BOUNDS AND ERROR CONTROL FOR EIGENVALUES

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SYNOPSIS

Control and estimation of errors are important but difficult aspects of any analysis from which the numerical results are necessarily approximate. The order of difficulty is greater for local or distributed quantities like stresses and displacements than for global or integrated parameters like eigenvalues and stiffnesses. To really bound a desired quantity between a pair of close upper and lower bounds one should obtain either an oscillatory but clear convergence or, preferably, two rapidly converging sequences one from above and the other from below. Application of the two complementary variational principles of energy and complimentary energy, when both are possible to apply, do yield upper and lower bound approximations. But these or other alternate methods are generally expensive. On the other hand it would be advantageous if one basic procedure could be perturbed in a simple manner to provide both lower and upper bounds and to refine the solution and control the errors without undue effort. This paper discusses this concept and presents three powerful methods to closely bound any desired parameter in a problem. These are particularly valuable for eigenvalue problems.

1. INTRODUCTION

Control of errors and assessment of accuracy are important but difficult aspects of any (numerically) satisfactory approximate analysis. The order of difficulty is greater for local or distributed quantities like stresses and displacements than for global or integrated parameters like eigenvalues or stiffnesses. An upper or lower bound is assured for global quantities (only), if a variational principle is applied in its "pure" form. Where the complementary principles of displacement and force can both be applied comfortably, one can bound a global quantity between two limits, as for example in Veubeke[1,2]. In principle such bounds may be refined to the desired degree of closeness by increasing the number of parameters in the variational formulation; in practice, the total effort involved is often considerable. In other cases, a reliable estimate of errors in a particular solution is often a matter of judgement. Where a proof of convergence exists and the convergence trend with increasing number of terms or elements is known, in principle, the true value can be estimated from the convergence curve. But in practice the information is generally insufficient to obtain an accurate value or to establish reliable bounds with reasonable computational effort. Thus, to really bound a desired quantity between an overestimate and an underestimate, one should obtain either an oscillatory (clear) convergence or, preferably, two converging sequences, one from above and the other from below. Such a procedure, as yet, involves a certain amount of search for a suitable technique (or pairs of techniques) and, even when possible, is expensive. In view of the above position a further search for economical techniques to control and estimate errors is desirable. In particular, it would be advantageous if the same basic procedure could be perturbed in a simple manner to provide both upper and lower bounds and to refine the solution and control the errors without undue additional effort.

In the first instance our attention was focussed on the analysis of two-dimensional elasticity problems by the direct (or boundary) method in which the differential equation is exactly satisfied, while those boundary conditions (or parts thereof) which cannot be identically satisfied are approximately satisfied. Techniques were successfully evolved for determining close bounds for both local and global values in various problems [3–5].

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In finite element formulations neither the differential equation in the entire field, nor the boundary conditions are identically satisfied. So the procedures developed for the direct method cannot be translated directly to the finite element methods. A different approach was evolved for the determination of eigenvalues by the FEM [5]. In this, one artificially introduces a parameter into the formulation of the problem and perturbs this parameter systematically so as to modify the kinetic (or potential) energy without affecting the strain energy of the system or vice versa.

A study of methods for obtaining bounds using the continuum and finite element methods and the results therefrom lent strength to a feeling that if one has a monotonically converging curve for a parameter in a problem it should be possible to start with this curve and operate on it directly to refine it or to obtain the other bound, so that close bounds could be achieved for that parameter with little further effort. This approach also proved fruitful and it would appear to be most useful and economical procedure available. It is possible to establish analytical justification for the foregoing procedures.

Lynn, Ramey and Dhillon have shown an independent appreciation of the line of thought in our approach and have proposed procedures to either obtain the opposite bound [7] or to refine the available bounds [8] for an eigenvalue. The technique of Lindberg and Olson [9] is also of interest in the context of bounding the eigenvalues. The approach to be described in the present paper will be seen to be more comprehensive and applicable to local and global quantities in a wide range of problems.

2. THE DIRECT (BOUNDARY) METHOD OF CONTINUUM ANALYSIS

We will first briefly consider the procedure for obtaining bounds by direct methods of continuum analysis. It is convenient to present the approach through a typical example. Consider a simply supported regular polygonal plate of n sides (each of length 'a') and flexural rigidity 'D', under uniform pressure q [5]. Referring to the inset in Fig. 1, and using the parameter $\bar{w} = Dw/qa^4$, the governing differential equation is

$$\nabla^4 \bar{w} = 1/a^4 \tag{1}$$

and the boundary conditions are

$$\bar{w} = 0, \quad \nabla^2 \bar{w} = 0 \text{ along the periphery.} \tag{2}$$

A series solution is conveniently written in polar coordinates as

$$\bar{w} = r^4/64a^4 + \sum_{m=0}^{M-1} (A_m + B_m r^2)\, r^{mn} \cos(mn\theta) \tag{3}$$

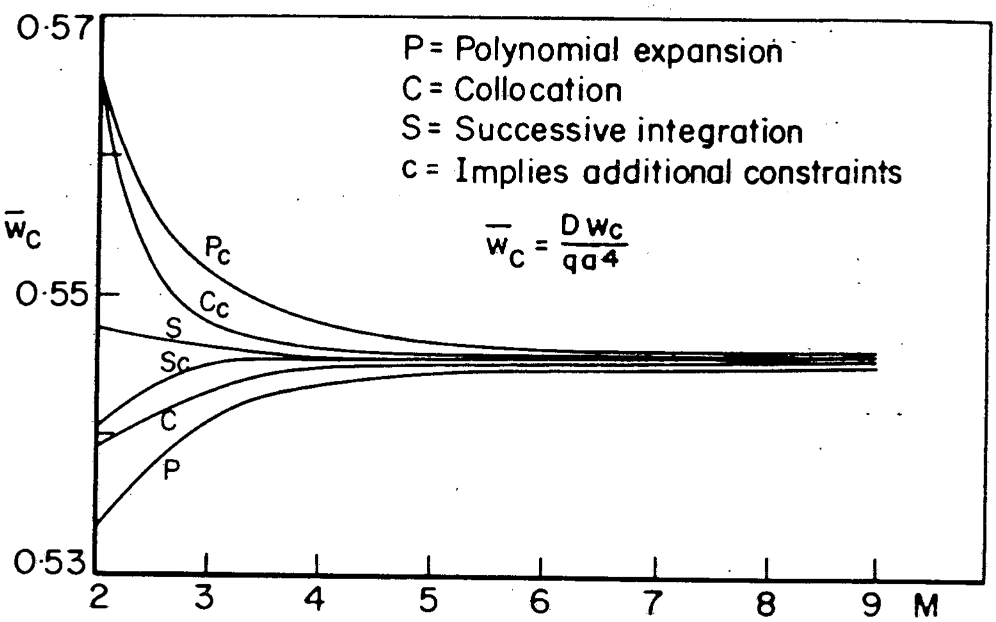
where A_m, B_m are unknown parameters to be determined. $\bar{w}$ satisfies identically the differential equation and the multiple symmetry of the problem. The constants A_m, B_m are to be evaluated by approximately satisfying the boundary conditions on the polygonal edge by a suitable procedure. Any procedure leads to a set of linear simultaneous equations in A_m, B_m. A converging sequence of solutions is obtained by systematically progressing the stage of truncation (i.e., by increasing M). Considering a parameter of the problem, each procedure yields a different converging trend. With a given procedure, the convergence trends are different for different parameters.

There are many standard procedures for such approximate satisfaction of boundary conditions. Collocation, Taylor expansion (also described in the present context as polynomial expansion), least squares and successive integration provide a sufficient range of basic techniques on account of the differences in the analytical and mathematical details and in the distribution of residual errors on the boundary. The Taylor expansion method results in the boundary error increasing away from the origin for expansion. The collocation method has zero error at predetermined locations. The other methods yield zero errors at points determined by the analysis.

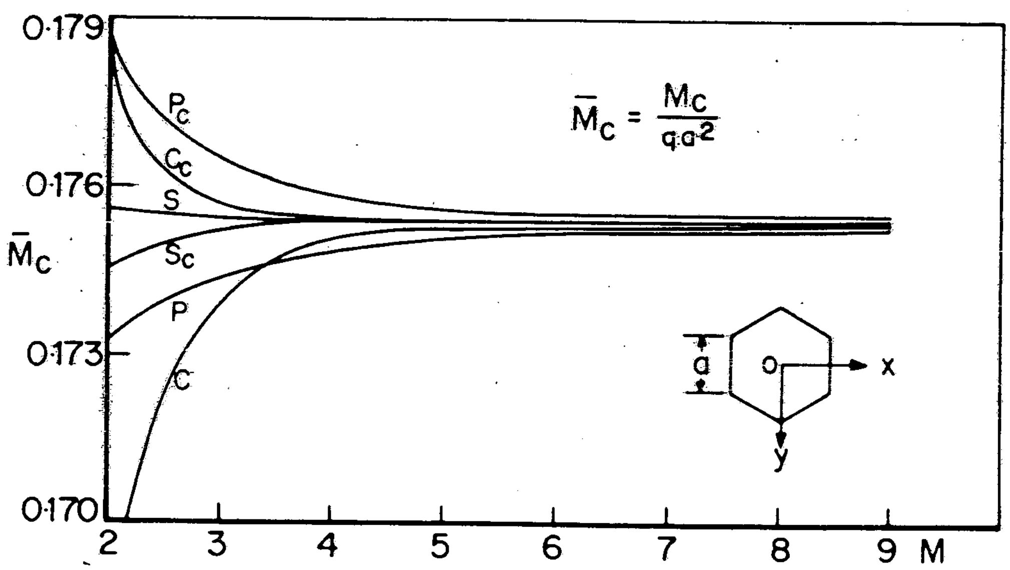
For the regular hexagon under study, convergence by the expansion (P), collocation (C) and integration (S) methods is shown in Figs. 1(a) and (b) for central deflection and central moment, respectively.

Consider now the Taylor expansion procedure used. The edge error in each of the two boundary conditions increases from zero at the mid point of the side to a maximum at the corners. This characteristic distribution of the edge error can be altered by imposing a constraint that the errors at the corners should also be zero. Correspondingly, the convergence curves will be altered from P to P_c in Fig. 1. In this case P and its variant P_c are both monotonic and provide lower and upper bounds respectively for the central deflection and moment. With equidistant collocation, the original solution (C) includes corner collocation and the convergence yields a lower bound. Relaxing the collocation at the corner by deleting the corresponding equations yields an upper bound curve (C_c). In the successive integration procedure the original solution (S) yields a converging over-estimate while the variant (S_c) yields underestimates.

The advantage with the technique used is obvious. The programme for the original procedure can include a simple instruction for inclusion or deletion of a particular set of equations to yield two converging sequences. The constraints are generally obvious from physical considerations. They may be natural (force) or kinematic in nature.



(a) CONVERGENCE OF CENTRAL DEFLECTION



(b) CONVERGENCE OF CENTRAL MOMENT

FIG. 1 CLAMPED HEXAGONAL PLATE UNDER UNIFORM LOAD q.

The foregoing method has been applied extensively with success. Early applications were to problems of St. Venant torsion, plate flexure and stress concentrations in perforated plates [3–5]. The results of a simple example of a square sectioned shaft under St. Venant torsion [3] is shown in Fig. 2. Here we notice that all the basic procedures applied and their variants (achieved by applying constraints) yield monotonic convergence for the torsion constant J which is a global quantity, but some of them yield oscillatory convergence for a local quantity like the maximum shear stress.

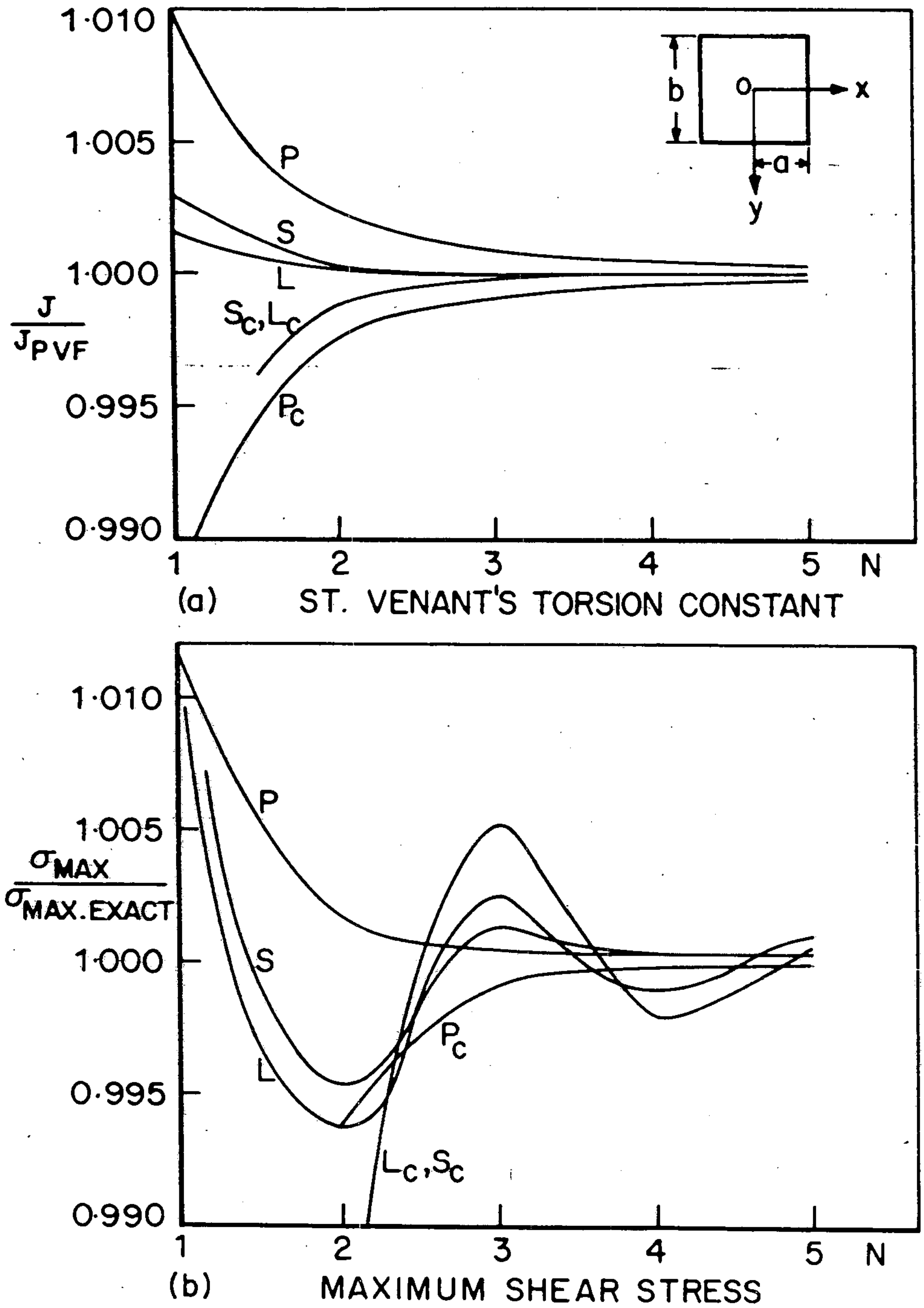


FIG. 2 ST. VENANT'S TORSION OF A SOLID SQUARE SECTION

3. AN ENERGY PERTURBATION METHOD FOR EIGENVALUES

For purposes of discussion, we can consider a vibration problem without any loss of generality. The governing equation for natural vibration takes the form,

$$m\ddot{x} + kx = 0 \tag{4}$$

which in matrix notation is written as

$$[K]\{V\} = \lambda[H]\{V\} \tag{5}$$

where $[K]$ is the elastic stiffness matrix, $[H]$ is the mass matrix, $\{V\}$ is the vector of kinematic freedoms and λ the eigenvalue. This equation provides as many eigenvalues as there are kinematic freedoms. In view of the errors implicit in the finite element formulation, the eigenvalues are obtained only approximately, the fractional errors increasing progressively with the order of the eigenvalue. If in a formulation, $\{V_p\}$ is the eigen mode corresponding to the p-th eigenvalue λ_p, the orthogonality between the modes leads to the relationship

$$\lambda_p = \frac{\{V_p\}^T[K]\{V_p\}}{\{V_p\}^T[H]\{V_p\}} = \frac{\text{Elastic Energy}}{\text{Kinetic Energy}} \tag{6}$$

An examination of this expression suggests a simple method of perturbing λ_p. If a suitable scalar parameter A is introduced into the formulation of either K or H, the value of λ_p is modified. If one takes an upper bound solution and introduces A such that the kinetic energy is increased, it leads to a reduction in λ_p. If one has a converging sequence of upper bounds for λ_p by systematically increasing the number of elements, then, by introducing the parameter A and giving it various values in steps, one may obtain refined upper bound sequences and also lower bound sequences. Thus from two close values of A, one can obtain close bounds for λ_p. This possibility is explored in the rest of the section. It is obvious that by a parallel procedure of decreasing the kinetic energy one can obtain an upper bound sequence from a lower bound sequence. It is also evident that in a similar manner one could perturb K to alter the elastic energy in the numerator of Eq. (6).

3.1 Torsional Vibration of a Uniform Cantilever Shaft

A preliminary understanding of the proposed method may be achieved by considering a lumped inertia finite element solution of the simple problem of torsional vibration of a uniform cantilever shaft [10]. For a shaft of length L, mass density p, torsional constant J, shear modulus G, and polar moment of inertia I, the governing equation is

$$GJ\theta'' + \rho\omega^2 I\theta = 0 \tag{7}$$

where θ is the angle of twist, ω is the angular frequency and the primes denote differentiation with respect to axial distnace. We define a frequency parameter λ as

$$\lambda = \rho\omega^2 I L^2/GJ \tag{8}$$

It is shown in Ref. 11 that if the shaft is divided into N elements of equal length ℓ_N, Eq. (7) may be represented by algebraic recurrence equations

$$(\theta_{n+1} - 2\theta_n + \theta_{n-1}) + (\rho\omega^2 \ell_N^2 I/6GJ)(\theta_{n+1} + 4\theta_n + \theta_{n-1}) = 0 \tag{9}$$

or

$$\theta_{n+1} - (2-\delta)\theta_n + \theta_{n-1} = 0\ ,\quad n < N \tag{10}$$

where

$$\delta = \frac{\lambda/N^2}{1+\lambda/6N^2} \quad \text{or} \quad \lambda = \frac{\delta N^2}{1-\delta/6}$$

together with the boundary conditions

$$\theta_0 = 0 \;;\; 2\theta_{N-1} - (2-\delta)\,\theta_N = 0 \tag{11}$$

The exact solution to Eqs. (10) and (11) yields (with $p \leq N$),

$$\delta_{p,N} = 2\left[1 - \cos((2p-1)\,\pi/2N)\right]$$

or

$$\lambda_{p,N} = 6N^2\left[1 - \cos\frac{(2p-1)\pi}{2N}\right] \Bigg/ \left[2 + \cos\frac{(2p-1)\pi}{2N}\right] \tag{12}$$

where $\lambda_{p,N}$ is the frequency parameter for the p-th mode obtained from an N-element solution.

We now propose that the second part of Eq. (9) be multiplied by

$$R = \left[1 - A\left\{\frac{(2p-1)\pi}{2N}\right\}^2\right]^{-1} \tag{13}$$

where A is a positive scalar parameter, in order to inflate the inertial matrix and thus modify the kinetic energy. This leads to a modified value for the eigenvalue parameter,

$$\lambda'_{p,N} = R^{-1}\lambda_{p,N}$$

$$= 6N^2\left[1 - A\left\{\frac{(2p-1)\pi}{2N}\right\}^2\right]\left[1 - \cos\frac{(2p-1)\pi}{2N}\right] \Bigg/ \left[2 + \cos\frac{(2p-1)\pi}{2N}\right] \tag{14}$$

As A is positive, $\lambda'_{p,N} \leq \lambda_{p,N}$

In Fig. 3, the first mode values $\lambda'_{1,N}$ are plotted against N for different values of A in the range A = 0 to 0.11. We note in the passing that the figure can be generalised for $\lambda'_{p,N}$ by labelling the ordinate as

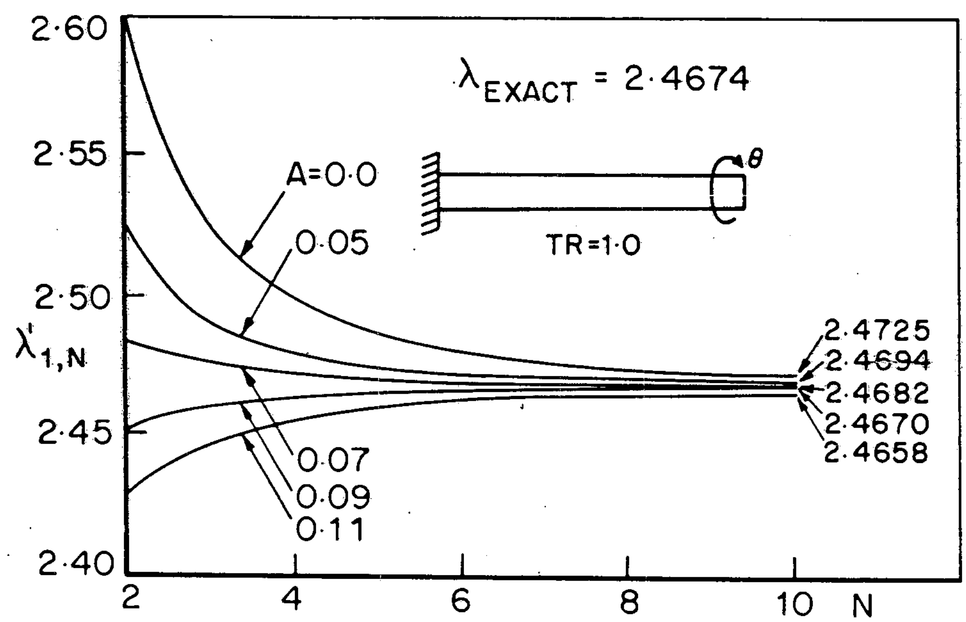


FIG. 3 FREQUENCY OF A UNIFORM SHAFT

NOTE: This Fig. is generalized for p^{th} mode by replacing $\lambda'_{1,N}$ by $\lambda'_{p,N}/(2p-1)^2$ for the ordinate & N by N/(2p−1) for the abscissa.

$\lambda'_{p,N}/(2P-1)^2$ and abscissa as $N/(2p-1)$. The curve $A = 0$, i.e., $\lambda_{1,N}$ by Eq. (12) shows monotonic convergence from above. The curves for $\lambda'_{1,N}$ using $A \leq 0.07$ all show the same trend. On the other hand for $A \geq 0.09$ the convergence is monotonic but from below. One may therefore expect that a horizontal-converging sequence should be achieved for a value between .07 and .09. But it can be seen intuitively and proved rigourously that a horizontal line cannot arise from Eq. (14). So we anticipate that there is a small range of A within which the convergence is non-monotonic. We can examine this directly because the exact solution for the problem is readily determined either directly from the differential equation or by a limiting process from Eq. (12) as

$$\lambda_p = \frac{(2p-1)^2\pi^2}{4} \qquad (15)$$

Thus $\lambda'_{p,N}$ is in error by

$$\begin{aligned} e_{p,N} &= \lambda'_{p,N} - \lambda_p \\ &= \left[\left\{6N^2 + \frac{(2p-1)^2\pi^2}{4} - \frac{3}{2}\,\frac{(2p-1)^2\pi^2}{4}\,A\right\}\left\{1-\cos\frac{(2p-1)\pi}{2N}\right\} - \frac{3\,(2p-1)^2\pi^2}{4}\right] \Bigg/ \left[2+\cos\frac{(2p-1)\pi}{2N}\right] \end{aligned} \qquad (16)$$

From this expression it follows that the error $e_{p,N}$ in the p-th eigenvalue estimated from the N element idealisation has the following dependence on the choice of A,

$$e_{p,N} \gtreqless 0 \text{ as } A \lesseqgtr \left[\frac{4N^2}{(2p-1)^2\pi^2}\right] + \frac{1}{6} - \frac{1}{4}\operatorname{cosec}^2\frac{(2p-1)\pi}{4N} \qquad (17)$$

Table I. Cross-over Values of the Parameter 'A'

N \ p	1	2
1	0.07195	–
2	0.08070	0.05383
3	0.08218	0.07195
4	0.08269	0.07628
5	0.08292	0.07990
6	0.08305	0.08070
7	0.08312	0.08138
8	0.08317	0.08187
9	0.08321	0.08218
10	0.08323	0.08239
∞	1/12	1/12

The crossover values of A for the errors in the first and second modes are given in Table I. It can readily be shown that for large $N/(2p-1)$, the crossover A approaches 1/12. The trends observed in Fig. 3 are now clearly explained. Further, it is seen that for values of A in the neighbourhood of 1/12 the convergence for $\lambda'_{P,N}/(2p-1)$ is monotonic beyond some $N/(2p-1)$. For instance considering the first mode, taking $A = .08$ the error is negative for $N = 1$ is positive for $N > 2$ and is monotonically convergent from above for $N \geq 3$. If the modifying function is well chosen then the convergence of $\lambda'_{P,N}$ should exhibit a monotonic trend from a relatively small number of elements, i.e., small $N/(2p-1)$. It is also to be noted that when the original solution is available for a large number of elements, it is preferable to obtain the bounds by using values of A which are not too close, for then one avoids the region in which the convergence trend is uncertain.

3.2 Choice of Modifying Functions

In the foregoing example we chose a simple modifying function which depends on the order of the eigenvalue and the order of approximation (N) but is identical for all elements. The factor (2p-1)/2N in the function can be recognised as a measure of the ratio of the element length to the wave length of the mode, a ratio which influences the error. In addition $R_{p,N}$, the modifying function for the p-th mode of an N-element idealisation, tends to unity monotonically with increasing N and therefore exhibits three characteristics:

1. $R_{p,N} > R_{p,N+1}$, for any N,
2. $R_{p,N} > 1$, for any N,
3. $R_{p,N} \to 1$ as $N \to \infty$.

For successful application of the method these three conditions are necessary to satisfy. But R can be made different for each element of the structure. In fact recognising that R is essentially a weighting function, it can be chosen to depend on the mode shape. A simple way of doing this is to link it to the end displacements θ_B, θ_C of each element BC obtained from the normalised eigenvector and write

$$R_{BC} = \left[1 - A^n (\theta_B - \theta_C)^n\right]^{-1} \qquad (18)$$

where A is constant for all elements of the shaft and N is a suitable positive number. As ab initio, θ_B, θ_C are unknown, a choice of this type involves iteration in the solution. The procedure can be extended to two dimensional problems such as vibration of plates. For instance the modifying function for parallelogram element idealisation can be written as

$$R = \left[1 - A^2 (V_{14} - V_{23})^2\right]^{-1} \left[1 - A^2 (V_{34} - V_{12})^2\right]^{-1} \qquad (19)$$

where $V_{ij} = (V_i + V_j)/2$ (i,j = 1,2,3,4) and V_i is the displacement for the i-th node of an element obtained from eigenvector.

Finally we recognise that a parallel procedure can be developed for obtaining close bounds from a solution which yields a lower bound sequence. In this case, the inertial terms in Eq. (5) for λ_p are to be deflated by dividing the mass matrix by functions of the type

$$R = \left[1 + A \frac{(2p-1)^2 \pi^2}{4 N^2}\right] > 1 \qquad (13a)$$

or

$$R = \left[1 + A^n (\theta_B - \theta_C)^n\right] > 1 \qquad (18a)$$

Application of the suggested procedure to two problems will be presented in the following subsections.

3.3 Procedure

As the modification function R given in Eqs. (18) or (19) contains the displacements, the method of numerical solution becomes iterative. The sequence of operatoins are as explained in the simplified flow chart in Fig. 4. It is our experience that normally one does not need many iterations. In the examples to be presented only 2 or 3 iterations were required.

To economise on the computational effort, it is necessary to first fix, without excessive effort a pair of values of A which will yield a pair of close over and under estimating sequences. This is best done by using solutoins with small numbers of elements (say 2 and 4 for beam type structures). Then one can obtain the converging sequences with these two values of A and use this information as the basis for chosing other values of A for further refinement of the bounds using larger values of N as needed.

3.4 Numerical Examples

The method of energy modification has been successfully applied to a large number of beam and palte vibration and stability problems [6]. We present here two examples, one of beam vibrations and the other of column stability.

3.4.1 Vibration of a uniform cantilever beam: The use of non-consistent elements yields a lower bound sequence for the first frequency of transverse vibrations of a uniform cantilever beam. For this problem

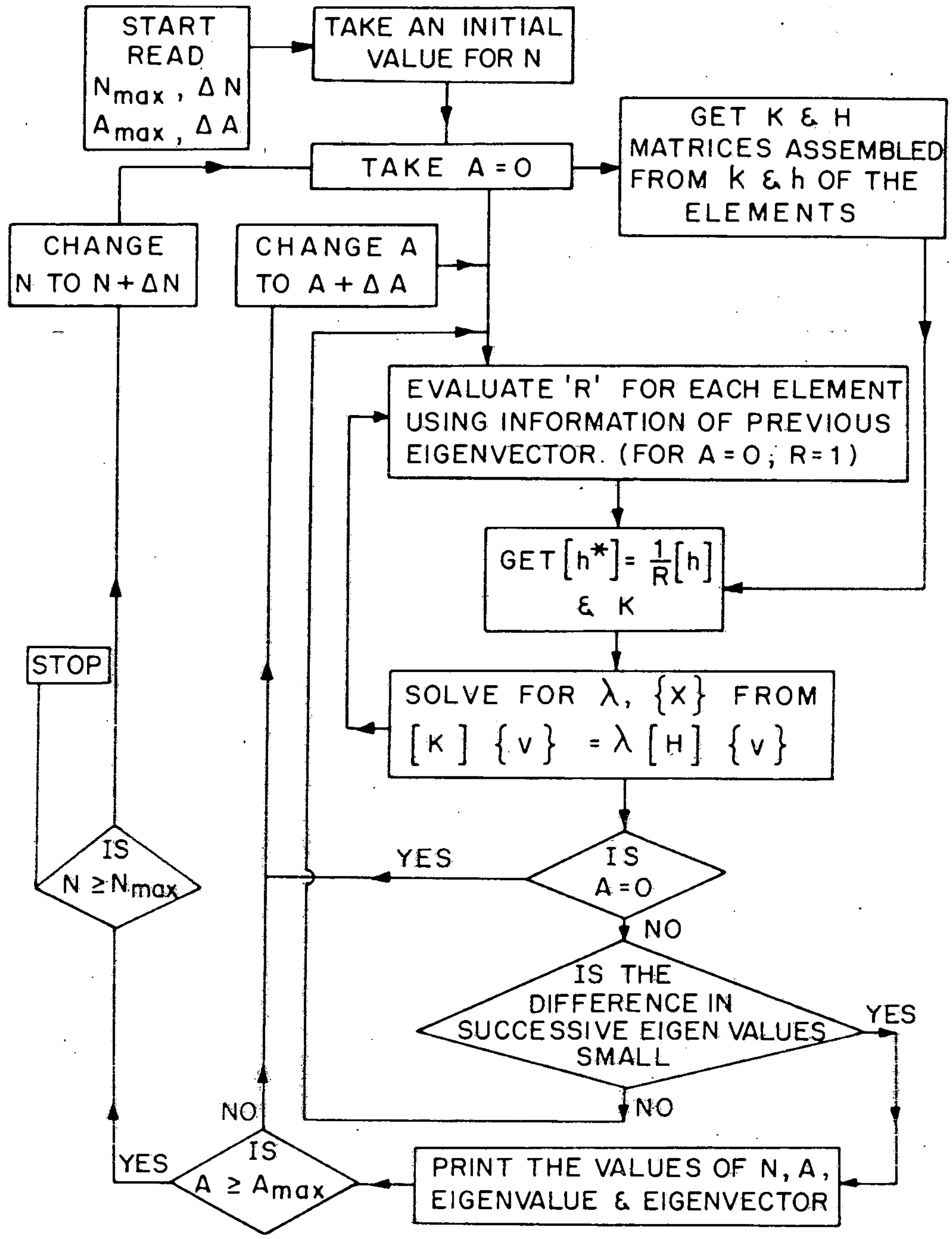


FIG. 4 FLOW CHART FOR OBTAINING BOUNDS FOR EIGENVALUES BY ENERGY PERTURBATION TECHNIQUE.

$$R = \left[1 + \frac{A^2}{2}(V_C - V_D)^2\right] \tag{20}$$

provides a simple modifying function. V_C and V_D are the elements of the eigenvector corresponding to the nodes C and D. Fig. 5 shows the original curve (A=0) and modified converging curves (A = 0.4, 0.6, 0.8) for frequency parameter λ_V. It is seen that A = 0.4 yields a substantially refined lower bound curve and A = 0.6 yields a good upper bound sequence. With 10 elements A = 0.4 and A = 0.6 yields a good bound between 12.3620 and 12.3841 where the true value is 12.3624.

3.4.2 Stability of linearly tapered circular column: In the stability problems the mass matrix of vibration problem is replaced by the geometric stiffness matrix. The buckling load parameter for a linearly tapered circular column (with taper ratio = 0.6) was initially analysed by FEM with nonconsistent elements. The result is an upper bound converging sequence. Using again the modifying function of Eq. (21) with A = 0.2, 0.3, 0.4, 0.5, 0.6 and 0.7 we obtained a series of converging curves yielding refined upper ($A \leq 0.4$) and lower ($A \geq 0.5$) bound sequences. These are shown in Fig. 6. With 10 elements A = 0.5 and 0.4 bound the parameter within 1/8% between 1.3082 and 1.3098.

$$R = \left[1 - \frac{A^2}{2}(V_C - V_D)^2\right]^{-1} \tag{21}$$

4. METHOD OF DIRECT MODIFICATIONS

A detailed study of the example in Section 3.1 suggests a more basic approach than that developed in Section 3 to obtain close bounds by direct modification of an available monotonically converging sequence. This method does not concern itself with the details of the actual problem, but concerns itself directly with properties of the convergence curve. As such it is applicable for both local and global values in a wide range of problems.

4.1 Basis

Consider the basic curve (A = 0) in Fig. 3. This is a monotonically converging lumped inertia finite element solution for the torsional frequency parameter of a uniform cantilever shaft. In view of the fact that the curve exhibits monotonic convergence, it can be expressed by an equation

$$\lambda_{p,N} = \lambda_p\left[1 \pm \frac{\alpha}{N^m} \pm \frac{\beta}{N^{m+1}} \pm \cdots\cdots\right] \tag{22}$$

with the provisos

1. m is a positive integer
2. $\lambda'_{p,N} \rightarrow \lambda_p$ monotonically as $N \rightarrow \infty$
3. $\alpha, \beta, \ldots$ are positive constants.

It is readily appreciated that with monotonic convergence from above and particularly with rapid convergence, the terms beyond α/N^m are negligible when N exceeds a reasonable value. Thus we can closely approximate $\lambda_{p,N}$ by

$$\lambda_{p,N} \doteq \lambda_p\left[1 + \frac{a}{N^m}\right] \tag{23}$$

where a/N^m may be treated as the principal error in the original solution. As yet a and m are not known. Let us modify the above expression by applying a deflating factor $[1 - A/N^n]$ which has two free constants A and n, so that we have a modified sequence

$$\begin{aligned} \lambda'_{p,N} &= \lambda_p\left[1 + \frac{a}{N^m}\right]\left[1 - \frac{A}{N^n}\right] \\ &\simeq \lambda_p\left[1 + \frac{a}{N^m} - \frac{A}{N^n}\right] \end{aligned} \tag{24}$$

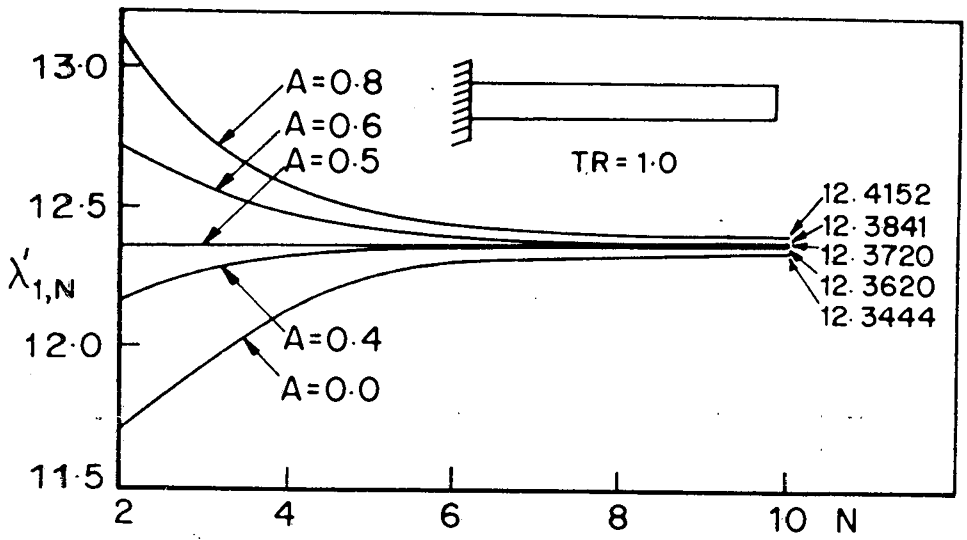


FIG. 5 FREQUENCY PARAMETER OF A UNIFORM CANTILEVER BEAM : ENERGY PERTURBATION APPLIED TO A LOWER BOUND SEQUENCE FOR FIRST MODE.

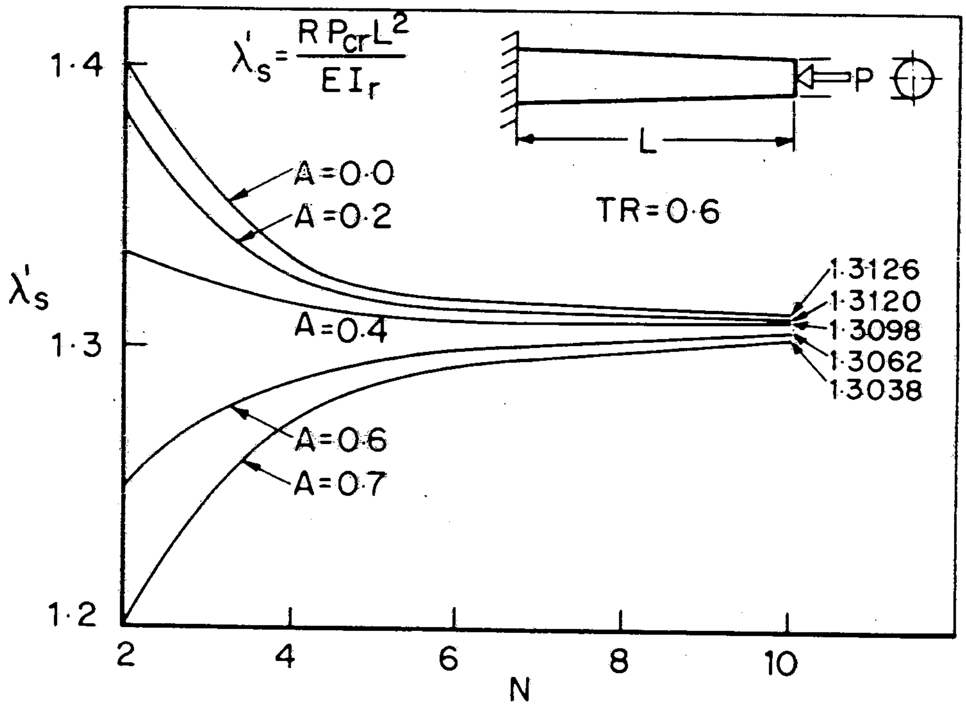


FIG. 6 STABILITY OF TAPERED COLUMN, ENERGY PERTURBATION APPLIED TO AN UPPER BOUND SEQUENCE.

The modified error is closely represented by

$$e'_{p,N} = \frac{a}{N^m} - \frac{A}{N^n} = \frac{1}{N^n}\left[\frac{a}{N^{m-n}} - A\right] = \frac{1}{N^n}\left[\frac{a}{N^k} - A\right] \quad (25)$$

The three possibilities $k = (m - n) \gtreqless 0$, which we designate as over, equal and under modifications yield three distinct types of modified sequences.

4.2 Overmodification ($k > 0,\ m > n$)

Let N_1 be a number (integral or fractional) for which $e = 0$, so that $a/N_1^k = A$, or

$$N_1 = (a/A)^{1/k} \quad (26)$$

We can readily see from the error equation (24) that for $N \lesseqgtr N_1$, $e \gtreqless 0$. Thus for any given combination of A and n, there exists a number N_1 beyond which the error in the modified sequences consistently negative and below which it is consistently positive. It can also be shown that the modified curve has a minimum at a value

$$N = N_1' = N_1\left(1 + \frac{k}{n}\right)^{1/k} > N_1 \quad (27)$$

To summarise:

(a) An over modified curve for an upper bound sequence is recognised by its exhibiting a minimum value followed by an asymptotic trend;

(b) Beyond the minimum point this overmodifying curve yields lower bound approximation.

4.3 Equal Modification ($k = 0$, $m = n$)

With equal modification the error $e = (a - A)/N^n$ and it is consistently positive ($a > A$) or negative ($a < A$) so that every modified curve is monotonic either from above or from below except that when $a - A \simeq 0$, in view of the approximation in the error Eq. (25), one obtains oscillatory trends and these curves should be ignored.

4.4 Undermodification ($k < 0,\ m < n$)

Following an argument parallel to that in section 4.2, we find that when an upper bound sequence is undermodified the error $e \gtreqless 0$ for $N \gtreqless N_2 = (A/a)^{-k}$ and a minimum occurs at $N_2' = N_2\ (1 + k/n)^{1/k} > N_2$. Thus

(a) An undermodified curve from an upper bound sequence is recognised by its exhibiting a maximum value followed by an asymptotic trend;

(b) Beyond the maximum this undermodified curve yields an upper bound value.

4.5 Lower Bound Approximations

When we start with a lower bound approximation for the quantity under study the original curve is a monotonically increasing one. The modified curves are obtained by applying an inflation factor $(1 + A/N^n)$. By extending the discussions of Sections 4.2 to 4.4 it is clear that one will again have over, equal and under modifications depending on $k = m - n \gtreqless 0$ respectively and that equal modification leads to purely monotonic curves. But here an overmodified curve will exhibit a maximum beyond which one is assured of upper bounds while the under modified curve shows a minimum beyond which it will generate refined lower bounds. But a modified curve will exhibit a maximum or minimum depending on whether it is over or under modified.

4.6 Consolidation

The results of our foregoing discussion are consolidated in Table II. In practice we may follow the following steps for determining close bounds to a desired quantity in a problem:

(a) Choose a procedure for the problem that yields a monotonically converging approximation for the desired quantity λ and obtain the converging sequence for a few values of N (terms or elements) in the range N_{min} to N_{max}.

(b) Apply a multiplying factor $R = [\ 1 \mp A/N^n]$ to this curve, where the negative or positive sign is used as the original curve is an upper or a lower bound curve, n is assigned a small integral value say 2 and A is assigned a convenient positive value say 1. If the modified curve does not exhibit clear characteristics, A is suitably altered until a curve is obtained exhibiting clear characteristics of a minimum-maximum and

Table II. Bounds by the Method of Direct Modifications

MODIFICATION / ORGINAL SEQUENCE	FACTOR	OVER $m > n$	EQUAL $m = n$	UNDER $m < n$
UPPER BOUNDS	$1 - \frac{A}{N^n}$	(a) MINIMUM	(a) MONOTONIC	(a) MAXIMUM
		(b) LOWER BOUNDS	(b) REFINED UPPER & LOWER BOUNDS	(b) REFINED UPPER BOUNDS
LOWER BOUNDS	$1 + \frac{A}{N^n}$	(a) MAXIMUM	(a) MONOTONIC	(a) MINIMUM
		(b) UPPER BOUNDS	(b) REFINED LOWER & UPPER BOUNDS	(b) LOWER BOUNDS

(a) : IDENTIFICATION CHARACTERISTIC

(b) : EFFECT OF MODIFICATION FOR LARGE N

ORIGINAL CURVE: $\lambda_{P,N} = \lambda_P\left[1 \pm \frac{a}{N^m}\right]$ (a,m not necessarily identified)

MODIFICATION CURVE: $\lambda_{P,N} = \lambda_P\left[1 \pm \frac{a}{N^m}\right]\left[1 \mp \frac{A}{N^n}\right]$

n : STIPULATED, GENERALLY A SMALL INTEGER.

A : A SCALAR PARAMETER. BY VARYING IT, OBTAIN DIFFERENT MODIFIED CURVES.

IN EACH CASE, THERE IS A MINIMUM VALUE OF A, WHICH WILL EXHIBIT THE ABOVE CHARACTERISTICS WITHIN THE DESIRED RANGE OF N.

an asymptotic approach beyond this mini-max point. Identify the nature of the curve and the bound obtained therefrom by reference to Table II.

(c) Repeat operation (b) with such judicious combinations of n and A as will yield a pair of upper and lower bounds which are sufficiently close.

The above procedure can be refined by first fitting a curve $\lambda_N = \lambda\ [1 \pm aN^{-m}]$ to the original sequence, preferably weighting the fit in favour of the larger values of N in the N_{min} to N_{max} range. With a knowledge of m (which need not be an integer) one can a priori choose the style of modification by fixing a suitable value for n. The value of 'a' obtained gives guidance for suitable initial values for A. Putting $A' = aN^{k}_{max}$, for equal modification the initial value of A should be on either side of A'. Similarly for under modification $A_{initial} > A'$ and for over modification $A_{initial} < A'$.

Finally there are two important points to note in recognising mini-max position on each curve. In each case the location of the mini-max point and the local curvature of the curve depend on the value of A. As A increases the value of N for the mini-max, increases for undermodified curves and decreases for overmodified curves. Also as A increases, the curvature at the mini-max location varies as A^{m-2n-2}. For undermodification $n > m$ and for a range of overmodification $m > n > m/2 - 1$ this curvature decreases with increasing A and so it becomes progressively difficult to recognise the mini-max location from the curves. On the other hand for overmodification in the range $n < m/2 - 1$ the curvature increases with increasing A and it becomes easier to recognise the mini-max location.

4.7 Numerical Examples

The direct modification has been applied to many problems in structural mechanics, starting with an initial converging sequence either from the literature or from our work. We now present five examples to bring out various aspects of the procedure and to indicate the method of interpreting the modified curves. The numerical results achieved confirm that this mehtod is strikingly simple and effective in a achieving close bounds from an initially monotonic converging sequence for any parameter.

4.7.1 Stability of uniform circular cantilever column: The buckling load parameter of a uniform circular column is known exactly; it is $\pi^2/4$ or 2.4674. An FEM solution [12] with nonconsistent equal length elements has resulted in an upper bound sequence converging monotonically with the number of elements (curve marked A = 0 in Fig. 7). The 10 element solution indicates an upper bound of 2.4724 (i.e., +0.2%) for the stability parameter. This example lends itself well to a demonstration of the effects of over equal and under modifications.

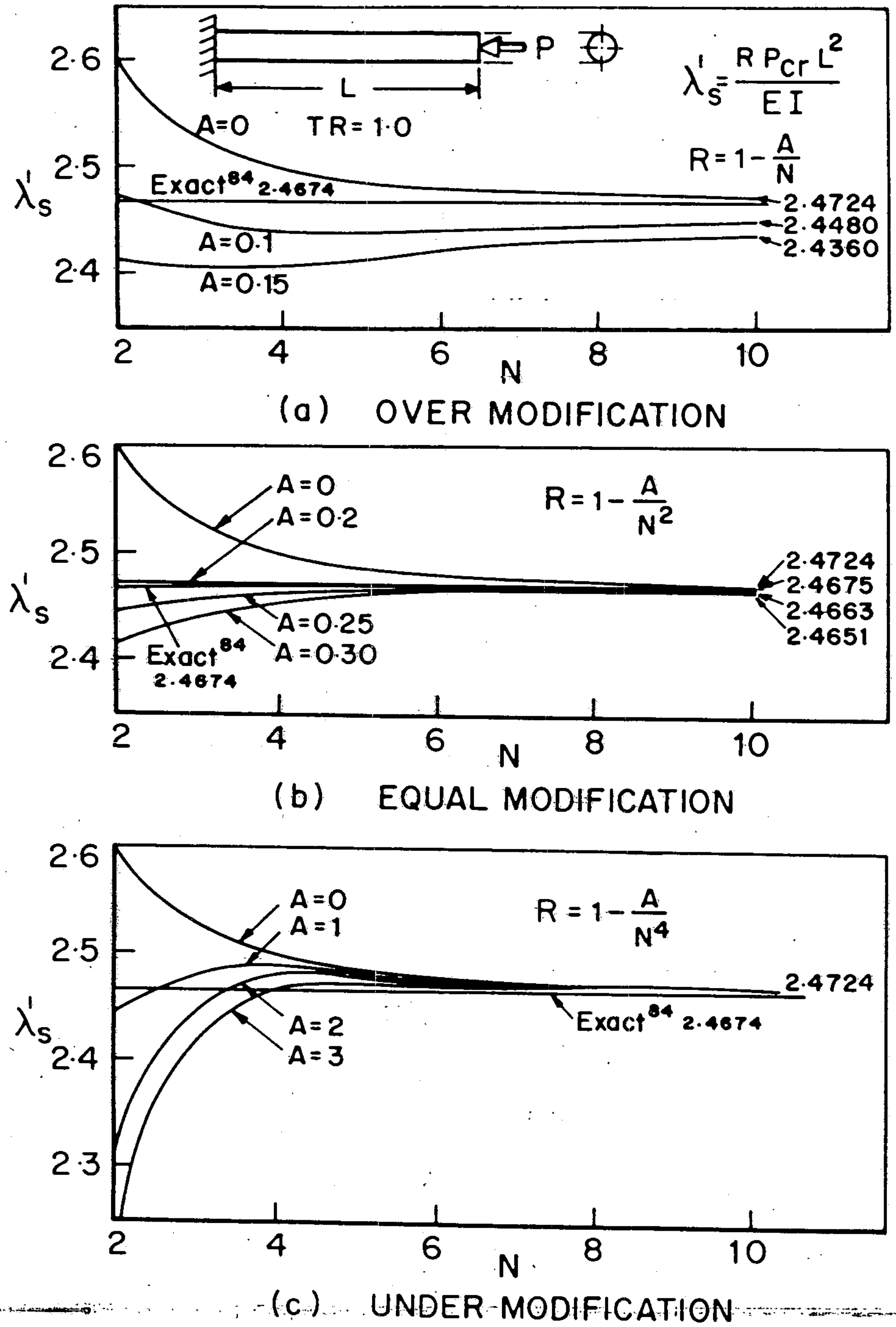


FIG. 7 STABILITY OF A UNIFORM CANTILEVER COLUMN

To the curve A = 0, these modifying functions (a) $1 - A/N$, (b) $1 - A/N^2$ and (c) $1 - A/N^4$ are applied and the corresponding results are shown in Figs. 7 (a), (b), and (c), respectively.

Refer first to Fig. 7 (a), in which the modifying function $R = 1 - A/N$ is applied with A = 0.10 and 0.15. These curves both show, minima around N = 4 and thereafter a monotonic rise with an asymptotic trend. Following Table II, we recognise this as a case of overmodification, yielding (with N = 10, A = 0.1) a lower bound of 2.448 for the stability parameter. The error at this stage is -0.8%.

Similarly by studying the trends in Fig. 7 (b) with $R = 1 - A/N^2$, we recognise that $R = 1 - A/N^2$ provides equalmodification, i.e., the principal error in the original FEM approximation is a/N^2. This can be independently confirmed by an analysis of the FEM formulation. Using 10 elements A = 0.2 refines upper bound to 2.4675 and A = 0.25 yields a close lower bound of 2.4663. Thus the true value is bounded between + .004% and -.045%.

Figure 7(c) confirms that $R = 1 - A/N^4$ results in undermodification. The curves with A = 1, 2, 3 exhibit maxima at about N = 4 and thereafter show asymptotic fall. However the 10 element upper bound of 2.4724 is not noticeably modified by any of them indicating that either that the upper bound is extremely close or there is excessive undermodification requiring compensation by the use of substantially large values for A. But using A = 10, it becomes difficult to locate mini-max location. In fact this figure indicates that one should apply a milder undermodification factor of $(1 - A/N^3)$ to seek clear indications and reliable results.

4.7.2 Stability of stiffened panel: Dawe [13] applied a finite element procedure and obtained a monotonically converging sequence for the stability parameter of a centrally stiffened panel Fig. 8. This sequence appears to be a lower bound curve. A modifying function $R = 1 + A/N^2$ is chosen to test this sequence and the stability parameter if possible. The curves for A = 0.1, 0.5 and 1.0 are shown in Fig. 8. The curve for A = 0.1 agrees with a known upper bound of 9.35 [14]. Thus the required parameter is seen to be bounded between 9.334 and 9.35, i.e., within 1/6%.

4.7.3 Stability of clamped rhombic plates: Mahabaliraja and Durvasula [15] have applied the Rayleigh-Ritz method and obtained an upper bound sequence for the stability parameters of clamped rhombic plates under inplane direct and shear loads. Figure 9 shows the results of applying the modifying function $R = 1 - A/N^2$ (where N is the number of arbitrary parameters) to the results for the highly swept 30° rhombus. It is seen from the curves for A = 0 to 8 that in both case (of direct loads and shear loads) $1 - A/N^2$ results in overmodification. The stability parameters are now bounded between 35.13 and 34.18 for direct loads and 51.75 and 50.35 for the shear load.

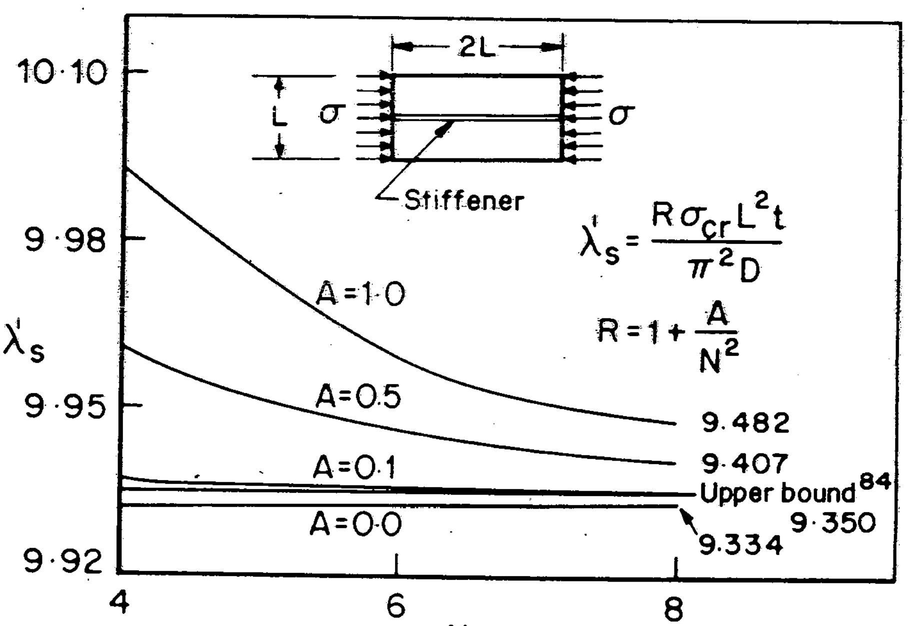
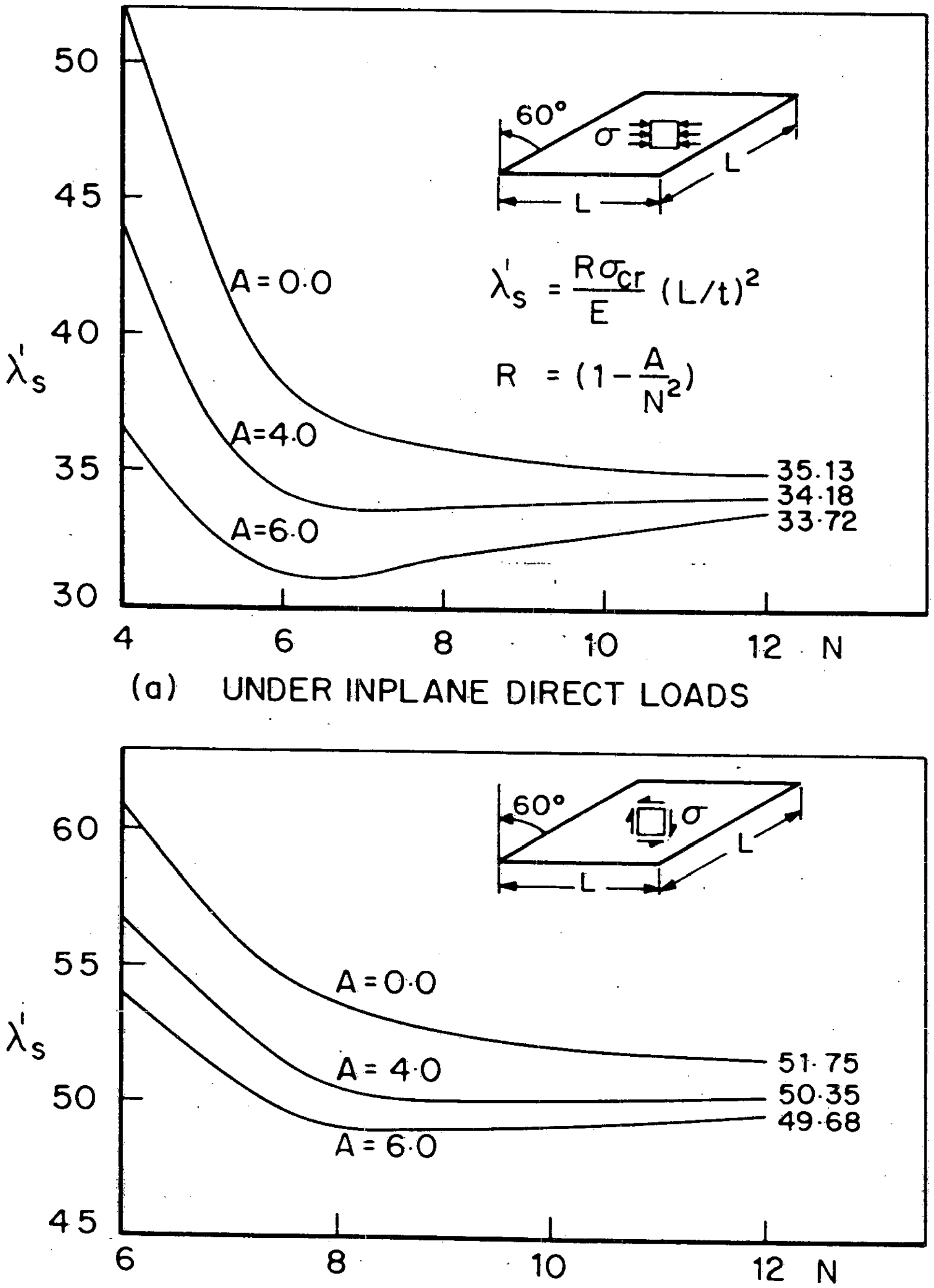


FIG. 8 STABILITY OF A SIMPLY SUPPORTED RECTANGULAR STIFFENED PLATE.



(a) UNDER INPLANE DIRECT LOADS

(b) UNDER INPLANE SHEAR LOADS

FIG. 9 BUCKLING OF CLAMPED RHOMBIC PLATE.

4.7.4 Supersonic flutter: Next, we study a lower bound converging sequence of coalescence values for supersonic flutter of a simply supported square plate obtained by an FEM [16]. The modifying function $R = 1 + A/N^4$ is applied and the resulting curves are shown in Fig. 10. The results indicate that we have achieved equal modification. The upper and lower bounds for coalescence value are 1848.99 and 1846.13, respectively, whereas the exact value is 1848.21.

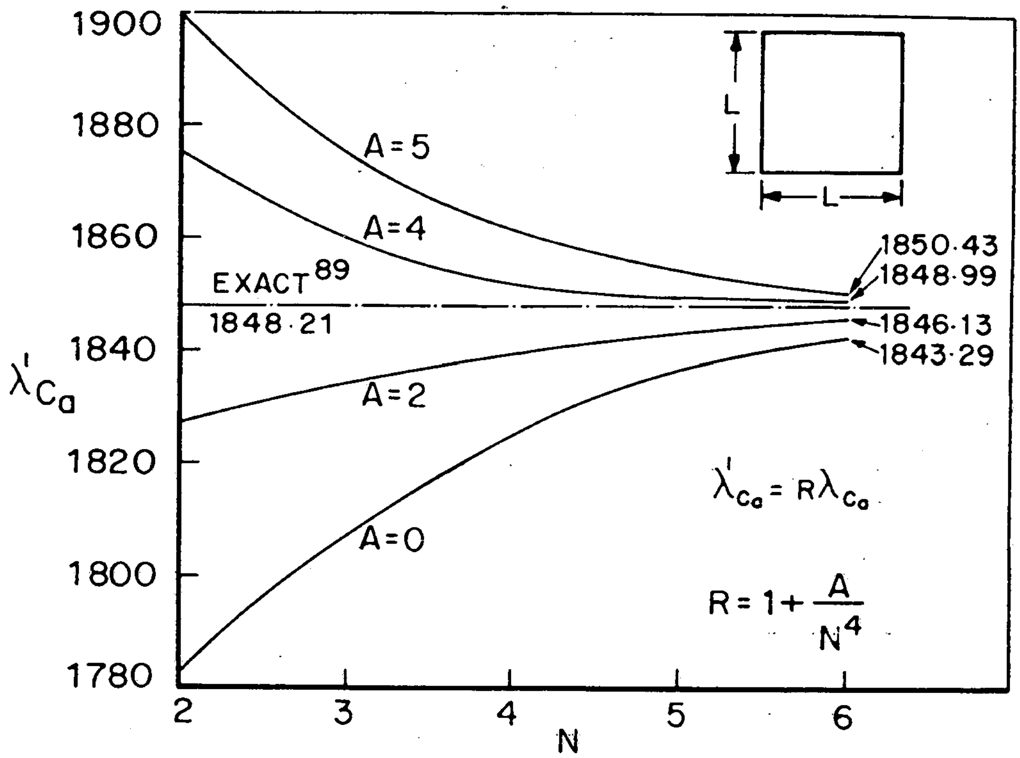


FIG. 10 COALESCENCE VALUE–SUPERSONIC FLUTTER OF SIMPLY SUPPORTED SQUARE PANEL

4.7.5 Bending of clamped rhombic plate under uniform pressure: Monforton [17] analysed the bending of the clamped rhombic plate under uniform pressure by FEM and obtained a smooth lower bound curve for the central deflection. This is now modified by the function $R = 1 + A/N^4$ as shown in Fig. 11. The trend of the curves indicates that we have effected equal modification.. In other words, the predominant error in the basic solution is of the order $1/N^4$. From this study the upper and lower bounds are found to be 7.691×10^{-4} and 7.683×10^{-4}. Rajaiah [5] applied a highly accurate analytical procedure and obtained 7.6898×10^{-4} for this parameter.

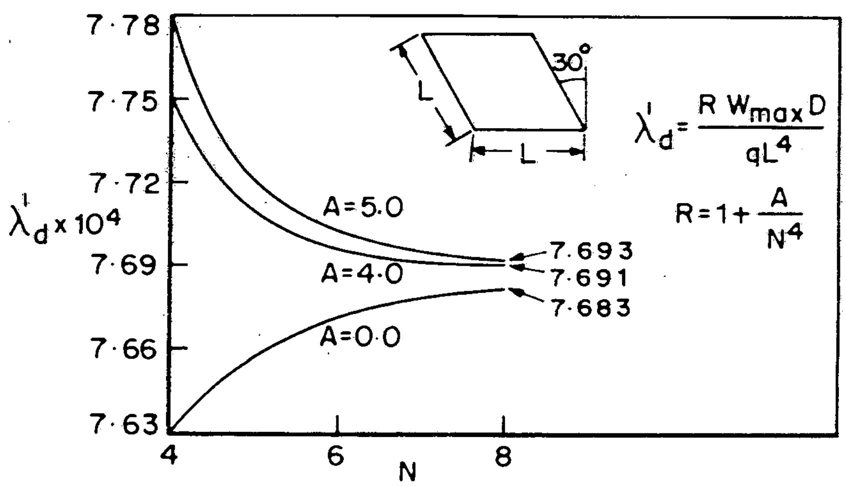


FIG. 11 DEFLECTION OF A CLAMPED RHOMBIC PLATE UNDER UNIFORM PRESSURE.

5. CONCLUDING REMARKS

The cost of an analytical solution increases rapidly with the degree of accuracy demanded. It is further multiplied when one desires close bounds for different parameters of the problem. In this paper we have presented three successful attempts to reduce the effort and cost for achieving close bounds in a wide range of problems. The first method has been developed in respect of the direct method of analysis for boundary value problems. The second is applicable to finite element methods. The third is more versatile and comprehensive in its applicability. Here we start with any monotonically convergent sequence of approximations, with no restrictions on the source of this sequence and operate on the properties of the converging curve (without necessarily determining them) to achieve close bounds with negligible effort. For application of this method, solutions with relatively small number of terms or elements are sufficient, the only restriction being that the original sequences for the desired quantities should be monotonically convergent. The method now opens up an opportunity to analyse a large amount of data on diverse problems already available in literature to obtain clear bounds for relevant quantities with insignificant effort.

The value of the last mentioned powerful method is vastly enhanced if it can be extended for application to oscillatory convergence. Preliminary studies have confirmed the possibility and further studies are currently under way to formulate the procedure and apply it to typical problems.

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