Curious Consequences of Simple Sequences

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Simple sequences and series of natural numbers, their reciprocals, integer powers of natural numbers and their reciprocals are considered. Some interesting physical and mathematical consequences of these are discussed.

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The simplest sequence is the sequence of natural numbers $1, 2, 3, \cdots n$. We all know that

$$S_n = 1 + 2 + 3 + ... + n = \frac{n}{2}(n+1). \qquad (1)$$

(See *Box* 1). It is obvious that $S_\infty = \lim_{n\to\infty} S_n$ tends to infinity. Next let us consider the sequence of reciprocals of natural numbers, i.e., 1, 1/2, 1/3, $\ldots 1/n$. It has not been possible to express the sum

$$H_n = 1 + 1/2 + 1/3 + ... + 1/n \qquad (2)$$

in closed form as in equation (1). The infinite series $H_n, n \to \infty$, is called the harmonic series. Since in series (2), each term is smaller than its predecessor, we may ask "Does $H_\infty = \lim_{n\to\infty} H_n$ exist?" The answer is "No". Nicholas Oresme (1323–1382), a multi-faceted genius, proved it as follows:

$$\begin{aligned} H_\infty &= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \cdots \\ &> 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}\right) + \cdots \\ &> 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \cdots \end{aligned}$$

So H_∞ obviously tends to infinity as we go on adding 1/2 indefinitely (see *Box* 2).

Keywords
Simple sequences, natural numbers, primes, Euler–Mascheroni constant.

Box 1.

Gauss virtually obtained this result when he was 7 years old. All the students of his class were asked to write the numbers 1 to 100 and add them up. Gauss quickly figured out 1 + 99 = 100, 2 + 98 = 100, 3 + 97 = 100, 49 + 51 = 100. That leaves a 100 at the end and a 50 in the middle. Thus the sum is fifty 100's (or hundred 50's) and one fifty. In other words, the sum is one hundred one fifty's (n = 100 in equation (1)) = 5,050.

Consequence of Divergence of H_∞

Let us consider the possible overhang of a stack of playing cards kept on a horizontal table. Assuming the cards to be uniform and of length L, the maximum possible overhang of a single card L_1 is obviously $L/2$, when the centre of gravity of the card coincides with the edge of the table. With two cards (see *Figure* 1), when the combined centre of gravity again coincides with the edge of the table, the total length of overhang is

$$L_2 = \frac{L}{2}\left(1 + \frac{1}{2}\right).$$

Box 2.

Another Proof:

$$H_\infty = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots$$

Multiplying and dividing each term by 2

$$\begin{aligned}
&= \frac{2}{2} + \frac{2}{4} + \frac{2}{6} + \frac{2}{8} + \ldots \\
&= \frac{1+1}{2} + \frac{1+1}{4} + \frac{1+1}{6} + \frac{1+1}{8} + \ldots \\
&= \left(\frac{1}{2} + \frac{1}{2}\right) + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{6} + \frac{1}{6}\right) + \left(\frac{1}{8} + \frac{1}{8}\right) + \ldots \\
&< 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \ldots
\end{aligned}$$

So we reach a contradiction $H_\infty < H_\infty$!

Thus, H_∞ is not a finite number and we cannot claim that multiplying and dividing each term by 2 leaves it unaltered.

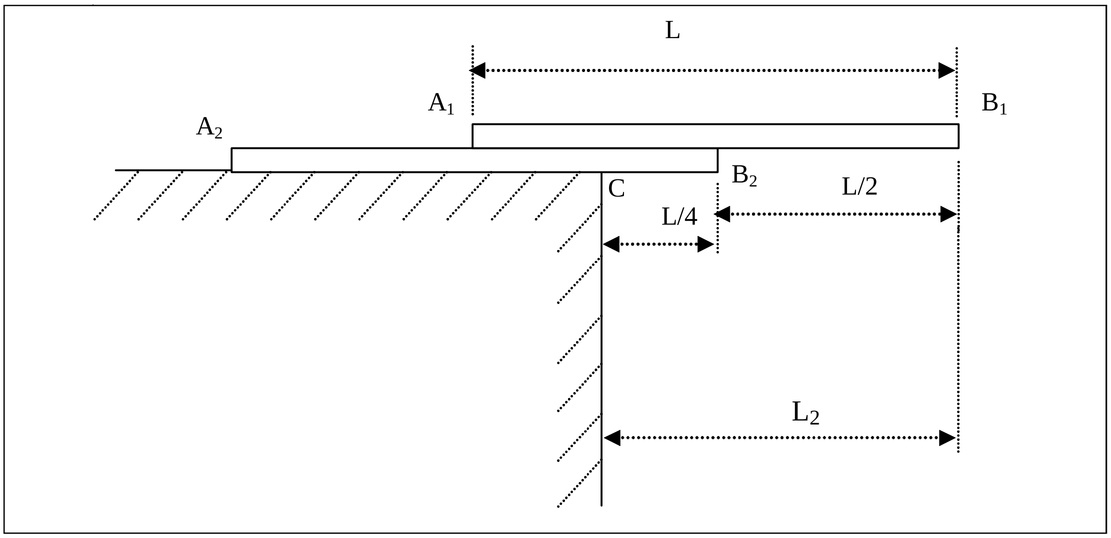


Figure 1.

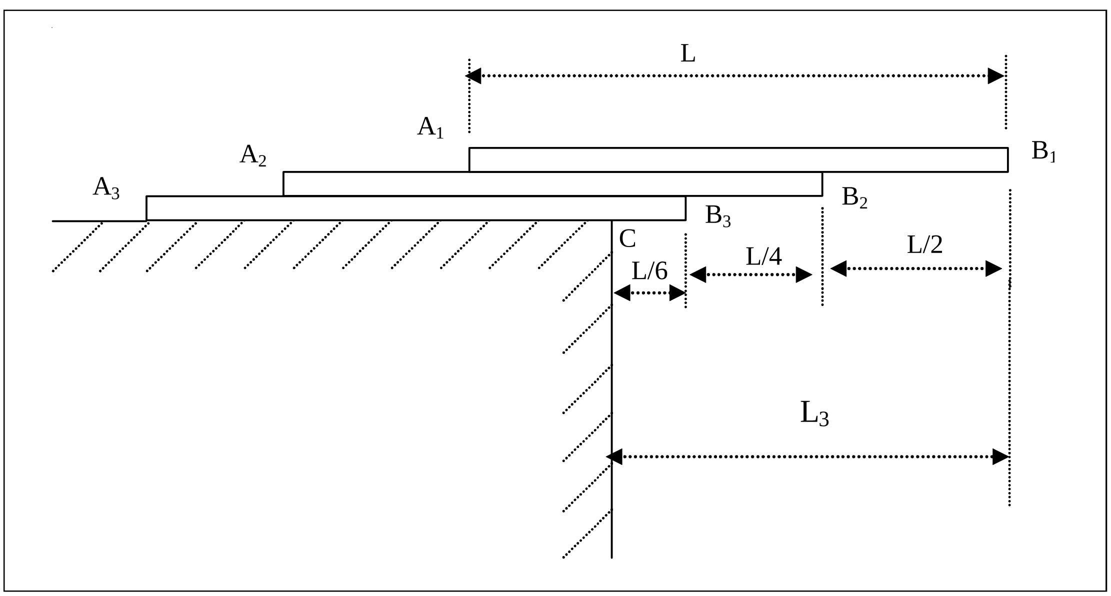
With three cards (see *Figure* 2), the total length of overhang is

$$L_3 = \frac{L}{2}\left(1 + \frac{1}{2} + \frac{1}{3}\right).$$

Thus, with n cards we can have an overhang of length

$$L_n = \frac{L}{2}\left(1 + \frac{1}{2} + \frac{1}{3} + ... + \frac{1}{n}\right) = \frac{L}{2}H_n.$$

Figure 2.



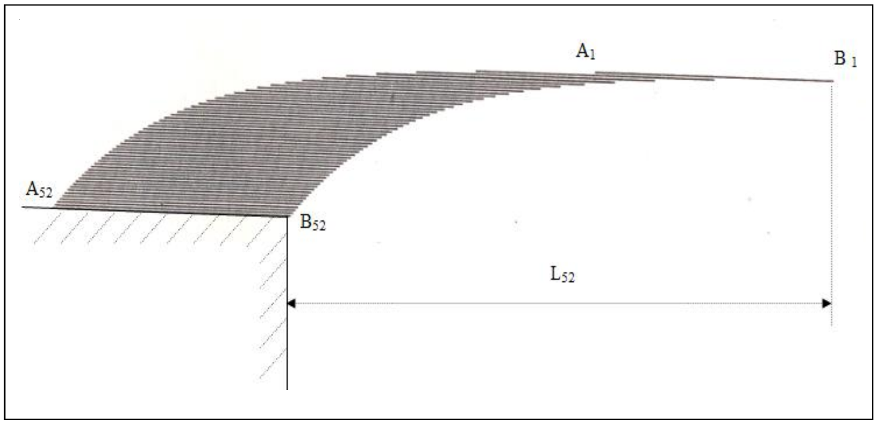


Figure 3.

Since H_n diverges with increasing n, we can have as long an overhang as we desire. By arranging a pack of 52 cards in this manner (see *Figure* 3), we get an overhang of length

$$L_{52} = 2.25940659073334L.$$

Notice that several cards (how many?) at the top are completely outside the table.

Now let us see what happens if one takes reciprocals of only the even or odd natural numbers. As shown below, in both cases the sum up to infinity does not tend to a finite limit.

$$\begin{aligned} H_\infty^E &= \frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \dots \\ &= \frac{1}{2}\left(1 + \frac{1}{2} + \frac{1}{3} + \cdots\right) = \frac{1}{2}H_\infty. \end{aligned}$$

So H_∞^E is not finite.

Similarly,

$$\begin{aligned} H_\infty^O &= 1 + \frac{1}{3} + \frac{1}{5} + \cdots \\ &> \frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \cdots \\ \text{or,} \qquad H_\infty^O &> H_\infty^E. \end{aligned}$$

Thus, H_∞^O also tends to infinity.

Consequence of Divergence of H_{∞}^{O}

Crossing a Desert: Let us now consider the following problem. Suppose we have a number of jeeps, each of which with its fuel tank full can cover a distance L. In this discussion, we shall refer to the volume of fuel also in terms of L. With N number of jeeps, what is the maximum length of a desert that can be covered by one jeep *without any jeep getting stranded in the desert*? Of course, the jeeps are allowed exchange of fuel within the desert.

With one jeep, obviously one can cross a desert of length $L_1 = L$. With two jeeps, first both jeeps travel a distance $L/3$. At this point, the second jeep transfers $L/3$ fuel to the first jeep and returns (has just sufficient fuel to do that). The first jeep is now full and can travel a further distance L. So the total distance that can be covered by the first jeep is $L_2 = L\left(1+\frac{1}{3}\right)$.

With 3 jeeps, all the jeeps travel up to $L/5$. At this point, jeep 3 transfers $L/5$ fuel to jeeps 1 and 2 each, which become full. Jeeps 1 and 2 travel a further distance $L/3$ when jeep 2 transfers $L/3$ fuel to jeep 1, which again becomes full.. At this point, jeep 2 returns empty to the location of jeep 3. Jeep 1 now has a full tank and can travel a further distance L. So jeep 1 travels a total distance $L_3 = L\left(1+\frac{1}{3}+\frac{1}{5}\right)$, with jeeps 2 and 3 returning to the base by sharing equally the fuel remaining in jeep 3.

Thus, with n jeeps (and, of course, with enough fuel), one jeep can cross a desert of length

$$L_n = L\left(1+\frac{1}{3}+\frac{1}{5}+....+\frac{1}{2n-1}\right) = LH_n^O.$$

Since H_n^O tends to infinity with increasing n, one can devise a strategy to cross a desert of any length without any jeep getting stranded in the desert.

Table 1.

n	H_n	$\ln(n)$	$H_n - \ln(n)$
100	5.187…	4.606…	0.581
1,000	7.486…	6.909…	0.575
1,000,000	14.392…	13.818…	0.574

Rate of Divergence of H_n

The sum H_n diverges very slowly.with increasing n. *Table* 1 convinces us that H_n diverges as $\ln(n)$.

In 1968, it was shown that for H_n to exceed 100, n must be a 44-digit number! If this result is to be obtained by brute force computation by adding a term and checking, then (assuming addition of one term takes 10^{-9} sec by a good computer) the addition of 10^{44} numbers will take 10^{35} sec, which is more than 10^{29} years (much more than the estimated age of the universe)!

Though H_n crosses all integers as n increases, it never equals any integer. In fact, it is not difficult to show that the sum of the reciprocals of any number of consecutive integers cannot be an integer, as the numerator will be odd and the denominator will be even.

Consequence of Slow Divergence of H_n

(i) *Record Rainfall*

Let us consider the data of annual rainfall at a place. We define a year as a 'record' year, if the rainfall in that year is greater than in all previous years. The expected number of such record years in n years is H_n. The first year, by definition, is a record year. In the second year, the rainfall (assuming different amount of rainfall in every year) may be more or less than that in the first year. So the expected number of record years at the end of the second year is $1 + \frac{1}{2}$. Considering the first three years, the probability of the third year having the maximum rainfall is $\frac{1}{3}$. This can be easily seen by considering different arrangements of three different amounts of rainfall in three years. Thus, the expected number of record years at the end of the third year is $1+\frac{1}{2}+\frac{1}{3}$. Extending this argument for n years, we get the expected number of record years in n years to be H_n.

The data for 160 years of rainfall in the Central Park of New York City shows 5 record years whereas $H_{160} = 5.65$. However, the data for 234 years of rainfall at Oxford shows only 5 record years ($H_{234} = 6.03$), probably confirming the age-old saying "English weather is most unpredictable". We might infer that unless the climate is tampered with, there should not be frequent appearances of record rainfall. The argument given above, obviously, cannot be applied to athletic records, which are dictated by so many other factors (which are not unbiased), like improved technique, sponsor's incentive, etc.

(ii) *Destructive Testing*

Suppose we have a sample of n nominally identical structural elements, and the lowest strength of the lot has to be determined by destructive testing. The test is conducted as follows. The first sample is broken and the limiting load is determined as L_1. The second specimen is loaded up to L_1. If it does not break, then it is set aside. If it breaks at a lower value, say L_2, then the third specimen is loaded up to only L_2. If the samples can be taken as a random sequence, then by following this destructive testing procedure, one expects to break only H_n specimens while testing n specimens. The logic is exactly the same as discussed in the rainfall record. Referring to *Table* 1, one expects to break about 7 or 8 members to determine the minimum strength for a sample of 1000 specimens.

Reciprocals of Primes

We know that the sequence of prime numbers never ends, but primes are sparsely distributed, so one can ask "If we sum the reciprocals of all primes, does *that* infinite series converge?" The answer is again "No". Obviously, this series H_n^p diverges even more slowly than H_n. In fact, the series of reciprocals of primes diverges as $\ln(\ln(n))$. Just like H_n, the series H_n^p also crosses every integer with increasing n, but never equals any integer. The proof is not very difficult. One might prove that the sum of reciprocals of any (even non-consecutive) number of primes cannot be an integer.

Next we consider a special set of primes, called 'twin primes'. When two consecutive odd numbers are primes, then these are called twin primes, e.g. (3, 5), (5, 7), (11, 13),…(1017, 1019), etc.. It is not yet known whether twin primes continue forever or not. But it is known that the sum of the reciprocals of twin primes converges:

$$H^{tp} = \left(\frac{1}{3}+\frac{1}{5}\right)+\left(\frac{1}{5}+\frac{1}{7}\right)+\ldots$$
$$+\left(\frac{1}{1019}+\frac{1}{1021}\right)+\ldots\ldots = 1.9021605824\ldots$$

The above number is known as 'Brun's constant'. If the above sum diverged, we could have concluded that twin primes definitely continue forever. Now we are still unable to decide whether twin primes continue forever or not! (See *Box* 3).

Box 3.

In 1994, while computing Brun's constant and evaluating the reciprocals of twin primes (824 633 702 441, 824 633 702 443), a flaw in the then Pentium numeric processor was detected, which was later rectified by Intel. In 1995, Intel announced a pre-tax expense of US$475 million – the cost of replacement of defective chips. This is probably the largest amount of money that has ever been associated with any mathematical activity.

Euler–Mascheroni Constant γ

From *Table* 1 we note that the difference between H_n and $\ln(n)$ is roughly equal to 0.57 for large n. To see this clearly, we consider *Figure* 4 where the function $y = 1/x$ is plotted for $1 \le x \le n$. Now, approximating the area under the curve (up to x–axis) by trapeziums (one such is marked between $x = 1$ and 2 in the figure), we can write

$$\begin{aligned}\int_1^n \frac{dx}{x} &= \ln n \approx \frac{1}{2}\left(1+\frac{1}{2}\right)+\frac{1}{2}\left(\frac{1}{2}+\frac{1}{3}\right)+ \\ &\qquad \frac{1}{2}\left(\frac{1}{3}+\frac{1}{4}\right)+\ldots+\frac{1}{2}\left(\frac{1}{n-1}+\frac{1}{n}\right) \\ &\approx \frac{1}{2}\left[1+2\left(H_n-1-\frac{1}{n}\right)+\frac{1}{n}\right] \\ &\approx H_n-\frac{1}{2}-\frac{1}{2n}.\end{aligned}$$

Thus,

$$H_n \approx \ln n + \frac{1}{2} + \frac{1}{2n}.$$

Let $\gamma_n = H_n - \ln n$; this is the difference of two diverging terms. It may be shown that γ_n converges. The constant $\gamma = \lim_{n\to\infty} \gamma_n$ is called 'Euler–Mascheroni Constant'. Euler first calculated it up to five and then to

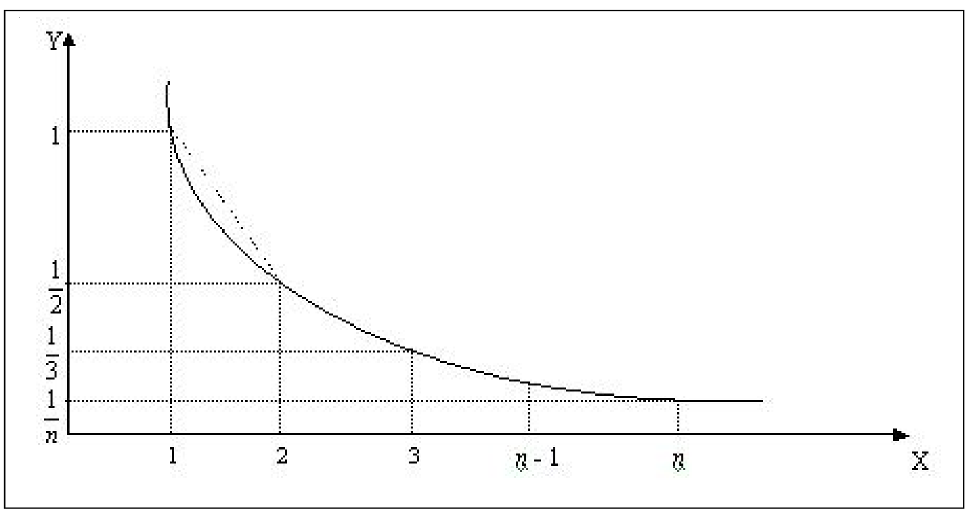


Figure 4.

sixteen decimal places as $\gamma = 0.5772156649015325\ldots$. In 1998, this constant was computed up to 108×10^6 decimal places. But it is not yet proved to be an irrational number.

Sums of Powers of Integers and Reciprocals

We have already noted (see equation(1)) that

$$S_n^{(1)} = 1 + 2 + 3 + ... + n = \frac{n}{2}\left(n+1\right).$$

Jakob Bernoulli calculated the sums of powers of integers in terms of what are known as Bernoulli numbers. For example, if we write

$$S_n^{(k)} = 1^k + 2^k + 3^k + \ldots + n^k,$$

then

$$S_n^{(1)} = \frac{n}{2}\left(n+1\right) = \frac{1}{2}\left(\begin{pmatrix} 2 \\ 0 \end{pmatrix} B_0 n^2 + \begin{pmatrix} 2 \\ 1 \end{pmatrix} B_1 n\right)$$

$$S_n^{(2)} = \frac{n}{6}\left(n+1\right)\left(2n+1\right)$$

$$= \frac{1}{3}\left(\begin{pmatrix} 3 \\ 0 \end{pmatrix} B_0 n^3 + \begin{pmatrix} 3 \\ 1 \end{pmatrix} B_1 n^2 + \begin{pmatrix} 3 \\ 2 \end{pmatrix} B_2 n\right)$$

$$S_n^{(3)} = \frac{n^2}{4}\left(n^2 + 2n + 1\right) = \left(S_n^{(1)}\right)^2,$$

where the Bernoulii numbers are

$$B_0 = 1, \quad B_1 = 1/2, \quad B_2 = 1/6,$$
$$B_4 = -1/30, \quad B_6 = 1/42, \; .....$$
$$B_3 = B_5 = B_7 = ... = 0.$$

We may note that except the first one, all other odd Bernoulii numbers are zero. Generalizing the above results for any value of k is left to the reader.

We have already noted that the Harmonic series H_∞ diverges. Now we show that if we consider the sum of the powers (more than 1) of reciprocals, then it converges. Towards this end, we follow the procedure of Nicholas Oresme outlined earlier. Let

$$\begin{aligned}\zeta(s) &= 1+\frac{1}{2^s}+\frac{1}{3^s}+\frac{1}{4^s}+\dots\\ &= 1+\left(\frac{1}{2^s}+\frac{1}{3^s}\right)+\left(\frac{1}{4^s}+\dots+\frac{1}{7^s}\right)+\dots\\ &< 1+\frac{2}{2^s}+\frac{4}{4^s}+\dots\\ &= 1+\frac{1}{2^{s-1}}+\left(\frac{1}{2^{s-1}}\right)^2+\dots\\ &= \frac{1}{1-\frac{1}{2^{s-1}}}\end{aligned}$$

Thus $\zeta(s)$ converges if $\frac{1}{2^{s-1}} < 1$, i.e., $s > 1$.

Euler was the first to calculate $\zeta(s)$ for even values of s up to 26. He also showed that for even values of s, one gets, in terms of Bernoulli numbers,

$$\zeta(2n) = (-1)^{n-1}\frac{(2\pi)^{2n}}{2\,(2n)!}B_{2n}.$$

No result is available for odd values of s (interestingly, as mentioned earlier, all odd Bernoulli numbers are zero for s more than 1). In 1978 it was shown that $\zeta(3)$ is irrational. (See *Box* 4).

Connection Between Two Infinite Universes of Natural Numbers and Primes

Euler produced the following remarkable result, starting from $\zeta(s)$, connecting all the natural numbers to all the prime numbers. This process is known as 'Eratosthenes sieve'. The final result is the starting point of a lot of later mathematics, especially in the theory of prime numbers. We start with

$$\zeta(s) = \frac{1}{1^s}+\frac{1}{2^s}+\frac{1}{3^s}+\dots; \quad s>1$$

and then multiplying both sides by $\frac{1}{2^s}$ we get

$$\frac{1}{2^s}\zeta(s) = \frac{1}{2^s} + \frac{1}{4^s} + \frac{1}{6^s} + \dots .$$

Subtracting this equation from the preceding one, we obtain

$$\left(1 - \frac{1}{2^s}\right)\zeta(s) = \frac{1}{1^s} + \frac{1}{3^s} + \frac{1}{5^s} + \frac{1}{7^s} + \frac{1}{9^s} + \dots .$$

Box 4.

Euler in 1735 (using a non-rigorous method) first obtained the value of $\zeta(2)$, which enhanced his reputation enormously as all great mathematicians before him failed at this problem. He started from the infinite series

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots .$$

Dividing both sides by x

$$\frac{\sin x}{x} = 1 - \frac{x^2}{6} + \frac{x^4}{120} - \frac{x^6}{5040} + \dots .$$

Substituting $y = x^2$,

$$1 - \frac{y}{6} + \frac{y^2}{120} - \frac{y^3}{5040} + \dots = \frac{\sin x}{x}.$$

Now the right hand side is zero for $x = n\pi$, $n = 1, 2, 3, \dots$, i.e. with $y = n^2\pi^2$. So Euler treated the *infinite* polynomial equation (with right hand side zero) as having roots $y_n = n^2\pi^2$. Then he used the result valid for a finite polynomial equation, which says that the sum of the reciprocal of the roots is governed by only the coefficient of y (with the leading term as unity). Thus,

$$\sum_n \frac{1}{y_n} = \frac{1}{6}, \qquad \text{or,} \qquad \zeta(2) = \sum_n \frac{1}{n^2} = \frac{\pi^2}{6}!$$

If we are surprised to see the appearance of π (the ratio of circumference to diameter of a circle) in this summation, then it will be quite astonishing to know that $6/\pi^2$ is also the probability of two positive integers taken at random to be coprime, i.e., not to have any common factor! Since the sum of the reciprocals of primes did not converge, Euler concluded that the number of primes is more than the number of perfect squares (of course, both are infinite).

Multiplying both sides by $\frac{1}{3^s}$ we get

$$\frac{1}{3^s}\left(1-\frac{1}{2^s}\right)\zeta\left(s\right)=\frac{1}{3^s}+\frac{1}{9^s}+\frac{1}{15^s}+\ldots\ .$$

Again subtracting the last equation from the preceding one, we obtain

$$\left(1-\frac{1}{2^s}\right)\left(1-\frac{1}{3^s}\right)\zeta\left(s\right)=\frac{1}{1^s}+\frac{1}{5^s}+\frac{1}{7^s}+\frac{1}{11^s}+\ldots\ .$$

Proceeding in this manner, all the prime numbers appear on the left hand side, and the right hand side becomes 1. Thus, finally

$$\prod_p\left(1-\frac{1}{p^s}\right)\zeta\left(s\right)=1$$

or

$$\sum_n\frac{1}{n^s}=\frac{1}{\prod_p\left(1-\frac{1}{p^s}\right)}\ .$$

The left hand side is summed over all the positive integers whereas, on the right hand side, we take the product over all primes.

Alternating Series

Let us consider the following infinite geometric series

$$1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\ldots=\frac{1}{1-\frac{1}{2}}=2.$$

Since this series converges with all the terms having positive sign, if the alternate terms are written with opposite signs, then also it will converge, i.e.,

$$1-\frac{1}{2}+\frac{1}{4}-\frac{1}{8}-\ldots=\frac{1}{1+\frac{1}{2}}=\frac{2}{3}.$$

Such a series is called absolutely convergent, and it always converges to the same value independent of ordering of the terms.

We have already seen that the harmonic series diverges. But if the sign in front of the alternate terms are made opposite then the alternating harmonic series

$$H_\infty^A = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \dots$$

which is the difference of two diverging series, viz. H_∞^O and H_∞^E, converges and is called a conditionally convergent series. A conditionally convergent series converges to a value which depends on the ordering of the terms. For example, if computed as written above, H_∞^A converges to $\ln 2$. To show this we start with

$$\ln(1+x) = 1 - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots \qquad -1 < x \leq 1$$

On substituting $x = 1$, we get the desired result. However, if we rewrite the same series with a different ordering of terms as

$$H_\infty^A = \left(1 - \frac{1}{2}\right) - \frac{1}{4} + \left(\frac{1}{3} - \frac{1}{6}\right) - \frac{1}{8} + \left(\frac{1}{5} - \frac{1}{10}\right) - \frac{1}{12} + \dots$$

then we get this to be

$$\begin{aligned} &= \frac{1}{2} - \frac{1}{4} + \frac{1}{6} - \frac{1}{8} + \frac{1}{10} - \frac{1}{12} + \dots \\ &= \frac{1}{2}\left(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \dots\right) \\ &= \frac{1}{2}\ln 2. \end{aligned}$$

Surprisingly, one can make H_∞^A converge to $\ln 3$ by the following ordering of the terms

$$\ln 3 = 1 + \left(\frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \frac{1}{9} + \frac{1}{11} - \frac{1}{6} + \frac{1}{13} + \frac{1}{15} - \frac{1}{8} + \frac{1}{17} + \frac{1}{19}\right)$$
$$\left(+\frac{1}{21} - \frac{1}{10} + \frac{1}{23} + \frac{1}{25} - \frac{1}{12} + \frac{1}{27} + \frac{1}{29} - \frac{1}{14} + \frac{1}{31} + \frac{1}{33} - \frac{1}{16} + \frac{1}{35} + \frac{1}{37}\right) + \ldots$$

Notice that in all the ordering mentioned above, the terms with the same sign appear in decreasing order. Such arrangements of the terms are called simple arrangements. There are theorems for such simple ordering. In fact, if we include even the non-simple ordering, then the conditionally convergent series can be made to converge to any number!

Suggested Reading

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[2] **J Derbyshire, *Prime Obsession*, Plume, Washington DC, 2004.**

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