambda$ , this closed neighborhood will still be in the basin of some sink. Since  $\Psi_\lambda(p_0)$  is holomorphic in  $\lambda$ , it follows that the set  $A$  is open and likewise that the set  $B$  is open. Since  $\Delta^n$  is connected, only one of these two sets can be nonempty, and since the point 0 is in one of them, the claim follows. This argument can be refined by further decomposing the set  $A$  into disjoint sets  $A_i$  of  $\lambda$  such that  $\Psi_\lambda(p_0)$  is contained in the basin of attraction of  $\Psi_\lambda(q^j)$  for each sink  $q^j$  of  $f_0$ . The conclusion in this case is that  $p_0$  is in the basin of attraction of  $q_0^j$  if and only if, for all  $\lambda$ ,  $\Psi_\lambda(p_0)$  is in the basin of attraction of  $q_\lambda^j$ .

To continue the proof of continuity, if  $q^\infty$  is a sink then we may assume without loss of generality that each  $q^j$  is contained in the basin of attraction of  $q^\infty$  but is not equal to  $q^\infty$ . Let  $U$  be a small neighborhood of  $J_0$  in  $J_0^-$ , and let  $N = \overline{f_0(U)} \setminus \overline{U}$ . Then  $N$  is compact and disjoint from  $J_0$ , and for each  $j$  there exists  $n_j$  such that  $f_0^{-n_j}(q^j) \in N$ . Moreover, since  $q^\infty$  is a sink, it follows that  $n_j \rightarrow \infty$ . Let

$$K = \overline{\{f_0^{-n_j}(q^j) : j \geq 1\}}.$$

Then  $K$  is a compact set contained in the intersection of  $N$  and the basin of  $q^\infty$ . The previous paragraph implies that  $\Psi_\lambda(K)$  is contained in the basin of attraction of  $q_\lambda^\infty$  for all  $\lambda$ . Hence, for fixed  $\lambda$ ,  $\Psi_\lambda(K)$  is a compact set in the basin of  $q_\lambda^\infty$ ; since  $n_j \rightarrow \infty$ , we see that  $f_\lambda^{n_j} \Psi_\lambda(K)$  converges uniformly to  $q_\lambda^\infty$ . Since  $\Psi_\lambda(q^j) \in \overline{f_\lambda^{n_j} \Psi_\lambda(K)}$ , we have  $\Psi_\lambda(q^j)$  converging to  $q_\lambda^\infty = \Psi_\lambda(q^\infty)$ . Thus  $\Psi_\lambda$  is continuous on all of  $J_0^-$ .

For properness, suppose  $p_0^j \in J_0^-$  with  $\|p_0^j\| \rightarrow \infty$  but  $\|\Psi_\lambda(p_0^j)\| \leq C$  for some large constant  $C$  and fixed  $\lambda$ . Since  $J_0$  is a bounded set, and since any  $p_0^j$  that is in  $K_0^+$  must be in  $J_0$ , we may assume without loss of generality that each  $p_0^j$  is in the complement of  $K_0^+$ . By the claim made previously,  $p_\lambda^j = \Psi_\lambda(p_0^j)$  is in the

complement of  $K_\lambda^+$ . After dropping to a subsequence, we can find a sequence  $n_j$  increasing to  $\infty$  such that  $q_0^j = f_0^{-n_j}(p_0^j)$  converges to a point  $q_0^\infty$  in  $J_0^- \setminus K_0^+$ . Let  $q_\lambda^j = \Psi_\lambda(q_0^j)$ . The conjugacy property of  $\Psi$  implies that  $q_\lambda^j = f_\lambda^{-n_j}(p_\lambda^j)$ . Let  $A_j = f_\lambda^{-n_j}((J_\lambda^- \setminus (\text{int } K_\lambda^+)) \cap \Delta_C^2)$ . Then each  $A_j$  is compact,  $A_{j+1} \subset A_j$ , and  $q_\lambda^j \in A_{n_j}$  for each  $j$ . Moreover, the continuity of  $\Psi_\lambda$  implies that  $q_\lambda^\infty$  is the limit of the sequence  $q_\lambda^j$ , so

$$q_\lambda^\infty \in \bigcap_{m>0} \overline{\bigcup_{j \geq m} \{q_\lambda^j\}} \subset \bigcap_{m>0} A_{n_m}.$$

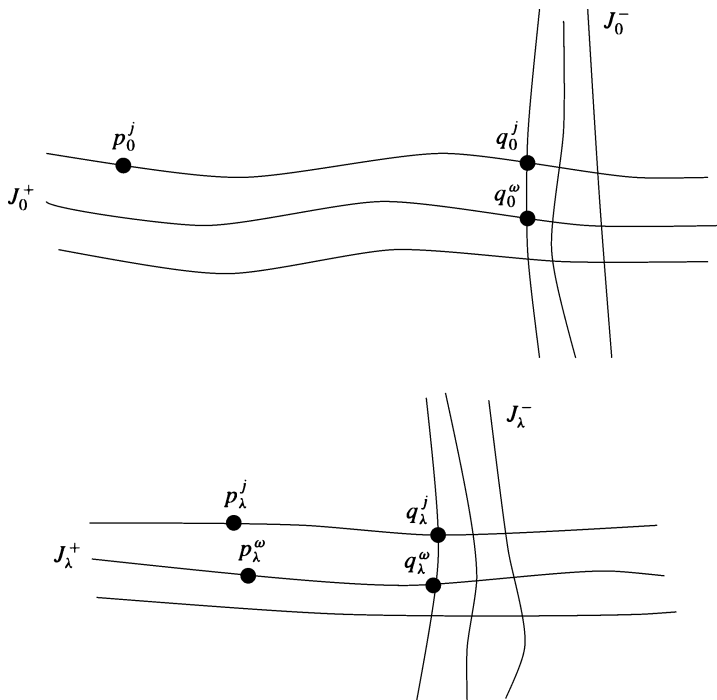
However, the intersection of all  $A_j$  is precisely  $J_\lambda$ , so  $q_\lambda^\infty$  must be in  $J_\lambda$ . But this is a contradiction because  $\Psi_\lambda$  is injective,  $\Psi_\lambda$  is a homeomorphism from  $J_0$  to  $J_\lambda$ , and  $q_0^\infty$  is not in  $J_0$ . It follows that  $\Psi_\lambda$  must be proper and hence is a homeomorphism of  $J_0^-$  to  $J_\lambda^-$ .

Applying this proof to  $f_\lambda^{-1}$  yields a conjugacy of  $f_0|_{J_0^+}$  to  $f_\lambda|_{J_\lambda^+}$  that agrees with the previously constructed map on  $J_0$ , so we get the map  $\Psi_\lambda$  defined on  $J_0^+ \cup J_0^-$ , as desired. This completes the proof of Theorem 1.1.  $\square$

*Proof of Theorem 1.2.* In the case when  $f_0$  is unstably connected and hyperbolic, Proposition 3.1 implies that  $f_\lambda$  is also unstably connected for  $\lambda$  near 0. Moreover, the previous construction applies to give  $\Psi$  on  $J_0^+ \cup J_0^-$ ; by replacing Theorem 5.6 with Theorem 5.9 and Theorem 6.5 with Theorem 6.6, the previous proof applies to show that  $\Psi$  is continuous.

For the properness, the previous proof does not apply directly, although it still implies that  $\Psi$  is proper on  $J_0^+ \cup J_0^-$ . To finish the proof, suppose that  $p_0^j$  is a sequence of points in  $U_0^+$  with  $\|p_0^j\| \rightarrow \infty$ . In this case, either  $G_0^+(p_0^j) \rightarrow \infty$  or  $G_0^-(p_0^j) \rightarrow \infty$ , and we suppose for now that the former applies. Note that the leaf of the lamination through  $p_0^j$ , which is a level set of  $G_0^+$ , is biholomorphic to the plane. Observe also that  $G_0^-$  is subharmonic on this leaf and is nonnegative, non-constant, and harmonic outside of the zero set; hence it must equal 0 somewhere. By definition of  $G_0^-$ , a zero of this function is precisely a point of  $J_0^-$ . Hence there exists a point  $q_0^j$  in  $J_0^-$  on the leaf through  $p_0^j$ . Then  $G_0^+(q_0^j) = G_0^+(p_0^j)$ , and since  $\Psi_\lambda$  is a homeomorphism on  $J_0^-$ , we must have  $G_\lambda^+(\Psi_\lambda(q_0^j)) \rightarrow \infty$ . Since  $\Psi_\lambda$  takes level sets of  $G_0^+$  to level sets of  $G_\lambda^+$ , we have  $G_\lambda^+(\Psi_\lambda(p_0^j)) \rightarrow \infty$  and so  $\|\Psi_\lambda(p_0^j)\| \rightarrow \infty$ .

Next, suppose  $G_0^-(p_0^j) \rightarrow \infty$  but  $G_0^+(p_0^j) < C$  and  $\|\Psi_\lambda(p_0^j)\| < C$  for some constant  $C$ . Again we can choose  $q_0^j$  in  $J_0^-$  on the leaf through  $p_0^j$ . Since the set of points in  $J_0^-$  with  $G_0^+ < C$  is bounded, we can drop to a convergent subsequence to obtain  $q_0^j$  converging to  $q_0^\infty \in J_0^-$ . See Figure 2. The continuity of  $\Psi_\lambda$  implies that  $q_\lambda^j = \Psi_\lambda(q_0^j)$  converges to  $q_\lambda^\infty = \Psi_\lambda(q_0^\infty)$ . Moreover,  $p_\lambda^j = \Psi_\lambda(p_0^j)$  is contained in the same leaf as  $q_\lambda^j$ , and dropping to a further subsequence, we may assume that  $p_\lambda^j$  converges to a point  $p_\lambda^\infty$  in  $U_\lambda^+ \cup J_\lambda^+$  in the same leaf as  $q_\lambda^\infty$ . Since  $\Psi_\lambda$  is a homeomorphism on  $J_0^-$ , a neighborhood  $Y$  of  $q_0^\infty$  in  $J_0^-$  maps onto a neighborhood of  $q_\lambda^\infty$  in  $J_\lambda^-$ . Since  $\Psi_\lambda$  maps each leaf of the lamination of  $J_0^+ \cup U_0^+$  bijectively to a leaf of the lamination of  $J_\lambda^+ \cup U_\lambda^+$ , the image of the leaves through points in  $Y$  contains a neighborhood in  $J_\lambda^+ \cup U_\lambda^+$  of the point  $p_\lambda^\infty$ . Theorem 6.6 implies



**Figure 2** Corresponding points under the map  $\Psi_\lambda$

that  $\Psi_\lambda$  converges uniformly when approaching a limit leaf. Together, these facts imply that  $\Psi_\lambda$  maps a small neighborhood in  $J_0^+ \cup U_0^+$  of  $p_0^\infty$  to a neighborhood of  $p_\lambda^\infty$  in  $J_\lambda^+ \cup U_\lambda^+$ . In particular, this image includes  $p_\lambda^j$  for all large  $j$ , so the preimage, the small neighborhood, includes  $p_0^j$  for all large  $j$ , which contradicts  $\|p_0^j\| \rightarrow \infty$ .  $\square$

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G. T. Buzzard  
 Department of Mathematics  
 Cornell University  
 Ithaca, NY 14853  
 gbuzzard@math.cornell.edu

K. Verma  
 Department of Mathematics  
 University of Michigan  
 Ann Arbor, MI 48109  
 kverma@umich.edu