General Theory of Vibrations of Cylindrical Tubes†

PART—II : UNCOUPLED TORSIONAL VIBRATIONS OF CLOSED TUBES

By

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Summary

The equations governing torsional vibrations of unstiffened cylindrical tubes have been presented. Methods proposed in Part I are illustrated by considering the torsional vibrations of doubly symmetric tubes. An exact solution of a simply supported tube with the boundary of the cross section given by $p = \frac{S}{2\pi} \cos \frac{2\pi s}{S}$ *is presented. A free-free tube of rectangular cross section is analysed by using first order approximation equations and the results are in good agreement with earlier work.*

ADDITIONAL NOTATION FOR PART II**

a	Half the width of the tube
b	Half the depth of the tube
k_w^2	$= \omega^2 \rho\, L^2/E$
k_T^2	$= \dfrac{\omega^2 \rho\, L^2 I_p}{G\, J_o}$

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**In addition to the notation given in Part I.

P	=	a/b
Q	=	L/a
δ (s—a)		Dirac-delta function
λ		Defined by Eq. (2.30)
λ_1^2, λ_2^2		Defined by Eq. (2.32)
μ^2	=	$\frac{\bar{S}_{\theta\theta}}{k^2 B_{\theta\theta}}$
υ^2		Defined by Eq. (2.28)

2.0 Introduction

Investigations on the torsional vibrations of doubly symmetric hollow thin-walled tubes 14, 16, 16, 22* reveal that the Bredth-Batho theory of torsion is inadequate for the determination of natural frequencies and mode shapes of such structures, particularly when the plan aspect ratio is small, because the influence of secondary effects such as shear lag and longitudinal inertia is considerable. Hence refined methods which include these secondary effects are essential for the accurate determination of the natural frequencies and mode shapes of such structures.

Kruszewski and Kordes[14] studied the torsional vibrations of rectangular tubes using Rayleigh-Ritz method. These studies clearly brought out that the influence of shear lag on natural frequency can be quite high. Hsu Lo[16] obtained the natural frequencies of a concentrated mass on a rectangular tube ignoring the mass of the tube. In Ref. (18) Hsu Lo and Goulard suggested a convenient method for approximate estimation of natural frequencies by assuming appropriate warp distribution on the peripheral direction. Mansfield[22] considered torsional vibrations of a fou boom rectangular tube with shear resistant webs. Most of the earlier workers considered rectangular cross section only. Adequate stress has not been imposed on the nature of frequency spectrum and on the mode shapes.

Recognising the necessity for a general theory which can consider an arbitrars cross section and can yield closed expressions at least for good approximate determination of frequencies and mode shapes, we proposed in Part I an unified mathematical theory for the prediction of natural frequencies and mode shapes of tubes with closely spaced rigid and massless diaphragms. Equations governing the torsional vibrations of doubly symmetric tubes are also deduced in (Ref. 32) Part I.

In this report torsional vibrations of doubly symmetric unstiffened closed tube are studied by using the authors' formulation presented in Part I. An exact solution for a simply supported tube with the cross section given by $p = \frac{S}{2\pi} \cos \frac{2\pi s}{S}$ is presented and this brings out the nature of the frequency spectrum. First order approxi-

*References are given in Part I.

mation equations[32] are used to obtain the natural frequency characteristics of free-free rectangular tubes. The results are compared with those given in Ref. (14). The influence of secondary effect on natural frequencies for various plan and cross sectional aspect ratios are presented in the form of graphs.

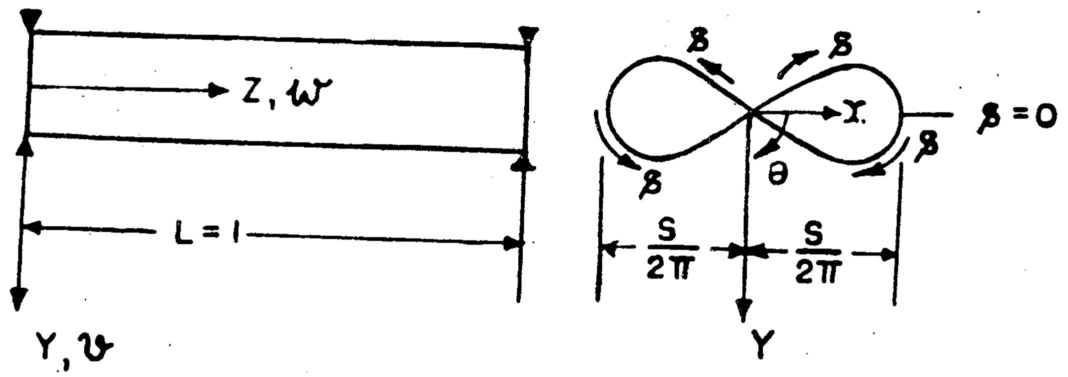

Fig. 2.1 : A simply supported tube with the boundary of the Cross Section given BY $P = \frac{S}{2\pi} \cos \frac{2\pi S}{S}$

2.1 Governing equations—rigorous formulation

The governing equations are given by Eqs. (1.39)* namely

$$\frac{d^2\theta}{dz^2} + k_\theta^2 \theta = -\frac{1}{S_{\theta\theta}} \frac{d}{dz} \oint \frac{\partial w}{\partial s} pt\, ds \quad \ldots (2.1)$$

$$k^2 \frac{\partial^2 w}{\partial z^2} + \frac{\partial^2 w}{\partial s^2} + k_s^2 w = -\frac{dp}{ds} \frac{d\theta}{dz} \quad \ldots (2.2)$$

The boundary conditions at each end are (Eqs. (1.38)).

$$\left.\begin{aligned} &\text{either } \theta = 0 \text{ or } \frac{d\theta}{dz} + \frac{1}{S_{\theta\theta}} \oint \frac{\partial w}{\partial s} pt\, ds = 0 \\ &\text{either } w = 0 \text{ or } \frac{\partial w}{\partial z} = 0, \end{aligned}\right\}$$

and the continuity condition is given by (Eq. 1.43))

$$\left(\frac{\partial w}{\partial s}\right)_{s=0} = \left(\frac{\partial w}{\partial s}\right)_{s=S} \quad \ldots (2.4)$$

2.2 A simply supported tube with boundury of the cross section given by $p = \frac{S}{2\pi} \cos \frac{2\pi s}{S}$ — exact solution

The boundary conditions in this case, are

$$\theta(0) = \theta(1) = 0 \quad \ldots (2.5)$$

$$\frac{\partial w(o,s)}{\partial z} = \frac{\partial w(1, s)}{\partial z} = 0 \quad \ldots (2.6)$$

* Sections and equations referred as (1.) can be seen in Part I.

and the continuity condition is given by

$$\frac{\partial w\,(z, O)}{\partial s} = \frac{\partial w\,(z, S)}{\partial s} \qquad \dots (2.7)$$

Examination of Eqs. (2.6) and (2.7) suggests the expression for w in the form

$$w = \sum_{m=1,2,3,\dots}^{\infty} \; \sum_{n=1,2,3,\dots}^{\infty} A_{mn} \cos m\pi z \sin \frac{4n\pi s}{S} \qquad \dots (2.8)$$

so that Eqs. (2.6) and (2.7) are satisfied; terms like $\sin \frac{4\pi s}{S}$, $\sin \frac{8\pi s}{S}$,...$\sin \frac{4n\pi s}{3}$ have to be used in the expansion of w in order to ensure zero warp at origin 0 as well as the doubly antiperiodic nature of warp. Substituting Eq. (2.8) in Eq (2.1) we have

$$\frac{d^2\theta}{dz^2} + k_\theta^2 \theta = \sum_{m=1,2,3,\dots}^{\infty} \; \sum_{n=1,2,3,\dots}^{\infty} A_{mn} \frac{4mn\,\pi^2 \sin m\pi z}{S\,S_{\theta\theta}} \oint pt \cos \frac{4n\,\pi s}{S} ds \qquad \dots (2.9)$$

The boundary of the cross section of the tube is given by

$$p = \frac{S}{2\pi} \cos \frac{2\pi s}{S} \qquad \dots (2.10)$$

Substituting Eq. (2.10) in (2.9) we have

$$\frac{d^2\theta}{dz^2} + k_\theta^2 \theta = 0 \qquad \dots (2.11)$$

Thus we see that in this case, torsional and warping vibrations are uncoupled.

Pure torsional vibrations

The solution of Eq. (2.11) is

$$\theta = A \sin k_\theta z + B \cos k_\theta z \qquad \dots (2.12)$$

Using boundary conditions (2.5) we have

$$B = 0 \qquad \dots (2.13)$$

and $k_\theta = m\,\pi, \quad m = 1, 2, 3,$ ... (2.14)

and the mode shape is

$$\theta = A \sin m\,\pi\, z$$
$$w = 0 \qquad \dots (2.15)$$

Pure warping vibrations

Substituting Eqs. (2.8) and (2.10) in Eq. (2.2) one has

$$\sum_{m=1,2,3,\dots}^{\infty} \; \sum_{n=1,2,3,\dots}^{\infty} A_{mn} \left\{ -k^2 m^2 \pi^2 - \frac{16\,n^2 \pi^2}{S^2} + k_s^2 \right\}$$
$$\cos m\,\pi\, z \sin \frac{4n\,\pi\, s}{S} = \sin \frac{2n\,\pi\, s}{S} \frac{d\theta}{dz}; \qquad \dots (2.16)$$

using orthogonal properties of Fourier series, we have

$$A_{ij}\left\{-k^2\, i^2 \pi^2 - \frac{16 j^2 \pi^2}{S^2} + k^2{}_s\right\} = 0 \qquad \dots (2.17)$$

For nontrivial solution one has

$$k^2{}_s = k^2\, i^2 \pi^2 + \frac{16 j^2 \pi^2}{S^2}\;;\; i, j = 1, 2, 3, \qquad \dots (2.18)$$

and the mode shapes are given by

$$\left.\begin{aligned} w &= A_{ij} \sin i \pi z \sin \frac{4 j \pi s}{S} \\ \theta &= 0 \end{aligned}\right\} \qquad \dots (2.19)$$

It is interesting to note that in this case, the torsional and warping vibrations are uncoupled. In Neuber tubes pt is constant ; in such cases uncoupled vibrations can logically be expected and this can also be seen from Eqs. (1.37). The tube considered in this report is not a Neuber tube. The uncoupled vibration in this case is due to the periodicty of p in s. A term of same periodicity namely sin $\frac{2 \pi s}{S}$ cannot occur in the expansion for w as this brings in nonzero warps of opposite signs at the origin, contradicting physical features of the problem. As such, the torsional and warping vibrations are uncoupled.

Neuber tubes do not warp under static loads; these have no coupling between torsional and warping vibration. The present example brings out the possibility of a new series of tubes having no coupling between torsional and warping vibration; in these tubes the periodicity of p in s is not admissible for w.

2.3 First order approximation equations

The governing equations in this case are obtained by putting $t_d = t_s = t$ in Eqs. (1.60) and (1.61). The equilibrium equations are

$$S_{\theta\theta} \frac{d^2 \theta}{dz^2} - b_{\theta\theta} \frac{d\phi_\theta}{dz} + k^2{}_s I_p \theta = 0 \qquad \dots (2.20)$$

and $$k^2 B_{\theta\theta} \frac{d^2 \phi_\theta}{dz^2} + (\bar{S}_{CC} \frac{d\theta}{dz} - \bar{b}_{\theta\theta}\, \phi_\theta) + k^2{}_s B_{\theta\theta}\, \phi_\theta = 0$$

and the boundary conditions at each end are

$$\text{either } \theta = 0 \text{ or } S_{\theta\theta} \frac{d\theta}{dz} - b_{\theta\theta} \phi_\theta = 0 \qquad (2.21)$$

$$\text{either } \phi_\theta = 0 \text{ or } \frac{d\phi_\theta}{dz} = 0$$

The cross sectional constants in Eqs. (2.20) are defined already in Eqs. (1.19), (1.55), (1.56), and (1.57) as

$$B_{\theta\theta} = \oint \bar{w}_1^2 \, t \, ds$$

$$\bar{b}_{\theta\theta} = \oint \left(\frac{d\bar{w}_1}{ds} \right)^2 t \, ds \qquad \ldots (2.22)$$

$$b_{\theta\theta} = \bar{S}_{\theta\theta} = \oint p \frac{d\bar{w}_1}{ds} t \, ds$$

$$S_{\theta\theta} = \oint p^2 \, t \, ds$$

$$I_p = \oint r^2 \, t \, ds$$

$\bar{w}_1$ has to be obtained from the relation (Eq. (1.49))

$$\bar{w}_1 = \int\int \frac{dp}{ds} \, ds \; ds \qquad \ldots (2.23)$$

The condition of continuity of $d\bar{w}_1/ds$ and the condition of zero net axial force determine $\bar{w}_1$ uniquely from Eq. (2.23). Since $\bar{w}_1$ has to satisfy (see Eqs. (1.38) and 1.50))

$$\left[-\frac{d\bar{w}_1}{ds} \phi_\theta + p \frac{d\theta}{dz} \right]_0^S = 0$$

which implies $\left(\frac{d\bar{w}_1}{ds} \right)_{s=0} = \left(\frac{d\bar{w}_1}{ds} \right)_{s=S}$

continuity of $(d\bar{w}_1/ds)$ is an essential condition to be satisfied. The equations of equilibrium may be written as

$$\theta'' + k^2_\theta \theta - \upsilon^2 \phi'_\theta = 0 \qquad \ldots (2.24)$$

$$(\phi''_\theta + k_w^2 \phi_\theta) + \mu^2 (\theta' - \phi_\theta) = 0 \qquad \ldots (2.25)$$

Using Eqs. (2.23) and (2.22) it can be shown that

$$\bar{b}_{\theta\theta} = \bar{S}_{\theta\theta}$$

and this is used in writing Eq. (2.25). Boundary conditions at each end are

either $\theta = 0$ or $\theta' - \upsilon^2 \phi_\theta = 0$ ... (2.26)

either $\phi_\theta = 0$ or $\phi'_\theta = 0$ ... (2.27)

Eqs. (2.24), (2.25) and (2.26) involve the following notation

$$k^2_\theta = k^2_s I_p / S_{\theta\theta}$$

$$k^2_w = k^2_s / k^2 \qquad \ldots (2.28)$$

$$\upsilon^2 = b_{\theta\theta} / S_{\theta\theta}$$

$$\mu^2 = \bar{S}_{\theta\theta} / k^2 B_{\theta\theta}$$

From Eqs. (2.24) and (2.25), we have

$$\left[\frac{d^4}{dz^4} + (k^2_\theta + k_w^2 - \lambda^2) \frac{d^2}{dz^2} + k^2_\theta (k_w^2 - \mu^2) \right] \left[\theta \text{ or } \phi_\theta \right] = 0 \qquad = (2.29)$$

where

$$\lambda^2 = \mu^2 (\upsilon^2 - 1) \qquad \ldots (2.30)$$

The solution of Eqs. (2.24) and (2.25) is of the form, as may be seen from Eq. (2.29),

$$\theta = A_1 \sin \lambda_1 z + A_2 \cos \lambda_1 z + A_3 \sinh \lambda_2 z + A_4 \cosh \lambda_2 z$$

$$\phi_\theta = A'_1 \sin \lambda_1 z + A'_2 \cos \lambda_1 z + A'_3 \sinh \lambda_2 z + A'_4 \cosh \lambda_2 z \qquad \text{... (2.31)}$$

where

$$-\lambda_1^2, \lambda_2^2 = \frac{1}{2}\left\{-(k_\theta^2 + k_w^2 - \lambda^2) \mp \sqrt{(k_\theta^2 + k_w^2 - \lambda^2)^2 - 4k_\theta^2 (k_w^2 - \mu^2)}\right\} \qquad \text{... (2.32)}$$

λ_1^2 is always positive while λ_2^2 may be positive or negative. Eqs. (2.31) are valid when λ_2^2 is positive. It is obvious that all the arbitrary constants involved in Eqs. (2.31) are not independent because they have to satisfy either one of Eqs. (2.24) and (2.25). Substituting Eq. (2.31) in Eq. (2.24) one obtains the relationship between the constants as

$$A_1 = -\frac{\lambda_1 \upsilon^2}{-\lambda_1^2 + k_\theta^2} A'_2$$

$$A_2 = \frac{\lambda_1 \upsilon^2}{-\lambda_1^2 + k_\theta^2} A'_1$$

$$A_3 = \frac{\lambda_2 \upsilon^2}{\lambda_2^2 + k_\theta^2} A'_4 \qquad \text{... (2.31a)}$$

$$A_4 = \frac{\lambda_2 \upsilon^2}{\lambda_2^2 + k_\theta^2} A'_3 ;$$

when λ_2^2 is negative the expressions for θ and ϕ_θ are of the form

$$\theta = \bar{A}_1 \sin \lambda_1 z + \bar{A}_2 \cos \lambda_1 z + \bar{A}_3 \sin \lambda_2 z + \bar{A}_4 \sin \lambda_2 z \qquad \text{(2.33)}$$

$$\phi_\theta = \bar{A}'_1 \sin \lambda_1 z + \bar{A}'_2 \cos \lambda_1 z + \bar{A}'_3 \sin \lambda_2 z + \bar{A}'_4 \sin \lambda_2 z$$

and the relationships between the constants are obtained by substituting them in Eq. (2.24) as

$$\bar{A}_1 = \frac{-\lambda_1 \upsilon^2}{-\lambda_1^2 + k_\theta^2} \bar{A}'_2$$

$$\bar{A}_2 = \frac{\lambda_1 \upsilon^2}{-\lambda_1^2 + k_\theta^2} \bar{A}'_1 \qquad \text{... (2.33a)}$$

$$\bar{A}_3 = -\frac{-\lambda_2 \upsilon^2}{-\lambda_2^2 + k_\theta^2} \bar{A}'_4$$

$$\bar{A}_4 = \frac{\lambda_2 \quad \upsilon^2}{-\lambda_2^2 + k_\theta^2} \quad \bar{A}_3'$$

Eqs. (2.31) and (2.31a) or Eqs. (2.33) and (2.33a) involve four arbitrary constants. The four boundary conditions of any problem (Eqs. (2.26) and (2.27)) will be sufficient to solve the problem.

2.4 Cross sectional constants of a rectangular closed tube—first order approximation equations.

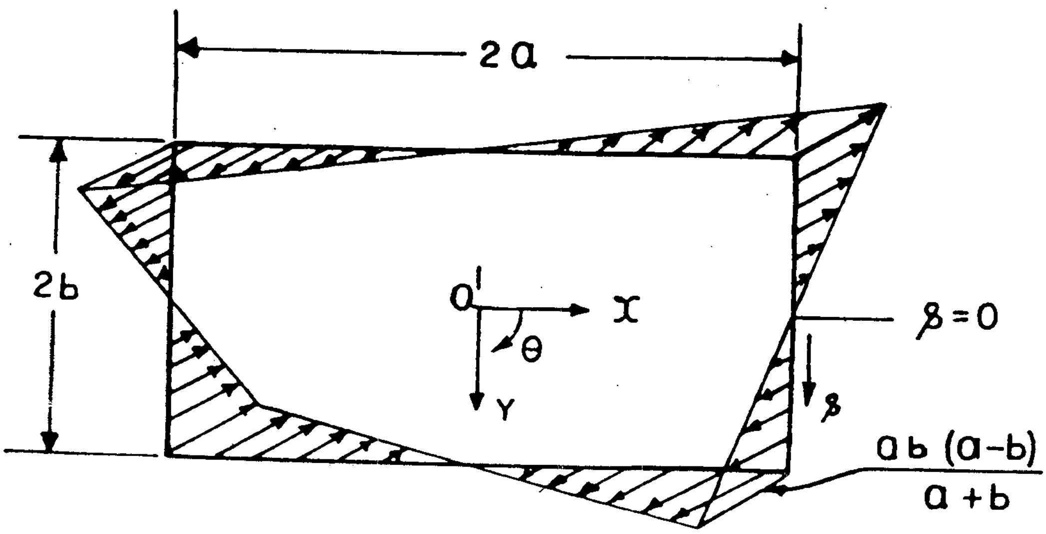

Fig. 2.2 : Cross Section of a rectangular Tube with $\bar{w}_1$ shown.

A cross section of a rectangular closed tube is shown in Fig. (2.2). In this cross section, p has jump discontinuities as shown in Fig. (2.3), and in such a case

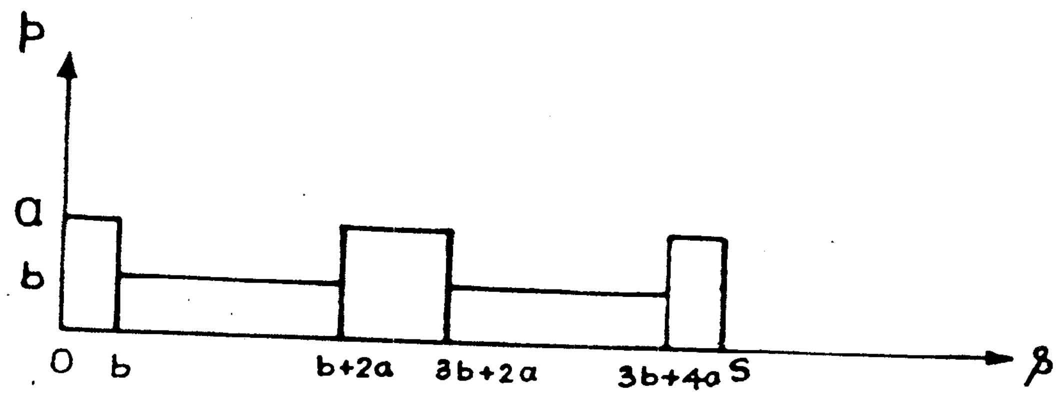

Fig. 2.3 : Variation of p along the periphery.

dp/ds can be conveniently represented as

$$\frac{dp}{ds} = (a-b) \left\{ -\delta_1 (s-b) + \delta_2 (s - \overline{b+2a}) - \delta_3 (s-\overline{3b+2a}) + \delta_4 \ (a-\overline{3b+4a}) \right\} \quad \ldots (2.34)$$

where δ_1, δ_2, δ_3 and δ_4 are Dirac-delta functions. From Eq. (1.49), we have

$$\bar{w}_1 = \iint \frac{dp}{ds}\, ds\, ds$$

and using the conditions, $\bar{w}_1(0) = 0$ and $\bar{w}_1(a+b) = 0$ we have

$$\bar{w} = \frac{ab\,(a-b)}{(a+b)}\,(s/b);\; 0 \leq s \leq b$$

$$= \frac{ab\,(a-b)}{(a+b)} \cdot \frac{(a+b-s)}{a};\; b \leq s \leq a+b \qquad (2.35)$$

and $\bar{w}_1$ is antiperiodic in s with a period $(a+b)$. As mentioned in Part I, $\bar{w}_1$ is similar to Bredth-Batho warp pattern at the free end of the tube under static tip torque. Using Eqs. (2.35), (2.22), (2.28) and (2.30), we have

$$\nu^2 = \frac{(a-b)^2}{(a+b)^2} = \left(\frac{P-1}{P+1}\right)^2$$

$$\mu^2 = \frac{3}{k^2\, ab} = \frac{3Q^2P}{k^2} \qquad \ldots (2.36)$$

$$\lambda^2 = \frac{12\, ab}{k^2\,(a+b)} = \frac{12\, Q^2\, P^2}{k^2\,(P+1)^2}$$

where P and Q are cross sectional and plan aspect ratios respectively and are given by

$$P = a/b,\; Q = L/a \qquad \ldots (2.37)$$

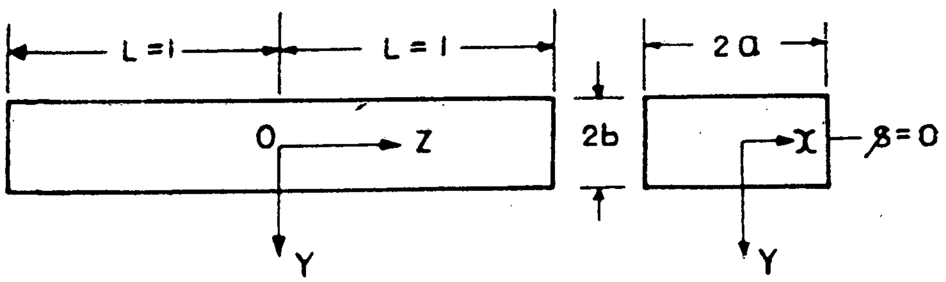

Fig. 2.4: Free-Free Rectangular Tube.

2.5 Free-free rectangular tube—symmetric modes

In this case θ is symmetric in z and ϕ_θ is antisymmetric in z while at the free end (Eqs. (2.26) and (2.27))

$$\theta'(1) - \nu^2\, \phi_\theta(1) = 0 \qquad \ldots (2.38)$$

and

$$\phi'_\theta(1) = 0 \qquad \ldots (2.39)$$

at $\lambda_2^2 > 0$

The appropriate expressions for θ and ϕ_θ are given in Eqs. (2.31) and (2.31a). Since θ is symmetric while ϕ_θ is antisymmetric in z it follows

$$A_1 = A_3 = A_2 = A_4 = 0 \qquad \ldots (2.40)$$

Satisfying the boundary conditions (2.39) and (2.40) and for nontrivial solution one obtains the characteristic equation

$$\frac{\tan \lambda_1}{\tanh \lambda_2} - \frac{\lambda_1}{\lambda_2}\left(\frac{-\lambda_1^2 + k_\theta^2}{\lambda_2^2 + k_\theta^2}\right) = 0 \qquad \dots (2.41)$$

from which eigen values can be computed. Imposing the normalising the condition of unit free end rotation namely

$$A_2 \cos \lambda_1 + A_4 \cosh \lambda_2 = 1, \qquad \dots (2.42)$$

the mode shapes are given by

$$\theta(z) = \frac{1}{\lambda_1^2 + \lambda_2^2}\left\{\frac{\lambda_2^2 + k_\theta^2}{\cos \lambda_1} \cos \lambda_1 z - \frac{-\lambda_1^2 + k_\theta^2}{\cos \lambda_2} \cosh \lambda_2 z\right\}$$

and

$$w(z, s) = -\bar{w}_1 \phi_\theta(z) = -\frac{(-\lambda_1^2 + k_\theta^2)(\lambda_2^2 + k_\theta^2)}{\nu^2(\lambda_1^2 + \lambda_2^2)} \bar{w}_1 \left\{\frac{\sin \lambda_1 z}{\lambda_1 \cos \lambda_1} - \frac{\sinh \lambda_2 z}{\lambda_2 \cosh \lambda_2}\right\} \qquad \dots (2.43)$$

(b) $\lambda_2^2 < 0$

Following the same procedure as above, we obtain the characteristic equation as

$$\frac{\tan \lambda_1}{\tan \lambda_2} - \frac{\lambda_1}{\lambda_2}\left(\frac{-\lambda_1^2 + k_\theta^2}{-\lambda_2^2 + k_\theta^2}\right) = 0 \qquad \dots (2.44)$$

and the mode shapes subject to the normalising condition of unit free end rotation are

$$\theta(z) = \frac{1}{\lambda_1^2 - \lambda_2^2}\left\{-\frac{-\lambda_2^2 + k_\theta^k}{\sin \lambda_1} \sin \lambda_1 z - \frac{-\lambda_1^2 + k_\theta^2}{\sin \lambda_2} \sin \lambda_2 z\right\}$$

and

$$w(z, s) = -w_1 \phi_\theta = \frac{(-\lambda_1^2 + k_\theta^2)(-\lambda_2^2 + k_\theta^2)}{\nu^2(\lambda_1^2 - \lambda_2^2)}$$

$$w_1 \left\{\frac{\cos \lambda_1 z}{\lambda_1 \sin \lambda z} - \frac{\cos \lambda_2 z}{\lambda_2 \sin \lambda_2 z}\right\} \qquad \dots (2.45)$$

Comparison with earlier results

Table 2.1 shows the comparison of the frequency parameter k_T. k_T by first approximation is obtained by using Eq. (2.41) and neglecting longitudinal inertia (putting $k_w^2=0$ in Eq. 2.25). These are compared with energy method solution of Kruszewski and Kordes[14] who also ignored longitudinal inertia.

Table 2.1—Comparison of k_T

P	Q	Mode	$k_T = \frac{\omega^2 \rho L^2 I_p}{G J_o}$ by First approximation equation	Ref. (14)
3.6	2	Fundamental	3.315	3.30
	6	,,	3.179	3.18
	10	,,	3.157	3.16
	14	,,	3.150	3.15
3.6	2	Second	7.105	6.98
	6	,,	6.541	6.53
	10	,,	6.398	6.39
	14	,,	6.347	6.35
3.6	2	Third	11.008	10.80
	6	,,	10.130	10.02
	10	,,	9.775	9.78
	14	,,	9.627	9.65

It can be seen from Table 2.1 that the differences are small. The computations involved in first approximation equations are very small compared to those in Ref. (14). As such, these may be strongly recommended for engineering purposes.

Influence of longitudinal inertia

The values of k_T are computed by using Eqs. (2.41). These include longitudinal inertia. These are compared with those obtained by ignoring longitudinal inertia. The contents of Table 2.2 are also shown in Fig. (2.5).

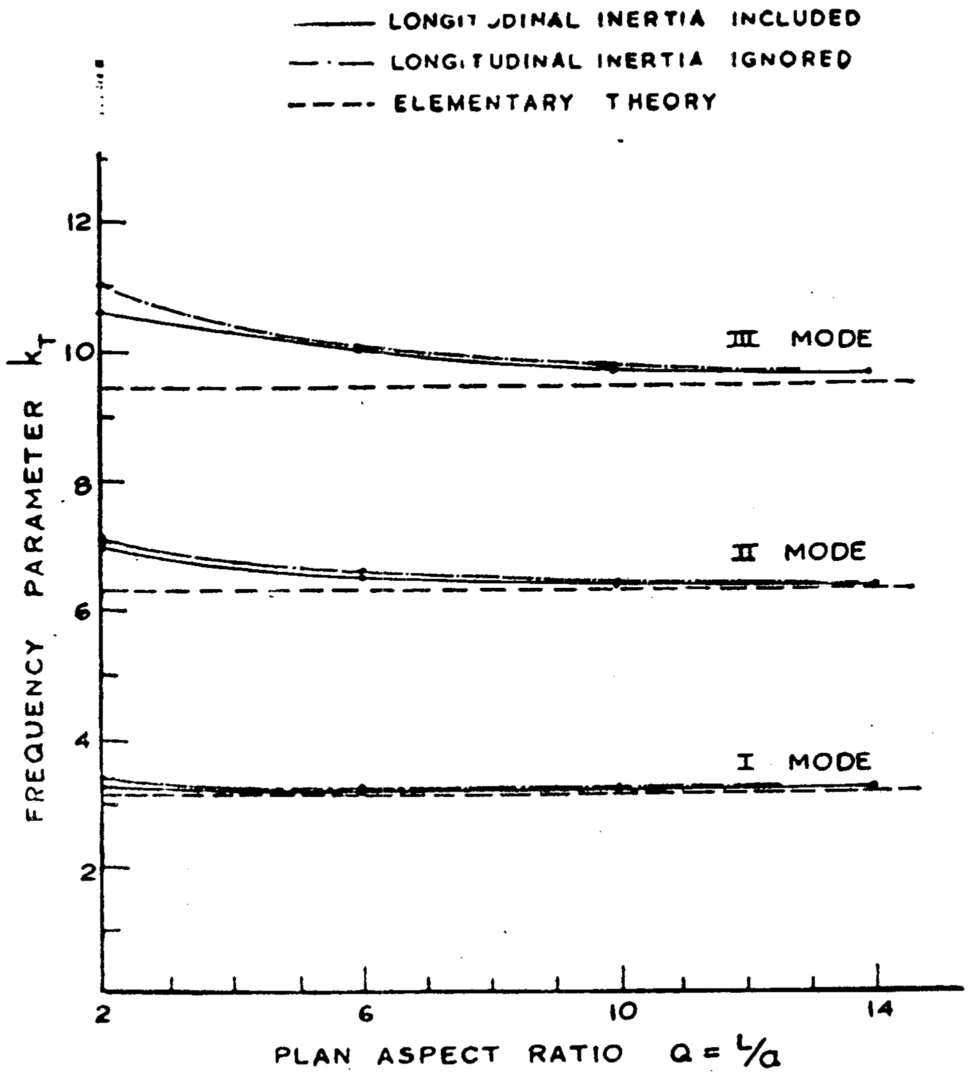

Fig. 2.5 : Influence of Plan Aspect Ratio on the Frequency Parameter k_T for Symmetric Modes of a Free Rectangular Tube a/b = 3.6

Table 2.2—Influence of lognitudinal inertia on natural frequencies.

P	Q	Mode	Longitudinal inertia Included k_T	Longitudinal inertia Ignored k_T
3.6	2	Fundamental	3.273	3.315
	6	,,	3.173	3.179
	10	,,	3.155	3.157
	14	,,	3.149	3.150
3.6	2	Second	6.957	7.105
	6	,,	6.502	6.541
	10	,,	6.382	6.398
	14	,,	6.338	6.347
3.6	2	Third	10.605	11.008
	6	,,	10.034	10.130
	10	,,	9.727	9.775
	14	,,	9.599	9.627

The mode shapes are obtained by using Eq. (2.43). The axial variation of w at the corner (s = a) of the tube is obtained by putting

$$[\bar{w}_1(s)]_{s=a} = \frac{a\,b\,(a-b)}{(a+b)} = \frac{(P-1)}{Q^2 P (P+1)} \qquad \ldots (2.46)$$

in second equation of Eqs. (2.43) as

$$w(z)_{\text{ at } s=a} = \frac{(P-1)}{Q^2 P (P+1)} \Sigma \frac{\left(-\lambda_1^2 + k_\theta^2\right)\left(\lambda_2^2 + k_\theta^2\right)}{\upsilon^2\left(\lambda_1^2 + \lambda_2^2\right)} \left\{ \frac{\sin \lambda_1 z}{\lambda_1 \cos \lambda_1} - \frac{\sinh \lambda_2 z}{\lambda_2 \cosh \lambda_2} \right\} \qquad \ldots (2.47)$$

The values of θ obtained by using the first equation of Eqs. (2.43) and the values of w obtained by using Eq. (2.47) (i) by including k_w^2, (ii) by putting $k_w^2 = 0$ (see Eq. 2.32) are compared in Figs. (2.6) to (2.8). These comparisons reveal that the

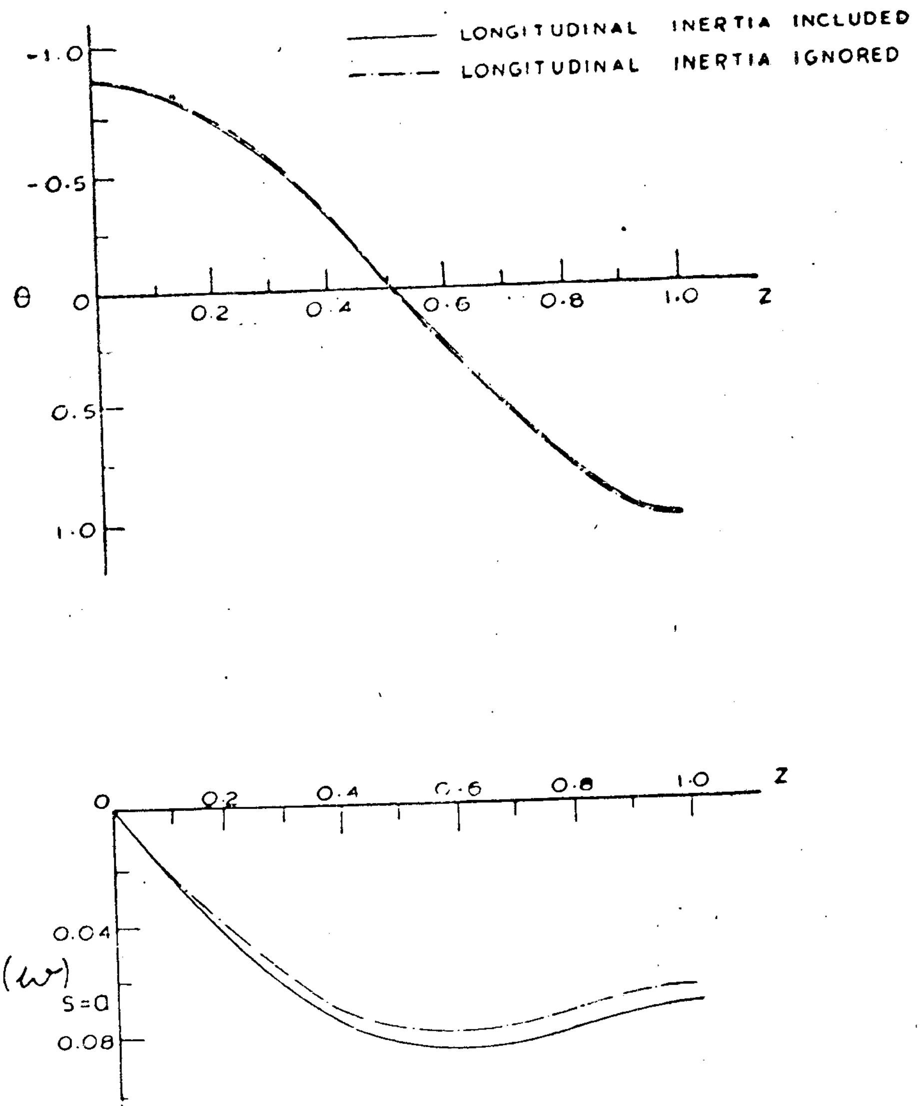

Fig. 2.6 :. Free Free Rectangular Tube Symmetric Modes L/a=2, a/b=3.6
kT—3.273 Longitudinal inertia included
kT 3.315 Longitudinal inertia ignored

neglect of longitudinal inertia effects warp considerably while there is negligible influence on θ distribution and on the natural frequency even in the shortest tube considered.

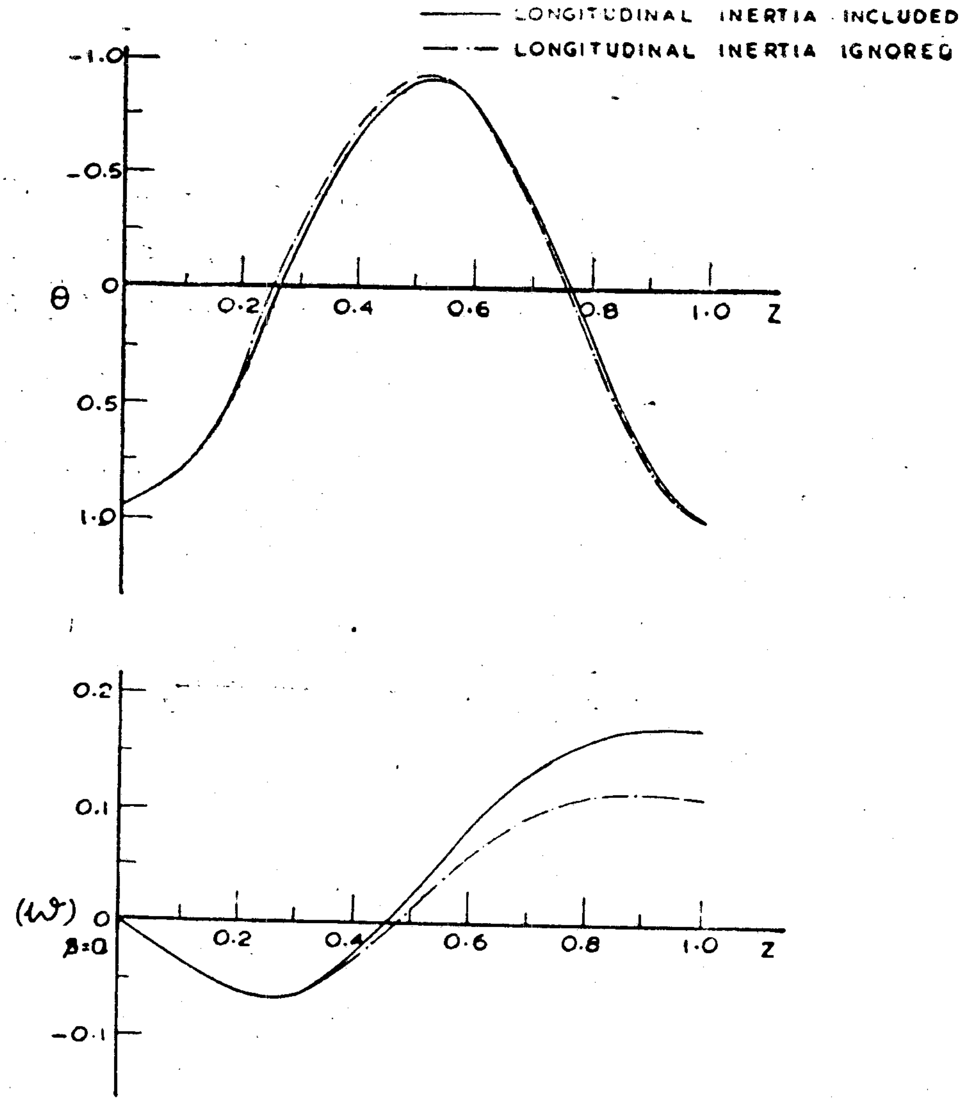

Fig. 2.7 : Free Free Rectangular Tube Symmetric Modes L/a = 2. a/b = 3.6
Second Mode
kT = 6.957 Longitudinal inertia included
kT = 7.105 Longitudinal inertia ignored

New frequencies

The rigorous equations bring out more frequencies than are accounted for in the elementary theory. The first order approximation equations bring out two sets of frequencies—one involving primarily rotational and the other primarily warping motions. The elementary theory brings out only the rotational set of frequencies. The following table shows results obtained by using the first order approximation equations. The number of nodal points has been obtained from computed mode shapes.

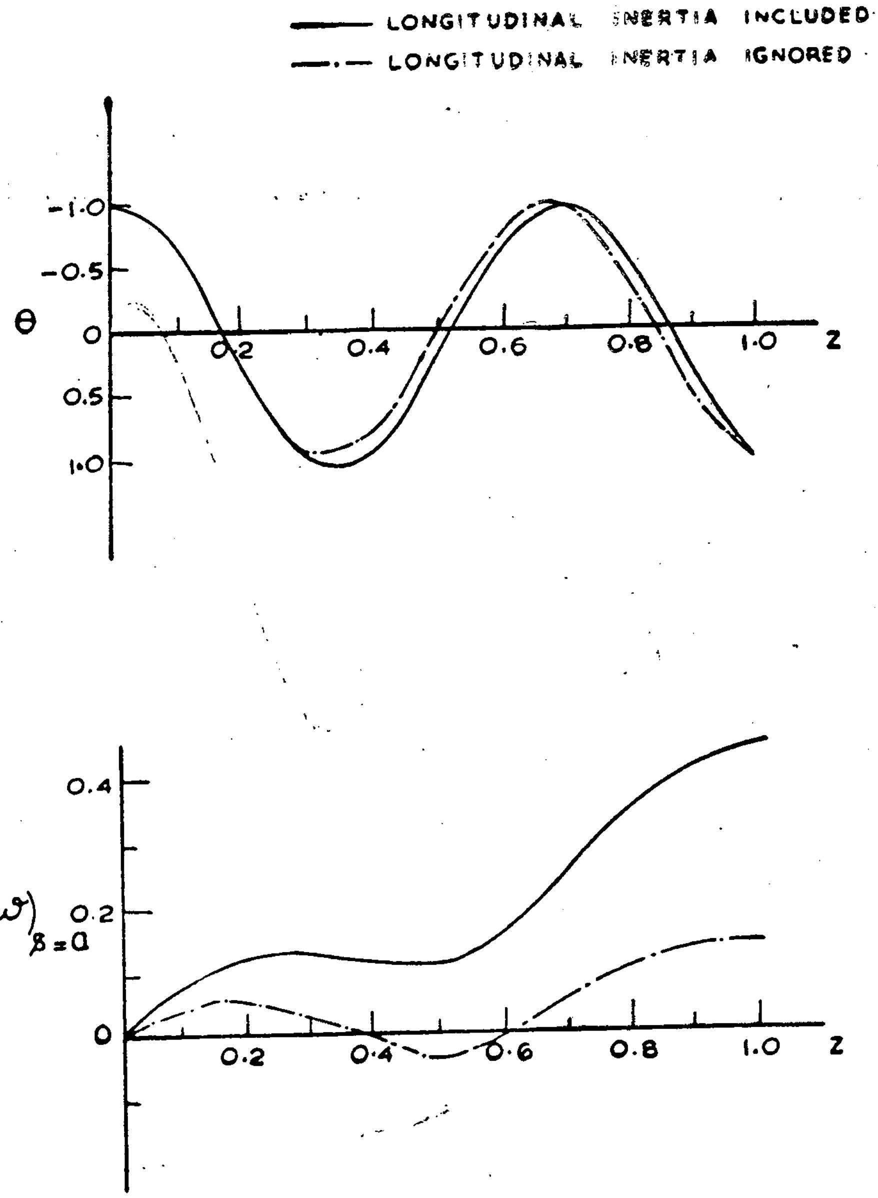

Fig. 2.8 Free Free Rectangular Tube Symmetric Modes L/a=2, a/b=3.6
Third Mode
k_T=10.505 Longitudinal inertia included
k_T=11.008 Longitudiual inertia ignored

Table 2.3—Frequency parameter k_T for symmetric vibrations of a free-free rectangular tube with P = 3.6 and Q = 2.

Mode	Longitudinal inertia			
	Included		Ignored	
	k_T	Number of nodal points in θ	k_T	Number of nodal points in θ
1	3.273	1	3.315	1
2	6.957	2	7.105	2
3	10.605	3	11.008	3
4	12.255	3	—	—
5	14.888	4	14.895	4

From Table 2.3, it can be seen that third and fourth frequeneies have same number of nodal points in θ. The mode shape reveals (Fig. (2.9)) that at the free end θ is small while w is large when $k_T = 12.255$, (while in all other cases θ is large but w is small). Thus it is the first frequency of the second set of frequencies of primarily warping modes. Such extra-frequencies do not appear in the solution if k_w^2 is

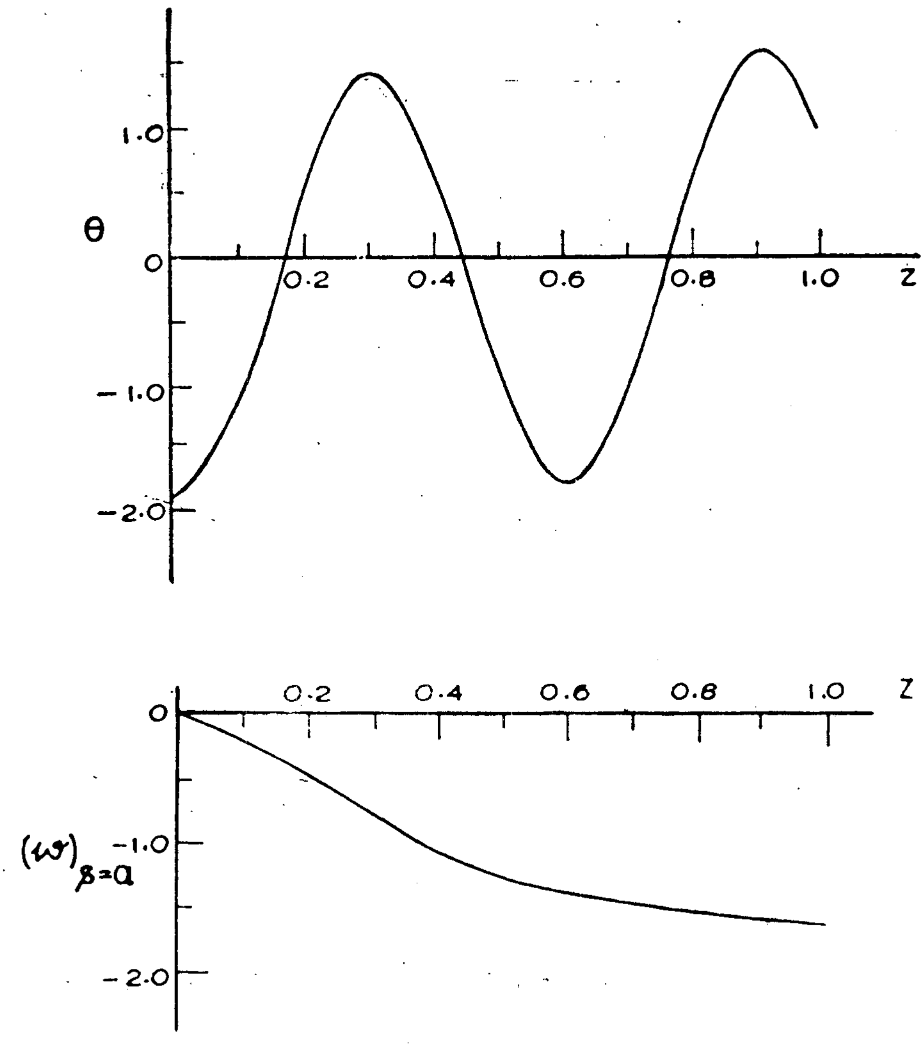

Fig. 2.9 : Free Free Rectangular Tube Symmetric Modes L/a = 2, a/b = 2
Primarily Warping Mode. K = 12.255
Longitudinal inertia included

ignored (see Table (2.3). The peculiar end condition in θ in mode shapes belonging to primarily warping modes may be noted (Fig. 2.9). A similar feature has been reported by Traill-Nash and Collar[7] in flexural vibrations for the first time.

Influence of aspect ratios

The values of frequency parameter for various values of plan and cross sectional aspect ratios are presented in Fig. (2.10). In these calculations longitudinal inertia is included. Since it has been established that the effect of longitudinal inertia is small, these reveal that the influence of shear lag is large in tubes with small plan aspect ratio (short tubes). The discrepancy from the elementary theory

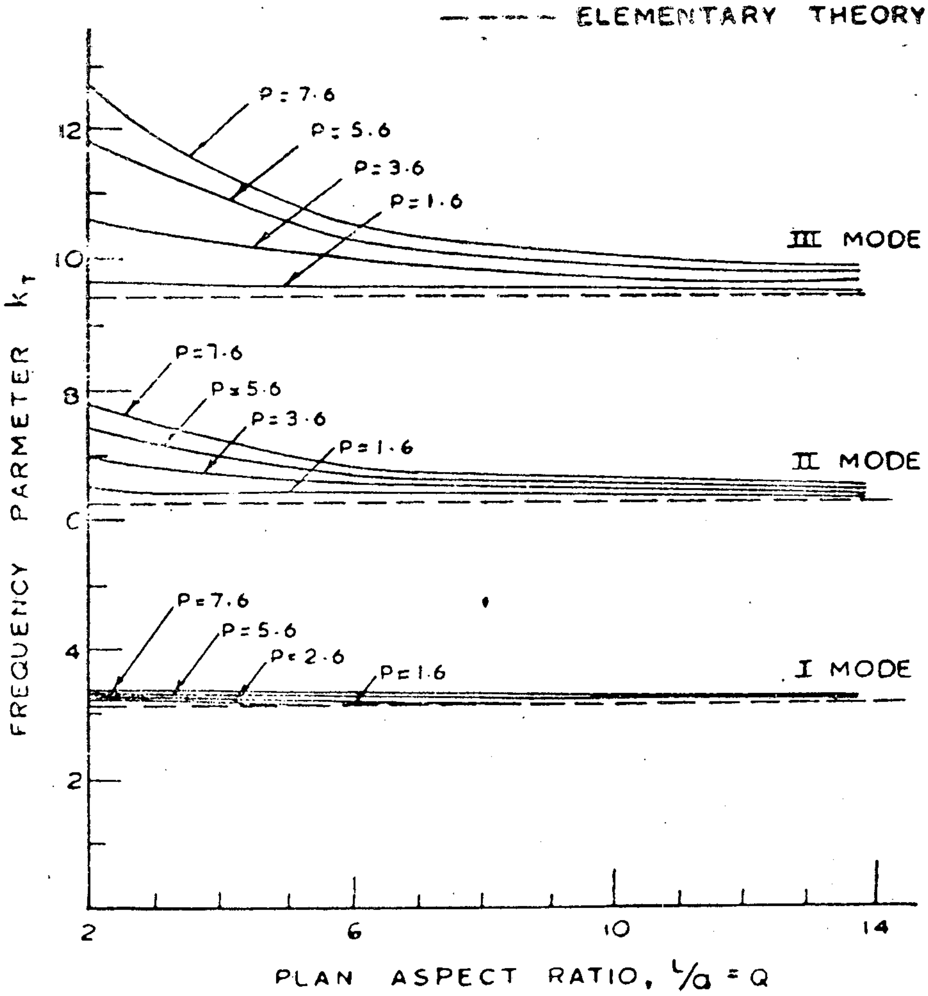

Fig. 2.10: Influence of Plan and Cross Sectional Aspect Ratio on the Frequency Parameter for Symmetric Vibrations of Free Free Rectangular Tube.

increases with the cross sectional aspect ratio. This discrepancy is even larger in higher modes.

2.6 Free-free rectangular tube—antisymmetric modes

In this case (Fig. 2.4) θ is antisymmetric in z while ϕ_θ is symmetric in z and the boundary conditions at the free end are (Eqs. (2.26), (2.27))

$$\theta'(1) - \nu^2 \phi_\theta(1) = 0 \qquad \ldots (2.48)$$

$$\phi'_\theta(1) = 0 \qquad \ldots (2.49)$$

(a) $\lambda_2^2 > 0$

The appropriate expressions for θ and ϕ_θ are given in Eqs. (2.31) and (2.31a). Since θ is antisymmetric and ϕ_θ is symmetric it follows

$$A_2 = A_4 = A_1' = A_3' = 0 \qquad \ldots (2.50)$$

Satisfying boundary conditions given by Eqs. (2.48) and (2.49) and for nontrivial solution one obtains the characteristic equation as

$$\frac{\tanh \lambda_2}{\tanh \lambda_1} + \frac{\lambda_1}{\lambda_2}\left(\frac{-\lambda_1^2 + k_\theta^2}{\lambda_2^2 + k_\theta^2}\right) = 0 \qquad \ldots (2.51)$$

The mode shapes subject to a normalising condition of unit rotation at the free end, are

$$\theta(z) = \frac{1}{\lambda_1^2 + \lambda_2^2}\left\{\frac{\lambda_2^2 + k_\theta^2}{\sin \lambda_1} \sin \lambda_1 z - \frac{-\lambda_1^2 + k_\theta^2}{\sinh \lambda_2} \sinh \lambda_2 z\right\}$$

and

$$w(z, s) = -\bar{w}_1 \phi_\theta = -\frac{(-\lambda_1^2 + k_\theta^2)(\lambda_2^2 + k_\theta^2)}{\nu^2(\lambda_1^2 + k_2^2)}\left\{\frac{\cos \lambda_1 z}{\lambda_1 \cos \lambda_1} + \frac{\cosh \lambda_2 z}{\lambda_2 \cosh \lambda_2}\right\} \bar{w}_1 \qquad \ldots (2.52)$$

(b) $\lambda_2^2 < 0$

Following the procedure as given above, one obtains the characteristic equation as

$$\frac{\tan \lambda_2}{\tan \lambda_1} - \frac{\lambda_1}{\lambda_2}\left(\frac{-\lambda_1^2 + k_\theta^2}{\lambda_2^2 - k_\theta^2}\right) = 0 \qquad \ldots (2.53)$$

and the mode shapes, subject to unit free end rotation, are

$$\theta(z) = \frac{1}{\lambda_1^2 - \lambda_2^2}\left\{\frac{\lambda_2^2 + k_\theta^2}{\sin \lambda_1} \sin \lambda_1 z - \frac{-\lambda_1^2 + k_\theta^2}{\sin \lambda_2} \sin \lambda_2 z\right\} \qquad \ldots (2.54)$$

$$w(z, s) = -\bar{w}_1 \phi_\theta = -\frac{(\lambda_1^2 + k_\theta^2)(-\lambda_2^2 - k_\theta^2)}{\nu^2(\lambda_1^2 - \lambda_2^2)}\left\{\frac{\cos \lambda_1 z}{\lambda_1 \sin \lambda_1} - \frac{\cos \lambda_2 z}{\lambda_2 \sin \lambda_2}\right\} \bar{w}_1$$

The values of the frequency parameter k_T are computed by using Eq. (2.51), for various values of plan and cross sectional aspect ratios. These are presented in Fig. (2.11). These indicate that the influence of shear lag is large for small plan aspect ratio tubes and increases with cross sectional aspect ratios. This influence is even larger for higher modes.

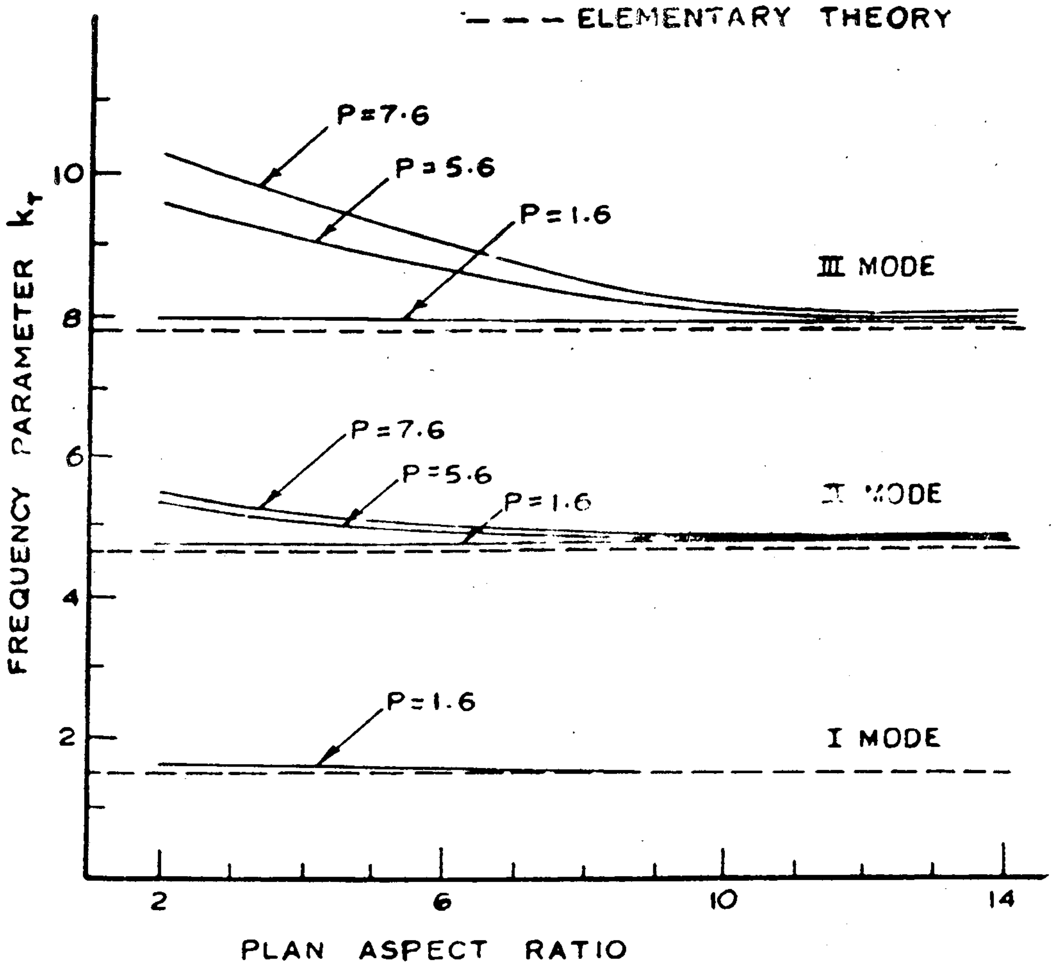

Fig. 2.11 : Influence of Plan and Cross Sectional Aspect Ratio on the Frequency Parameter for anti-symmetric Vibrations of Free Free Rectangular tube.

2.7 Conclusions

In this chapter problems of torsional vibrations of doubly symmetric unstiffened tubes are considered. The exact solution of a simply supported tube with cross section given by $p = \frac{S}{2\pi} \cos \frac{2\pi s}{S}$ is presented. Although pt is not constant here, we have uncoupled torsional and warping vibration. This is because of the periodicity of p in s which is not present in w. This example brings out the possibility of a new series of tubes having no coupling between torsional and warping vibration.

First order approximation equations are simple and are most suitable for engineering use. Comparison of the natural frequencies of a free-free rectangular tube performing symmetric modes of vibration with the results of Kruszewski and Kordes[14] show good agreement. Hence the solution by second approximation equations has not been attempted.

Numerical results reveal that the influence of longitudinal inertia is very small on natural frequency and the rotational mode shape while the effect is considerable on warping, the influence of shear lag is to increase natural frequency with cross sectional aspect ratio. The effect of shear lag is large for small plan aspect ratios and increases for higher modes.

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