Calculation of the micromaser spectrum. I. Green's-function approach and approximate analytical techniques

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We first calculate the "exact" micromaser spectrum by solving numerically for the Green's function of the appropriate master equation for the one-atom (micro-) maser. We then proceed to calculate approximate analytical expressions for the maser linewidth using (i) a phase-operator approach, (ii) an ansatz similar to the standard quantum theory of the laser, and (iii) a linear expansion of the correlation function. Novel features, quite distinct from the familiar Schawlow-Townes linewidth, are found, e.g., the linewidth *decreases* as the number of thermal photons increases.

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I. INTRODUCTION

The micromaser [1] is a fundamental system in quantum optics of special interest because on the one hand it is simple enough to be theoretically tractable [2,3] and on the other hand it shows all important quantum phenomena of the matter-light interaction. Examples include collapse and revivals of the Rabi nutation [4], the generation of sub-Poissonian photon statistics [2,3,5], to the extreme case of a number state [6], trapping states [7], and a quantum-nondemolition measurement of the photon number [8]. Moreover, the micromaser allows us to prepare and measure the off-diagonal elements of the field density operator. Atoms injected in a superposition of the two atomic states create [9] nonvanishing off-diagonal elements; that is, they create a preferred field phase. A Ramsey arrangement [10] provides information [11,12] about their time evolution and the evolution of the field as it manifests itself in the spectrum of the micromaser.

In a previous paper [11] we have investigated this spectrum. The approach pursued in Ref. [11] rests on an approximate analytical expression for the maser linewidth as well as on a numerical evaluation of the two-time correlation function. That paper was intended as a short note and thus we omitted there all details of the calculations.

This pair of papers presents the calculations necessary to understand the analysis, and in so doing, yields additional insights. In the present paper we compute the two-time correlation function of the micromaser field via a Green's-function approach. We compare the so-calculated linewidth to approximate analytical expressions. In the following paper [13] we investigate the eigenvalues of the master equation for the off-diagonal elements of the micromaser density matrix. We discuss the contribution of various eigenvalues to the micromaser linewidth.

The present paper is organized as follows. In Sec. II we define the steady-state spectrum of the electric field as the Fourier transform of the two-time correlation function $K(t) = \langle a^\dagger(t)a(0)\rangle$. We express $K(t)$ in terms of the Green's function and the steady-state photon statistics. In Sec. III we apply our general considerations to the micromaser. We summarize the equation of motion for the density matrix of the maser field and obtain an exact numerical solution for the Green's function based on continued fractions. A detailed discussion of the spectrum and its width, that is, the maser linewidth [11,14], and its dependence on the pump parameter and the number of thermal photons follows. We show that in a broad region of the micromaser parameters the linewidth has a tendency to decrease with increasing temperature. Such a property is quite distinct from the usual spectral properties of the laser or maser. To gain insight into this behavior we derive in Sec. IV an approximate analytical expression for the micromaser linewidth based on the London phase operator. This demonstrates that novel features of the micromaser spectrum result from interferences between various quantum Rabi oscillations. We conclude by summarizing our results in Sec. V.

II. DEFINITIONS OF THE SPECTRUM

A. General considerations

There exist various definitions of the spectrum [15]. One relies on the decay of the expectation value of the electric field

$$\langle E(t)\rangle = \mathcal{E}_0\langle a^\dagger(t)\rangle + \text{c.c.} = \mathcal{E}_0\sum_n \sqrt{n+1}\,\rho_{n,n+1}(t) + \text{c.c.} \quad (2.1)$$

from an initial value of $\langle E(t=0)\rangle$ governed by the off-diagonal element $\rho_{n,n+1}(t=0)$ of the field-density operator. The spectrum is then given by the Fourier transform of Eq. (2.1). This of course implies a measurement scheme which first creates nonvanishing off-diagonal elements and then monitors their decay. For the micromaser this can be achieved by first injecting atoms into the maser cavity which are initially in a coherent superposition of the excited state and the ground state [9]. When we now interrupt the flow of coherently excited atoms, the so-prepared off-diagonal elements of the density matrix decay.

Another method of measuring the spectrum does not require the initial preparation of off-diagonal elements. It is based on the Fourier transform of the two-time correlation function

$$\langle E^{(+)}(t)E^{(-)}(0)\rangle \sim \langle a^\dagger(t)a(0)\rangle \equiv K(t)\,. \tag{2.2}$$

Here we compare the diffusion of the electric field from an arbitrary initial field $E(t=0)$ to the field $E(t)$ at a later time t. In the remainder of the paper we assume that at $t=0$ our field has reached steady state. At $t=0$ the correlation function $K(t=0)=\langle a^\dagger(0)a(0)\rangle$ is then ruled by the the steady-state photon statistics $P_n=\rho_{nn}$, that is, by the diagonal elements of the field-density operator. The two-time correlation function $K(t)$ is governed by the decay of the off-diagonal elements $\rho_{n,n+1}(t)$ starting from very specific initial conditions, i.e., $\rho_{n,n+1}(t=0)=\sqrt{n+1}\,P_{n+1}$.

In the present two papers—with the exception of Sec. IV—we follow the second approach. In this paper we pursue the Green's-function technique to calculate the time evolution of the $K(t)$ whereas the following paper is devoted to an eigenvalue and eigenvector analysis.

B. Green's-function approach

The focus of the present papers is the calculation of the micromaser spectrum

$$S(\omega-\omega_c)=\mathrm{Re}\int_0^\infty K(t)\,e^{-i(\omega-\omega_c)t}\,dt\,, \tag{2.3}$$

that is, the Fourier transform of the two-time correlation function

$$K(t)=\langle a^\dagger(t)a(0)\rangle\,. \tag{2.4}$$

We first express this correlation function $K(t)$ by the Green's function of the master equation for the field-density operator. The expression obtained is rather general. In Sec. III we apply this result to the case of the micromaser.

The time dependence of the field operator

$$a^\dagger(t)=U^\dagger(t)\,a^\dagger(0)\,U(t) \tag{2.5}$$

expressed by the time-evolution operator $U(t)$ casts $K(t)$ into the form

$$\begin{aligned}K(t)&=\mathrm{Tr}_{f,r}[U^\dagger(t)a^\dagger(0)U(t)a(0)\rho_{f,r}(0)]\\&=\mathrm{Tr}_{f,r}[a^\dagger(0)U(t)a(0)\rho_{f,r}(0)U^\dagger(t)]\,.\end{aligned} \tag{2.6}$$

Here $\rho_{f,r}$ denotes the density matrix for the total system consisting of the field (f) and the reservoir (r). When we trace over the photon-number states $|n\rangle$ of the field we find

$$\begin{aligned}K(t)&=\mathrm{Tr}_r\left[\sum_{l,m,n=0}^{\infty}\langle l|a^\dagger(0)U(t)|m\rangle\langle m|a(0)\rho_{f,r}(0)|n\rangle\langle n|U^\dagger(t)|l\rangle\right]\\&=\sum_{l,m,n=0}^{\infty}\sqrt{l(m+1)}\,\mathrm{Tr}_r[U_{l-1,m}(t)\langle m+1|\rho_{f,r}(0)|n\rangle\,U^\dagger_{n,l}(t)]\,,\end{aligned} \tag{2.7}$$

where the time-evolution operators

$$U_{j,k}(t)=\langle j|U(t)|k\rangle\,,\quad U^\dagger_{j,k}(t)=\langle j|U^\dagger(t)|k\rangle \tag{2.8}$$

still contain the reservoir operators.

At steady state, that is at time $t=0$, we factorize the density operator $\rho_{f,r}$ into that of the field $\rho_f^{(s)}$ and the reservoir $\rho_r^{(s)}$. This implies for the matrix elements

$$\begin{aligned}\langle m+1|\rho_{f,r}(0)|n\rangle&=\rho_r^{(s)}\langle m+1|\rho_f^{(s)}|n\rangle\\&=\rho_r^{(s)}\delta_{m+1,n}\langle n|\rho_f^{(s)}|n\rangle\\&\equiv\rho_r^{(s)}\delta_{m+1,n}P_n\,.\end{aligned} \tag{2.9}$$

Here we have assumed that the steady-state field-density matrix is diagonal; that is, we confine ourselves to systems without a preferred phase. When we substitute Eq. (2.9) into Eq. (2.7) we arrive at

$$K(t)=\sum_{l,m}G_{l,m}(t)\sqrt{(l+1)(m+1)}\,P_{m+1}\,, \tag{2.10}$$

where

$$G_{l,m}(t)=\mathrm{Tr}_r[U_{l,m}(t)\rho_r^{(s)}U^\dagger_{m+1,l+1}(t)]\,. \tag{2.11}$$

The initial condition for $G_{l,m}(t)$ reads

$$\begin{aligned}G_{l,m}(t=0)&=\mathrm{Tr}_r[\langle l|m\rangle\rho_r^{(s)}\langle m+1|l+1\rangle]\\&=\delta_{l,m}\mathrm{Tr}_r[\rho_r^{(s)}]=\delta_{l,m}\,,\end{aligned} \tag{2.12}$$

where we have used the orthogonality of the number states and $\mathrm{Tr}_r[\rho_r^{(s)}] = 1$. We note that this result also follows from the quantum regression theorem [16] as shown in Appendix A.

In order to obtain the time dependence of the Green's function $G_{l,m}$ we interpret the $G_{l,m}$ as the the off-diagonal element $\langle l|G^{(m)}|l+1\rangle$ of the operator

$$G^{(m)}(t) = \mathrm{Tr}_r[U(t)(|m\rangle\langle m+1|)\rho_r^{(s)}U^\dagger(t)]\,. \tag{2.13}$$

Since the time-evolution operator $U(t)$ enters into $G^{(m)}(t)$ in the same way as into the field-density operator,

$$\rho_f(t) = \mathrm{Tr}_r[U(t)\rho_{f,r}(0)U^\dagger(t)]\,, \tag{2.14}$$

the operator $G^{(m)}(t)$ must obey the same master equation as the field-density operator $\rho_f(t)$. In particular, for every m the Green's function $G_{l,m} = \langle l|G^{(m)}|l+1\rangle$ must satisfy the same equation of motion as $\rho_{l,l+1}(t) = \langle l|\rho_f(t)|l+1\rangle$.

III. APPLICATION TO THE MICROMASER

So far our considerations are quite general. Before we apply the Green's-function technique to the micromaser system we briefly summarize [2,3,9] the equation of motion for the density matrix $\rho_n^{(k)}(t) \equiv \langle n|\rho_f(t)|n+k\rangle$ of the micromaser field. In the interaction picture the density matrix elements $\rho_n^{(k)}(t)$ obey the three term recurrence relation

$$\dot\rho_n^{(k)}(t) = \mathcal{A}_n^{(k)}\,\rho_{n-1}^{(k)} + \mathcal{B}_n^{(k)}\,\rho_n^{(k)} + \mathcal{C}_n^{(k)}\,\rho_{n+1}^{(k)}\,, \tag{3.1}$$

where

$$\mathcal{A}_n^{(k)} = r\sin(g\tau\sqrt{n}\,)\sin(g\tau\sqrt{n+k}) + \gamma n_b\sqrt{n(n+k)}\,, \tag{3.2a}$$

$$\mathcal{B}_n^{(k)} = -r[1-\cos(g\tau\sqrt{n+1}\,)\cos(g\tau\sqrt{n+1+k})] - \gamma(n_b+1)\left(n+\frac{k}{2}\right) - \gamma n_b\left(n+1+\frac{k}{2}\right)\,, \tag{3.2b}$$

$$\mathcal{C}_n^{(k)} = \gamma(n_b+1)\sqrt{(n+1)(n+1+k)}\,. \tag{3.2c}$$

Here r is the injection rate of excited Rydberg atoms whose time of flight through and coupling strength with the cavity field are given by τ and g, respectively. The cavity decay rate is denoted by γ and n_b is the mean thermal photon number.

Since we want to calculate the correlation function $K(t)$ via Eq. (2.10) we have to determine (i) the steady-state photon-number distribution and (ii) the time dependence of the Green's function $G_{n,m}$. The steady-state photon-number distribution follows from Eq. (3.1) using the condition of detailed balance [2] and reads

$$P_n = \rho_n^{(0)} = P_0\prod_{\nu=1}^{n}\left(\frac{n_b}{n_b+1} + \frac{N\sin^2(g\tau\sqrt{\nu}\,)}{\nu(n_b+1)}\right)\,, \tag{3.3}$$

where P_0 denotes a normalization constant and $N = r/\gamma$ represents the number of atoms passing through the cavity in a time γ^{-1}.

Since for every m the Green's function $G_{n,m}$ must satisfy the equation of motion for $\rho_n^{(1)}(t)$ we have to solve

$$\dot G_{n,m} = \mathcal{A}_n^{(1)}\,G_{n-1,m} + \mathcal{B}_n^{(1)}\,G_{n,m} + \mathcal{C}_n^{(1)}\,G_{n+1,m} \tag{3.4}$$

subject to the initial condition Eq. (2.12).

For the Laplace transform of $G_{n,m}(t)$,

$$\tilde G_{n,m}(s) = \int_0^\infty e^{-st}\,G_{n,m}(t)\,dt\,, \tag{3.5}$$

we obtain from Eq. (3.4) the following recurrence relation:

$$\mathcal{A}_n^{(1)}\,\tilde G_{n-1,m} + (\mathcal{B}_n^{(1)} - s)\,\tilde G_{n,m} + \mathcal{C}_n^{(1)}\,\tilde G_{n+1,m} = -\delta_{nm}\,. \tag{3.6}$$

The solution for $\tilde G_{n,m}(s)$ can be expressed in terms of continued fractions [17]. Equation (2.3) together with Eqs. (2.10) and (3.5) yields for the spectrum of the micromaser

$$S(\omega-\omega_c) = \mathrm{Re}\sum_{n,m}\tilde G_{n,m}(i(\omega-\omega_c)) \times\sqrt{(m+1)(n+1)}\,P_{m+1}\,. \tag{3.7}$$

We note that the spectrum of the micromaser field is directly related to the steady-state photon statistics and to the spectrum of the Green's function.

In Fig. 1 we show the normalized spectrum $S(\omega-\omega_c)/S(0)$ for various values of the interaction parameter $g\tau$. It contains only one peak centered at the cavity frequency ω_c and is Lorentzian for a wide range of the pump parameter $\theta = \sqrt{N}g\tau$.

We obtain the linewidth D, that is the full width at

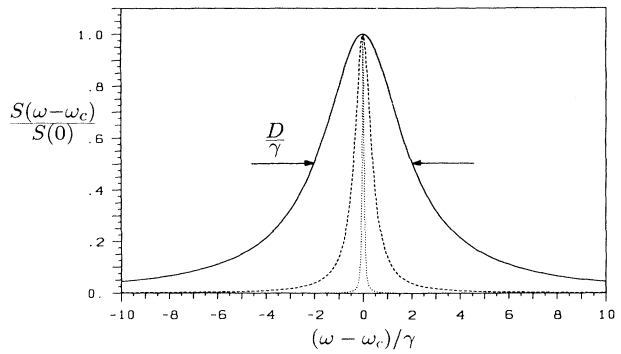

FIG. 1. Normalized spectrum $S(\omega-\omega_c)/S(0)$ as a function of the parameter $(\omega-\omega_c)/\gamma$ for $N = 20$, $n_b = 1$ and for $g\tau = 0.3$ (dotted curve), $g\tau = 1$ (broken curve), and $g\tau = 3$ (solid curve).

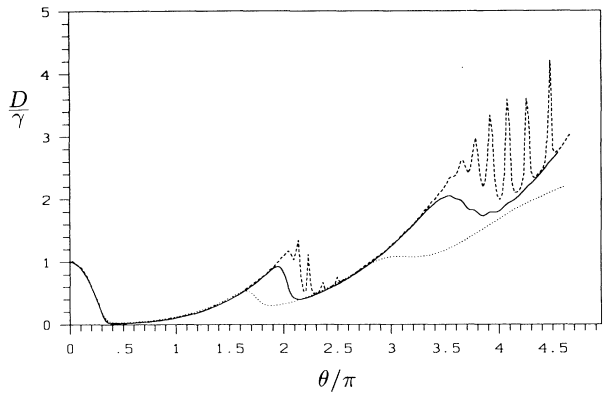

FIG. 2. Exact numerical linewidth D/γ as a function of the pump parameter $\theta = \sqrt{N}g\tau$ for $N = 50$, $n_b = 10^{-4}$ (broken curve), $n_b = 0.05$ (solid curve), and $n_b = 1$ (dotted curve).

half maximum of the function $S(\omega - \omega_c)/S(0)$, numerically. We now discuss the dependence of D on various maser parameters such as interaction parameter $g\tau$, mean thermal photon number n_b, and atomic injection rate N. In Figs. 2 and 3 we depict the dependence of the micromaser linewidth on the interaction parameter $g\tau$, that is, as a function of the pump parameter θ. In the vicinity of the threshold region, $\theta = 1$, the linewidth of the micromaser is much narrower than the cavity damping constant γ. For the cavity at very low temperatures (broken curve) the linewidth exhibits sharp resonances which are reminiscent of the trapping states [7]. We note that the phase diffusion is especially large there. Moreover, Fig. 2 shows that in a large region of the pump parameter θ the linewidth D even reduces as the thermal mean photon number n_b increases, a phenomenon quite different from a standard laser. In the limit of large-θ values the linewidth decreases as indicated in the lower part of Fig. 3. This phenomenon is alien to the monotonic dependence of the Schawlow-Townes linewidth.

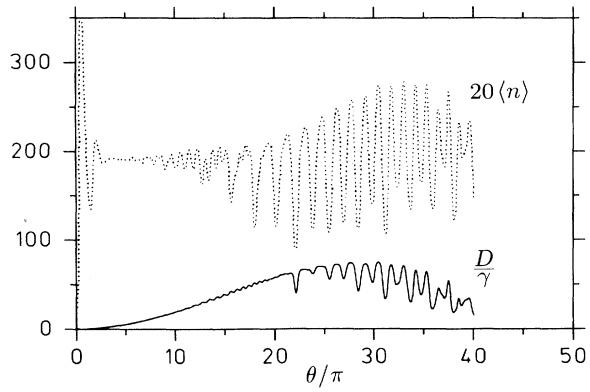

FIG. 3. Exact numerical linewidth D/γ as a function of the pump parameter $\theta = \sqrt{N}g\tau$ for $N = 20$, $n_b = 1$ compared to the corresponding average photon number.

IV. APPROXIMATE ANALYTICAL APPROACH

The numerical calculations of the micromaser spectrum via continued fractions make it difficult to gain insight into the dependence of the maser linewidth on the parameters of interest such as n_b and θ depicted in Figs. 1–3. We therefore devote this section to present an approach based on the London phase operator [18,19] which allows us to obtain approximate analytical expressions for D [20]. For an application of the ordinary quantum theory of the laser to the micromaser and a linear expansion of the correlation function we refer to the Appendices B and C.

The treatment of Secs. II and III takes the phase as well as the intensity fluctuations into account. However, above threshold, when there is a rather well-defined number of photons present, that is, when the fluctuations in the intensity are small compared to the average intensity, one would expect that the decay of the expectation value $\langle E(t)\rangle$, Eq. (2.1), is mainly governed by phase fluctuations. We therefore consider in the present section an approach in which we represent the operator $a^\dagger$ by the rather simplified phase operator [18]

$$a^\dagger(t) = r_0 \widehat{e^{i\varphi(t)}}\,, \tag{4.1}$$

and thus

$$\langle E(t)\rangle \sim \langle a^\dagger(t)\rangle \approx r_0 \langle \widehat{e^{i\varphi(t)}}\rangle\,. \tag{4.2}$$

We now derive an equation of motion for the expectation value $\langle \widehat{e^{i\varphi}}\rangle$ and compare the corresponding analytical phase-diffusion constant with the exact numerical calculation based on computing the two-time correlation function of the micromaser field. From the definition of the phase operator

$$\widehat{e^{i\varphi}} = \sum_{n=0}^{\infty} |n+1\rangle\langle n|\,, \tag{4.3}$$

we find for its expectation value

$$\langle \widehat{e^{i\varphi}}\rangle = \sum_n \rho_n^{(1)}\,. \tag{4.4}$$

With the help of Eq. (3.1) we obtain

$$\frac{d}{dt}\langle \widehat{e^{i\varphi}}\rangle = \sum_n \mathcal{A}_n^{(1)}\rho_{n-1}^{(1)} + \mathcal{B}_n^{(1)}\rho_n^{(1)} + \mathcal{C}_n^{(1)}\rho_{n+1}^{(1)}$$

$$= \sum_n [\mathcal{A}_{n+1}^{(1)} + \mathcal{B}_n^{(1)} + \mathcal{C}_{n-1}^{(1)}]\,\rho_n^{(1)}\,, \tag{4.5}$$

that is

$$\frac{d}{dt}\langle \widehat{e^{i\varphi}}\rangle \equiv -\tfrac{1}{2}\sum_n \mu_n \rho_n^{(1)}\,. \tag{4.6}$$

Here the quantity μ_n reads

$$\begin{aligned}\tfrac{1}{2}\mu_n &= -\mathcal{A}_{n+1}^{(1)} - \mathcal{B}_n^{(1)} - \mathcal{C}_{n-1}^{(1)} \\ &= 2r\sin^2\left(\frac{g\tau(\sqrt{n+2}-\sqrt{n+1})}{2}\right) \\ &\quad +\gamma(n_b+1)[(n+\tfrac{1}{2}-\sqrt{n(n+1)}] \\ &\quad +\gamma n_b[n+\tfrac{3}{2}-\sqrt{(n+1)(n+2)}]\,. \end{aligned} \tag{4.7}$$

We obtain an approximate solution of Eq. (4.6) when we assume that μ_n is a slowly varying function of n. In this case we can replace μ_n by its mean value

$$\langle\mu_n\rangle = \sum_{n=0}^{\infty}\mu_n P_n \tag{4.8}$$

and obtain

$$\frac{d}{dt}\langle\widehat{e^{i\varphi}}\rangle \approx -\tfrac{1}{2}\langle\mu_n\rangle\sum_n \rho_n^{(1)} = -\tfrac{1}{2}\langle\mu_n\rangle\langle\widehat{e^{i\varphi}}\rangle\,. \tag{4.9}$$

When we further assume that P_n is significantly different from zero only for $n \gg 1$ we may expand the square roots in the definition of μ_n, Eq. (4.7), for large n and obtain

$$\begin{aligned} D &\equiv \langle\mu_n\rangle \\ &\approx \left\langle 4r\sin^2\left(\frac{g\tau}{4\sqrt{n}}\right)\right\rangle + \left\langle\frac{\gamma(2n_b+1)}{4n}\right\rangle \end{aligned} \tag{4.10}$$

as an approximation for the phase-diffusion constant D [21].

We can directly relate D to the average photon number when we assume that $\langle\mu_n\rangle \approx \mu_{n=\langle n\rangle}$. We then obtain from Eq. (4.10)

$$\begin{aligned} D &\approx \mu_{n=\langle n\rangle} \\ &\approx 4r\sin^2\left(\frac{g\tau}{4\sqrt{\langle n\rangle}}\right) + \frac{\gamma(2n_b+1)}{4\langle n\rangle}\,. \end{aligned} \tag{4.11}$$

This result has been derived in our previous paper [11] with the help of a detailed balance approach.

We note that the linewidth (4.11) is a generalization of the usual laser linewidth. In the limit of short interaction times or large photon numbers, that is, when $g\tau/(4\sqrt{\langle n\rangle}) \ll 1$, we expand the sine function in Eq. (4.11) and arrive at the familiar Schawlow-Townes linewidth [22]

$$D = \frac{\alpha+\gamma(2n_b+1)}{4\langle n\rangle}\,, \tag{4.12}$$

where

$$\alpha = \gamma(\sqrt{N}g\tau)^2 = \gamma\theta^2\,. \tag{4.13}$$

For small pump parameters, that is $\theta/\pi < 5$ this Schawlow-Townes type of expression for the maser linewidth is indistinguishable from the expression for D given in Eq. (4.11). However, for larger values of θ the expressions do not agree as manifest in Fig. 4. In Figs. 5 and 6 we compare the analytical expressions for D, Eq. (4.10) and Eq. (4.11), to the exact numerical linewidth for small values of θ (Fig. 5) and large values of θ (Fig. 6). We note that both approximations describe the qualitative features of the exact linewidth rather well. The sine expressions of Eqs. (4.10) and (4.11) smooth the quadratic divergence of the Schawlow-Townes expression, Eq. (4.12). However, both approximations overestimate the peaks

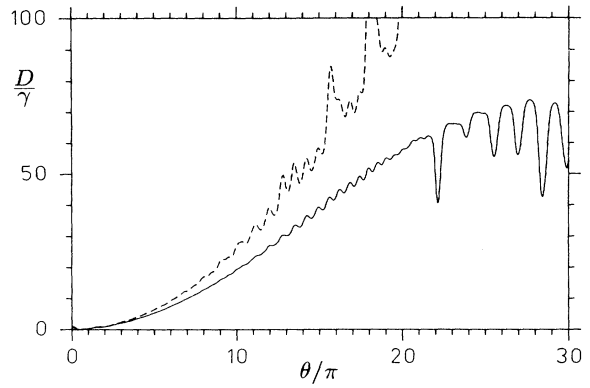

FIG. 4. Comparison between the exact linewidth D/γ (solid curve) and the Schawlow-Townes analog (broken curve), Eq. (4.12), for $N = 20$ and $n_b = 1$.

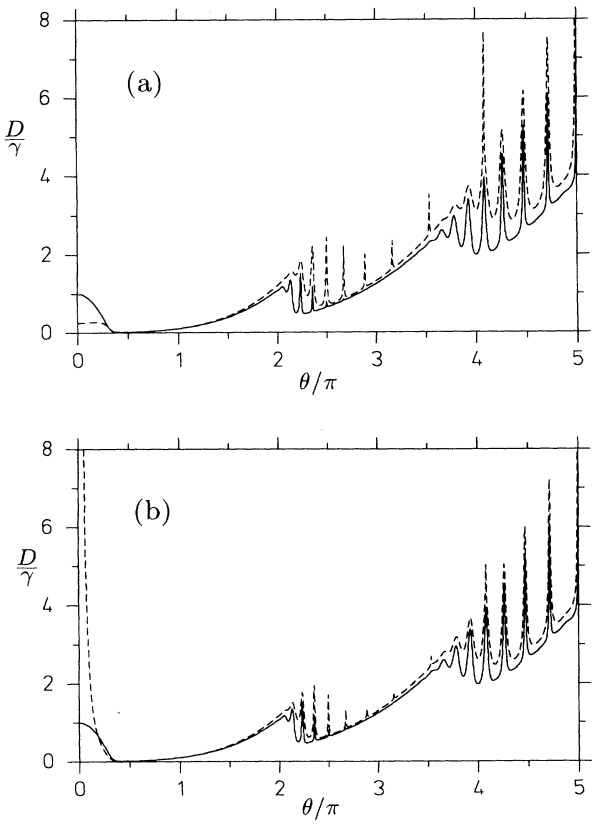

FIG. 5. Comparison between the exact linewidth D/γ (solid curves) and the approximate analytical expressions (broken curves), (a) Eq. (4.10) and (b) Eq. (4.11). The parameters are $N = 50$ and $n_b = 10^{-4}$.

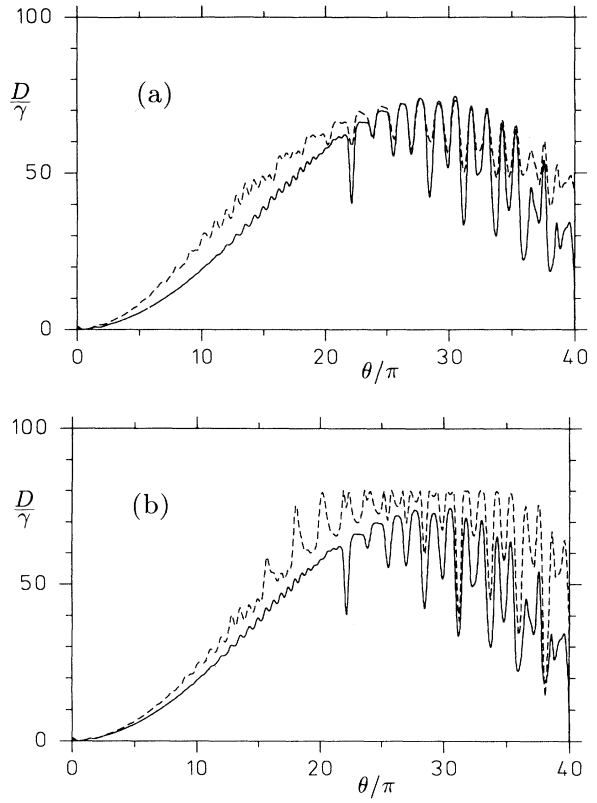

FIG. 6. Comparison between the exact linewidth D/γ (solid curves) and the approximate analytical expressions (broken curves), (a) Eq. (4.10) and (b) Eq. (4.11). The parameters are $N = 20$ and $n_b = 1$.

caused by the trapping states. For large values of θ the approximation (4.10) describes the linewidth better than Eq. (4.11).

We conclude this section with a discussion of the physical interpretation of the spectral properties of the micromaser. Equation (4.11) shows that the complicated pattern of the micromaser linewidth results from the complicated dependence of $\langle n \rangle$ on the pump parameter [3,23] which enters the denominator in the argument of the sine function. Equations (4.7) and (4.11) show that the linewidth of the micromaser field is determined by the interference between various quantum Rabi oscillations via their relative phase, that is, $(\sqrt{n+2} - \sqrt{n+1})g\tau \approx g\tau/(2\sqrt{n})$. Hence the number of Rabi oscillations and the photon-number distribution P_n enter the expression for the micromaser linewidth. For a pump parameter θ close to its threshold value the mean photon number $\langle n \rangle$ is large (see Fig. 1 of Refs. [3,23]). Consequently the phase difference between various Rabi oscillations, i.e., the value $(\sqrt{n+2} - \sqrt{n+1})g\tau$, is small. Therefore, in this region of the pump parameter θ the linewidth D is much narrower than the cavity damping constant γ. In the quiescent periods in which the mean photon number remains at a quasi-steady-state (see Fig. 1 of Ref. [23]) the mean photon number in the cavity increases substantially as n_b increases. Hence for a fixed value of $g\tau$ the phase differences between Rabi frequencies decrease with n_b, therefore (although the photon distribution is broader) the linewidth D is reduced as n_b increases, see Fig. 2. In the region of large interaction times $g\tau$, such that the mean value of the phase difference $\langle(\sqrt{n+2} - \sqrt{n+1})g\tau\rangle$ is larger than π, the linewidth D tends to decrease as this difference is increased.

V. CONCLUSION

In this paper we give a numerical approach to calculate the micromaser spectrum via a two-time correlation function of the micromaser field and present a comparison with approximate analytical results. We consider influences of the mean thermal photon number and the pump parameter on the micromaser spectrum. Novel features quite distinct from the familiar Schawlow-Townes laser linewidth are found. It is shown that they result from the interference between various quantum Rabi oscillations. We emphasize that the micromaser spectrum can be measured using the multiple fields method [11], proposed in our previous paper. The role of the pumping statistics on the micromaser spectrum will be considered elsewhere.

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APPENDIX A: QUANTUM REGRESSION THEOREM

In this appendix we derive the expression Eq. (2.10) for the two-time correlation function, Eq. (2.4), with the help of the quantum regression theorem. The mean value of the field operator $a^\dagger$ reads

$$\langle a^\dagger(t_0)\rangle = \sum_n \rho_{n,n+1}(t_0)\sqrt{n+1}\,. \tag{A1}$$

With the help of the Green's function $G_{m,n}(t_0+t;t_0)$ we find from Eq. (A1)

$$\langle a^\dagger(t_0+t)\rangle = \sum_{n,m} G_{n,m}(t_0+t;t_0)\rho_{m,m+1}(t_0)\sqrt{n+1}\,, \tag{A2}$$

where $G_{n,m}$ satisfies for every m the equation of motion for the matrix element $\rho_{n,n+1}$ with the initial condition

$$G_{n,m}(t_0;t_0) = \delta_{n,m}\,. \tag{A3}$$

When we interpret $\rho_{m,m+1}$ in Eq. (A2) as the mean value of the operator $|m+1\rangle\langle m|$ and use the quantum regression theorem [16] we can write the two-time correlation function $\langle a^\dagger(t_0+t)a(t_0)\rangle$ as

$$\langle a^\dagger(t_0+t)a(t_0)\rangle = \sum_{n,m} G_{n,m}(t_0+t;t_0)\sqrt{(m+1)(n+1)}\,\rho_{m+1,m+1}(t_0)\,. \tag{A4}$$

In the steady-state limit, that is, for $t_0 \to \infty$, the two-time correlation function

$$K(t) = \lim_{t_0\to\infty}\langle a^\dagger(t_0+t)a(t_0)\rangle$$

is identical to Eq. (2.10).

APPENDIX B: LASER THEORY APPROACH

The quantum theory of the laser [22] calculates the off-diagonal elements based upon the ansatz

$$\rho_n^{(k)} = F(k)\sqrt{\rho_n^{(0)}\rho_{n+k}^{(0)}}\,e^{-\tilde{\mu}_n t/2}\,, \tag{B1}$$

where $F(k)$ is some arbitrary function of k. This ansatz assumes that the off-diagonal elements $\rho_n^{(k)}$ resemble the diagonal elements $\rho_n^{(0)}$. According to this ansatz we try a solution of the Eq. (3.1) for $\rho_n^{(1)}(t)$ in the form

$$\rho_n^{(1)}(t) = Ce^{-\tilde{\mu}_n t/2} \times \prod_{\nu=1}^{n}\left(\frac{n_b}{n_b+1}+\frac{N\sin^2(g\tau\sqrt{\nu})}{(n_b+1)\nu}\right)^{1/2} \times \prod_{\nu=1}^{n+1}\left(\frac{n_b}{n_b+1}+\frac{N\sin^2(g\tau\sqrt{\nu})}{(n_b+1)\nu}\right)^{1/2}, \tag{B2}$$

where C is a constant which determines $\rho_n^{(1)}(t=0)$ and $\tilde{\mu}_n$ is the decay parameter which we want to calculate. Equation (B2) casts the elements $\rho_{n-1}^{(1)}(t)$ and $\rho_{n+1}^{(1)}(t)$ in the form

$$\rho_{n-1}^{(1)}(t) = \rho_n^{(1)}(t)\,B_n^-\,e^{(\tilde{\mu}_n-\tilde{\mu}_{n-1})t/2}\,, \tag{B3a}$$

$$\rho_{n+1}^{(1)}(t) = \rho_n^{(1)}(t)\,B_n^+\,e^{(\tilde{\mu}_n-\tilde{\mu}_{n+1})t/2}\,, \tag{B3b}$$

where

$$B_n^- = (n_b+1)\left[n_b+\frac{N}{n}\sin^2(g\tau\sqrt{n})\right]^{-1/2} \times\left[n_b+\frac{N}{n+1}\sin^2(g\tau\sqrt{n+1})\right]^{-1/2}, \tag{B4a}$$

$$B_n^+ = \frac{1}{n_b+1}\left[n_b+\frac{N}{n+1}\sin^2(g\tau\sqrt{n+1})\right]^{1/2} \times\left[n_b+\frac{N}{n+2}\sin^2(g\tau\sqrt{n+2})\right]^{1/2}. \tag{B4b}$$

We substitute the ansatz (B2) into the equation of motion for $\rho_n^{(1)}(t)$, Eq. (3.1), and find with the help of Eqs. (B3)

$$\dot{\rho}_n^{(1)}(t) = -\tfrac{1}{2}\tilde{\mu}_n\rho_n^{(1)}(t) = [\mathcal{B}_n^{(1)} + \mathcal{A}_n^{(1)}\,B_n^- e^{(\tilde{\mu}_n-\tilde{\mu}_{n-1})t/2} + \mathcal{C}_n^{(1)}\,B_n^+ e^{(\tilde{\mu}_n-\tilde{\mu}_{n+1})t/2}]\rho_n^{(1)}(t)\,. \tag{B5}$$

We note that Eq. (B5) is consistent with the ansatz Eq. (B2) only in the case when the decay rate $\tilde{\mu}_n$ is independent of n or at least is a slowly varying function of n, that is, when the exponentials in the right-hand side of Eq. (B5) can be neglected. In this case the solution of Eq. (B5) reads

$$\rho_n^{(1)}(t) = e^{-\frac{1}{2}\tilde{\mu}_n t}\rho_n^{(1)}(0)\,, \tag{B6}$$

where

$$\tfrac{1}{2}\tilde{\mu}_n = -\mathcal{B}_n^{(1)} - \mathcal{A}_n^{(1)}\,B_n^- - \mathcal{C}_n^{(1)}\,B_n^+\,. \tag{B7}$$

Here the quantities $\mathcal{A}_n^{(1)}$, $\mathcal{B}_n^{(1)}$, and $\mathcal{C}_n^{(1)}$ are defined by Eqs. (3.2). Algebraic manipulations allow us to rewrite Eq. (B7) in the form

$$\tfrac{1}{2}\tilde{\mu}_n = r[1-\cos(g\tau\sqrt{n+1})\cos(g\tau\sqrt{n+2})] + \gamma(n_b+1)(n+\tfrac{1}{2}) + \gamma n_b(n+\tfrac{3}{2}) - \gamma(n_b+1)\sqrt{n(n+1)}\,(1+C_n)^{-1/2} - [\gamma n_b\sqrt{(n+1)(n+2)} + r\sin(g\tau\sqrt{n+1})\sin(g\tau\sqrt{n+2})](1+C_{n+1})^{1/2}\,, \tag{B8}$$

where

$$C_n = \frac{n_b N[\sqrt{n+1}\sin(g\tau\sqrt{n}) - \sqrt{n}\sin(g\tau\sqrt{n+1})]^2}{[N\sin(g\tau\sqrt{n})\sin(g\tau\sqrt{n+1}) + n_b\sqrt{n(n+1)}]^2}\,. \tag{B9}$$

Since C_n essentially scales with n_b we have for $n_b \ll 1$ in Eq. (B8)

$$1 + C_n \approx 1 + C_{n+1} \approx 1 \tag{B10}$$

and obtain

$$\tfrac{1}{2}\tilde{\mu}_n \approx r[1-\cos(g\tau\sqrt{n+1})\cos(g\tau\sqrt{n+2})] - r\sin(g\tau\sqrt{n+1})\sin(g\tau\sqrt{n+2}) + \gamma(n_b+1)[n+\tfrac{1}{2}-\sqrt{n(n+1)}] + \gamma n_b[n+\tfrac{3}{2}-\sqrt{(n+1)(n+2)}]\,. \tag{B11}$$

We now interpret $\tilde{\mu}_n$ as the phase-diffusion coefficient for fixed n. The effective phase-diffusion coefficient is then

obtained by averaging $\tilde{\mu}_n$ over the steady-state photon statistics. Since the quantities $\tilde{\mu}_n$ given in Eq. (B11) and μ_n given in Eq. (4.7) agree, this averaging procedure leads to Eqs. (4.10) and (4.11) as described in Sec. IV.

APPENDIX C: SHORT-TIME APPROXIMATION

In Sec. III we have presented the equation of motion for the density operator of the micromaser field in the number state representation. For the following discussion it is more convenient to use the density operator ρ_f itself. In the interaction picture the equation of motion for $\rho_f(t)$ reads

$$\frac{\partial \rho_f}{\partial t} = r[M(\tau) - 1]\rho_f(t) + \frac{\gamma}{2} L_{\rm cav}\rho_f(t)\,, \tag{C1}$$

where

$$M(\tau)\rho_f(t) = \cos(\sqrt{aa^\dagger}\, g\tau)\rho_f(t)\cos(\sqrt{aa^\dagger}\, g\tau) + a^\dagger \frac{\sin(\sqrt{aa^\dagger}\, g\tau)}{\sqrt{aa^\dagger}}\rho_f(t)\frac{\sin(\sqrt{aa^\dagger}\, g\tau)}{\sqrt{aa^\dagger}} a \tag{C2a}$$

and

$$L_{\rm cav}\rho_f(t) = (n_b+1)[2a\rho_f(t)a^\dagger - a^\dagger a\rho_f(t) - \rho_f(t)a^\dagger a] + n_b[2a^\dagger\rho_f(t)a - aa^\dagger\rho_f(t) - \rho_f(t)aa^\dagger]\,. \tag{C2b}$$

We express the correlation function $K(t) = \langle a^\dagger(t)a(0)\rangle$ by

$$K(t) = {\rm Tr}_f\{a^\dagger\tilde{\rho}_f(t)\}\,, \tag{C3}$$

where—similar to the Green's function of Sec. II—the operator

$$\tilde{\rho}_f(t) = {\rm Tr}_r[U(t)a(0)\rho_{f,r}(0)U^\dagger(t)] \tag{C4}$$

satisfies the equation of motion for the maser field-density operator, Eq. (C1), with the initial condition

$$\tilde{\rho}_f(0) = a(0)\rho_f(0)\,. \tag{C5}$$

For small t we approximate the correlation function $K(t)$ by [24]

$$K(t) \approx K(0) + t\left.\frac{dK(t)}{dt}\right|_{t=0} \approx K(0)\left(1 - \frac{D}{2}t\right) \approx K(0)e^{-Dt/2}\,, \tag{C6}$$

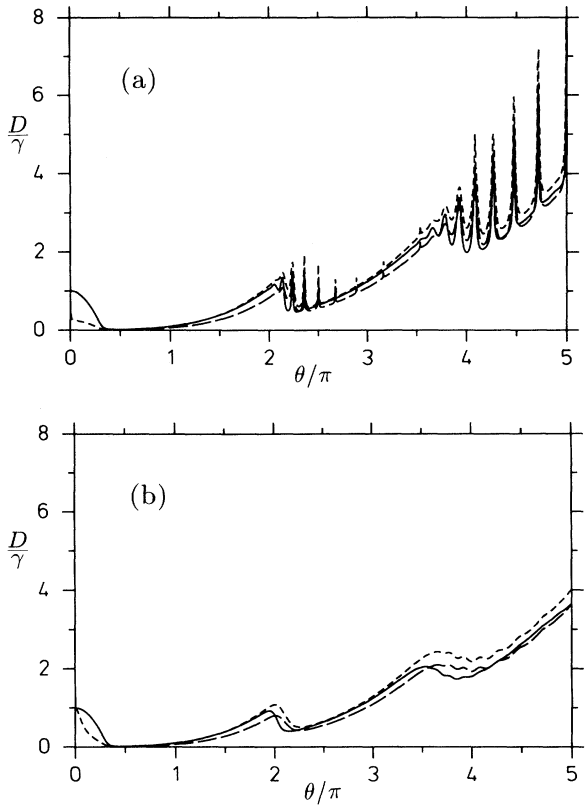

FIG. 7. Comparison between the exact linewidth D/γ (——) and the approximate analytical expressions, Eq. (C10) (— —) and Eq. (C14) (- - -) for $N = 50$. For n_b we have chosen (a) $n_b = 10^{-4}$ and (b) $n_b = 0.05$.

that is, we approximate $K(t)$ by an exponential decay corresponding to a linewidth

$$D \approx -2\frac{\dot{K}(0)}{K(0)}\,. \tag{C7}$$

Equation (C7) together with the equation of motion for $\tilde{\rho}_f(t)$, Eq. (C1), yields

$$-\frac{D}{2}{\rm Tr}_f\{a^\dagger a\rho_f(0)\} \approx r{\rm Tr}_f\{a^\dagger[M(\tau)-1][a\rho_f(0)]\} + \frac{\gamma}{2}{\rm Tr}_f\{a^\dagger L_{\rm cav}[a\rho_f(0)]\}\,. \tag{C8}$$

Cyclic permutations and the relation

$$f(aa^\dagger)a = af(a^\dagger a)\,, \tag{C9}$$

which is valid for an arbitrary function f, lead to

$$D \approx \gamma + 4r\frac{\left\langle \sin^2\left[\frac{g\tau}{2}(\sqrt{a^\dagger a+1} - \sqrt{a^\dagger a})\right] a^\dagger a\right\rangle}{\langle a^\dagger a\rangle} - 2r\frac{\langle \sin(\sqrt{a^\dagger a+1}\, g\tau)\sin(\sqrt{a^\dagger a}\, g\tau)[\sqrt{(a^\dagger a+1)a^\dagger a} - a^\dagger a]\rangle}{\langle a^\dagger a\rangle}\,. \tag{C10}$$

This expression simplifies when we replace $\langle f(a^\dagger a)\rangle$ by $f(\langle a^\dagger a\rangle)$ and when we use for $\langle n\rangle \equiv \langle a^\dagger a\rangle \gg 1$ the approximation $\sin(\sqrt{\langle a^\dagger a\rangle}\, g\tau) \approx \sin(\sqrt{\langle a^\dagger a\rangle + 1}\, g\tau)$. We then arrive at

$$D \approx \gamma + 4r \sin^2\left(\frac{g\tau}{4\sqrt{\langle n\rangle}}\right) - \frac{r}{\langle n\rangle} \sin^2(\sqrt{\langle n\rangle + 1}\, g\tau)\,. \tag{C11}$$

In order to cast the second term in Eq. (C11) into a form similar to the one in approximation (4.11) we derive from Eq. (C1) an equation of motion for $\langle n\rangle$. From

$$\frac{d}{dt}\langle n\rangle = r \sin^2(\sqrt{\langle n\rangle + 1}\, g\tau) - \gamma[\langle n\rangle - n_b] \tag{C12}$$

we find in steady state

$$r\langle \sin^2(\sqrt{\langle n\rangle + 1}\, g\tau)\rangle = \gamma[\langle n\rangle - n_b]\,. \tag{C13}$$

Again we have replaced $\langle f(a^\dagger a)\rangle$ by $f(\langle a^\dagger a\rangle)$. We substitute this result into Eq. (C11) and obtain

$$D \approx 4r \sin^2\left(\frac{g\tau}{4\sqrt{\langle n\rangle}}\right) + \frac{\gamma n_b}{\langle n\rangle}\,. \tag{C14}$$

In Fig. 7 we compare and contrast the unfactorized expression for D, Eq. (C10), and the factorized one, Eq. (C14), to the exact linewidth obtained from the Green's function. Note that for $r = 0$, that is for an empty cavity with $\langle n\rangle = n_b$ in steady state, both approximations Eq. (C10) and Eq. (C14) reduce to the exact expression

$$D = \gamma\,. \tag{C15}$$

This is in contrast to the approximation Eq. (4.11) which for $r = 0$ and $n_b \ll 1$ reads

$$D = \frac{\gamma}{4n_b} \gg \gamma\,. \tag{C16}$$

This sharp rise in D for small values of θ is apparent in Fig. 5(b).

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