On measurable relations

by

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Abstract. The results in this paper supplement those in "Measurable relations" by C. J. Himmelberg [H]. Some results of that paper are extended and examples are given showing the sharpness of earlier results. In particular, examples are given showing (i) that the intersection of two weakly measurable, closed-valued relations may fail to be weakly measurable, and (ii) that Filippov's implicit function theorem may fail without appropriate restrictions either on the domain of the relation or on its values.

1. Introduction. Measurable relations, i.e., set valued functions which assign to each element t of a measurable space T a subset of a topological space X in a manner satisfying any one of several possible definitions of measurability, have been studied extensively in recent years by many authors. In this paper, which is somewhat of a sequel to the paper of Himmelberg [H], we extend some earlier results and give several examples that indicate the sharpness of some theorems dealing with measurable relations and their properties.

Generally, the notation follows that in [H]. Also, as in [H], we are most concerned with the case when T is an arbitrary, abstract measurable space. For general references to recent literature the reader is frequently referred to the excellent survey paper by Wagner [W].

In Section 2 we give the notation and terminology that will be used. Also, several propositions, which give known properties of measurable relations, are stated without proof. All of these will be used later.

In Section 3 we present some extensions of known results. Several of these extend results in [H] by noting that a separability hypothesis on X can be deleted.

Section 4 is devoted to examples. In particular, there is an example showing that the intersection of two closed-valued measurable multifunctions may fail to be measurable. Also there is an example showing that the Filippov type implicit function lemma may fail without appropriate restrictions either on the domain of the relation or on its values. The last example is of a countable (but not closed)-

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valued measurable relation that has no measurable selector. These examples indicate some of the limitations that are inherent in dealing with measurable relations.

2. Preliminaries. Throughout this paper T will denote a measurable space with σ-algebra $\mathscr{A}$. Unless otherwise specified, that is all we will assume about T.

X will be a metrizable space, and, following Bourbaki, we will call X: *Polish*, if X is separable and metrizable by a complete metric; *Lusin*, if X is the bijective continuous image of a Polish space; *Souslin*, if X is the continuous image of a Polish space.

As usual, a relation F is a subset of $T\times X$. Alternatively, F may be regarded as a function from T to the set of all subsets of X, $\mathscr{P}(X)$. The set $\{t\in T|\ F(t)\neq\emptyset\}$ is called the *domain* of F: $T\to\mathscr{P}(X)$. If the domain of F is T, then F is called a *multifunction* (or correspondence) from T to X and we often write $F\subset T\times X$ instead of F: $T\to\mathscr{P}(X)$. If $B\subset X$, then $F^{-1}(B)=\{t\in T|\ F(t)\cap B\neq\emptyset\}$. Relations are composed in the usual way.

A relation $F\subset T\times X$ is measurable (weakly measurable, $\mathscr{C}$-measurable) iff $F^{-1}(B)$ is measurable for each closed (respectively, open, compact) subset B of X. If $F\subset Y\times X$, where Y is a topological space, then the assertion that F is measurable (weakly measurable, etc.) means that F is measurable (weakly measurable, etc.) when Y is assigned the σ-algebra $\mathscr{B}$ of Borel subsets of Y. Likewise, if $F\subset(T\times Y)\times X$, then the various kinds of measurability of F are always defined in terms of the product σ-algebra $\mathscr{A}\times\mathscr{B}$ on $T\times Y$ generated by the sets $A\times B$ where $A\in\mathscr{A}$ and $B\in\mathscr{B}$.

The following propositions are known and we state them here for ready reference.

PROPOSITION 2.1. *If $F\subset T\times X$ is measurable or weakly measurable, then domain of F is measurable. If X is perfectly normal, then measurability implies weak measurability.*

PROPOSITION 2.2. *Let J be an at most countable set and let $F_n\subset T\times X$ be a relation for each $n\in J$.*

(i) *If each F_n is measurable (weakly measurable, etc.) so is the relation $\bigcup_n F_n\subset T\times X$ defined by $(\bigcup_n F_n)(t)=\bigcup_n F_n(t)$.*

(ii) *If X is second countable and each F_n is weakly measurable, then so is the relation $\prod_n F_n\subset T\times X^J$ defined by $(\prod_n F_n)(t)=\prod_n F_n(t)$.*

PROPOSITION 2.3. *$F\subset T\times X$ is weakly measurable if and only if the relation $\bar{F}\subset T\times X$ defined by $\bar{F}(t)=\overline{F(t)}$ is weakly measurable, where $\overline{F(t)}$ denotes the closure of $F(t)$.*

3. Some extension of known results. The following result was proved in [H, Theorem 3.2(i)] when X is a separable metric space using an embedding technique.

THEOREM 3.1. *Let X be a metric space and let $F\subset T\times X$ be a relation with closed values, that is, $F(t)$ is closed for each t. Then measurability of $F\Rightarrow$ weak measurability of $F\Rightarrow\mathscr{C}$-measurability of F.*

Proof. The first implication is well known and, in particular, follows from Proposition 2.1. So suppose F is weakly measurable and let K be a compact subset of X. Then $K=\bigcap_n B_n$ where $B_n=\{x|\ d(x,K)<1/n\}$. To complete the proof of Theorem 3.1 it suffices to show that $F^{-1}(K)=\bigcap_n F^{-1}(B_n)$. So suppose $t\in F^{-1}(B_n)$ for each n. Then $F(t)\cap B_n\neq\emptyset$ for each n. Choose $x_n\in F(t)\cap B_n$. Since $x_n\in B_n$, there is a $y_n\in K$ such that $d(x_n,y_n)<1/n$. Since K is compact, the sequences (x_n) and (y_n) have a common limit point which must belong to both $F(t)$ and K. That is, $t\in F^{-1}(K)$ and hence $F^{-1}(K)\supset\bigcap_n F^{-1}(B_n)$. Since it is trivial to verify the opposite inclusion, the proof of Theorem 3.1 is complete.

Remark. Kaniewski has recently given an example showing that the first implication cannot be reversed without additional assumptions (cf. [W, Example 2.4]). In Section 4 we will modify Kaniewski's example to obtain counterexamples to other desirable properties of multifunctions. Nishiura [Ni] has given an example showing that the second implication cannot be reversed even if X is separable metric, $(T,\mathscr{A})$ is complete, and $F(t)$ has compact values.

The next theorem gives sufficient conditions for the equivalence of measurability and weak measurability provided T and X are both Borel spaces. (Recall that a Borel space is a Borel subset of a Polish space.)

THEOREM 3.2. *Let T and X be Borel spaces, and let $F\subset T\times X$ be a multifunction with $F(t)$ closed and σ-compact for each $t\in T$. Then F is measurable if and only if F is weakly measurable.*

The proof of Theorem 3.2 depends on the following result due to Brown and Purves [BP, Corollary 1]: If T and X are Polish spaces and if $E\subset T\times X$ is a Borel set such that $E(t)=\{x|\ (t,x)\in E\}$ is σ-compact for each $t\in T$, then the projection of E on T is a Borel set. The Brown–Purves Theorem generalizes a result due to Kunugui [Kun] and Novikov [No-2] where the set $E(t)$ was required to be compact.

Proof of Theorem 3.2. By Proposition 2.1 we need only prove the "if" part. So assume F is weakly measurable. Then it follows, as in the proof of [H, Theorem 3.3], that F is a Borel subset of $T\times X$. Next, let T^* and X^* be the metric completions of T and X, respectively. Then F is also a Borel subset of $T^*\times X^*$ since T and X are Borel. Let B be closed in X and let $\bar{B}$ be the X^* closure of B. Then $B=\bar{B}\cap X$ and so

$$F^{-1}(B)=\{t\in T|\ F(t)\cap\bar{B}\neq\emptyset\}$$
$$=\text{projection on } T^* \text{ of } E=(T^*\times\bar{B})\cap F.$$

The set E satisfies the hypothese of the result of Brown and Purves mentioned above and hence $F^{-1}(B)$ is a Borel set. Thus F is a measurable multifunction.

Remarks. In [H] Theorem 3.2 is proved when T is a measurable space and either X is metric and F has compact values [H, Theorem 3.1] or X is σ-compact, metrizable and F has closed values [H, Theorem 3.2(ii)].

Dauer and Van Vleck [DV, Proposition 2] have given an example of a multifunction F with $F(t)$ σ-compact, but not closed, which is weakly measurable but which is not measurable.

The proof of the preceding theorem actually yields the following.

PROPOSITION 3.3. *Let T and X be Borel spaces, and let $F\subset T\times X$ be a multifunction with $F(t)$ σ-compact for each t. (Note that we do not assume $F(t)$ to be closed.) If F is a Borel set in $T\times X$, then F is a measurable multifunction.*

Remark. There is an example, [DV, Example 1], which shows that the converse of Proposition 3.3 is not true.

Proposition 2.2 shows that as a general rule the property of measurability of relations is preserved by (at most) countable unions and products. Unfortunately, the situation is more complicated for intersections; see [W, p. 863] for a summary of positive results. In Section 4 we give an example showing that the intersection of two closed-valued weakly measurable multifunctions need not be weakly measurable. However, for $\mathscr{C}$-measurable multifunctions the situation is better as the following result, which removes the hypothesis of separability from a result of Himmelberg ([H, Corollary 4.3]), shows.

THEOREM 3.4. *Let X be a metric space, and let $F_n\subset T\times X$ be a $\mathscr{C}$-measurable relation with closed values for each n in an at most countable set. Then $F = \bigcap_n F_n$ is $\mathscr{C}$-measurable.*

Proof. Let K be an arbitrary but fixed compact subset of X. Define $G_n\subset T\times K$ by $G_n(t) = F_n(t)\cap K$. Let B be any closed subset of K. Since K is compact, B is also closed in X. Consequently each G_n is measurable since F_n is $\mathscr{C}$-measurable. By [H, Theorem 4.1], $\bigcap_n G_n$ is measurable. Hence it follows from Proposition 2.1 that $\{t\in T|\ \bigcap_n G_n(t)\neq\emptyset\}$ is measurable. But

$$\{t\in T|\ \bigcap_n G_n(t)\neq\emptyset\} = \{t\in T|\ \bigcap F_n(t)\cap K\neq\emptyset\} = F^{-1}(K).$$

Thus $F = \bigcap_n F_n$ is $\mathscr{C}$-measurable.

As an application of Theorem 3.4, we show that the boundary of a $\mathscr{C}$-measurable multifunction is $\mathscr{C}$-measurable. For related results see [H, Theorems 4.6 and 4.7].

COROLLARY 3.5. *Let X be a metric space, and let $F\subset T\times X$ be a $\mathscr{C}$-measurable relation with compact values. Then the relation $\mathrm{Bd}F\subset T\times X$, defined by $(\mathrm{Bd}F)(t)$ = boundary of $F(t)$, is $\mathscr{C}$-measurable.*

Proof. First, the relation $G\subset T\times X$, defined by $G(t) = X-F(t)$, is measurable. To see this, let B be a closed subset of X. Then

$$G^{-1}(B) = \{t|\ (X-F(t))\cap B\neq\emptyset\} = T-\{t|\ B\subset F(t)\}.$$

If the set $\{t|\ B\subset F(t)\}\neq\emptyset$, then B must be compact since F has compact values. In this case, let A be a countable dense subset of B. Then

$$G^{-1}(B) = T-\{t|\ A\subset F(t)\} = T-\bigcap_{a\in A}F^{-1}(\{a\}).$$

Since F is $\mathscr{C}$-measurable, it follows, in this case, that G^{-1} is measurable. On the other hand, if $\{t|\ B\subset F(t)\}=\emptyset$, then $G^{-1}(B) = T$ and so is measurable. Thus G is measurable and hence, by Proposition 2.1 and 2.3, $\bar{G}$ is weakly measurable. Thus, by Theorem 3.1, $\bar{G}$ is $\mathscr{C}$-measurable. The result now follows from Theorem 3.4 since

$$\mathrm{Bd}F(t) = F(t)\cap\bar{G}(t).$$

4. Some counterexamples. In this section we give five examples. The first four serve to delineate the sharpness of some of the results in [H]; the fifth answers negatively a question raised in [DV]. The first example is of a $\mathscr{C}$-measurable multifunction which is not weakly measurable. Nishiura [Ni] has already given such an example. However, ours, which is based on the existence of a Borel set whose projection is not Borel, will be used in the later examples. The second example shows that the intersection of two closed-valued weakly measurable multifunctions may fail to be weakly measurable. The known sufficient conditions all involve some type of compactness hypothesis either on X or on the values $F(t)$; see [W, p. 863] for a summary and references. The third example shows that the multifunction defined by $t\to\{x|\ f(t,x) = 0\}$ need not be weakly measurable even if f is a continuous function. Himmelberg [H, Theorem 6.4] has given a positive result for $(T,\mathscr{A})$ complete. The fourth example deals with measurable implicit function theorems of the type first proved by Filippov [F]: Given a function $f\colon T\times X\to Y$, a multifunction $F\subset T\times X$, and a function $g\colon T\to Y$ such that $g(t)\in f(\{t\}\times F(t))$ for all $t\in T$, when does there exist a measurable function $r\colon T\to X$ such that $r(t)\in F(t)$ and $g(t) = f(t, r(t))$ for all $t\in T$? Several results for this problem are given in [H]. (Note that Theorems 7.1, 7.2, and 7.4 in [H] have not been established unless Y is additionally required to be separable.) For a summary of these and other results see [W, Section 7]. The last example is that of a measurable, countable (but not closed)-valued multifunction F which does not have a measurable selector. Dauer and Van Vleck, [DV, Proposition 2], have given such an example for F weakly measurable.

In the following examples, let $T = [0, 1]$ equipped with the σ-algebra of Borel sets, let Z be the irrationals, and let B be a closed subset of $T\times Z$ such that the projection $p_T(B)$ is not Borel and the projection $p_Z(B)\neq Z$. (Recall that Z is Polish.)

EXAMPLE 1. A $\mathscr{C}$-measurable multifunction that is not weakly measurable. Choose $z_0\in Z-p_Z(B)$ and define $F\subset T\times Z$ by $F(t) = B(t)\cup\{z_0\}$. Then F has closed graph and is $\mathscr{C}$-measurable. For let K be a compact subset of Z. Then $F^{-1}(K) = p_T((T\times K)\cap F)$ is a Borel set by the Kunugi–Novikov Theorem. On the other hand, F is not weakly measurable. To see this, let G be a neighborhood of $p_Z(B)$ not containing z_0. Then $F^{-1}(G) = p_T(B)$.

EXAMPLE 2. The intersection of two weakly measurable multifunctions F, G: $T \to X$ with closed values may fail to be weakly measurable, even if X is Polish and $\mathrm{dom}(F \cap G) = T$.

Let $X = T \times Z$, and define $F_* \subset T \times X$ by $F_*(t) = p_T^{-1}(\{t\})$. Clearly F_* is weakly measurable since p_T is an open map. However F_* is not measurable since $F_*^{-1}(B) = p_T(B)$ is not a Borel set. (This example of a weakly measurable, but not measurable, multifunction was given by J. Kaniewski and appears in [W, p. 865].) Now let $x_0 \in X - B$ and define F, $G \subset T \times X$ by

$$F(t) = F_*(t) \cup \{x_0\} \quad \text{and} \quad G(t) = B \cup \{x_0\}.$$

Then F is weakly measurable, G is continuous, and $\mathrm{dom}(F \cap G) = T$. But $F \cap G$ is not weakly measurable, for let U be any open set containing B but not x_0. Then

$$\begin{aligned}(F \cap G)^{-1}(U) &= \{t \mid F(t) \cap G(t) \cap U \neq \emptyset\}\\ &= \{t \mid F_*(t) \cap B \neq \emptyset\}\\ &= F_*^{-1}(B) = p_T(B).\end{aligned}$$

EXAMPLE 3. There is a continuous function f: $T \times Z \to \mathbf{R}^1$ such that the multifunction $F \subset T \times Z$ defined by $F(t) = \{x \mid f(t, x) = 0\}$ is not weakly measurable.

Let F be given as in Example 1. Then, since F has closed graph, F is a closed subset of the metric space $T \times Z$. Let f: $T \times Z \to \mathbf{R}^1$ be any continuous function with F as its zero set.

EXAMPLE 4. There is a continuous function f: $T \times Z \to \mathbf{R}^1$ such that $\{x \mid f(t, x) = 0\} \neq \emptyset$ for each t, but such that there is no measurable function r: $T \to Z$ such that $f(t, r(t)) = 0$ for each $t \in T$.

Note that here the given function g: $T \to \mathbf{R}^1$ in the Filippov problem is the zero function and that the given multifunction $F \subset T \times Z$ is the constant multifunction $F(t) = Z$ for each t.

To find f it is sufficient to first find a closed subset Γ of $T \times Z$ such that $p_T(\Gamma) = T$ and Γ has no measurable selector. For then let f be any continuous function on $T \times Z$ with Γ as zero set.

Such a set Γ has essentially been given by Novikov [No-1]. In his example Γ is measurable rather than closed, but it is not difficult to modify his argument to require Γ closed. The following modification is due to R. Darst. Let A_1, A_2 be analytic subsets of T such that $T - A_1$, $T - A_2$ are disjoint complementary analytic sets that cannot be separated by Borel sets, and let h_1, h_2 be (continuous) maps of Z onto A_1, A_2, respectively. Then $\Gamma = \mathrm{Gr}(h_1^{-1} \cup h_2^{-1})$ is the desired set. (Such sets as A_1, A_2 have been shown to exist by Sierpiński, Lusin, and Novikov; cf. [Kur, p. 485].)

EXAMPLE 5. A countable, but not closed, -valued measurable multifunction may fail to have a measurable selector.

This example is based on the following result of Blackwell and Dubins (BD, Corollary 2]: There is no Borel measurable function g defined on the space Ω of sequences of real numbers such that $g(\omega_1) = g(\omega_2)$ whenever ω_1 and ω_2 have the same range and such that, for all $\omega \in \Omega$, $g(\omega)$ is an element of the range $\varrho(\omega)$ of ω.

To use this, let $\mathcal{S}$ be the set of all nonempty, at most countable sets of real numbers. Let $\mathcal{A}$ be the smallest σ-algebra of subsets of $\mathcal{S}$ generated by sets of the form $\{F \in \mathcal{S} \mid F \cap B \neq \emptyset\}$ for B closed in $\mathbf{R}^1$. ($\mathcal{A}$ is the Borel σ-algebra on $\mathcal{S}$ generated by the upper semi-finite topology on $\mathcal{S}$; see [M, Definition 9.1].) Let $F \subset \mathcal{S} \times \mathbf{R}^1$ be the multifunction defined by $F(S) = S$ for each $S \in \mathcal{S}$. Clearly F is measurable.

We now claim that F does not have a measurable selector. For suppose f: $\mathcal{S} \to \mathbf{R}^1$ is a measurable selector and consider the function $g = f \circ \varrho$: $\Omega \to \mathbf{R}^1$, where ϱ: $\Omega \to \mathcal{S}$ is the function whose value $\varrho(\omega)$ at ω is defined to be the range of ω. It is not difficult to show that the function ϱ is measurable; hence g is measurable. Clearly $g(\omega_1) = g(\omega_2)$ when ω_1 and ω_2 have the same range, and, further, for each $\omega \in \Omega$, $g(\omega) = f(\varrho(\omega)) \in F(\varrho(\omega)) = \varrho(\omega)$. This contradicts the Blackwell-Dubins result and hence f cannot exist.

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