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MEASURING THE STATISTICAL ISOTROPY OF THE COSMIC MICROWAVE BACKGROUND ANISOTROPY

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ABSTRACT

The statistical expectation values of the temperature fluctuations of the cosmic microwave background (CMB) are assumed to be preserved under the rotations of the sky. This assumption of the *statistical isotropy* (SI) of the CMB anisotropy should be observationally verified since detection of a violation of SI could have profound implications for cosmology. We propose a set of measures, κ_ℓ ($\ell = 1, 2, 3, \ldots$), for detecting a violation of SI in an observed CMB anisotropy sky map indicated by nonzero κ_ℓ. We define an estimator for the κ_ℓ spectrum and analytically compute its cosmic bias and cosmic variance. The results match those obtained by measuring κ_ℓ using simulated sky maps. Nonzero (bias-corrected) κ_ℓ larger than the SI cosmic variance will imply a violation of SI. The SI measure proposed in this Letter is an appropriate statistic to investigate a preliminary indication of SI violation in the recently released *Wilkinson Microwave Anisotropy Probe* data.

Subject headings: cosmic microwave background — cosmology: observations

On-line material: color figure

Cosmic microwave background (CMB) anisotropy is a very powerful observational probe of cosmology. In standard cosmology, CMB anisotropy is expected to be statistically isotropic; i.e., the statistical expectation values of the temperature fluctuations $\Delta T(\hat{\boldsymbol{q}})$ are preserved under the rotations of the sky. In particular, the angular correlation function $C(\hat{\boldsymbol{q}}, \hat{\boldsymbol{q}}') \equiv \langle \Delta T(\hat{\boldsymbol{q}}) \Delta T(\hat{\boldsymbol{q}}')\rangle$ is rotationally invariant for Gaussian fields. In spherical harmonic space, where $\Delta T(\hat{\boldsymbol{q}}) = \sum_{lm} a_{lm} Y_{lm}(\hat{\boldsymbol{q}})$, this translates to a diagonal $\langle a_{lm} a^*_{l'm'}\rangle = C_l \delta_{ll'} \delta_{mm'}$, where C_l, the widely used angular power spectrum of CMB anisotropy, is a complete description of (Gaussian) CMB anisotropy. Hence, it is important to be able to determine whether the observed CMB sky is a realization of a statistically isotropic process or not.[1]

We propose a set of measures κ_ℓ ($\ell = 1, 2, 3, \ldots$) that for nonzero values indicate and quantify a violation in statistical isotropy (SI) in a CMB map. A null detection of κ_ℓ will be a direct confirmation of the *assumed* SI of the CMB sky. It will also justify a model comparison based on the angular power spectrum C_l only (Bond, Pogosyan, & Souradeep 1998, 2000a, 2000b; Souradeep 2000). The detection of an SI violation can have exciting and far-reaching implications for cosmology. In particular, an SI violation in CMB anisotropy is the most generic signature of nontrivial geometrical and topological structure of space on ultralarge scales. Nontrivial cosmic topology is a theoretically well-motivated possibility that is only recently being observationally probed on the largest scales (Ellis 1971; Lachieze-Rey & Luminet 1995; Starkman 1998; Levin 2002).

For a statistically isotropic CMB sky, the correlation function

$$C(\hat{\boldsymbol{n}}_1, \hat{\boldsymbol{n}}_2) \equiv C(\hat{\boldsymbol{n}}_1 \cdot \hat{\boldsymbol{n}}_2) = \frac{1}{8\pi^2} \int d\mathcal{R}\, C(\mathcal{R}\hat{\boldsymbol{n}}_1, \mathcal{R}\hat{\boldsymbol{n}}_2), \quad (1)$$

where $\mathcal{R}\hat{\boldsymbol{n}}$ denotes the direction obtained under the action of a rotation $\mathcal{R}$ on $\hat{\boldsymbol{n}}$, and $d\mathcal{R}$ is a volume element of the three-dimensional rotation group. The invariance of the underlying statistics under rotation allows the estimation of $C(\hat{\boldsymbol{n}}_1 \cdot \hat{\boldsymbol{n}}_2)$ using the average of the temperature product $\widetilde{\Delta T}(\hat{\boldsymbol{n}})\widetilde{\Delta T}(\hat{\boldsymbol{n}}')$ between all pairs of pixels with the angular separation θ. In particular, for a CMB temperature map $\widetilde{\Delta T}(\hat{\boldsymbol{q}}_i)$ defined on a discrete set of points on a celestial sphere (pixels) $\hat{\boldsymbol{q}}_i$ ($i = 1, \ldots, N_p$),

$$\tilde{C}(\theta) = \sum_{i,j=1}^{N_p} \widetilde{\Delta T}(\hat{\boldsymbol{q}}_i)\widetilde{\Delta T}(\hat{\boldsymbol{q}}_j)\delta(\cos\theta - \hat{\boldsymbol{q}}_i \cdot \hat{\boldsymbol{q}}_j) \quad (2)$$

is an estimator of the correlation function $C(\theta)$ of an underlying SI statistic.[2]

In the absence of SI, $C(\hat{\boldsymbol{q}}, \hat{\boldsymbol{q}}')$ is estimated by a single product $\widetilde{\Delta T}(\hat{\boldsymbol{q}})\widetilde{\Delta T}(\hat{\boldsymbol{q}}')$ and hence is poorly determined from a single realization. Although it is not possible to estimate each element of the full correlation function $C(\hat{\boldsymbol{q}}, \hat{\boldsymbol{q}}')$, some measures of the statistical anisotropy of the CMB map can be estimated through suitably weighted angular averages of $\widetilde{\Delta T}(\hat{\boldsymbol{q}})\widetilde{\Delta T}(\hat{\boldsymbol{q}}')$. The angular averaging procedure should be such that the measure involves averaging over a sufficient number of independent "measurements," but it should ensure that the averaging does not erase all of the signature of statistical anisotropy (as would happen in eq. [1] or eq. [2]). Another important desirable property is that the measures be independent of the overall orientation of the sky. Based on these considerations, we propose a set of measures κ_ℓ of SI violation given by

$$\kappa_\ell = \int d\Omega \int d\Omega' \left[\frac{(2\ell+1)}{8\pi^2} \int d\mathcal{R}\, \chi_\ell(\mathcal{R}) C(\mathcal{R}\hat{\boldsymbol{q}}, \mathcal{R}\hat{\boldsymbol{q}}')\right]^2, \quad (3)$$

where $C(\mathcal{R}\hat{\boldsymbol{q}}, \mathcal{R}\hat{\boldsymbol{q}}')$ is the two-point correlation between $\mathcal{R}\hat{\boldsymbol{q}}$ and $\mathcal{R}\hat{\boldsymbol{q}}'$ obtained by rotating $\hat{\boldsymbol{q}}$ and $\hat{\boldsymbol{q}}'$ by an element $\mathcal{R}$ of the rotation group. The measures κ_ℓ involve an angular average of the correlation weighed by the characteristic function of the rotation group $\chi_\ell(\mathcal{R}) = \sum_M D^\ell_{MM}(\mathcal{R})$, where $D^\ell_{MM'}$ are the Wigner *D*-functions (Varshalovich, Moskalev, & Khersonskii 1988). When $\mathcal{R}$ is expressed as rotation by an angle ω (where $0 \le \omega \le \pi$) about an axis $\hat{\boldsymbol{r}}(\Theta, \Phi)$, the characteristic function $\chi_\ell(\mathcal{R}) \equiv$

[1] Statistical isotropy of CMB anisotropy and its measurement have been discussed in the literature (Ferreira & Magueijo 1997; Bunn & Scott 2000).

[2] This simplified description does not include optimal weights to account for observational issues, such as instrument noise and nonuniform coverage. However, this is well studied in the literature, and therefore we avoid discussing them here in order to keep our presentation clear.

$\chi_\ell(\omega) = \sin[(2\ell+1)\omega/2]/\sin(\omega/2)$ is completely determined by ω, and the volume element of the three-dimensional rotation group is given by $d\mathcal{R} = 4\sin^2\omega/2\, d\omega \sin\Theta\, d\Theta\, d\Phi$. Using the identity $\int d\mathcal{R}'\chi_\ell(\mathcal{R}')\chi_\ell(\mathcal{R}\mathcal{R}') = \chi_\ell(\mathcal{R})$, equation (3) can be simplified to

$$\kappa_\ell = \frac{(2\ell+1)}{8\pi^2}\int d\Omega \int d\Omega' C(\hat{\boldsymbol{q}}, \hat{\boldsymbol{q}}') \int d\mathcal{R}\,\chi_\ell(\mathcal{R}) C(\mathcal{R}\hat{\boldsymbol{q}}, \mathcal{R}\hat{\boldsymbol{q}}'), \tag{4}$$

containing only one integral over the rotation group. For a statistically isotropic model, $C(\mathcal{R}\hat{\boldsymbol{q}}_1, \mathcal{R}\hat{\boldsymbol{q}}_2) \equiv C(\hat{\boldsymbol{q}}_1, \hat{\boldsymbol{q}}_2)$ is invariant under rotation, and equation (4) gives $\kappa_\ell = \kappa^0\delta_{\ell 0}$ because of the orthonormality of $\chi_\ell(\omega)$. Hence, κ_ℓ defined in equation (3) is a measure of SI.

The measure κ_ℓ has a clear interpretation in harmonic space. The two-point correlation $C(\hat{\boldsymbol{q}}, \hat{\boldsymbol{q}}')$ can be expanded in terms of the orthonormal set of bipolar spherical harmonics as

$$C(\hat{\boldsymbol{q}}, \hat{\boldsymbol{q}}') = \sum_{ll'\ell M} A^{\ell M}_{ll'}\{Y_l(\hat{\boldsymbol{q}}) \otimes Y_{l'}(\hat{\boldsymbol{q}}')\}_{\ell M}, \tag{5}$$

where $A^{\ell M}_{ll'}$ are the coefficients of the expansion. These coefficients are related to an "angular momentum" sum over the covariances $\langle a_{lm}a^*_{l'm'}\rangle$ as

$$A^{\ell M}_{ll'} = \sum_{mm'}\langle a_{lm}a^*_{l'm'}\rangle(-1)^{m'}\mathfrak{C}^{\ell M}_{lml'-m'}, \tag{6}$$

where $\mathfrak{C}^{\ell M}_{lml'm'}$ are Clebsch-Gordan coefficients. The bipolar functions transform just like an ordinary spherical harmonic function $Y_{\ell M}$ under rotation (Varshalovich et al. 1988). Substituting the expansion equation (5) into equation (3), we can show that

$$\kappa_\ell = \sum_{ll'M}|A^{\ell M}_{ll'}|^2 \geq 0 \tag{7}$$

is positive semidefinite and can be expressed in the form

$$\kappa_\ell = \frac{2\ell+1}{8\pi^2}\int d\mathcal{R}\,\chi_\ell(\mathcal{R})\sum_{lml'm'}\langle a_{lm}a^*_{l'm'}\rangle\langle a_{lm}a^*_{l'm'}\rangle^R, \tag{8}$$

where $\langle\ldots\rangle^R$ is computed in a frame rotated by $\mathcal{R}$. When SI holds, $\langle a_{lm}a^*_{l'm'}\rangle = C_l\delta_{ll'}\delta_{mm'}$, implying $A^{\ell M}_{ll'} = (-1)^l C_l(2l+1)^{1/2}\delta_{ll'}\delta_{\ell 0}\delta_{M0}$. The coefficients A^{00}_{ll} represent the statistically isotropic part of a general correlation function. The coefficients $A^{\ell M}_{l_1 l_2}$ are the inverse transform of the two-point correlation

$$A^{\ell M}_{l_1 l_2} = \int d\Omega \int d\Omega'\, C(\hat{\boldsymbol{n}}, \hat{\boldsymbol{n}}')\{Y_{l_1}(\hat{\boldsymbol{n}}) \otimes Y_{l_2}(\hat{\boldsymbol{n}}')\}^*_{\ell M}. \tag{9}$$

The symmetry $C(\hat{\boldsymbol{n}}, \hat{\boldsymbol{n}}') = C(\hat{\boldsymbol{n}}', \hat{\boldsymbol{n}})$ implies

$$A^{\ell M}_{l_2 l_1} = (-1)^{(l_1+l_2-\ell)}A^{\ell M}_{l_1 l_2}, \quad A^{\ell M}_{ll} = A^{\ell M}_{ll}\delta_{\ell, 2k}, \quad k = 0, 1, 2, \ldots. \tag{10}$$

Recently, the *Wilkinson Microwave Anisotropy Probe* (*WMAP*) has provided high-resolution (almost) full sky maps of CMB anisotropy (Bennet et al. 2003) from which κ_ℓ can be measured. Given a single independent CMB map, $\widetilde{\Delta T}(\hat{\boldsymbol{q}})$, we need to look for a violation of SI. Formally, the estimation procedure involves averaging the product of temperature at pairs of pixels obtained by rotating a given pair of pixels by an angle ω around a sufficiently large sample of rotation axes. The integral in the brackets in equation (3) is estimated by summing up the terms for different values of ω weighed by the characteristic function. We can define an estimator for κ_ℓ as

$$\tilde{\kappa}_\ell = \tilde{\kappa}^B_\ell - \mathfrak{B}_\ell,$$
$$\tilde{\kappa}^B_\ell = \frac{(2\ell+1)}{8\pi^2}\sum_{i,j=1}^{N_p}\widetilde{\Delta T}(\hat{\boldsymbol{q}}_i)\widetilde{\Delta T}(\hat{\boldsymbol{q}}_j)\sum_{m=1}^{N_w}\chi_\ell(w_m) \times \sum_{n=1}^{N_r}\widetilde{\Delta T}(\mathcal{R}_{mn}\hat{\boldsymbol{q}}_i)\widetilde{\Delta T}(\mathcal{R}_{mn}\hat{\boldsymbol{q}}_j), \tag{11}$$

where, as described below, $\mathfrak{B}_\ell \equiv \langle\tilde{\kappa}^B_\ell\rangle$ accounts for the "cosmic bias" for the biased estimator $\tilde{\kappa}^B_\ell$. As with the sky, the rotation group is also discretized as $\mathcal{R}_{mn}$, where $m = 1, \ldots, N_w$ is an index of equally spaced intervals in rotation angle w and $n = 1, \ldots, N_r$ indexes a set of equally spaced directions in the sky. While we have also implemented this real-space computation, practically, we find it faster to estimate κ_ℓ in the harmonic space by taking advantage of fast methods of the spherical harmonic transform of the map. In harmonic space, we first define an unbiased estimator for the bipolar harmonic coefficients based on equation (6) and then estimate κ_ℓ using equation (7),

$$\tilde{A}^{\ell M}_{ll'} = \sum_{mm'}a_{lm}a_{l'm'}\mathfrak{C}^{\ell M}_{lml'm'}, \quad \tilde{\kappa}_\ell = \sum_{ll'M}\left|\tilde{A}^{\ell M}_{ll'}\right|^2 - \mathfrak{B}_\ell. \tag{12}$$

Assuming Gaussian statistics of the temperature fluctuations, the cosmic bias is given by (A. Hajian & T. Souradeep 2003, in preparation)

$$\langle\tilde{\kappa}^B_\ell\rangle - \kappa_\ell = \sum_{l_1,l_2}\sum_{m_1,m_1'}\sum_{m_2,m_2'}[\langle a^*_{l_1m_1}a_{l_1m_1'}\rangle\langle a^*_{l_2m_2}a_{l_2m_2'}\rangle + \langle a^*_{l_1m_1}a_{l_2m_2'}\rangle\langle a^*_{l_2m_2}a_{l_1m_1'}\rangle] \times \sum_M \mathfrak{C}^{\ell M}_{l_1m_1l_2m_2}\mathfrak{C}^{\ell M}_{l_1m_1'l_2m_2'}. \tag{13}$$

Given a single CMB sky map, the individual elements of the $\langle a_{lm}a^*_{l'm'}\rangle$ covariance are poorly determined. So we can correct for the bias $\mathfrak{B}_\ell$ that arises from the SI part of correlation function where

$$\mathfrak{B}_\ell \equiv \langle\tilde{\kappa}^B_\ell\rangle_{\rm SI} = (2\ell+1)\sum_{l_1}\sum_{l_2=|\ell-l_1|}^{\ell+l_1}C_{l_1}C_{l_2}[1+(-1)^\ell\delta_{l_1l_2}]. \tag{14}$$

Hence, for an SI correlation, the estimator $\tilde{\kappa}_\ell$ is unbiased, i.e., $\langle\tilde{\kappa}_\ell\rangle = 0$.

Assuming Gaussian CMB anisotropy, the cosmic variance of the estimators $\tilde{A}^{\ell M}_{ll'}$ and $\tilde{\kappa}_\ell$ can be obtained analytically for full sky maps. The cosmic variance of the bipolar coefficients

$$\sigma^2(\tilde{A}^{\ell M}_{l_1l_2}) = \sum_{m_1,m_1'}\sum_{m_2,m_2'}[\langle a_{l_1m_1}a_{l_1m_1'}\rangle\langle a_{l_2m_2}a_{l_2m_2'}\rangle + \langle a_{l_1m_1}a_{l_2m_2'}\rangle\langle a_{l_2m_2}a_{l_1m_1'}\rangle] \times \mathfrak{C}^{\ell M}_{l_1m_1l_2m_2}\mathfrak{C}^{\ell M}_{l_1m_1'l_2m_2'}, \tag{15}$$

which, for an SI correlation, further simplifies to

$$\sigma^2_{\rm SI}(\tilde{A}^{\ell M}_{l_1 l_2}) = C_{l_1} C_{l_2} [1 + (-1)^{\ell} \delta_{l_1 l_2}] \times \sum_{m_1, m_2} (-1)^{m_1+m_2} \mathfrak{C}^{\ell M}_{l_1 m_1 l_2 m_2} \mathfrak{C}^{\ell M}_{l_1 m_1' l_2 m_2'}. \quad (16)$$

Note that for $l_1 = l_2$, the cosmic variance is zero for odd ℓ as a result of equation (10) arising from the symmetry of $C(\hat{\boldsymbol{q}}, \hat{\boldsymbol{q}}')$.

A similar but more tedious computation of 105 terms of the eight-point correlation function yields an analytic expression for the cosmic variance of $\tilde{\kappa}_\ell$ (A. Hajian & T. Souradeep 2003, in preparation). For the SI correlation, the cosmic variance for $\ell > 0$ is given by

$$\begin{aligned}\sigma^2_{\rm SI}(\tilde{\kappa}_\ell) = &\sum_{l:2l\geq\ell} 4C_l^4 \{2\frac{(2\ell+1)^2}{2l+1} + (-1)^{\ell}(2\ell+1) \\ &+ [1 + 2(-1)^{\ell}] F^{\ell}_{ll}\} \\ &+ \sum_{l_1} \sum_{l_3=|\ell-l_1|}^{\ell+l_1} 4C_{l_1}^2 C_{l_3}^2 [(2\ell+1) + F^{\ell}_{l_1 l_3}] \\ &+ 8\sum_{l_1} \frac{(2\ell+1)^2}{2l_1+1} C_{l_1}^2 \left[\sum_{l_2=|\ell-l_1|}^{\ell+l_1} C_{l_2}\right]^2 \\ &+ 16(-1)^{\ell} \sum_{l_1:2l_1\geq\ell} \frac{(2\ell+1)^2}{2l_1+1} \sum_{l_2=|\ell-l_1|}^{\ell+l_1} C_{l_1}^3 C_{l_2}, \end{aligned} \quad (17)$$

where

$$F^{\ell}_{l_1 l_3} = \sum_{m_1 m_2 = -l_1}^{l_1} \sum_{m_3 m_4 = -l_3}^{l_3} \sum_{M, M' = -\ell}^{\ell} C^{\ell M}_{l_1 -m_1 l_3 -m_3} C^{\ell M}_{l_1 m_2 l_3 m_4} \times C^{\ell M'}_{l_3 m_4 l_1 m_1} C^{\ell M'}_{l_3 -m_3 l_1 -m_2}. \quad (18)$$

Numerically, it is advantageous to rewrite $F^{\ell}_{ll'}$ in a series involving 9-j symbols. The expressions for variance and bias are valid for full sky CMB maps. For observed maps, one has to contend with incomplete or nonuniform sky coverage. In such cases, one could estimate the cosmic bias and variance by averaging over many independent realizations of simulated CMB sky from the same underlying correlation function. Figure 1 shows the measurement of κ_ℓ in an SI model with a flat-band power spectrum. The bias and variance are estimated by taking measurements of 50 independent random full sky maps using the HEALPix software package.[3] The cosmic bias and variance obtained from these realizations match the analytical results. Just as in the case of cosmic bias, the cosmic variance of κ_ℓ at odd multipoles is smaller. The figure clearly shows that the envelope of cosmic variance for odd and even multipoles converge at large ℓ. For a constant $l(l+1)C_l$ angular power spectrum, the $\sigma_{\rm SI}(\tilde{\kappa}_\ell)$ falls off with ℓ. (The absence of the dipole and monopole in the maps affects κ_ℓ for $\ell < 4$, leading to the apparent rise in cosmic variance at $\ell < 4$ seen in Fig. 1.)

The bias and cosmic variance depend on the total SI angular power spectrum of the signal and noise, $C_l = C_l^S + C_l^N$. Hence, where possible, prior knowledge of the expected κ_ℓ signal should be used to construct multipole space windows to weigh down

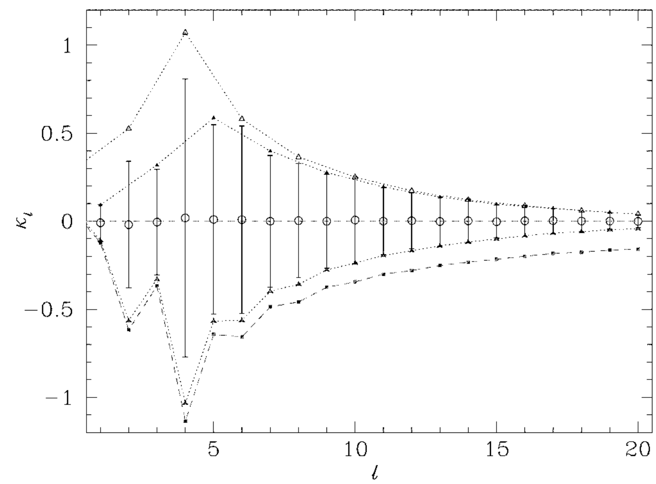

FIG. 1.—Bias-corrected "measurement" of κ_ℓ of an SI CMB sky with a flat-band power spectrum smoothed by a Gaussian beam $[l(l+1)C_l = \exp(-l^2/18^2)]$. The cosmic error $\sigma(\kappa_\ell)$, obtained using 50 independent realizations of the CMB (full) sky map, match the analytic results shown by the lower dotted curve with stars. The upper dotted curves separately outline the cosmic error envelope for odd multipoles (*filled triangles*) and for even multipoles (*open triangles*). Violation of SI will be indicated by nonzero κ_ℓ measured in an observed CMB map in excess of $\sigma(\kappa_\ell)$ given by the C_l of the map. The lower dashed curve (*filled squares*) shows the cosmic error for an ideal unit flat-band power spectrum $[l(l+1)C_l = 1]$ with no beam smoothing. The curve falls off roughly at $1/\ell$ at large ℓ. [*See the electronic edition of the Journal for a color version of this figure.*]

the contribution from the region of multipole space where the SI violation is not expected, e.g., the generic breakdown of SI due to cosmic topology. The underlying correlation patterns in the CMB anisotropy in a multiply connected universe is related to the symmetry of the Dirichlet domain (Wolf 1984; Vinberg 1993). In a companion paper, we study the κ_ℓ signal expected in flat, toroidal models of the universe and connect the spectrum to the principle directions in the Dirichlet domain (Hajian & Souradeep 2003). The SI violation arising from cosmic topology is usually limited to low multipoles. A wise detection strategy would be to smooth CMB maps to a low angular resolution. When searching for a specific form of SI violation, linear combinations of κ_ℓ can be used to optimize the signal-to-noise ratio. Before ascribing the detected breakdown of statistical anisotropy to cosmological or astrophysical effects, one must carefully account for and model into the SI simulations other mundane sources of SI violation in real data, such as incomplete and nonuniform sky coverage, beam anisotropy, foreground residuals, and statistically anisotropic noise.

In summary, the κ_ℓ statistics quantifies the breakdown of SI into a set of numbers that can be measured from the single CMB sky available. The κ_ℓ spectrum can be measured very fast, even for high-resolution CMB maps. The statistics have a very clear interpretation as quadratic combinations of off-diagonal correlations between a_{lm} coefficients. The signal SI violation is related to underlying correlation patterns. The angular scale on which the off-diagonal correlations (patterns) occur is reflected in the κ_ℓ spectrum. As a tool for detecting cosmic topology (more generally, cosmic structures on ultra-large scales), the κ_ℓ spectrum has the advantage of being independent of the overall orientation of the correlation pattern. This is particularly suited for searching for cosmic topology since the signal is independent of the orientation of the Dirichlet domain. (However, orientation information is available in the $A^{\ell M}_{l_1 l_2}$.) The recent all-sky CMB map from *WMAP* is an ideal

[3] Publicly available at http://www.eso.org/science/healpix (Górski, Hivon, & Wandelt 1999).

data set where one can measure the SI. Interestingly, there are hints of SI violation in the low multipole of *WMAP* (Tegmark, de Oliveira-Costa, & Hamilton 2003; de Oliveira-Costa et al. 2003; Eriksen et al. 2003). Hence, it is of great interest to make a careful estimation of the SI violation in the *WMAP* data via the κ_ℓ spectrum. This work is in progress, and results will be reported elsewhere (A. Hajian et al. 2003, in preparation). This approach complements the direct search for the signature of cosmic topology (Cornish, Spergel, & Starkman 1998).

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