FINITE ELEMENT MODELLING OF NATURAL VIBRATION PROBLEMS

A. V. KRISHNA MURTY
Professor, Department of Aeronautical Engineering, IIT, Bombay
A. K. RAO
Professor, Department of Aeronautical Engineering, IISc, Bangalore
K.S.R.K. PRASAD
Research Associate, Department of Aeronautical Engineering, IIT, Bombay
and
G. V. RAO
Structural Engineer, Structures Division
Vikram Sarabhai Space Center, Trivandrum, India

SYNOPSIS

Finite element models of Modified Rayleigh-Ritz, Galerkin, Least Square, Hybrid (Pian's) and Collocation methods are presented for natural vibration problems. A comparative assessment is made of these methods with Rayleigh-Ritz finite element models. This study brings out that the Galerkin model is relatively superior. A method of improving the accuracy in the estimation of eigenvalues is also included.

1. INTRODUCTION

The classical finite element formulation can be considered to be an analogue of the Rayleigh-Ritz procedure in continuum mechanics. Since its introduction, several other formulations have appeared in the literature. In general, it is possible to develop finite element analogues for most of the continuum methods [1-11]. For static analysis of structures, Szabo and Lee [3] have given a finite element analogue of the Galerkin method. It has also been used for nonlinear problems. Hicks Jr. [7] has used a finite element analogue of the Collocation method for studying a stability problem. For vibration problems, Rayleigh-Ritz type finite element models have been used extensively so far [13-15].

It is only recently, some investigators, including the authors, have started examining the possibility of using alternate models [2, 4-6, 8-10]. In this paper we present five models based on Modified Rayleigh-Ritz, Galerkin, Least Square, Pian's Hybrid and Collocation methods. Through a comparative discussion, the salient features of these models are highlighted.

As the main objective of the paper is to study certain basic features of the models, numerical examples are limited to relatively simple cases and through an examination of various models aided by quantitative comparison of the errors in eigenvalues, the relative merits of these models are brought out.

A serious drawback in the finite element analysis of vibration problems is the size of the dynamical matrix to be handled. Comparatively large order dynamical matrices are required to give a satisfactory degree of accuracy even in the first few frequencies. Condensation techniques are often used to reduce the size of the matrices which in turn introduce certain errors so that the results are unreliable beyond certain limits. In this paper we present a method of using initial approximations (original as well as perturbed) based on relatively smaller sized dynamical matrices, to generate accurate eigenvalues.

2. RAYLEIGH-RITZ METHOD

Basis

In this method one starts with an assumed displacement function, in a series form, with a number of free constants, satisfying geometric boundary conditions. The strain energy and kinetic energy expressions are obtained in quadratic forms in these free constants, by substituting the assumed displacement functions. Minimisation of the Lagrangian function yields requisite equations from which eigenvalues and eigenvectors can be computed.

Finite Element Analogue

For the sake of completeness, we shall briefly describe here the conventional finite element displacement formulation for natural vibration problems, which is, in fact, a finite element analogue of the Rayleigh-Ritz method. In this method the stationarity property of the Lagrangian, i.e.,

$$\delta (U - T) = 0 \tag{1}$$

is used. The expressions for the strain energy U and the kinetic energy T are

$$U = \frac{1}{2} \{\rho\}^T [K] \{\rho\} \tag{2}$$

and

$$T = \frac{1}{2} \omega^2 \{\rho\}^T [M] \{\rho\} \tag{3}$$

In writing Eq. (3) the oscillations have been assumed to be sinusoidal. The global elastic stiffness matrix $[K]$ and the mass matrix $[M]$ are 'assembled' from element matrices $[k]$ and $[m]$ respectively as

$$[K] = [a]^T \overset{\nwarrow}{\underset{\searrow}{[k]_{ij}}} [a] \tag{4}$$

$$[M] = [a]^T \overset{\nwarrow}{\underset{\searrow}{[m]_{ij}}} [a] \tag{5}$$

where $[a]$ is the displacement transformation matrix, and is a Boolean if a coordinate transformation is not involved. The elements of the stiffness and the mass matrices, k_{ij} and m_{ij} are obtained using the relationships

$$k_{ij} = \frac{\partial^2 U_e}{\partial \rho_{e_i} \partial \rho_{e_j}} \tag{6}$$

$$\omega^2 m_{ij} = \frac{\partial^2 T_e}{\partial \rho_{e_i} \partial \rho_{e_j}} \tag{7}$$

The expressions for U_e and T_e are

$$U_e = \frac{1}{2} \int_e \{\bar{e}\}^T [\bar{E}] \{\bar{e}\} \, dv \tag{8}$$

$$T_e = \frac{\omega^2}{2} \int_e \{\bar{w}\}^T [\mu] \{\bar{w}\} \, dv \tag{9}$$

with

$$\{\bar{e}\} = [\Gamma] \{\bar{w}\} \tag{10}$$

where $\{\bar{w}\}$ is the displacement field assumed in the element in terms of the nodal displacement vector $\{\rho_e\}$, as

$$\{\bar{w}\} = [\psi]\{\alpha\} = [\psi][A]^{-1}\{\rho_e\} \tag{11}$$

The relationship between $\{\rho_e\}$ and the constants $\{\alpha\}$ is given by

$$\{\rho_e\} = [A] \{\alpha\} \tag{12}$$

Substituting Eqs. (2), (3) in Eq. (1) one obtains the governing equation as,

$$[K] \{\rho\} - \omega^2 [M] \{\rho\} = 0 \tag{13}$$

from which eigenvalues and eigenvectors are computed.

3. MODIFIED RAYLEIGH-RITZ METHOD

Basis

In the modified Rayleigh-Ritz method, to start with, a deformed shape is assumed, as in the Rayleigh-Ritz method, in the form of a series with a number of free constants, with each component of the series satisfying the geometric boundary conditions. Assuming sinusoidal oscillations, in the case of beams, the inertia loading on the structure is taken as the product of the square of the natural frequency, the local displacement and the mass per unit length. Minimisation of a function, defined as the difference between the strain energy corresponding to the inertia loading and the kinetic energy corresponding to the assumed displacement distributions, provides the basis for a procedure for evaluating natural frequencies.

Finite Element Analogue

The structure is divided into elements and in each element a suitable displacement distribution is chosen in terms of the nodal values. The expression for kinetic energy is written as in the case of the Rayleigh-Ritz method as

$$T = \frac{1}{2} \omega^2 \{\rho\}^T [M] \{\rho\} \tag{14}$$

The inertia loading $\{H\}$ in the structure corresponding to the assumed displacements is

$$\{H\} = \omega^2 \overbrace{[\mu]_{ij}}^{\searrow} [a] \{\rho\} \tag{15}$$

and the corresponding strain energy U_1 is

$$U_1 = \frac{1}{2} \{H\}^T [b]^T \overbrace{[f]_{ij}}^{\searrow} [b] \{H\} \tag{16}$$

where $[f]$ is the element flexibility matrix based on appropriately chosen (also see Ref. [4]) element stress distributions $\{S\}$, and $[b]$ is the load transformation matrix defined by

$$\{S\} = [b] \{H\} \tag{17}$$

Defining matrix $[B]$ relating $\{S\}$ and $\{\rho\}$ as

$$\{S\} = \omega^2 [B] \{\rho\} \tag{18}$$

it can be shown that

$$[B] = [b] \overline{[\mu]_{ij}} [a] \tag{19}$$

Using $[B]$, the expression U_1 may now be written as

$$U_1 = \frac{1}{2} \omega^4 \{\rho\}^T [F] \{\rho\} \tag{20}$$

where

$$[F] = [B]^T \overline{[f]_{ij}} [B] \tag{21}$$

Using the condition

$$\delta (U_1 - T) = 0 \tag{22}$$

the governing equations are obtained as

$$\omega^2 [F] \{\rho\} - [M] \{\rho\} = 0 \tag{23}$$

We may note that ω^2 appears, in this method, with the strain energy term, in contrast to its affiliation with kinetic energy term in the Rayleigh-Ritz method. The method of writing the expression for U_1 involves the use of the concepts of the matrix force method [2]. Hence although the method may give better results than the Rayleigh-Ritz method, the scope of its application can be restrictive.

4. GALERKIN METHOD

Basis

For the Galerkin method, the dependent variable in the governing differential equation is assumed in the form of a series, with each of the component functions satisfying all the boundary conditions. Substitution of this in the governing differential equation yields an expression for error. Integrals of the products of the error in the differential equation and the component functions of the assumed series are set equal to zero, to obtain the necessary set of algebraic equations, from which eigenvalues and eigenvectors can be computed.

Finite Element Analogue

The differential equation governing the free vibration problems may be written as

$$L_1(\bar{w}) - \lambda L_2(\bar{w}) = 0 \tag{24}$$

where L_1 and L_2 are differential operators and λ is the eigenvalue. Considering one dimensional problems, the displacement distributions in the i-th element may be taken as in Eq. (11) as

$$\bar{w}_i = [\psi]_i \{\alpha\}_i = [\psi]_i [A]_i^{-1} \{\rho_e\}_i \tag{25}$$

Substituting $\bar{w}_i$ in the governing differential equation, the expression for the error in the differential equation in the i-th element, $\bar{\epsilon}_i$ becomes

$$\bar{\epsilon}_i = (L_{1i} - \lambda L_{2i}) [\psi]_i [A]_i^{-1} \{\rho_e\}_i \tag{26}$$

Following the Galerkin procedure the governing equation can be obtained as

$$[E_k]\{\rho\} - \lambda [E_m]\{\rho\} = 0 \tag{27}$$

where

$$\{\rho_e\} = [a]\{\rho\} \tag{28}$$

$$[E_k] = [a]^T \overset{\nwarrow}{\underset{\searrow}{[e_k]_i}} [a] \tag{29}$$

and

$$[E_m] = [a]^T \overset{\nwarrow}{\underset{\searrow}{[e_m]_i}} [a] \tag{30}$$

with

$$[e_k]_i = [A]_i^{-1T} \left(\int_e [\psi]_i^T L_{1i} [\psi]_i \, d\xi \right) [A]_i^{-1} \tag{31}$$

and

$$[e_m]_i = [A]_i^{-1T} \left(\int_e [\psi]_i^T L_{2i} [\psi]_i \, d\xi \right) [A]_i^{-1} \tag{32}$$

The eigenvalues and eigenvectors can be computed from Eq. (27).

5. LEAST SQUARE METHOD

Basis

In this method also, the dependent variable in the governing differential equation is assumed as a series. The error function is obtained as described in section 4. Minimisation of the integral of the square of the error function provides the necessary set of algebraic equations from which eigenvalues and eigenvectors can be computed.

Finite Element Analogue

As in the case of the Galerkin method (section 4), the error function for the i-th domain is given by

$$\bar{\epsilon}_i = (L_{1i} - \lambda L_{2i}) [\psi]_i [A]_i^{-1} \{\rho_e\}_i \tag{33}$$

The vector of errors in all elements is

$$\{\bar{\epsilon}_i\} = \overset{\nwarrow}{\underset{\searrow}{(L_{1i} - \lambda L_{2i})[\psi]_i [A]_i^{-1}}} [a]\{\rho\} \tag{34}$$

Using the minimisation condition that

$$\frac{\partial}{\partial \rho_j} \int_D \{\bar{\epsilon}_i\}^T \{\bar{\epsilon}_i\} \, dv = 0 \tag{35}$$

We get

$$[P]\{\rho\} - \lambda [Q]\{\rho\} + \lambda^2 [R]\{\rho\} = 0 \tag{36}$$

where

$$[P] = [a]^T \overset{\nwarrow}{[A]_i^{-1T}} \left(\int_e (L_{1i}[\psi]_i)^T (L_{1i}[\psi]_i) d\xi \right) \underset{\searrow}{[A]_i^{-1}} [a] \tag{37}$$

$$[Q] = [a]^T \overline{[A]_i^{-1}}^T \left(\int_e (L_{1i}[\psi]_i)^T (L_{2i}[\psi]_i)\, d\xi + \int_e (L_{2i}[\psi]_i)^T (L_{1i}[\psi]_i)\, d\xi \right) \underline{[A]_i^{-1}} [a] \tag{38}$$

and

$$[R] = [a]^T \overline{[A]_i^{-1}}^T \left(\int_e (L_{2i}[\psi]_i)^T (L_{2i}[\psi]_i)\, d\xi \right) \underline{[A]_i^{-1}} [a] \tag{39}$$

It is to be noted that, in contrast to other methods described in this paper, this method yields a quadratic eigenvalue equation giving rise to complex eigenvalues and hence involves more computational effort than the three earlier methods.

6. COLLOCATION METHOD

Basis

In the collocation method considered here, we start with deflection assumed in terms of some admissible functions containing undetermined constants.

Deflections of the structure, subjected to the inertia loading corresponding to the assumed deflections, are evaluated and are equated to the assumed deflections at a certain number of points. This provides the necessary algebraic equations from which eigenvalues and eigenvectors can be computed.

Finite Element Analogue

In section 3 , we have given the expression for the strain energy in the structure subjected to the inertia loading $\{H\}$ as

$$U_1 = \frac{1}{2} \{H\}^T [b]^T \overline{[f]_i} [b] \{H\} \tag{40}$$

where $\{H\}$ corresponds to assumed displacement fields and is given by

$$\{H\} = \omega^2 \overline{[\mu]_i} [a] \tag{41}$$

Using Castigliano's theorem (Part II) , the deflections $\{\rho_1\}$ due to $\{H\}$ can be obtained as

$$\{\rho_1\} = \frac{\partial U_1}{\partial \{H\}} = [b]^T \overline{[f]_i} [b] \{H\} \tag{42}$$

To use the method of collocation, $\{\rho_1\}$ is equated to $\{\rho\}$. Hence

$$\{\rho\} = [b]^T \overline{[f]_i} [b] \{H\} \tag{43}$$

Substituting Eq. (41) in the above, one obtains the final governing equation as

$$\{\rho\} = \omega^2 [D] \{\rho\} \tag{44}$$

where

$$[D] = [b]^T \overline{[f]_i} [b] \overline{[\mu]_i} [a] \tag{45}$$

7. HYBRID METHOD (PIAN'S TYPE)

Basis

Pian has given an effective hybrid concept for static analysis wherein, stress fields satisfying equilibrium equations are assumed in the interior of the element, compatible displacements are assumed at the inter-element boundaries and the complementary energy principle is used for formulating the problem. Analogous to this, it is possible to develop a formulation for natural vibration problems. But this will have certain disadvantages which will be discussed later.

Finite Element Analogue

To bring out the possibility of a formulation based on this hybrid concept as well as to unfold certain difficulties involved, we shall develop the method in detail with the aid of a simple example of torsional oscillations of a uniform shaft. The governing differential equation is

$$\theta'' + \bar{\omega}^2 \theta = 0 \tag{46}$$

We use the solution of this equation to assume equilibrating stress fields in the element. For the i-th element, the expression for θ satisfying equilibrium which we shall denote as θ_e, can be obtained as

$$\theta_{ei} = \left[\alpha_1 \cos \omega\xi, -\alpha_2 \sin \omega\xi\right]_i = \left[\psi\right]\left[A\right]_i^{-1} \left\{\rho_e\right\}_i \tag{47}$$

where

$$\left[\psi\right] = \left[\cos \omega\xi, -\sin \omega\xi\right] \tag{48}$$

and

$$\begin{Bmatrix}\rho_{e_1} \\ \rho_{e_2}\end{Bmatrix}_i = \left[A\right]_i \begin{Bmatrix}\rho_1 \\ \rho_2\end{Bmatrix}_i \tag{49}$$

The corresponding internal torque satisfying equilibrium can be taken as

$$T_i = -GJ\omega \left[\sin \omega\xi, \cos \omega\xi\right]\left[A\right]_i^{-1}\left\{\rho_e\right\}_i$$

$$= -GJ\omega \left[\phi\right]\left[A\right]_i^{-1}\left\{\rho\right\}_i \tag{50}$$

where $$\left[\phi\right] = \left[\sin \omega\xi, \cos \omega\xi\right] \tag{50a}$$

The complementary strain energy U_i^* in the element becomes

$$U_i^* = \frac{1}{2}\int_e \frac{T_i^2 d\xi}{GJ} = GJ\omega^2 \left\{\rho_e\right\}_i^T \left[f(\omega)\right]_i \left\{\rho_e\right\}_i \tag{51}$$

where $$\left[f(\omega)\right] = \left[A\right]_i^{-1T}\left(\int_e \left[\phi\right]^T\left[\phi\right] d\xi\right)\left[A\right]_i^{-1} \tag{52}$$

The inertia loading in the element

$$f_{si} = \sigma I_p \omega^2 \theta_e = \sigma I_p \left[\psi\right]_i \left[A\right]_i^{-1}\left\{\rho_e\right\}_i \tag{53}$$

To facilitate evaluation of complementary work done, a suitable displacement distribution is chosen independent of θ_e and will be denoted as θ_d. For example θ_d can be taken as

$$\theta_{di} = \begin{bmatrix} 1 & , & 1-\xi \end{bmatrix} \{\rho_e\}_i = [\psi_d]_i \{\rho_e\}_i \tag{54}$$

The complementary work W^* can be written as

$$W_i^* = \frac{1}{2} \int_e t_s \theta_d \, d\xi = \omega^2 \sigma I_p \{\rho_e\}_i^T [m(\omega)]_i \{\rho_e\}_i \tag{55}$$

where

$$[m(\omega)]_i = \frac{1}{2} [A]_i^{-1} \left(\int_e [\psi]_i^T [\psi_d]_i \, d\xi \right) \tag{56}$$

Applying the condition

$$\delta(U^* - W^*) = 0 \tag{57}$$

we obtain the final equation

$$[F(\omega)] \{\rho\} - \frac{1}{\lambda_0} [M(\omega)] \{\rho\} = 0 \tag{58}$$

where

$$[F(\omega)] = [a]^T \left[[f(\omega)]_i \right] [a] \tag{59}$$

and

$$[M(\omega)] [a]^T \left[[m(\omega)]_i \right] [a] \tag{60}$$

Here $[a]$ is the displacement transformation matrix as defined in earlier sections.

As different from other methods discussed in this paper, elements of $[F(\omega)]$ and $[M(\omega)]$ are functions of the frequency ω. Hence the computer programming becomes more complicated. In view of this conspicuous disadvantage, in this paper, this method has not been investigated further. However it is worth mentioning that this shows a possibility of applying the complementary energy principle to eigenvalue problems. As the complementary energy principle gives lower bounds to stiffness parameters, this mixed method may, with certain modifications, provide a method of generating lower bounds to eigenvalues.

8. COMPARATIVE STUDY OF VARIOUS FORMULATIONS

Table I shows a qualitative comparison of six methods discussed in this paper. Salient remarks about each of these methods are also included. Modified Rayleigh-Ritz, Galerkin and Collocation methods, result in governing equations similar to conventional formulations. The scope or application of the Modified Rayleigh-Ritz method is restricted because of the classical limitations of the force method, which is to be used in this formulation. The collocation method suffers from the disadvantage of vagueness in the method of matching and its effective usage is dependent on the ingenuity of the user. Therefore the Galerkin method can be considered to be a promising alternative for the Rayleigh-Ritz method.

In view of the conspicuous complications, Pian's Hybrid model cannot be considered useful for vibration problems unless it is made to yield lower bounds. The Least Square Method, gives a quadratic equation in eigenvalues. In general, it is expected this will give complex eigenvalues. The imaginary quantity must vanish when the exact solution is numerically approached, because from physical considerations, the eigenvalues for a natural vibration problem are real quantities. Therefore, at any stage of approximation, the imaginary part may give an indication of error. The possibility of this feature makes the study of this method interesting.

Table I. Some Finite Element Schemes for Natural Vibration Problems : Qualitative Comparison

Basis	Rayleigh-Ritz Method	Modified Rayleigh-Ritz Method	Galerkin Method	Least-Square Method	Pian's type Hybrid Method	Collocation Method
Form of the final Matrix Equation	$[K]\{\rho\} - \lambda[M]\{\rho\} = 0$	$[F]\{\rho\} - \frac{1}{\lambda}[M]\{\rho\} = 0$	$[E_k]\{\rho\} - \lambda[E_m]\{\rho\} = 0$	$[P]\{\rho\} - \lambda[Q]\{\rho\} + \lambda^2[R]\{\rho\} = 0$	$[F(\omega)]\{\rho\} - \frac{1}{\lambda_0}[M(\omega)]\{\rho\} = 0$	$[D]\{\rho\} - \frac{1}{\lambda}\{\rho\} = 0$
Upper or lower bound for eigenvalue	upper bound	upper bound	cannot be predicted	cannot be predicted	cannot be predicted	cannot be predicted
		1 Requires use of concepts of matrix force method 2 More suited for one - dimensional problems	1 Element degrees of freedom are more 2 Programming is simple as for RR method 3 Can consider non-linear problems for which energy principle may not exist 4 Can be extended to initial value problems	1 Results in quadratic eigenvalue equations 2 Eigenvalues are complex in general 3 Programming more complicated and requires more computational time	1 Results in transcendental matrix equation 2 Conventional computer programmes cannot be used directly 3 In general it is not a convenient formulation	1 Requires concepts of the matrix force method 2 More suited for one-dimensional problems

9. NUMERICAL STUDIES

Using various methods discussed in earlier sections, eigenvalues have been computed and these are presented in Tables II to V . Table II shows a comparison of the percentage error in eigenvalue parameter in the case of torsional oscillations of a shaft. This comparison clearly brings out that the Modified Rayleigh-Ritz method gives better results than the Rayleigh-Ritz method and, the Galerkin method gives better results than these two. Collocation has given poor results. This trend can also be noticed in the cases of beam flexural vibration, the results for which are reported in Tables III to V. Thus, these numerical comparisons confirm our expectation that the Galerkin method can give higher degree of accuracy for the eigenvalues for the same degrees of freedom of the structure.

10. IMPROVEMENT OF EIGENVALUES

We notice from earlier sections that most of the methods yield the governing equation for natural vibration problems in the form

$$[K]\{\rho\} - \lambda [M]\{\rho\} = 0 \tag{61}$$

Normally, one is required to consider large number of degrees of freedom, i.e. large order matrices $[K]$ and $[M]$ in order to achieve a satisfactory degree of accuracy in the eigenvalue.

In Refs. [16 & 17] we have developed a method of improving the eigenvalues as well as generating bounds with reference to the Rayleigh-Ritz Finite Element Method. In another paper [18] at this seminar, we are presenting this work in some detail. Here we intend to show that it can be used with any method leading to a governing equation of the type given by Eq. (61).

From the orthogonality of principal modes it follows that

$$\lambda = \frac{\{\rho\}^T [K] \{\rho\}}{\{\rho\}^T [M] \{\rho\}} \tag{62}$$

In a finite element scheme, $[K]$ and $[M]$ are normally obtained as

$$[K] = [a]^T \Big[[k]_{ij} \Big] [a] \tag{63}$$

$$[M] = [a]^T \Big[[m]_{ij} \Big] [a] \tag{64}$$

Let us define a modified matrix $[\bar{m}]$, as

$$[\bar{m}] = [R(N)]^p [m] \tag{65}$$

where $R(N)$ is a modification function, containing a scalar parameter A, and satisfying the conditions

$$R(N) \geq 1 , \quad R(N+1) \leq R(N) , \quad R(N) \to 1 , \text{ as } N \to \infty \tag{66}$$

One may choose $R(N)$ as function of $\{\rho\}$ say ,

$$R(N) = \left[1 + \frac{A^2}{2} (\rho_i - \rho_{i-1})^2 \right] \tag{67}$$

Table II. Accuracies in Eigenvalues of Cantilever Shafts

Mode	Order of Dynamical Matrix	% Error in Eigenvalue Parameters Collocation	Rayleigh-Ritz	Modified Rayleigh-Ritz	Galerkin
First	2	47.0148	5.2387	0.0600	0.0261
	4	16.9632	1.2916	0.0034	0.0006
	10	5.6401	0.2058	0.0001	0.0000
Second	2	280.8549	42.7010	8.8067	6.1068
	4	46.1413	12.0032	0.3508	0.1855
	10	9.2844	1.8639	0.0072	0.0018
	20	3.5152	0.4635	0.0004	0.0000
Third	4	128.8178	33.0513	3.8423	2.2089
	10	16.9889	5.2387	0.0600	0.0261
	20	5.2599	1.2916	0.0034	0.0006
	40	1.9195	0.3217	0.0002	0.0000
Fourth	4	302.7720	41.9556	13.1430	13.3356
	10	29.6672	10.4187	0.2578	0.1341
	20	7.9376	2.5434	0.0135	0.0041
	40	2.5553	0.6313	0.0008	0.0001

Table III. Accuracies in Eigenvalues of Cantilever Beams

Mode	Order of Dynamical Matrix	% Error in Eigenvalue Parameters Collocation	Rayleigh-Ritz [14]	Modified Rayleigh-Ritz	Galerkin
First	4	30.9159	0.0977	0.0152	0.0000
	10	10.8746	0.0000	0.0003	-
Second	4	59.5831	1.71	0.5465	0.0082
	8	-	0.23	0.0268	0.0000
	20	6.0616	0.0051	0.0005	0.0000
Third	4	128.6834	48.4	4.1697	0.2280
	8	-	1.55	0.2156	0.0007
	20	7.7238	0.051	0.0042	0.0000
Fourth	4	205.8996	226.0	4.4203	27.0543
	8	-	2.92	0.8921	0.0604
	20	10.3583	0.190	0.0165	0.0000

Table IV. Accuracies in Eigenvalues of Simply Supported Beams

Mode	Order of Dynamical Matrix	% Error in Eigenvalue Parameters Rayleigh-Ritz [14]	Modified Rayleigh-Ritz	Galerkin
First	4	0.791	0.0610	0.00005
	8	0.0517	0.0034	0.00000
	20	-	0.0001	0.00000
Second	4	23.1	1.4521	0.00601
	8	0.791	0.0610	0.00005
	20	-	0.0014	0.00000
Third	4	53.74	12.0609	5.67573
	8	3.68	0.3668	0.00396
	20	-	0.0072	0.00000
Fourth	4	61.70	-	12.90444
	8	23.20	1.4521	0.00603
	20	-	0.0238	0.00001

Table V. Accuracies in Eigenvalues of Encastered Beams

Mode	Order of Dynamical Matrix	% Error in Eigenvalue Parameters Rayleigh-Ritz [14]	Modified Rayleigh-Ritz	Galerkin
First	4	0.821	0.7927	0.00415
	8	-	0.0376	0.00000
	20	-	0.0007	0.00000
Second	4	4.033	4.7585	0.06633
	8	-	0.277	0.00054
	20	-	0.0051	0.00000
Third	4	46.400	2.4834	11.44143
	8	-	1.0756	0.03174
	20	-	0.0195	0.00000
Fourth	4	112.0	-	23.20796
	8	-	3.028	0.05751
	20	-	0.0537	0.00001

in which case, it varies from element to element. If the original method gives an upper bound (lower bound) p is to be taken as unity with positive (negative) sign.

Instead of $[m]$, we use the modified matrix $[\bar{m}]$ in the formulation, in Eq. (61). We shall call this a modified formulation, and the solution for various values of A exhibits very interesting characteristics. As $N \to \infty$, $R(N) \to 1$ and so $[\bar{m}] \to [m]$ and the solution of the modified formulation tends to the solution of the original problem. For any finite value of N , $R_N > 1$ and so the term $[a]^T [m] [a]$ is consistently overestimated (underestimated) for positive (negative) values of p . Hence the rate of convergence of the eigenvalue changes leading to improved values for λ , if the scalar parameter is properly chosen. By similar arguments it can be shown that by a suitable choice of A , it is possible to generate accurate bounds on both sides. In Refs. [16], [17] and [18] , details regarding the choice of A have been given and the procedure is confirmed by numerical experiments and by evaluation of discretisation errors in some cases.

11. CONCLUSIONS

In this paper, five finite element models for vibration problems, based on, Modified Rayleigh-Ritz , Galerkin, Least Square, Pian's Hybrid and Collocation methods have been presented. A comparative study of these methods with the classical Rayleigh-Ritz type finite element model indicates that the Galerkin method is relatively superior. Further investigation is needed to usefully exploit the potential capability of its extension to nonlinear vibration problems.

REFERENCES

1. Pian, T. H. H. & TONG, P., "Basis of Finite Element Method for Solid Continua", Vol. 1, No. 1, 1969, Int. J. Numerical Methods in Engineering, pp 3-38.

2. Govinda Raju, S. P. & Rao, A. K., "A Matrix Method for Vibration and Stability Problems", Vol. 8, No. 3, 1969, J. Aeronautical Society of India, pp 90-99.

3. Szabo, B. A. & Lee, G. C., "Derivation of Stiffness Matrices in Plane Elasticity by Galerkin Method", Vol. 1, No. 3, 1969, Int. J. Numerical Methods in Engineering, pp 301-310.

4. Prasad, K. S. R. K., Krishna Murty, A. V. & Rao, A. K., "A Finite Element Aanlogue of the Modified Rayleigh-Ritz Method for Vibration Problems", Vol. 5, No. 2, 1972, Int. J. Numerical Methods in Engineering, pp 163-169.

5. Prasad, K. S. R. K., Krishna Murty, A. V. & Rao, A. K., "Finite Element Formulations and Some Adaptations from the Analysis of Solid Continua", Presented at a Symposium on Finite Element Techniques in Structural Vibrations, ISVR, Southampton, UK, March 1971.

6. Prasad, K. S. R. K., "On Finite Element Formulations for Vibration Problems", Ph. D. Thesis to be submitted to Faculty of Engg., IISc., Bangalore.

7. Gardener, W. Hicks, Jr., "Finite Element Elastic Buckling Analysis", Vol. 93, 1967, J. Struct. Div., Proceedings of ASCE, p 5662, pp 71-86.

8. Tabarrok, B., "A Variational Principle for Dynamic Analysis of Continua by Hybrid Finite Element Method", Vol. 7, 1971, Int. J. Solids Structures, pp 251-268.

9. Sakaguchi, R. L. & Tabarrok, B., "Calculations of Plate Frequencies from Complementary Energy Formulation", Vol. 2, 1970, Int. J. Numerical Methods in Engineering, pp 283-293.

10. Hutton, S. G. & Anderson, D. L., "Finite Element Method : A Galerkin Approach", Vol. 9, 1972, J. Engg. Mech. Div., Proceedings of ASCE, pp 1503-1520.

11. Aral, M., Mayer, P. G. & Smith, Jr. C. V., "Finite Element Galerkin Method, Solution to Selected Elliptic and Parabolic Differential Equations", Proc. 3rd Conf. on Matrix Methods in Structural Mechanics, WPAFB, Ohio, 1971.

12. Prasad, K. S. R. K. & Krishna Murty, A. V., "The Galerkin Finite Element Method for Vibration Problems", Vol. 11, No. 4, 1973, J. AIAA pp. 544-546.

13. Archer, J. S., "Consistent Matrix Formulation for Structural Analysis Using Finite Element Techniques", Vol. 3, 1965, J. AIAA, pp 1910-1918.

14. Leckie, F. A. & Lindberg, G. M., "The Effect of Lumped Parameters on Beam Frequencies", Vol. 14, 1963, Aero. Quarterly, pp 1910-1918.

15. Argyris, J. H., "Continua and Discontinua", Proc. Ist Conf. on Matrix Methods in Structural Mechanics, WPAFB, Ohio, 1965, pp 11-189.

16. Venkateswara Rao, G., Krishna Murty, A.V. & Rao, A.K., "Bounds for Eigenvalues in Some Vibration and Stability Problems", Vol.5, 1972, Int. J. Numerical Methods in Engineering, pp. 227-242.

17. Venkateswara Rao, G., "Finite Element Analysis of Vibration and Stability Problems", 1972, Ph.D. Thesis, Faculty of Engg., IISc., Bangalore.

18. Rao, A.K., Ramamurthy, T.S., Krishna Murty, A.V. & Venkateswara Rao, G., "Bounds and Error Control for Eigenvalues", to be presented at 1975 TOKYO Seminar on Finite Element Analysis.

APPENDIX A : NOTATIONS

$[a]$: displacement transformation matrix
A : scalar parameter in Eq. (67)
$[A]$: matrix defined in Eq. (12)
$[b]$: load transformation matrix, Eq. (17)
$[B]$: load transformation matrix, Eq. (19)
$[D]$: dynamical matrix
$\{e\}$: vector of element strains
E, $[E]$: Young's modulus and elasticity matrix respectively
$[E_k]$, $[E_m]$: matrices defined by Eqs. (29,30) respectively
$[e_k]$, $[e_m]$: matrices as defined by Eqs. (31,32) respectively
I, I_p : moment of inertia and polar moment of inertia respectively
$[f]$: element flexibility matrix
$[f(\omega)]$: a flexibility matrix whose elements depend on ω as defined in Eq. (52)
$[F]$: flexibility matrix of the structure
$[F(\omega)]$: flexibility matrix in the hybrid method as defined in Eq. (59)
GJ : torsional rigidity
$\{H\}$: vector of inertia loads
$[k]$, $[K]$: stiffness matrices of element and structure respectively
L, l : total length and element length respectively
L_1, L_2 : differential operators
$[m]$, $[M]$: mass matrices of element and structure respectively
$[\bar{m}]$: modified mass matrix
$[m(\omega)]$: mass matrix in the hybrid method defined by Eq. (56)
N : number of elements into which the structure/domain is divided
$[P]$, $[Q]$, $[R]$: matrices as defined by Eqs. (37-39) respectively
$R(N)$: perturbation function defined by Eq. (65)
$\{S\}$: supervector of element stresses as defined by Eq. (17)
t_s : inertia loading on the shaft, as defined in Eq. (53)
T : kinetic energy of the structure
T_i : torque in the i-th element and also used for kinetic energy of the i-th element
U, U_1 : potential energies of the structure corresponding to displacements $\{\rho\}$ & $\{\rho_1\}$
$\bar{w}$: displacement (or state variable) distribution function
W^* : complementary work
x : coordinate
$\{\alpha\}$: vector of arbitrary constants
δ : operator denoting variation
$\bar{\epsilon}$: error function obtained by substituting the assumed displacement in governing differential equation
$\theta, \theta_e, \theta_d$: rotational displacements as in Eqs. (46, 47, 54) respectively
λ : non-dimensional eigenvalue defined by $\lambda = \omega^2/\omega_o^2$

λ_o : reference eigenvalue defined by $\lambda_O = \omega_o^2$, where $\omega_o^2 = EI/mL^4$ for flexure and $\omega_o^2 = GJ/I_\rho L^2$ for torsion

$[\mu]$: element mass property matrix as in Eq. (41)

$[\bar{\mu}]$: a matrix of element mass distribution functions

ξ : non-dimensional local coordinate

σ : density

$\{\rho\}$: displacement (state variable) vectors

$\{\rho_e\}$: supervector of element nodal displacement vectors $\{\rho_e\}_i$

$\{\rho_1\}$: displacement vector as defined in Eq. (42)

$[\Gamma]$: matrix relating strain vector to the displacement field as in Eq. (10)

$\omega, \omega_e, \omega_O$: circular frequency, it's exact value and reference value respectively

$\bar{\omega}$: non-dimensional frequency, given by $\bar{\omega} = \omega / \omega_O$

$[\psi]$ $[\phi]$: matrices of displacement and stress distribution functions in local co-ordinates

$\ulcorner \quad \lrcorner$: diagonal matrix

$\int_e , \int_O$: integral over the element and over the entire domain respectively.