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EQUILIBRIA OF CONTINUOUS TWO-PERSON GAMES

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Some sufficient conditions are given to show the existence of equilibrium points with finite spectrum for nonzero-sum two-person continuous games on the unit square. We also examine the question of uniqueness of the equilibrium point for such games.

1. Players I and II choose secretly an x and a y in the closed interval [0, 1]. Player I receives $K_1(x, y)$ and player II receives $K_2(x, y)$ where K_1, K_2 are continuous on the unit square. The following theorem is classical in game theory: ([3], see page 156).

There exists a pair of probability distributions (F^0, G^0), called Nash equilibrium points satisfying

$$K_1(F^0, G^0) \geqq K_1(x, G^0) \text{ for all } x \text{ in } 0 \leqq x < 1$$

and

$$K_2(F^0, G^0) \geqq K_2(F^0, y) \text{ for all } y \text{ in } 0 \leqq y \leqq 1$$

where $K_1(F, G) = \iint K_1(x, y)dF(x)dG(y)$ and

$$K_1(x, G) = \int K_1(x, y)dG(y) \text{ etc.}$$

Let $\mathscr{E}$ be the set of such pairs (F^0, G^0).

One can ask the following questions. When does an $(F^0, G^0) \in \mathscr{E}$ with the spectrums of F^0 and G^0 being finite? For what class of games, $\mathscr{E}$ has a unique point? Here we try to answer these questions by giving some sufficient conditions.

2. In this section we prove the following results.

THEOREM 1. *Let $K_1(x, y)$ and $K_2(x, y)$ be continuous on $0 \leqq x, y \leqq 1$. Let $K_2(x, y)$ be concave in y for each x. Then there is an equilibrium (F^0, G^0) such that G^0 is a degenerate probability distribution and F^0 is concentrated at most at two points.*

THEOREM 2. *Let $K_1(x, y)$ and $K_2(x, y)$ be continuous on $0 \leqq x, y \leqq 1$. Let $(\partial^n/\partial y^n)K_2(x, y) \leqq 0$. Then there is an $(F^0, G^0) \in \mathscr{E}$ with the spectrum of F^0 and G^0 finite.*

We need the following theorem of Bohnenblust, Karlin, and Shapley [1] in the sequel.

PROPOSITION. *Let K be compact convex in R^n. Let g_α be a family of continuous convex functions on K. Let $\sup_\alpha g_\alpha(x) > 0$ for each $x \in K$. Then there exists $(n+1)$ induces $\alpha_1, \alpha_2, \cdots, \alpha_{n+1}$ such that $\sum_{i=1}^{n+1} \lambda_i g_{\alpha_i}(x) > 0$ for all x. Here $(\lambda_1, \lambda_2, \cdots, \lambda_{n+1})$ is a probability vector.*

REMARK. Theorems 1 and 2 are known for zero-sum games [1], [2]. Our proof follows similar lines.

Proof of Theorem 1. We first prove the result when $K_2(x, y)$ is strictly concave in y for each x.

Let $(F^0, G^0) \in \mathscr{E}$. Since $K_2(x, y)$ is strictly concave in y, $\sigma(G^0)$, the spectrum of G^0 contains just one element, say y^0. Let $C = \sigma(F^0) =$ the spectrum of F^0. Consider

$$\max_C \{K_2(x, y^0) - K_2(x, y)\} = \psi(y) .$$

We claim $\psi(y) \geqq 0$ for all y, for otherwise

$$K_2(x, y^0) - K_2(x, y') < 0 \text{ for all } x \text{ in } C \text{ and for some } y' .$$

Thus $K_2(F^0, y^0) - K_2(F^0, y') < 0$ contradicting $(F^0, G^0) \in \mathscr{E}$. Since $K_2(x, y^0) - K_2(x, y)$ is convex for each x and continuous in x for each y over the compact set C it follows from the above proposition that for some $0 \leqq \lambda \leqq 1$

$$\lambda K_2(x_1, y^0) + (1-\lambda)K_2(x_2, y^0) \geqq \lambda K_2(x_1, y) + (1-\lambda)K_2(x_2, y)$$

for all y. Define $F^* = \lambda I_{x_1} + (1-\lambda)I_{x_2}$. Here I_x stands for the degenerate distribution at x. Clearly $(F^*, y^0) \in \mathscr{E}$. The general case is handled by approximating $K_2(x, y)$ by a sequence of strictly concave functions in y. For the proof of Theorem 2 we need the following lemma of Glicksberg [2].

LEMMA. *Let $h(y) \geqq 0$ on the unit interval with $h^n(y) > 0$. Then there exists a polynomial $p(y) \geqq 0$ of degree at most $(n-1)$ such that $h(y) - p(y) \geqq 0$ on $[0, 1]$ and has exactly n roots counting multiplicities.*

For the sake of completeness we reproduce the proof.

Proof. By Rolle's theorem we know that $h(y)$ has at most n roots. If it has exactly n roots $p(y) \equiv 0$ is a choice. If $h(y)$ has fewer than n roots, then $p(y)$ is constructed as follows. Let $y_1, y_2, \cdots, y_k$ be the roots of $h(y)$. Let $q(y)$ be a polynomial with the same roots and multiplicities. Then $q(y) \geqq 0$ and of degree at most $(n-1)$. Let m_i be the multiplicity of y_i. Then

$$\lim_{y \to y_i} \frac{h(y)}{q(y)} = \lim_{y \to y_i} \frac{h^{m_i}(y)}{q^{m_i}(y)} = \delta_i > 0 \ .$$

Hence we have open sets E_i around y_i with

$$\frac{h(y)}{q(y)} \geqq \frac{\delta_i}{2} \quad \text{in} \quad E_i \ .$$

The complement of $\cup E_i$ is a compact set, hence h/q is the ratio of two continuous nonvanishing functions and that it achieves its minimum δ_0. Hence for $\varepsilon < \min(\delta_0, \delta_1/2)$ $h(y) - \varepsilon q(y) \geqq 0$. Let ε_0 be the supremum of all ε for which $h(y) - \varepsilon q(y) \geqq 0$. Then $h(y) - \varepsilon_0 q(y)$ has at least one more root for otherwise $h(y) - \varepsilon_0 q(y)$ would satisfy all the conditions that $h(y)$ satisfies and we cound find an $\varepsilon' > 0$ with

$$[h(y) - \varepsilon_0 q(y)] - \varepsilon' q(y) \geqq 0$$

which contradicts that ε_0 is the supremum. Therefore we either have a new root or at least the multiplicity of a former root is increased. The function $h - \varepsilon_0 q \geqq 0$ satisfies

$$\frac{\partial^n}{\partial y^n}(h - \varepsilon q) > 0 \ .$$

We may continue the process until we arrive at a $p(y)$ satisfying the conditions of the lemma. Further one easily checks that this polynomial is unique.

Proof of Theorem 2. First we prove the theorem when $K_2(x, y)$ is of the form $K_2(x, y) = \sum_{i=0}^{n} a_i(x)y^i$ with $(\partial^n/\partial y^n)K_2(x, y) < 0$. Let $(F^0, G^0) \in \mathscr{E}$. Since $K_2(F^0, G^0) = \max K_2(F^0, y)$ and $(\partial^n/\partial y^n)K_2(F^0, y) < 0$ we see that $\sigma(G^0)$ is finite. We will produce an F^* with $\sigma(F^*)$ finite and $(F^*, G^0) \in \mathscr{E}$. Consider $S = \{\alpha = (\alpha_0, \alpha_1, \cdots, \alpha_n) \colon \alpha_i = \int y^i dG, i = 0, 1, 2, \cdots, n$, for some probability distribution $G\}$. Let $M(x, \alpha) = \sum_i a_i(x)\alpha_i$. Clearly S is compact and convex in R^{n+1} and $M(x, \alpha)$ is affine in α for each x. Let $C = \sigma(F^0)$ and $K_2(x, G^0) = \sum a_i(x)\alpha_i^0 = M(x, \alpha^0)$.

Define $\psi(\alpha) = \max_{x \in C} \{M(x, \alpha^0) - M(x, \alpha)\}$. As in Theorem 1 we have $\psi(\alpha) \geqq 0$ for every α in S and by the proposition of Bohnenblust, Karlin, and Shapley we have a finite number of points $x_1, x_2, \cdots, x_m$ in C such that

$$\sum \lambda_i M(x_i, \alpha^0) \geqq \sum \lambda_i M(x_i, \alpha) \text{ for all } \alpha \in S$$

where $\lambda = (\lambda_1, \cdots, \lambda_m)$ is a probability vector. Let $F^* = \sum_i \lambda_i I_{x_i}$. Clearly $(F^*, G^0) \in \mathscr{E}$. This proves Theorem 2 for the special case.

Proof of the theorem in the general case. Let $(F^0, G^0) \in \mathscr{E}$ and $h(y) = \int K_2(x, y) dF^0(x)$. Without loss of generality $K_2(F^0, G^0) = 0$, and hence $h(y) \leqq 0$ for all y. From the above lemma of Glicksberg we have a polynomial $P(y)$ with $h(y) \leqq P(y) \leqq 0$ and $P(y) - h(y)$ has exactly n roots counting multiplicities. For each x we can find coefficients $a_i(x)$, $i = 0, 1, \cdots, n-1$ with $\sum a_i(x) y^i - K_2(x, y)$ having the same roots and multiplicities as that of $p(y) - h(y)$.

Now we will show that the coefficients $a_i(x)$ are continuous in x, for $i = 0, 1, 2, \cdots, n-1$. Fix $x = x_0$ and consider $\sum_{i=0}^{n-1} a_i(x_0) y^i - K_2(x_0, y)$. The polynomial $\sum_{i=0} a_i(x_0) y^i$ is unique for otherwise we will have 2 distinct polynomials of degree less than n whose difference will have n roots counting multiplicities. This is clearly not possible. Thus for each fixed x the coefficients $a_i(x)$ are uniquely determined. In fact we may write n linear equations for the unknown $a_0(x), \cdots, a_{n-1}(x)$. The matrix of coefficients in these equations is nonsingular and since this matrix is independent of x, it follows from the continuity of $K_2(x, y)$ in x, that the $a_i(x)$'s are continuous in x. Now since the roots interior to $[0, 1]$ are of even multiplicity

$$\sum a_i(x) y^i - K_2(x, y) \geqq 0 \quad \text{for all} \quad (x, y)$$

or

$$\sum a_i(x) y^i - K_2(x, y) \leqq 0 \quad \text{for all} \quad (x, y) .$$

But $\int \sum a_i(x) y^i dF^0(x) - K_2(F^0, y)$ has roots and multiplicities as that of $P(y) - h(y)$. Since $P(y)$ is unique $P(y) = \int \sum a_i(x) y^i dF^0(x)$. Since $P(y) > h(y)$ for some y the first inequality holds.
Since $K_2(x, y) \leqq \sum a_i(x) y^i = P(x, y)$

$$0 = K_2(F^0, G^0) \leqq \iint \sum a_i(x) dF^0(x) y^i dG^0(y) = P(F^0, G^0)$$
$$\leqq \int P(y) dG^0(y) \leqq 0 .$$

Thus $P(F^0, G^0) = 0 \geqq P(F^0, y) = P(y)$ for all y. We therefore have (F^0, G^0) as an equilibrium point of the auxiliary game. We know from the first part that this can be replaced by (F^*, G^0) where $\sigma(F^*)$ is a finite subset of $\sigma(F^0)$. We claim (F^*, G^0) is also an equilibrium point for the original game. To show this it is enough to prove $K_2(F^*, G^0) = P(F^*, G^0)$ for then $K_2(F^*, G^0) = P(F^*, G^0) \geqq$

$P(F^*, G) \geqq K_2(F^*, G)$ for all G. Suppose $K_2(F^*, G^0) < P(F^*, G^0)$ (the other inequality cannot hold). Then $K_2(x, y) < P(x, y)$ for some $x \in \sigma(F^*) \subset \sigma(F^0)$ and $y \in \sigma(G^0)$ and $K_2(x, y) \leqq P(x, y)$ for all (x, y). Combining these two statements we have $0 = K_2(F^0, G^0) < P(F^0, G^0) \leqq 0$ a contradiction. Hence the assertion. This completes the proof of Theorem 2.

3. In this section we give a set of sufficient conditions for the uniqueness of optimal strategies and equilibrium points for continuous games.

DEFINITION. Let μ_1, μ_2 be two probability measures on the unit interval. We say $\mu_1 \leqq \mu_2$ if for some $k > 0$ $\mu_1(E) \leqq k\mu_2(E)$ for all E. We say $\mu_1 \sim \mu_2$ if $\mu_1 \leqq \mu_2$ and $\mu_2 \leqq \mu_1$.

THEOREM 3. *Let $K(x, y)$ be a continuous payoff on $0 \leqq x, y \leqq 1$ for a zero-sum two-person game. Suppose every optimal strategy for each player has the spectrum the entire unit interval. Further assume every pair of optimal strategies for each player be equivalent. Then there is only one optimal strategy for each player.*

THEOREM 4. *Let $K_1(x, y)$, $K_2(x, y)$ be continuous payoffs on $0 \leqq x, y \leqq 1$ for a nonzero-sum two-person game. Let for every $(\mu_1, \nu_1) \in \mathscr{E}$, $(\mu_2, \nu_2) \in \mathscr{E}$ $\sigma(\mu_1) = \sigma(\mu_2) = \sigma(\nu_1) = \sigma(\nu_2) = [0, 1]$ and $\mu_1 \sim \mu_2$, $\nu_1 \sim \nu_2$. Then $\mathscr{E}$ has just one element.*

Proof of Theorem 3. It suffices to prove that the compact convex set of optimal strategies for each player has exactly one extreme point.

Let if possible μ_1, μ_2 be two distinct extreme optimal strategies, say for player I. Since $\mu_1 \sim \mu_2$ we have a $k > 2$ such that $\mu_1(E) \leqq k\mu_2(E)$ for all E. Define

$$\mu'(E) = (1 + \theta)\mu_2(E) - \theta\mu_1(E)$$

and

$$\mu''(E) = (1 - \theta)\mu_2(E) + \theta\mu_1(E)$$

where

$$0 < \theta = \frac{1}{k-1} < 1 .$$

When the spectrum of an optimal strategy for player II is the whole unit interval every optimal strategy for player I is an equalizer. That is

$$\int K(x, y)d\mu_1(x) \equiv v \quad \text{for all} \quad y$$

and

$$\int K(x, y)d\mu_2(x) \equiv v \quad \text{for all} \quad y\ .$$

Thus it is easily seen that μ', μ'' are two distinct optimal strategies for player I and that $\mu_2 = (\mu' + \mu'')/2$. This contradicts the fact that μ_2 is an extreme optimal for player I. Hence the theorem.

Proof of Theorem 4. Since for any pair $(\mu_1, \nu_1) \in \mathscr{E}$, $(\mu_2, \nu_2) \in \mathscr{E}$ we have

$$\int K_1(x, y)d\nu_2(y) \equiv c_2,\ \int K_2(x, y)d\mu_1(x) \equiv \alpha_1$$

$$\int K_1(x, y)d\nu_1(y) \equiv c_1,\ \int K_2(x, y)d\mu_2(x) \equiv \alpha_2$$

(μ_1, ν_2) and (μ_2, ν_1) also belong to $\mathscr{E}$ and that $\mathscr{E}$ is a convex set. Of course $\mathscr{E}$ is compact. The rest of the proof is as in Theorem 3.

REMARK 1. For example if every optimal strategy for each player possesses a continuous strictly positive density, then one can check that the conditions of Theorem 3 are satisfied; hence such games will have unique optimal strategies.

REMARK 2. For matrix games the notion of equivalence coincides with the notion of completely mixed strategies.

REMARK 3. It would be interesting to know whether Theorem 3 is valid if we just assume that the spectrum of every optimal strategy for each player is the unit interval.

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