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Conservation law with discontinuous flux

By

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1. Introduction

In this paper we study the following scalar conservation law:

$$\begin{aligned} \frac{\partial u}{\partial t} + \frac{\partial}{\partial x}(F(x,u)) &= 0, \qquad x \in \mathbb{R}, \quad t > 0, \\ u(x,0) &= u_0(x), \qquad x \in \mathbb{R}, \end{aligned} \tag{1.1}$$

where the flux function $F(x,u)$ is a discontinuous function of x given by $F(x,u) = H(x)f(u) + (1 - H(x))g(u)$, H is the Heaviside function, f and g are smooth functions on $\mathbb{R}$. Conservation laws with discontinuous flux function in x appears for example in the modelling of two phase flow in a porous media [6], [9], in sedimentation problem [3], [4] and in traffic flow [13].

It is well known that after a finite time, solution of (1.1) do not in general posses a continuous solution even if u_0 is sufficiently smooth. Hence a solution u of (1.1) ~~we mean~~ a solution in the weak sense. That is $u \in L^\infty_{\text{loc}}$ such that for all $\varphi \in C^1_0(\mathbb{R} \times \mathbb{R}_+)$,

$$\int_{-\infty}^{\infty} \int_0^{\infty} \left(u \frac{\partial \varphi}{\partial t} + F(x,u) \frac{\partial \varphi}{\partial x} \right) dt\, dx + \int_{-\infty}^{\infty} u(x,0)\varphi(x,0) dx = 0. \tag{1.2}$$

It is easy to see that (1.2) is the weak formulation of the following problem. Denoting $u_t = \partial u/\partial t$, $u_x = \partial u/\partial x$, then in the weak sense u satisfies

$$\begin{aligned} &u_t + f(u)_x = 0 \qquad \text{for} \quad x > 0, \quad t > 0, \\ &u_t + g(u)_x = 0 \qquad \text{for} \quad x < 0, \quad t > 0, \\ &u(x,0) = u_0(x), \end{aligned} \tag{1.3}$$

and at $x = 0$, u satisfies the Rankine-Hugoniot condition i.e., for almost all $t > 0$,

$$f(u(0+,t)) = g(u(0-,t)), \tag{1.4}$$

where $u(0+,t) = \lim_{x \to 0+} u(x,t)$ and $u(0-,t) = \lim_{x \to 0-} u(x,t)$.

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Kruzkov [11] proved that if F is continuous in u and $\partial F/\partial x$ is bounded, then (1.1) admits a weak solution. If F is discontinuous in x, Kruzkov's method does not guarantee a solution. For example in (1.3) if we take $g(u) = u$, $f(u) = -u$, $u_0(x) = 2$ if $x < 0$ and $u_0(x) = 3$ if $x > 0$, then it is easy to see that $u(x,t) = 2$ if $x < 0$ and $u(x,t) = 3$ if $x > 0$, is a solution for (1.3) but do not satisfy (1.4). Hence it is not a weak solution of (1.1). The discontinuity of the flux function at $x = 0$ causes a discontinuity of a solution which in general not uniquely determined by the initial data. When there is no discontinuity of a flux function at $x = 0$, that is $f = g$ and strictly convex, this problem was studied by Lax and Oleinik [5], [12], [14]. Using the Hamilton-Jacobi equation they obtain an explicit formula for the solution and derive an entropy condition so that the solution they obtained is unique.

For a general f, Kruzkov [11] proves the uniqueness of an entroy solution. Kruzkov [11] and Keyfitz [10] showed that the entropy solution can be represented by L^1-contraction semigroup.

When $f \neq g$, this problem is studied by Gimse and Risebro [6] and Diehl [1], [2]. In the case of two phase flow problem, Gimse and Risebro [4] obtain a unique solution of the Riemann problem for (1.3) and (1.4) by minimizing $|u(0+,t) - u(0-,t)|$. Using this they construct a sequence of approximate solutions converging to a weak solution for bounded initial data. Later it was pointed out by Diehl [3], [4] that "to minimize $|u(0+,t) - u(0-,t)|$" may not be a suitable choice. Instead of this one has to look for the solution which has smaller variation (he puts a condition called Γ condition). In this class Diehl gives an explicit formula for a solution in the case of a Riemann problem and proves the uniqueness.

Now the question is "**Whether the solution obtained from Diehl can be represented by a contraction semigroup in L^1-norm in the sense of Kruzkov [11] and Keyfitz [9]?**".

By looking at Diehl's work it is not clear that solution can be represented by a contraction semigroup. The main difficulty is to obtain a proper entropy condition at $x = 0$.

In this paper under the following condition (see Section 3 for details):

(H) f, $g \in C^1(\mathbb{R})$ **and are strictly convex and super linear growth,**

we settle this question affirmatively for arbitrary bounded initial data (see also [1]). Also we show that, in general, our solution differ from the solution obtained by Diehl (see Example 5.3). Here we give an explicit formula for the solution of (1.3) satisfying (1.4). This agrees with the Lax-Oleinik [5] formula when $f = g$. Also we give a correct entropy condition at $x = 0$ so that the problem (1.3) and (1.4) admits a unique solution determined by the initial condition like in Kruzkov [11]. In [2] we have relaxed the condition **(H)** and proved the existence by making use of the Riemann problem solution given in Section 5 and construct a Godunov type scheme satisfying the boundary entropy condition.

The plan of the paper is as follows. In Section 2, we state entropy conditions E_i and E_b, and prove uniqueness results. In Section 3, we state explicit formula for the solution of the corresponding Hamilton-Jacobi equation of (1.3) and (1.4) (Theorem 3.1) and also formula for the solution of (1.3) and (1.4) (Theorem 3.2). In Section 4, we prove these theorems and also show that these formulas satisfy the entropy conditions E_i and E_b given below. In Section 5, we derive the solution for the Riemann problem.

2. Entropy conditions and Uniqueness results

In this section we state the interior and boundary entropy conditions and prove the uniqueness of a solution, using the method of Kruzkov [11].

Interior entropy condition (E_i)**.** u is said to satisfy the entropy condition (E_i) (Lax-Oleinik entropy conditions) if for all $t > 0$

$$\lim_{0<z\to 0} u(x+z,t) \le \lim_{0<z\to 0} u(x-z,t) \qquad \text{if} \quad x>0, \tag{2.1}$$

$$\lim_{0<z\to 0} u(x+z,t) \le \lim_{0<z\to 0} u(x-z,t) \qquad \text{if} \quad x<0\,. \tag{2.2}$$

Boundary entropy condition (E_b)**.** At $x = 0$, $u(0+,t) = \lim_{x\to 0+} u(x,t)$, $u(0-,t) = \lim_{x\to 0-} u(x,t)$ exist for almost all $t > 0$. Furthermore for all most all $t > 0$ one of the following condition must hold:

$$f'(u(0+,t)) \ge 0 \qquad \text{and} \qquad g'(u(0-,t)) \ge 0, \tag{2.3}$$

$$f'(u(0+,t)) \le 0 \qquad \text{and} \qquad g'(u(0-,t)) \le 0, \tag{2.4}$$

$$f'(u(0+,t)) \le 0 \qquad \text{and} \qquad g'(u(0-,t)) \ge 0. \tag{2.5}$$

Entropy pairs. Let φ_1, φ_2 be convex functions. Let $\psi_1'(s) = f'(s)\varphi_1'(s)$, $\psi_2'(s) = g'(s)\varphi_2'(s)$. Then (φ_i, ψ_i), $i = 1, 2$ are called entropy pairs associated to (1.1).

Kruzkov entropy condition (E_K)**.** A weak solution $u \in L^\infty_{\text{loc}}$ of (1.3) and (1.4) is said to satisfy (E_K) if for every entropy pairs (φ_i, ψ_i) $i = 1, 2$ and for every $\rho \in C_0^1(\mathbb{R} \times \mathbb{R}_+), \rho \ge 0$

$$\int_0^\infty \int_0^\infty \left(\phi_1(u) \frac{\partial \rho}{\partial t} + \psi_1(u) \frac{\partial \rho}{\partial x} \right) dt\, dx \ge - \int_0^\infty \psi_1(u(0+,t))\rho(0,t)dt, \tag{2.6}$$

$$\int_{-\infty}^0 \int_0^\infty \left(\phi_2(u) \frac{\partial \rho}{\partial t} + \psi_2(u) \frac{\partial \rho}{\partial x} \right) dt\, dx \ge \int_0^\infty \psi_2(u(0-,t))\rho(0,t)dt. \tag{2.7}$$

Then we have the following

Theorem 2.1. *Let $u, v \in L^\infty$ be two weak solutions of* (1.3) *and* (1.4)*. Assume that u, v satisfy the entropy conditions (E_b) and (E_K) and satisfies initial condition in the following sense*:

$$\lim_{t\to 0^+} ||u(\cdot,t) - u_0||_{L^1} = \lim_{t\to 0^+} ||v(\cdot,t) - u_0||_{L^1} = 0.$$

Then $u \equiv v$.

As an immediate consequence of this we have the following

Theorem 2.2. *Let $u, v \in L^\infty \cap BV_{\text{loc}}$ be two solutions of* (1.3) *and* (1.4) *satisfying* (E_i) *and* (E_b). *Further more assume that the set of discontinuities of u and v are discrete set of Lipschitz curves. Then $u \equiv v$.*

Proof of the Uniqueness results. Using the idea of Kruzkov, we prove the following inequalities (2.11) and (2.12). In order to prove the uniqueness these inequalities are not sufficient. In addition to these, we need (E_b) to prove the uniqueness.

For the sake of completeness we provide the proof of these inequalities. Let u and v satisfies the hypothesis of Theorem 2.1. Like in continuous flux function ([8, p. 24], [7, p. 23]) by approximation one obtains, for every $k \in \mathbb{R}, 0 \le \rho \in C_0^1(\mathbb{R}\times\mathbb{R}_+)$

(2.8)
$$\int_0^\infty \int_0^\infty \left(|u(x,t)-k|\frac{\partial\rho}{\partial t} + f(u(x,t),k)\frac{\partial\rho}{\partial x}\right) dx\, dt \ge -\int_0^\infty f(u^+(t),k)\rho(0,t)dt,$$

(2.9)
$$\int_{-\infty}^0 \int_0^\infty \left(|u(x,t)-k|\frac{\partial\rho}{\partial t} + g(u(x,t),k)\frac{\partial\rho}{\partial x}\right) dx\, dt \ge \int_0^\infty g(u^-(t),k)\rho(0,t)dt,$$

where $f(a,b) = ((f(a)-f(b))/(a-b))|a-b|$, $g(a,b) = ((g(a)-g(b))/(a-b))|a-b|$, $u^+(t) = u(0+,t)$, $u^-(t) = u(0-,t)$. Let $\rho \ge 0$ be a compactly supported smooth function of x,t,y,s and by taking $k = v(y,s)$, $\rho \in C_0^1(\mathbb{R}\times\mathbb{R}_+\times\mathbb{R}\times\mathbb{R}_+)$, $D_+ = \mathbb{R}_+\times\mathbb{R}_+$, $D_+^2 = D_+ \times D_+$ and integrating (2.8) with respect to y and s to obtain

$$\int_{D_+^2} \left(|u(x,t)-v(y,s)|\frac{\partial\rho}{\partial t} + f(u(x,t),v(y,s))\frac{\partial\rho}{\partial x}\right) dx\, dy\, ds\, dt$$
$$\ge -\int_{D_+\times R_+} f(u^+(t),v(y,s))\rho(0,t,y,s)dt\, dy\, ds.$$

Let $A(x,t,y,s) = (f(u(x,t))-f(v(y,s)))/(u(x,t)-v(y,s))$ and interchanging u and v in the above formula and add to obtain

$$\int_{D_+^2} |u(x,t)-v(y,s)| \left\{ \left(\frac{\partial\rho}{\partial t}+\frac{\partial\rho}{\partial s}\right) + A(x,t,y,s)\left(\frac{\partial\rho}{\partial x}+\frac{\partial\rho}{\partial y}\right)\right\} dx\, dy\, ds\, dt$$
$$\ge -\int_{D_+\times\mathbb{R}_+} f(u^+(t),v(y,s))\rho(0,t,y,s)dt\, dy\, ds$$
$$-\int_{D_+\times\mathbb{R}_+} f(u(x,t),v^+(s))\rho(x,t,0,s)dt\, dx\, ds.$$

Let $\tilde\rho(X,T_1,Y,T_2) = \rho(X+Y,T_1+T_2,Y,T_1)$, then after a change of variables $t = T_1+T_2$, $s = T_1$, $x = X+Y$, $y = Y$, $D_+^2 = \tilde D$, the above inequality reduces to

$$\int_{\tilde D} |u(X+Y,T_1+T_2)-v(Y,T_1)| \left[\frac{\partial\tilde\rho}{\partial T_1} + A(X+Y,T_1+T_2,Y,T_1)\frac{\partial\tilde\rho}{\partial Y}\right] dX\,dY\,dT_1\,dT_2$$
$$\geq -\int_{D_+\times\mathbb{R}_+} f(u^+(t),v(y,s))\rho(0,t,y,s)dt\,dy\,ds - \int_{D_+\times\mathbb{R}_+} f(u(x,t),v^+(s))\rho(x,t,0,s)dx\,dt\,ds.$$

Let $\varepsilon_1>0, \varepsilon_2>0$ and $\alpha\in C_0^1((-1,0)\times(-1,0))$ with $0\leq\alpha$, $\int_{\mathbb{R}^2}\alpha(X,T_2)dX\,dT_2 = 1$. Let $\alpha_{\varepsilon_1,\varepsilon_2}(X,T_2) = (1/\varepsilon_1\varepsilon_2)\alpha(X/\varepsilon_1,T_2/\varepsilon_2)$. Let $\beta\in C_0^1(\overline{\mathbb{R}}_+\times\mathbb{R}_+)$ and take $\rho(x,t,y,s)=\alpha_{\varepsilon_1,\varepsilon_2}(x-y,t-s)\beta(y,s)$. Then $\tilde\rho(X,T_1,Y,T_2) = \alpha_{\varepsilon_1,\varepsilon_2}(X,T_2)\beta(Y,T_1)$. Let

$$\begin{aligned}
I_1 &= \int_{D_+\times\mathbb{R}_+} f(u^+(t),v(y,s))\rho(0,t,y,s)dt\,dy\,ds\\
&= \int_{D_+\times\mathbb{R}_+} f(u^+(t),v(y,s))\alpha_{\varepsilon_1,\varepsilon_2}(-y,t-s)\beta(y,s)dy\,dt\,ds\\
&= \int_0^\infty\int_{-1}^0\int_0^1 f(u^+(t),v(\varepsilon_1 z,\ t-\varepsilon_2 T))\alpha(-z,T)\beta(\varepsilon_1 z,\ t-\varepsilon_2 T)dz\,dT\,dt.
\end{aligned}$$

Since $\lim_{\varepsilon_1\to 0} v(\varepsilon_1 z,t) = v^+(t)$ for almost all t and $\int f(u^+(t),v^+(t-\varepsilon_2 T))\alpha(-z,T)\beta(0,t-\varepsilon_2 T)dz\,dT$ converges in L^1 as $\varepsilon_2\to 0$ to obtain

$$\lim_{\varepsilon_2\to 0}\lim_{\varepsilon_1\to 0} I_1 = \int_0^\infty f(u^+(t),v^+(t))\beta(0,t)dt.$$

Let

$$\begin{aligned}
I_2 &= \int_{D_+\times\mathbb{R}_+} f(u(x,t),v^+(s))\rho(x,t,0,s)dx\,dt\,ds\\
&= \int_0^\infty\int_0^\infty\int_0^\infty f(u(x,t),v^+(s))\alpha_{\varepsilon_1,\varepsilon_2}(x,t-s)\beta(0,s)dx\,dt\,ds\\
&= 0,
\end{aligned}$$

since $\alpha_{\varepsilon_1,\varepsilon_2}(x,t-s)=0$ for $x\geq 0$. Let

$$\begin{aligned}
I_3 &= \int_{\tilde D} |u(X+Y,T_1+T_2)-v(Y,T_1)| \left[\frac{\partial\tilde\rho}{\partial T_1} + A(X+Y,T_1+T_2,Y,T_1)\frac{\partial\tilde\rho}{\partial Y}\right] dX\,dY\,dT_1\,dT_2\\
&= \int_{\tilde D} |u(X+Y,T_1+T_2)-v(Y,T_1)| \left[\frac{\partial\beta}{\partial T_1}\right.
\end{aligned}$$

$$+ A(X+Y, T_1+T_2, Y, T_1)\,\frac{\partial \beta}{\partial Y}\Bigg]\,\alpha_{\varepsilon_1,\varepsilon_2}(X,T_2)dX\,dY\,dT_1\,dT_2$$

$$= \int_{D_+}\int_{-Y/\varepsilon_1}^{\infty}\int_{-T_1/\varepsilon_2}^{\infty} |u(Y+\varepsilon_1 X, T_1+\varepsilon_2 T_2) - v(Y,T_1)|\left[\frac{\partial \beta}{\partial T_1}\right.$$

$$\left. + A(Y+\varepsilon_1 X, T_1+\varepsilon_2 T_2, Y, T_1)\,\frac{\partial \beta}{\partial Y}\right]\alpha(X,T_2)dT_2\,dX\,dY\,dT_1\,.$$

For $M > 0$, there exist a $C(M) > 0$ such that for $a_1, a_2, b \le M$

$$\begin{aligned}|f(a_1,b) - f(a_2,b)| &= \Bigg|\left(\int_0^1 f'(\theta a_1 + (1-\theta)b)d\theta\right)|a_1 - b| \\ &\qquad - \left(\int_0^1 f'(\theta a_2 + (1-\theta)b)d\theta\right)|a_2 - b|\Bigg| \\ &\le \int_0^1 |f'(\theta a_1 + (1-\theta)b)|d\theta\,||a_1-b| - |a_2-b|| \\ &\qquad + |a_2 - b|\int_0^1 |f'(\theta a_1 + (1-\theta)b) - f'(\theta a_2 + (1-\theta)b)|d\theta \\ &\le C(M)|a_1 - a_2|,\end{aligned}$$

hence

$$\lim_{\varepsilon_2\to 0}\lim_{\varepsilon_1\to 0} I_3 = \int_{D_+} |u(Y,T_1) - v(Y,T_1)|\left\{\frac{\partial \beta}{\partial T_1} + A(Y,T_1,Y,T_1)\frac{\partial \beta}{\partial Y}\right\}dY\,dT_1.$$

Therefore combining all the inequality to obtain

$$\begin{aligned}&\int_{D_+} |u(x,t) - v(x,t)|\left\{\frac{\partial \beta}{\partial t} + A(x,t,x,t)\frac{\partial \beta}{\partial x}\right\}dx\,dt \\ &\qquad \ge -\int_0^\infty f(u^+(t), v^+(t))\beta(0,t)dt.\end{aligned}$$

Let $\beta(x,t) = \varphi(t)C(x,t)$ where $C(x,t)$ is such that $\partial C/\partial t + M|\partial C/\partial x| \le 0$, then

$$\begin{aligned}&\int_{D_+} |u(x,t) - v(x,t)|\varphi'(t)C(x,t)dx\,dt \\ &\qquad + \int_{D_+} |u(x,t) - v(x,t)|\left\{\frac{\partial C}{\partial t} + A\frac{\partial C}{\partial x}\right\}\varphi(t)dx\,dt \\ &\ge -\int_0^\infty f(u^+(t), v^+(t))\varphi(t)C(0,t)\,dt\,.\end{aligned}$$

Now $\partial C/\partial t + A(\partial C/\partial x) \le \partial C/\partial t + M|\partial C/\partial x| \le 0$, hence

(2.10)

$$\int_{D_+} |u(x,t) - v(x,t)|\varphi'(t)C(x,t)dx\,dt \ge -\int_0^\infty f(u^+(t), v^+(t))\varphi(t)C(0,t)\,dt\,.$$

Let $a < b$, $a + b \geq 0$, $x_0 = 1/2(a + b)$, $G = \{(x, t), |x - x_0| + Mt \leq 1/2(b - a)\}$ and $\rho_\varepsilon(s)$ be a decreasing C^1-function in $\mathbb{R}^+$ such that $\rho_\varepsilon(s) \to \chi_{[0,(b-a)/2]}(s)$ in $L^1(\mathbb{R}^+)$. Let $C(x, t) = \rho_\varepsilon(|x - x_0| + Mt)$. Then $\partial C/\partial t + M|\partial C/\partial x| \leq M(\rho' + |\rho'|) = 0$. Hence substituting this in (2.10) and letting $\varepsilon \to 0$ to obtain

$$\int_{D_+ \cap G} |u(x, t) - v(x, t)|\varphi'(t)dx\, dt \geq - \int_0^{\max(\frac{-a}{M}, 0)} f(u^+(t), v^+(t))\varphi(t)dt\,.$$

Let $b - a \geq 2MT$ and $0 \leq \varphi \in C_0^\infty(0, T)$, then the above inequality becomes

$$\begin{aligned}\int_0^\infty \varphi'(t) \int_{\max(a+Mt,0)}^{b-Mt} |u(x, t) - v(x, t)|\, dx\, dt \\ \geq - \int_0^{\max(\frac{-a}{M}, 0)} f(u^+(t), v^+(t))\varphi(t)dt.\end{aligned} \tag{2.11}$$

Similarly the inequality (2.9) gives that for any $\varphi \in C_0^\infty(0, T)$ when $T \leq (d - c)/2M, c < d$ with $c + d \leq 0$,

$$\begin{aligned}\int_0^\infty \varphi'(t) \int_{c+Mt}^{\min(d-Mt,0)} |u(x, t) - v(x, t)|\, dx\, dt \\ \geq \int_0^{\max(\frac{d}{M}, 0)} g(u^-(t), v^-(t))\varphi(t)\, dt\,.\end{aligned} \tag{2.12}$$

□

Proof of Theorem 2.1. From (2.11) for any $a \geq 0$, $\max(-a/M, 0) = 0$ and hence $\int_{a+Mt}^{b-Mt} |u(x, t) - v(x, t)|dx$ is a decreasing function and hence $\int_{a+Mt}^{b-Mt} |u(x, t) - v(x, t)|dx \leq \int_a^b |u_0(x) - v_0(x)|dx = 0$. Therefore $u = v$ for $x \geq Mt$. Let $a < 0 < b$, $a + b \geq 0$, $c < 0 < -a$, $c - a \leq 0$, $T_0 = -a/M = d/M$. Then by the choice of a, b, c, $T_0 \leq \min((b - a)/2M, -(c + a)/2M)$. Let $\varphi \in C_0^\infty(0, T_0)$, Then from (2.11) and (2.12) we have

$$\begin{aligned}\int_0^{T_0} \varphi'(t) \int_{c+Mt}^{b-Mt} |u(x, t) - v(x, t)|dx\, dt \\ \geq \int_0^{T_0} (g(u^-(t), v^-(t)) - f(u^+(t), v^+(t)))\varphi(t)dt.\end{aligned} \tag{2.13}$$

Claim. For almost all t, $g(u^-(t), v^-(t)) - f(u^+(t), v^+(t)) \geq 0$.

Let

$$\begin{aligned}I = g(u^-(t), v^-(t)) - f(u^+(t), v^+(t)) = |u^-(t) - v^-(t)|\frac{g(u^-(t)) - g(v^-(t))}{u^-(t) - v^-(t)} \\ - |u^+(t) - v^+(t)|\frac{f(u^+(t)) - f(v^+(t))}{u^+(t) - v^+(t)}.\end{aligned}$$

Without loss of generality assume that $u^+(t) \geq v^+(t)$. Suppose $(f(u^+(t)) - f(v^+(t)))/(u^+(t) - v^+(t)) \geq 0$, then $f(u^+(t)) \geq f(v^+(t))$ and hence from (1.4) $g(u^-(t)) \geq g(v^-(t))$. If $f'(u^+(t)) \geq 0$ then by (E_b), $g'(u^-(t)) \geq 0$ and hence $v^-(t) \leq u^-(t)$ since g is convex. This together with (1.4) gives $I = g(u^-(t)) - g(v^-(t)) - f(u^+(t)) + f(v^+(t)) \geq 0$. If $f'(u^+(t)) < 0$, then by convexity of f $f(u^+(t)) < f(v^+(t))$ which is not possible by assumption.

Suppose $(f(u^+(t)) - f(v^+(t)))/(u^+(t) - v^+(t)) \leq 0$, then $f(u^+(t)) \leq f(v^+(t))$ and hence from (1.4) $g(u^-(t)) \leq g(v^-(t))$. If $u^-(t) \leq v^-(t)$, then $I \geq 0$. If $u^-(t) > v^-(t)$, then $I = 0$. This proves the claim.

From the above claim and (2.13), it follows that $\int_0^{T_0} \varphi'(t) \int_{c+Mt}^{b-Mt} |u(x,t) - v(x,t)|dx\, dt \geq 0$, for all $\varphi \geq 0, \varphi \in C_0^\infty(0,T)$. Hence $t \to \int_{c+Mt}^{b-Mt} |u(x,t) - v(x,t)|dx$ is a non-increasing function. This implies that $\int_{c+Mt}^{b-Mt} |u(x,t) - v(x,t)|dx \leq \int_a^b |u_0(x) - v_0(x)|dx = 0$ and therefore $u = v$ for $0 \leq x \leq Mt$. Similarly for $x < 0$ and hence $u \equiv v$. This proves the Theorem. $\square$

Proof of Theorem 2.2. Let φ be smooth function $Y'(s) = f'(s)\varphi'(s)$. Let

$$I(s,t) = (s-t)(Y(s) - Y(t)) - (\varphi(s) - \varphi(t))(f(s) - f(t)) \tag{2.14}$$

Claim. $s \mapsto I(s,t)$ is convex and positive for each fixed t.

$$\begin{aligned}
\frac{\partial I}{\partial s} &= (Y(s) - Y(t)) + (s-t)Y'(s) - \varphi'(s)(f(s) - f(t)) - (\varphi(s) - \varphi(t))f'(s),\\
\frac{\partial^2 I}{\partial s^2} &= 2Y'(s) + (s-t)Y''(s) - 2\varphi'(s)f'(s)\\
&\quad - \varphi''(s)(f(s) - f(t)) - (\varphi(s) - \varphi(t))f''(s)\\
&= (s-t)(f''(s)\varphi'(s) + f'(s)\varphi''(s))\\
&\quad - \varphi''(s)(f(s) - f(t)) - (\varphi(s) - \varphi(t))f''(s)\\
&= \varphi''(s)[f(t) - f(s) - (t-s)f'(s)] + f''(s)[\varphi(t) - \varphi(s) - (t-s)\varphi'(s)]\\
&= (s-t)^2 \left[\varphi''(s) \int_0^1 (1-\theta) f''(\theta t + (1-\theta)s)\, d\theta \right.\\
&\quad \left. + f''(s) \int_0^1 (1-\theta)\varphi''(\theta t + (1-\theta)s) d\theta \right]\\
&\geq 0.
\end{aligned}$$

Furthermore, $I(t,t) = \partial I/\partial s(t,t) = 0$. Hence $I(s,t) \geq 0$. This proves the claim.

Using this claim and Theorem 2.1 we deduce Theorem 2.2. In order to do this, it is enough to show that (E_i) implies (E_K). Let φ be a smooth convex function on $\mathbb{R}$ and let $\psi'(s) = f'(s)\varphi'(s)$. Since u is a solution of (1.3) and set of discontinuities of u are discrete set of lipschitz curves, hence by integration by parts we have for any $\rho \in C_0^1(\mathbb{R} \times \mathbb{R}_+)$, $\rho \geq 0$,

$$\int_0^\infty \int_0^\infty \left(\varphi(u) \frac{\partial \rho}{\partial t} + \psi(u) \frac{\partial \rho}{\partial x} \right) dx\, dt = \sum_{j=1}^{\infty} \int_{\Gamma_j} \{[\varphi(u)]\nu_1 + [\psi(u)]\nu_2\}\rho\, d\sigma - \int_0^\infty \psi(u^+(t))\rho(0,t)dt,$$

where Γ_j are curves of discontinuities of u in $x > 0$, (ν_1, ν_2) the unit outward normal to Γ_j and $[\varphi(u)] = \phi(u^-) - \phi(u^+)$, jump across the curves Γ_j. Since u is a solution of (1.3), hence by Rankine-Hugoniot condition, $\nu_2 > 0$ and $\nu_1/\nu_2 = -[f(u)]/[u]$. Hence on Γ_j,

$$\begin{aligned} [\varphi(u)]\nu_1 + [\psi(u)]\nu_2 &= \nu_2 \left\{ -\frac{[\phi(u)][f(u)]}{[u]} + [\psi(u)] \right\} \\ &= \frac{\nu_2}{(u^- - u^+)} I(u^-, u^+)\,. \end{aligned}$$

Since u satisfies (E_i) and hence $u^- \geq u^+$. Therefore from the above claim $[\phi(u)]\nu_1 + [\psi(u)]\nu_2 \geq 0$. This proves that u satisfies (2.8). Similarly u satisfies (2.9). Hence u satisfies (E_K) and therefore from Theorem 2.1, $u \equiv v$. This proves the Theorem. □

3. Explicit formula for the solution

Before going to the explicit formula, let us recall some well known results on convex functions without proof (for example see [5, p. 112]).

$f : \mathbb{R} \to \mathbb{R}$ is said to be a strictly convex and superlinear growth if for $a \neq b, t \in (0,1)$

$$f(ta + (1-t)b) < tf(a) + (1-t)f(b) \qquad \text{and} \qquad \lim_{a\to\infty} \frac{f(a)}{|a|} = \infty.$$

Associate to f define its convex dual f^* by

$$f^*(x) = \sup_{y\in\mathbb{R}} \{xy - f(y)\}.$$

If f is strictly convex and super linear growth then f and f^* satisfies the following:

(a) $f^*(0) = -\min f$, is finite,

(b) f^* is strictly convex and super linear growth and satisfies

$$f(y) = \sup_{x\in\mathbb{R}} \{xy - f^*(x)\}.$$

Definition 3.1 (Admissible curves). Let $0 \leq s < t$ and $\xi \in c([s,t], \mathbb{R})$. ξ is called an admissible curve if the following holds.

(1) ξ consists of at most three linear curves (see Figs. 3.1a, 3.1b and 3.1c for $x \geq 0$, Figs. 3.2a, 3.2b and 3.2c for $x \leq 0$) and each segment lies completely in either $x \geq 0$ or $x \leq 0$.

(2) Let $s = t_3 \leq t_2 \leq t_1 \leq t_0 = t$ be such that for $i = 1, 2, 3$, $\xi_i = \xi|_{[t_i, t_{i-1}]}$ be the linear parts of ξ. If ξ consists of three linear curves then $\xi_2 = 0$ (see Figs. 3.1c and 3.2c).

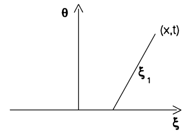

Fig.3.1a. $\xi = (\xi_1, \phi, \phi)$

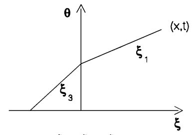

Fig.3.1b. $\xi = (\xi_1, \phi, \xi_3)$

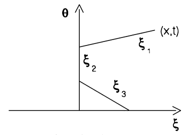

Fig.3.1c. $\xi = (\xi_1, \xi_2, \xi_3)$

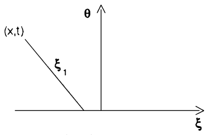

Fig.3.2a. $\xi = (\xi_1, \phi, \phi)$

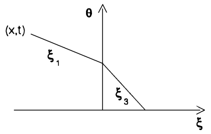

Fig.3.2b. $\xi = (\xi_1, \phi, \xi_3)$

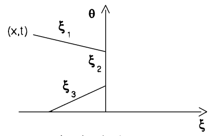

Fig.3.2c. $\xi = (\xi_1, \xi_2, \xi_3)$

Represent an admissible curve $\xi = \{\xi_1, \xi_2, \xi_3\}$. Let

$$\begin{aligned} c(x,t,s) &= \{\xi \in c([s,t], \mathbb{R}); \xi(t) = x, \xi \text{ is an admissible curve}\}, \\ c(x,t) &= c(x,t,0). \end{aligned} \tag{3.1}$$

Divide $c(x, s, t)$ into three categories defined as below.

(3.2)

$$c_0(x,t,s) = \{\xi \in c(x,t,s); \xi \text{ is linear and } x\xi(\theta) \geq 0 \,\forall\, \theta \in [s,t]\}.$$
(see Figs. 3.1a and 3.2a)

$$c_r(x,t,s) = \{\xi \in c(x,t,s); \xi \text{ consists of three pieces and } x\xi(\theta) \geq 0 \,\forall\, \theta \in [s,t]\}.$$
(see Figs. 3.1c and 3.2c)

$$c_b(x,t,s) = c(x,t,s) - \{c_r(x,t,s) \cup c_0(x,t,s)\}.$$
(see Figs. 3.1b and 3.2b)

$$c_l(x,t) = c_l(x,t,0) \text{ for } l = 0, r, b.$$

Let f and g satisfies the hypothesis (H). Let f^*, g^* denotes the convex duals of f and g respectively. Let w be a function on $\mathbb{R}$ and $\xi \in c(x,t,s)$. Define

$$\begin{aligned} \rho_{\xi,w}(x,t,s) = w(\xi(s)) &+ \int_{\{\theta \in [s,t]; \xi(\theta) > 0\}} f^*\left(\frac{d\xi}{d\theta}\right) d\theta \\ &+ \int_{\{\theta \in [s,t]; \xi(\theta) < 0\}} g^*\left(\frac{d\xi}{d\theta}\right) d\theta \\ &+ meas\{\theta \in [s,t]; \xi(\theta) = 0\} \min\{f^*(0), g^*(0)\}. \end{aligned} \tag{3.3}$$

Theorem 3.1. *Let v_0 be an uniformly Lipschitz continuous function on $\mathbb{R}$ and $\rho_\xi(x,t) = \rho_{\xi,v_0}(x,t,0)$. Define*

$$v(x,t) = \inf\{\rho_\xi(x,t), \quad \xi \in c(x,t)\}. \tag{3.4}$$

Then v is an uniformly Lipschitz continuous function satisfying the following:

(i)

$$\begin{aligned} v_t + f(v_x) &= 0 \qquad \text{in} \quad x>0,\ t>0,\\ v_t + g(v_x) &= 0 \qquad \text{in} \quad x<0,\ t>0. \end{aligned} \tag{3.5}$$

(ii) *For almost every t, $v_x(0+,t) = \lim_{x\to 0+} v_x(x,t)$ and $v_x(0-,t) = \lim_{x\to 0-} v_x(x,t)$ exist and satisfy $f(v_x(0+,t)) = g(v_x(0-,t))$. Furthermore there exist disjoint sets V, S_1, S_2 such that $(0,\infty) = V\cup S_1\cup S_2$, V an open set, $meas(S_2) = 0$ with the property that for almost every $t \in V$, one of the following pair of inequalities holds*:

$$f'(v_x(0+,t)) \geq 0, \quad g'(v_x(0-,t)) \geq 0, \tag{3.6}$$

$$f'(v_x(0+,t)) \leq 0, \quad g'(v_x(0-,t)) \leq 0. \tag{3.7}$$

If $t \in S_1$, then

$$f'(v_x(0+,t)) \leq 0, \quad g'(v_x(0-,t)) \geq 0\,. \tag{3.8}$$

(iii) *There exist a constant $M>0$ and Lipschitz continuous functions $R_1(t)\geq 0$, $L_1(t) \geq 0$ on $[0,\infty)$ with $R_1(0) = L_1(0) = 0$ such that for all $z>0$*

$$\begin{aligned} &f'(v_x(x+z,t)) - f'(v_x(x,t))\\ &\quad\leq \begin{cases} \dfrac{Mz}{z+x} & \text{if} \quad 0<x<x+z<R_1(t),\\ \dfrac{z}{t} & \text{if} \quad x>0 \text{ and not in the above range.} \end{cases} \end{aligned} \tag{3.9}$$

$$\begin{aligned} &g'(v_x(x,t)) - g'(v_x(x-z,t))\\ &\quad\leq \begin{cases} \dfrac{mz}{z-x} & \text{if} \quad L_1(t)<x-z \text{ and } x<0,\\ \dfrac{z}{t} & \text{if} \quad x<0 \text{ and not in the above range.} \end{cases} \end{aligned} \tag{3.10}$$

Theorem 3.2. *Let $u_0 \in L^\infty(\mathbb{R})$ and $v_0(x) = \int_0^x u_0(\theta)d\theta$. Let v be as in* (3.4), *then $u = \partial v/\partial x$ is a weak solution of* (1.3) *and* (1.4). *Also u satisfies the entropy condition* (E_i) *and* (E_b) *and the solution can be given explicitly as follows*:

There exist Lipschitz continuous functions $R_1(t)\geq 0$ and $L_1(t)\leq 0$ on $(0,\infty)$ and bounded variation functions $y_+(x,t)$ for $x\geq 0$ (non increasing in $(0,R_1(t))$ and non decreasing in $[R_1(t),\infty)$) and $y_-(x,t)$ for $x\leq 0$ (non decreasing in $(-\infty,0)$) such that

(i) *For* $x > 0$,

$$
u(x,t) = \begin{cases} (f')^{-1}\left(\dfrac{x}{t - y_+(x,t)}\right) & \textit{if} \quad x \le R_1(t), \\ (f')^{-1}\left(\dfrac{x - y_+(x,t)}{t}\right) & \textit{if} \quad x > R_1(t). \end{cases} \tag{3.11}
$$

(ii) *For* $x < 0$,

$$
u(x,t) = \begin{cases} (g')^{-1}\left(\dfrac{x}{t - y_-(x,t)}\right) & \textit{if} \quad x \le L_1(t), \\ (g')^{-1}\left(\dfrac{x - y_-(x,t)}{t}\right) & \textit{if} \quad x > L_1(t). \end{cases} \tag{3.12}
$$

Furthermore u *is unique in the class of all solutions for which the set of discontinuities is a discrete set of Lipschitz curves.*

Remark 3.3. Like in the single flux case, under suitable assumptions on f and g, following decay estimates holds. Let $u, v, R_1(t), L_1(t)$ be as in Theorem 3.2. Then

(1) Assume that $|v_0|_\infty < \infty$, $f^*(0) \ge g^*(0)$, $g(0) = 0$, $g'(0) \ge 0$ and there exist $c > 0$ such that $1/c \le g^{*''}(\xi) \le c$. Let $U_+ = \{t; R_1(t) > 0\}$. Then there exist a constant $M > 0$ such that for almost all $t \in U_+$

$$
|f(u(0+,t))| \le M/t^{1/2}\,.
$$

(2) Assume that $|v_0|_\infty < \infty$, $f^*(0) \le g^*(0)$, $f(0) = 0$, $f'(0) \le 0$ and there exist a $c > 0$ such that $1/c \le f^{*''}(\xi) \le c$. Let $U_- = \{t; L_1(t) < 0\}$. Then there exisst a constant $M > 0$ such that for almost all $t \in U$.

$$
|g(u(0-,t))| \le M/t^{1/2}\,.
$$

4. Proof of Theorems of 3.1 and 3.2

Let v, v_0 and $c(x,t)$ be defined as in Section 3.

Definition 4.1. For $x \in \mathbb{R}$, $t > 0$, define the set of characteristic curves $ch(x,t)$ by

$$
ch(x,t) = \{\xi \in c(x,t); \quad \rho_\xi(x,t) = v(x,t)\}. \tag{4.1}
$$

Then we have the following

Lemma 4.2. *For* $(x,t) \in \mathbb{R} \times \mathbb{R}_+$, $v \in L^\infty_{loc}(\overline{\mathbb{R} \times \mathbb{R}_+})$ *and* $ch(x,t) \ne \phi$. *Furthermore there exist a constant* $M > 0$ *depending only on Lipschitz constant of* v_0, f *and* g *such that* $|d\xi/ds|_\infty \le M$ *for all* $\xi \in ch(x,t)$.

Proof.

Step 1. $v \in L^\infty_{loc}(\mathbb{R} \times \overline{\mathbb{R}}_+)$.

Let $L = \text{Lip}\ (v_0)$. It is enough to prove for $x \geq 0$. Let $\xi = (\xi_1, \xi_2, \xi_3) \in c(x,t)$ defined on the partition $0 = t_3 \leq t_2 \leq t_1 \leq t_0 = t$. Let $\dot{\xi}_i = d\xi_i/ds$ and $(\dot{\xi}_1, \dot{\xi}_2, \dot{\xi}_3) = (p_1, p_2, p_3)$. We have to consider three cases. Suppose $\xi \in c_0(x,t)$, then $\xi(\theta) = x + p_1(\theta - t)$ and hence $\rho_\xi(x,t) = v_0(\xi(0)) + tf^*(p_1) \geq v_0(0) - L(x - p_1 t) + tf^*(p_1) = v_0(0) - Lx + t(Lp_1 + f^*(p_1)) \geq v_0(0) - Lx - tL_1$ where $L_1 = |\inf_p(Lp + f^*(p))| < \infty$. Suppose $\xi \in c_b(x,t)$, then $t_1 = t_2 = t - x/p_1$ and $0 \leq t - x/p_1 \leq t$ and hence

$$\begin{aligned}
\rho_\xi(x,t) &= v_0\left(-p_3\left(t - \frac{x}{p_1}\right)\right) + \frac{x}{p_1} f^*(p_1) + \left(t - \frac{x}{p_1}\right) g^*(p_3) \\
&\geq v_0(0) + \left(t - \frac{x}{p_1}\right)\{-Lp_3 + g^*(p_3)\} + \frac{x}{p_1} f^*(p_1) \\
&\geq v_0(0) + \left(t - \frac{x}{p_1}\right)\inf_p\{-Lp + g^*(p)\} + \frac{x}{p_1}\inf_p f^*(p) \\
&\geq v_0(0) - L_2 t,
\end{aligned}$$

where $L_2 = |\inf_p\{-Lp + g^*(p)\}| + |\inf_p f^*(p)| + |\inf_p g^*(p)|$. Suppose $\xi \in c_r(x,t)$, then $\xi(0) = -t_2 p_3$ and with L_2 as above

$$\begin{aligned}
\rho_\xi(x,t) &= v_0(-t_2 p_3) + (t - t_1) f^*(p_1) + (t_1 - t_2)\min(f^*(0), g^*(0)) + t_2 f^*(p_3) \\
&\geq v_0(0) - L_2\left[t_2 + (t - t_1) + (t_1 - t_2)\right] \\
&= v_0(0) - L_2 t\,.
\end{aligned}$$

Since $\xi_0(\theta) = x$ for $\theta \in [0,t]$ is in $c(x,t)$ and hence by the above estimates, we obtain a constant $\tilde{L} > 0$ such that for all $(x,t) \in \mathbb{R}_+ \times \mathbb{R}_+$,

$$v_0(0) - \tilde{L}(x + t) \leq v(x,t) \leq \rho_{\xi_0}(x,t) = v_0(x) + tf^*(0)\,.$$

This proves that $v \in L^\infty_{loc}(\overline{\mathbb{R}_+ \times \mathbb{R}_+})$.

Step 2. $ch(x,t) \neq \phi$.

Let $\xi_n \in c(x,t)$ be a minimizing sequence for $v(x,t)$. Let $\xi_n = (\xi_{1n}, \xi_{2n}, \xi_{3n})$ defined on $0 = t_3 \leq t_{2n} \leq t_{1n} \leq t_0 = t$. Let $\dot{\xi}_{in} = p_{in}, i = 1,2,3$. Let $\varepsilon > 0$ and choose $N(\varepsilon) > 0$ such that for all $n \geq N(\varepsilon)$, $\rho_{\xi_n}(x,t) - \varepsilon \leq v(x,t)$. We claim that for some subsequence, $\{|\dot{\xi}_n|_\infty\}$ is bounded or a minimizer is achieved. To prove this we have to consider several cases. Suppose for some subsequence, $\xi_n \in c_0(x,t)$, then ξ_n is a straight line and hence $v_0(x - p_{1n}t) + tf^*(p_{1n}) - \varepsilon \leq v(x,t) \leq v_0(x) + tf^*(0)$. Therefore $t(f^*(p_{1n}) - f^*(0)) - \varepsilon \leq v_0(x) - v_0(x - p_{1n}t) \leq Lp_{1n}t$. This implies that either $\{p_{1n}\}$ is bounded or $\{(f^*(p_{1n}) - f^*(0))/p_{1n}\}$ is bounded. Hence by superlinearity $\{p_{1n}\}$ is bounded and therefore $\{|\dot{\xi}_n|_\infty\}$ is bounded.

Suppose for some subsequence, $\xi_n \in c_b(x,t)$, then $t_{2n} = t - x/p_{1n}$. Let $\alpha_n = (\xi_{1n}, 0, 0)$ defined on the same partition as of ξ_n, then

$$v_0\left(p_{3n}\left(\frac{x}{p_{1n}} - t\right)\right) + \frac{x}{p_{1n}} f^*(p_{1n}) + \left(t - \frac{x}{p_{1n}}\right) g^*(p_{3n}) - \varepsilon \leq v(x,t) \tag{4.2}$$

$$\leq v_0(0) + \frac{x}{p_{1n}} f^*(p_{1n}) + \left(t - \frac{x}{p_{1n}}\right) \min(f^*(0), g^*(0)) \tag{4.3}$$

and hence

$$p_{3n}\left(t - \frac{x}{p_{1n}}\right)\left(\frac{g^*(p_{3n}) - \min(f^*(0), g^*(0))}{p_{3n}} - L\right) \leq \varepsilon\,. \tag{4.4}$$

Let $\lim_{n\to\infty}(t - x/p_{1n}) > 0$. Either $p_{3n} \leq 1$ or $p_{3n} \geq 1$. Suppose $p_{3n} \geq 1$, then letting $n \to \infty$, $\varepsilon \to 0$ in the above equation gives $\{p_{3n}\}$ is bounded. Since $t - x/p_{1n} \leq t$, hence $\{p_{3n}(t - x/p_{1n})\}$ is bounded and

$$\frac{x}{p_{1n}} f^*(p_{1n}) \leq v(x,t) + \varepsilon - v_0\left(p_{3n}\left(\frac{x}{p_{1n}} - t\right)\right) - \left(t - \frac{x}{p_{1n}}\right) g^*(p_{3n})\,.$$

This implies that $\{p_{1n}\}$ is bounded. Hence $\{|\dot{\xi}|_\infty\}$ is bounded.

Suppose $\lim_{n\to\infty}(t - x/p_{1n}) = 0$, then $\lim_{n\to\infty} p_{1n} = x/t$. If $\{p_{3n}\}$ is bounded, then $\{|\dot{\xi}_n|_\infty\}$ is bounded. If $p_{3n} \to \infty$, then letting $n \to \infty$, $\varepsilon \to 0$ in (4.4) to obtain $\lim_{t\to\infty}(t - x/p_{1n}) g^*(p_{3n}) = 0$. This implies that

$$v(x,t) = \lim_{n\to\infty} \rho_{\xi_n}(x,t) = v_0(0) + t f^*\left(\frac{x}{t}\right)\,,$$

and hence $\xi(\theta) = x + (x/t)(\theta - t) \in ch(x,t)$.

Suppose for some subsequence $\xi_n \in c_r(x,t)$, then

$$\begin{aligned} v_0(-p_{3n}t_{2n}) + (t - t_{1n}) f^*(p_{1n}) + t_{2n} f^*(p_{3n}) + (t_{1n} - t_{2n}) \min(f^*(0), g^*(0)) \\ \leq v(x,t) + \varepsilon \\ \leq v_0(0) + (t - t_{1n}) f^*(p_{1n}) \\ + t_{1n} \min(f^*(0), g^*(0)) + \varepsilon\,, \end{aligned}$$

this implies that

$$|p_{3n}|\, t_{2n} \left\{\frac{f^*(p_{3n})}{|p_{3n}|} - L - \min(f^*(0), g^*(0))\right\} \leq \varepsilon\,. \tag{4.5}$$

Let $\lim_{n\to\infty} t_{2n} > 0$. Then either $|p_{3n}| \leq 1$ or $|p_{3n}| \geq 1$. Suppose $|p_{3n}| \geq 1$, then letting $n \to \infty$, $\varepsilon \to 0$ in (4.5) to obtain that $\{p_{3n}\}$ is bounded. Since $x/p_{1n} = t - t_{1n}$, hence

$$\begin{aligned} \frac{x f^*(p_{1n})}{p_{1n}} &\leq v(x,t) + \varepsilon - v_0(-p_{3n}t_{2n}) - t_{2n} f^*(p_{3n}) \\ &\quad - (t_{1n} - t_{2n}) \min(f^*(0), g^*(0))\,, \end{aligned} \tag{4.6}$$

and this implies that $\{p_{1n}\}$ is bounded and therefore $\{|\dot{\xi}_n|_\infty\}$ is bounded.

Let $\lim_{n\to\infty} t_{2n} = 0$, then either $\{p_{3n}\}$ is bounded or $p_{3n} \to \infty$. If $\{p_{3n}\}$ is bounded, then it follows from (4.6) that $\{p_{1n}\}$ is bounded. Hence $\{|\dot{\xi}_n|_\infty\}$ is bounded. If $p_{3n} \to \infty$, then from (4.5) it follows that $t_{2n} f^*(p_{3n}) \to 0$ and hence $p_{3n}t_{2n} \to 0$. Hence from (4.6) $\{p_{1n}\}$ is bounded. Let for a subsequence $p_{1n} \to p$, $t_{1n} \to t_1$, then

$$\begin{aligned} v(x,t) &= \lim_{n\to\infty} \rho_{\xi_n}(x,t) \\ &= v_0(0) + (t-t_1)f^*(p) + t_1 \min(f^*(0), g^*(0)) . \end{aligned}$$

Hence if $\xi_1(\theta) = x + p(\theta - t)$ for $\theta \in [t_1, t]$ and $\xi_2(\theta) = 0$ for $\theta \in [0, t_1]$, then $\xi = (\xi_1, \xi_2, \phi) \in ch(x,t)$. This proves the claim.

Now from the above claim we can choose a subsequence still denoted by $\{\xi_n\}$ converging to ξ uniformly and $\dot{\xi}_{in} \to \dot{\xi}_i$ to obtain $\xi \in ch(x,t)$. This proves $ch(x,t) \neq \phi$.

Step 3. $|d\xi/ds|_\infty \leq M$, $\forall\, \xi \in ch(x,t)$.

Let $\xi = (\xi_1, \xi_2, \xi_3) \in ch(x,t)$ defined on the partition $0 = t_3 \leq t_2 \leq t_1 \leq t_0 = t$. Let $\dot{\xi}_i = p_i$ $i = 1,2,3$. Let $\xi \in c_0(x,t)$, then $\xi(\theta) = x + p_1(\theta - t)$ and hence $v_0(x - p_1 t) + t f^*(p_1) = v(x,t) \leq v_0(x) + t f^*(0)$. This implies that $f^*(p_1) \leq L p_1$ and hence $\{|\dot{\xi}|_\infty\}$ is bounded by a constant independent of (x,t).

Let $t_1 > 0$. Suppose $\xi \in c_b(x,t)$, then $t_2 = t_1 = t - x/p_1$. Let $\alpha = (\xi_1, 0, \phi)$, then $v_0(p_3(x/p_1 - t)) + (x/p_1) f^*(p_1) + (t - x/p_1) g^*(p_3) = v(x,t) \leq v_0(0) + (x/p_1) f^*(p_1) + (t - x/p_1)\min(f^*(0), g^*(0))$. This implies that $(1/p_3)(g^*(p_3) - \min(f^*(0), g^*(0))) \leq L$ and hence $\{p_3\}$ is bounded independently of (x,t). If $p_1 = 0$, then there is nothing to be proved. Hence assume that $p_1 > 0$. Since ξ is a minimizer and hence $(\partial/\partial p_1)\rho_\xi(x,t) = 0$. This implies that

$$p_3 v_0'\left(p_3\left(\frac{x}{p_1} - t\right)\right) - f^*(p_1) + p_1 f^{*\prime}(p_1) - g^*(p_3) = 0 .$$

Since $f(f^{*\prime}(p_1)) = -f^*(p_1) + p_1 f^{*\prime}(p_1)$ to obtain $f(f^{*\prime}(p_1)) = g^*(p_3) - p_3 v_0'(p_3(x/p_1 - t))$. Since p_3 is bounded independently of (x,t), $|v_0'|_\infty < \infty$ and f is of superlinear growth implies that p_1 is bounded independently of (x,t). This gives that $\{|\dot{\xi}|_\infty\}$ is bounded independently of (x,t).

Suppose $\xi \in c_r(x,t)$. Then

$$v(x,t) = v_0(-t_2 p_3) + \frac{x}{p_1} f^*(p_1) + \left(t - \frac{x}{p_1} - t_2\right)\min(f^*(0), g^*(0)) + t_2 f^*(p_3) .$$

Let $p_1 > 0$, then $(\partial/\partial p_1)\rho_\xi(x,t) = 0$ implies that $-f(f^{*'}(p_1)) = \min(f^*(0), g^*(0))$ and hence $\{p_1\}$ is bounded independently of (x,t). Let $p_3 > 0$, then $(\partial/\partial p_3)\rho_\xi(x,t) = 0$ and hence $f^{*\prime}(p_3) = v_0'(-t_2 p_3)$ and hence p_3 is bounded independently of (x,t). This proves the Lemma. □

Lemma 4.3. *$x \mapsto v(x,t)$ is a Lipschitz continuous function with Lipschitz constant independent of t.*

Proof. In order to prove this, we have to consider several cases. In the sequel we denote L_1 to be a generic constant depending only on Lipschitz constant of v_0, f^* and g^*.

Step 1. Let $0 \le x_1 < x_2$, then $v(x_2,t) - v(x_1,t) \le L_1|x_2 - x_1|$.

From Lemma 4.2, there exist a $\xi = (\xi_1, \xi_2, \xi_3)$ in $ch(x_1,t)$ defined on the partition $0 = t_3 \le t_2 \le t_1 \le t_0 = t$. Let $(p_1,p_2,p_3) = (\dot{\xi}_1, \dot{\xi}_2, \dot{\xi}_3)$ and $p_0 = (f^*(0) - g^*(0))^+$.

Suppose $\xi \in c_0(x_1,t)$. If $x_1 = 0$ and $p_1 \le 0$ or $x_1 > 0$, define $\alpha(\theta) = x_2 + p_1(\theta - t), \theta \in [0,t]$, then $\alpha \in c_0(x_2,t)$ and

$$\begin{aligned} v(x_2,t) - v(x_1,t) &\le v_0(x_2 - p_1 t) + tf^*(p_1) - v_0(x_2 - p_1 t) - tf^*(p_1) \\ &\le L(x_2 - x_1)\,. \end{aligned}$$

Suppose $\xi \in c_b(x_1,t)$. Let $\tilde{p} = p_3 + (f^*(0) - g^*(0))^+ = p_3 + p_0$. If $(x_2 - x_1) - \tilde{p}(t - x_1/p_1) \ge 0$, define $\lambda = x_1 + (\tilde{p}(t - x_1/p_1))/t$ be the slope of the line joining (x_2,t) and $(x_2 - x_1 - \tilde{p}(t - x_1/p_1), 0)$ and $\alpha(\theta) = x_2 + \lambda(\theta - t)$, $\theta \in [0,t]$, then α is in $c_0(x_2,t)$ and by convexity of f^*

$$\begin{aligned} v(x_2,t) - v(x_1,t) &\le v_0(\alpha(0)) - v_0(\xi(0)) + tf^*(\lambda) - \frac{x_1}{p_1} f^*(p_1) \\ &\quad - \left(t - \frac{x_1}{p_1}\right) g^*(p_3) \\ &\le L\left(x_2 - x_1 - (\tilde{p} - p_3)\left(t - \frac{x_1}{p_1}\right)\right) \\ &\quad + \frac{x_1}{p_1} f^*(p_1) + \left(t - \frac{x_1}{p_1}\right) f^*(\tilde{p}) - \frac{x_1}{p_1} f^*(p_1) \\ &\quad - \left(t - \frac{x_1}{p_1}\right) g^*(p_3) \\ &\le \left(L + \frac{|\tilde{p} - p_3|}{\tilde{p}}\right)(x_2 - x_1) - \left(t - \frac{x_1}{p_1}\right)\tilde{p}\left(\frac{f^*(\tilde{p}) - g^*(p_3)}{\tilde{p}}\right) \\ &\le \tilde{L}(x_2 - x_1), \end{aligned}$$

since $\tilde{p}(t - x_1/p_1) \le x_2 - x_1, |(f^*(\tilde{p}) - g^*(p_3))/\tilde{p}| \le |(f^*(\tilde{p}) - g^*(p_3))/p_0|$ if $f^*(0) > g^*(0)$ and $(f^*(\tilde{p}) - g^*(p_3))/\tilde{p} = (f^*(p_3) - f^*(0))/p_3 + (g^*(0) - g^*(p_3))/p_3 + (f^*(0) - g^*(0))/p_3$ if $f^*(0) \le g^*(0)$.

If $x_2 - x_1 - \tilde{p}(t - x_1/p_1) < 0$, define $\lambda = x_2 p_1 \tilde{p}/(\tilde{p}x_1 + p_1(x_2 - x_1))$ the slope of the line joining between (x_2,t) and $(0, t - x_1/p_1 - (x_2 - x_1)/\tilde{p}), \alpha_1(\theta) = x_2 + \lambda(\theta - t), \theta \in [t - x_1/p_1 - (x_2 - x_1)/\tilde{p}, t]$ and $\alpha_3(\theta) = p_3(\theta - t + x_1/p_1 + (x_2 - x_1)/\tilde{p}), \theta \in [0, t - (x_2 - x_1)/\tilde{p} - x_1/p_1]$, then $\alpha = (\alpha_1, \phi, \alpha_3) \in c_b(x_2,t)$ (see Fig. 4.1) and by convexity of f^* we have

$$\begin{aligned} v(x_2,t) - v(x_1,t) &\le v_0(\alpha(0)) - v_0(\xi(0)) + \frac{x_2}{\lambda} f^*(\lambda) + \left(t - \frac{x_2}{\lambda}\right) g^*(p_3) \\ &\quad - \frac{x_1}{p_1} f^*(p_1) - \left(t - \frac{x_1}{p_1}\right) g^*(p_3) \end{aligned}$$

$$\begin{aligned}
&\le L\frac{p_3}{\tilde p}(x_2-x_1)+\frac{x_1}{p_1}f^*(p_1)+\frac{x_2-x_1}{\tilde p}f^*(\tilde p)\\
&\quad+\left(t-\frac{x_2}{\lambda}\right)g^*(p_3)-\frac{x_1}{p_1}f^*(p_1)-\left(t-\frac{x_1}{p_1}\right)g^*(p_3)\\
&\le \left(L+\frac{f^*(\tilde p)-g^*(p_3)}{\tilde p}\right)(x_2-x_1)\\
&\le \tilde L(x_2-x_1)\,,
\end{aligned}$$

since $t-x_2/\lambda=(t-x_1/p_1)-(x_2-x_1)/\tilde p$, $(f^*(\tilde p)-g^*(p_3))/\tilde p\le |(f^*(p_3)-f^*(0))/p_3|+|(g^*(p_3)-g^*(0))/p_3|$ if $f^*(0)\le g^*(0)$ and if $f^*(0)>g^*(0)$, then $|(f^*(\tilde p)-g^*(p_3))/\tilde p|\le \tilde L$.

Let $\xi\in c_r(x_1,t)$. If $f^*(0)\le g^*(0)$, then by strict convexity of f^*, ξ cannot be a minimizer. Hence assume that $f^*(0)>g^*(0)$ and $p_0=f^*(0)-g^*(0)$ and $\tilde p=p_1+p_0$. If $x_2-x_1+\tilde p(t_2-t+x_1/p_1)\ge 0$, define $\lambda=(x_1/p_1t)p_1+(t_2/t)p_3+((t-t_2-x_1/p_1)/t)\tilde p$ and $\alpha(\theta)=x_2+\lambda(\theta-t)$ for $\theta\in[0,t]$, then $\alpha\in c_0(x_2,t)$ and by convexity of f^*,

$$\begin{aligned}
v(x_2,t)-v(x_1,t)&\le v_0(\alpha(0))-v_0(\xi(0))+tf^*(\lambda)-\frac{x_1}{p_1}f^*(0)-(t_1-t_2)g^*(0)\\
&\quad-t_2f^*(p_3)\\
&\le L\left(x_2-x_1+\tilde p\left(t_2-t+\frac{x_1}{p_1}\right)\right)+\frac{x_1}{p_1}f^*(p_1)\\
&\quad+t_2f^*(p_3)+(t-t_2-\frac{x_1}{p_1})f^*(\tilde p)\\
&\quad-\frac{x_1}{p_1}f^*(p_1)-(t_1-t_2)g^*(0)-t_2f^*(p_3)\\
&\le 2L(x_2-x_1)+\left(t-\frac{x_1}{p_1}-t_2\right)\tilde p\frac{(f^*(\tilde p)-g^*(0))}{\tilde p}\\
&\le \left(2L+\left|\frac{f^*(\tilde p)-g^*(0)}{\tilde p}\right|\right)\,(x_2-x_1),
\end{aligned}$$

since $\tilde p(t-x_1/p_1-t_2)\le x_2-x_1$.

Suppose $\tilde p(t-x_1/p_1-t_2)>x_2-x_1$, define $\lambda=x_2p_1\tilde p/(\tilde p x_1+p_1(x_2-x_1))$, $\alpha_1(\theta)=x_2+\lambda(\theta-t)$, $\theta\in[t-x_1/p_1-(x_2-x_1)/\tilde p,t]$ $\alpha_2(\theta)=0$ for $\theta\in[t_2,t-x_1/p_1-(x_2-x_1)/\tilde p]$, $\alpha_3=\xi_3$ in $[0,t_2]$. Then $\alpha=(\alpha_1,\alpha_2,\alpha_3)\in c_r(x_2,t)$ (see Fig. 4.2) and

$$\begin{aligned}
v(x_2,t)-v(x_1,t)&\le \frac{x_2}{\lambda}f^*(\lambda)+\left(t-\frac{x_1}{p}-\frac{x_2-x_1}{\tilde p}-t_2\right)g^*(0)\\
&\quad-\frac{x_1}{p_1}f^*(p_1)-\left(t-\frac{x_1}{p_1}-t_2\right)g^*(0)\\
&\le \frac{x_1}{p_1}f^*(p_1)+\frac{x_2-x_1}{\tilde p}f^*(\tilde p)-\frac{x_1}{p_1}f^*(p_1)\\
&\quad-\frac{x_2-x_1}{\tilde p}g^*(0)
\end{aligned}$$

$$\leq \left|\frac{f^*(\tilde{p}) - g^*(0)}{\tilde{p}}\right| (x_2 - x_1)$$
$$\leq \tilde{L}(x_2 - x_1)\,.$$

Now Step 1 follows from all the above estimates.

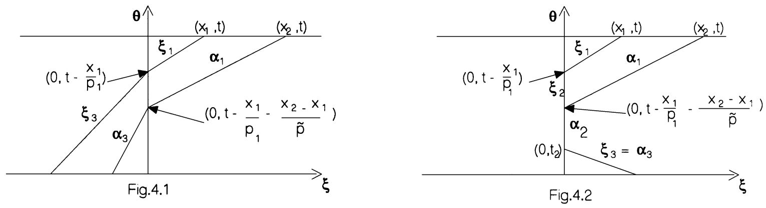

Fig.4.1 Fig.4.2

Step 2. $v(x_2,t) - v(x_1,t) \geq -\tilde{L}(x_2 - x_1)\,.$

Let $\xi \in ch(x_2,t)$ with $(p_1,p_2,p_3) = (\dot{\xi}_1,\dot{\xi}_2,\dot{\xi}_3)$ defined on $0 \leq t_3 \leq t_2 \leq t_1 \leq t_0 = t$. Suppose $\xi \in c_0(x_2,t)$. If $x_1 - p_1 t \geq 0$, define $\alpha(\theta) = x_1 + p_1(\theta - t), \theta \in [0,t]$, then $\alpha \in c_0(x_1,t)$ and

$$v(x_2,t) - v(x_1,t) \geq v_0(x_2 - p_1 t) - v_0(x_1 - p_1 t) \geq -L(x_2 - x_1)\,.$$

If $x_1 - p_1 t < 0$, define $\alpha(\theta) = x_1 + (x_1/t)(\theta - t), \theta \in [0,t]$ then $\alpha \in c_0(x_1,t)$ and by convexity of f^*.

$$\begin{aligned} v(x_2,t) - v(x_1,t) &\geq v_0(x_2 - p_1 t) - v_0(0) + t f^*(p_1) - \frac{x_1}{t} f^*\left(\frac{x_1}{t}\right) \\ &\geq -L(x_2 - p_1 t) + (t p_1 - x_1)\left(\frac{f^*(p_1) - f^*(0)}{p_1}\right) \\ &\geq -\left[L + \left|\frac{f^*(p_1) - f^*(0)}{p_1}\right|\right](x_2 - x_1)\,, \end{aligned}$$

since $x_1 - p_1 t \leq 0$ implies that $x_2 - p_1 t \leq x_2 - x_1$, $x_1 - p_1 t \geq x_1 - x_2$. Let $\xi \in c_r(x_2,t) \cup c_b(x_2,t)$. Define $\lambda = (x_2/x_1)p_1, \alpha_1(\theta) = x_1 + \lambda(\theta - t)$ for $\theta \in [t_1,t]$, $\alpha = (\alpha_1,\xi_2,\xi_3)$, then $\alpha \in c_r(x_1,t) \cup c_b(x_2,t)$ and

$$\begin{aligned} v(x_2,t) - v(x_1,t) &\geq \frac{x_2}{p_1} f^*(p_1) - \frac{x_1}{\lambda} f^*(\lambda) \\ &= \frac{x_2}{p_1} f^*(p_1) - \frac{x_1}{p_1} f^*(p_1) - \frac{(x_2 - x_1)}{p_1} f^*(0) \\ &= (x_2 - x_1)\frac{(f^*(p_1) - f^*(0))}{p_1} \\ &\geq -\tilde{L}(x_2 - x_1). \end{aligned}$$

This proves Step 2.

By Step 1 and 2 it follows that for $0 \leq x_1 < x_2, |v(x_2,t) - v(x_1,t)| \leq \tilde{L}|x_2 - x_1|$. Similarly if $x_2 < x_1 \leq 0, |v(x_2,t) - v(x_1,t)| \leq \tilde{L}|x_2 - x_1|$. Let

$x_1 < 0 < x_2$, then $|v(x_1,t) - v(x_2,t)| \leq |v(x_1,t) - v(0,t)| + |v(x_2,t) - v(0,t)) \leq \tilde{L}(|x_1| + |x_2|) = \tilde{L}|x_2 - x_1|$. This proves the Lemma. $\square$

Lemma 4.4 (Dynamic Programming Principle). *Let* $0 \leq s < t$, $w(x) = v(x,s)$ *and define*

$$W(x,t,s) = \inf\left\{\rho_{\xi,w}(x,t,s), \quad \xi \in c(x,t,s)\right\}, \tag{4.7}$$

then $v(x,t) = W(x,t,s)$.

Proof. From Lemma 4.3, $v(x,s)$ is uniformly continuous function in x and hence by imitating Lemma 4.2, minimizers for $W(x,t,s)$ exist. Denote $ch(x,t,s)$ the set of minimizing curves. In order to prove the Lemma, we have to consider several cases.

Let $x \geq 0$ and $\xi = (\xi_1, \xi_2, \xi_3) \in ch(x,t,s)$ with $(p_1,p_2,p_3) = (\dot{\xi}_1, \dot{\xi}_2, \dot{\xi}_3)$ defined on the partition $s = t_3 \leq t_2 \leq t_1 \leq t_0 = t$. Let $y = \xi(s)$ and $\alpha = (\alpha_1, \alpha_2, \alpha_3) \in ch(y,s)$ with $(q_1,q_2,q_3) = (\dot{\alpha}_1, \dot{\alpha}_2, \dot{\alpha}_3)$ defined on the partition $0 = s_3 \leq s_2 \leq s_1 \leq s_0 = s$.

Case 1. Let $\xi \in c_0(x,t,s)$.

Let $\alpha \in c_0(y,s)$. Define $\lambda = (1 - s/t)p_1 + (s/t)q_1$ and $\beta(\theta) = x + \lambda(\theta - t)$, $\theta \in [0,t]$. Then $\beta \in c_0(x,t)$ and

$$\begin{aligned}
v(x,t) &\leq v_0(x - \lambda t) + t f^*(\lambda) \\
&\leq v_0(x - q_1 t) + s f^*(q_1) + (t-s) f^*(p_1) \\
&= v(y,s) + (t-s) f^*(p_1) \\
&= W(x,t,s).
\end{aligned}$$

Let $\alpha \in c_b(y,s) \cup c_r(y,s)$ and $\lambda = x/((t-s)p_1 + (sq_1 - x))$. Define $\gamma_1(\theta) = x - \lambda(\theta - t)$, $\theta \in [s_1, t]$, $\gamma_2 = \alpha_2, \gamma_3 = \alpha_3$, then $\gamma = (\gamma_1, \gamma_2, \gamma_3) \in c_b(x,t) \cup c_r(x,t)$ and

$$\begin{aligned}
v(x,t) &\leq v_0(\gamma(0)) + \frac{x}{\lambda} f^*(\lambda) + (s_1 - s_2)\min(f^*(0), g^*(0)) \\
&\quad + s_2 g^*(q_1) \\
&\leq v_0(\alpha(0)) + \frac{y}{q_1} f^*(q_1) + (s - s_2)\min(f^*(0), g^*(0)) \\
&\quad + s_2 g^*(q_1) + (t-s) f^*(p_1) \\
&= v(y,s) + (t-s) f^*(p_1) \\
&= W(x,t,s).
\end{aligned}$$

Case 2. Let $\xi \in c_b(x,t,s)$.

Let $\alpha \in c_0(y,s)$. Define $\lambda = (q_1 s - y)/(t - x/p_1) = s/(t - x/p_1) + ((t - s - x/p_1)/(t - s/p_1))p_3$, $\gamma_1 = \xi_1, \gamma_2 = \phi, \gamma_3(\theta) = \lambda(\theta - t + x_1/p_1)$, then $\gamma =$

$(\gamma_1, \gamma_2, \gamma_3) \in c_b(x,t)$ and

$$\begin{aligned} v(x,t) &\leq v_0(\gamma(0)) + \frac{x}{p_1} f^*(p_1) + \left(t - \frac{x}{p_1}\right) g^*(\lambda) \\ &\leq v_0(\alpha(0)) + \frac{x}{p_1} f^*(p_1) + \left(t - s - \frac{x}{p_1}\right) g^*(p_3) + s g^*(q_1) \\ &= v(y,s) + \frac{x_1}{p_1} f^*(p_1) + \left(t - s - \frac{x}{p_1}\right) g^*(p_3) \\ &\leq W(x,t,s)\,. \end{aligned}$$

Let $\alpha \in c_b(y,s) \cup c_r(y,s)$. Define $\gamma_1 = \xi_1$ $\gamma_2(\theta) = 0$ for $\theta \in [t_1, s_2]$, $\gamma_3 = \alpha_3$, then $\gamma = (\gamma_1, \gamma_2, \gamma_3) \in c_r(x,t) \cup c_b(x,t)$. If $\gamma \in c_b(x,t)$, then

$$\begin{aligned} v(x,t) &\leq v_0(\gamma(0)) + \frac{x}{p_1} f^*(p_1) + (t_1 - s_2)\min(f^*(0), g^*(0)) \\ &\quad + s_2 g^*(q_3) \\ &\leq v_0(\gamma(0)) + \frac{x}{p_1} f^*(p_1) + (t_1 - s_1) g^*(0) + (s_1 - s_2)\min(f^*(0), g^*(0)) \\ &\quad + s_2 g^*(q_3) \\ &\leq v_0(\gamma(0)) + \frac{x}{p_1} f^*(p_1) + (t_1 - s) g^*(p_3) + \frac{y}{q_1} g^*(q_1) \\ &\quad + (s_1 - s_2)\min(f^*(0), g^*(0)) + s_2 g^*(q_3) \\ &= v(y,s) + \frac{x}{p_1} f^*(p_1) + (t_1 - s) g^*(p_3) \\ &\leq W(x,t,s)\,, \end{aligned}$$

since $0 = (1/(t_1 - s_1))(t_1 - s + y/q_1)$. Similarly the same inequality holds if $\gamma \in c_r(x,t)$.

Case 3. $\xi \in c_r(x,t,s)$.

Suppose $\alpha \in c_0(y,s)$, define $\lambda = -\alpha(0)/t_2, \gamma_1 = \xi_1, \gamma_2 = \xi_2, \gamma_3(\theta) = \lambda(\theta - t_2)$ for $\theta \in [0.t_2]$. Then $\gamma = (\gamma_1, \gamma_2, \gamma_3) \in c_r(x,t)$ and

$$\begin{aligned} v(x,t) &\leq v_0(\gamma(0)) + \frac{x}{p_1} f^*(p_1) + (t_1 - t_2)\min(f^*(0), g^*(0)) \\ &\quad + t_2 f^*(\lambda) \\ &\leq v_0(\gamma(0)) + \frac{x}{p_1} f^*(p_1) + (t_1 - t_2)\min(f^*(0), g^*(0)) \\ &\quad + (t_2 - s) f^*(p_3) + s f^*(q_1) \\ &= v(y,s) + \left\{\frac{x}{p_1} f^*(p_1) + (t_1 - t_2)\min(f^*(0), g^*(0)) + (t_2 - s) f^*(p_3)\right\} \\ &\leq W(x,t,s), \end{aligned}$$

since $\lambda = (1/t_2)((t_2 - s)p_3 + s q_1)$. Suppose $\alpha \in c_b(y,s)$, define $\lambda = -\alpha(0)/t_1$, $\gamma_1 = \xi_1$, $\gamma_2 = \phi$, $\gamma_3(\theta) = \lambda(\theta - t_1)$ for $\theta \in [0, t_1]$. Then $\gamma = (\gamma_1, \gamma_2, \gamma_3) \in c_b(x,t)$.

By strict convexity of $f^*, \xi \in c_r(x,t,s)$ implies that $f^*(0) > g^*(0)$. Hence

$$\begin{aligned} v(x,t) &\le v_0(\gamma(0)) + \frac{x}{p_1} f^*(p_1) + t_1 g^*(\lambda) \\ &\le v_0(\alpha(0)) + \frac{x}{p_1} f^*(p_1) + s_1 g^*(q_3) + (t_1 - s_1) g^*(0) \\ &\le v_0(\alpha(0)) + \frac{x}{p_1} f^*(p_1) + (t_1 - t_2) g^*(0) + (t_2 - s_1) f^*(0) \\ &\quad + s_1 g^*(q_3) \\ &\le v_0(\alpha(0)) + \frac{x}{p_1} f^*(p_1) + (t_1 - t_2) g^*(0) + (t_2 - s) f^*(p_3) \\ &\quad + \frac{y}{s_1} f^*(q_1) + s_1 g^*(q_3) \\ &= v_0(y,s) + \frac{x}{p_1} f^*(p_1) + (t_1 - t_2) \min(f^*(0), g^*(0)) \\ &\quad + (t_2 - s) f^*(p_3) \\ &\le W(x,t,s)\,. \end{aligned}$$

Suppose $\alpha \in c_r(y,s)$, define $\gamma_1 = \xi_1, \gamma_2 = 0$ in $[t_1, s_2]$ and $\gamma_3 = \alpha_3$. Then $\gamma = (\gamma_1, \gamma_2, \gamma_3) \in c_r(x,t)$ and $f^*(0) > g^*(0)$. Hence

$$\begin{aligned} v(x,t) &\le v_0(\gamma(0)) + (t_1 - s_2) g^*(0) + s_1 g^*(q_3) \\ &\le v_0(\gamma(0)) + (t_1 - t_2) g^*(0) + (t_2 - s_1) f^*(0) + (s_1 - s_2) g^*(0) \\ &\quad + s_1 g^*(q_3) \\ &\le v_0(\alpha(0)) + (t_1 - t_2) g^*(0) + (s_1 - s_2) g^*(0) + (t_2 - s) f^*(p_3) \\ &\quad + \frac{y}{q_1} f^*(q_1) + s_1 g^*(q_3) \\ &= v(y,s) + \frac{x}{p_1} f^*(p_1) + (t_1 - t_2) \min(f^*(0), g^*(0)) + (t_2 - s) g^*(p_3) \\ &\le W(x,t,s)\,. \end{aligned}$$

Combining all the 3-cases to conclude that

$$v(x,t) \le W(x,t,s)\,. \tag{4.8}$$

In order to prove the reverse inequality, let $\xi = (\xi_1, \xi_2, \xi_3) \in ch(x,t)$ with $(p_1, p_2, p_3) = (\dot{\xi}_1, \dot{\xi}_2, \dot{\xi}_3)$ defined on the partition $0 = t_3 \le t_1 \le t_2 \le t_1 \le t_0 = t$.

Case 4. Let $\xi \in c_0(x,t)$.

Let $y = \xi(s)$ and $\alpha(\theta) = x + p_1(\theta - t)$ for $\theta \in [s,t]$. Then $\alpha \in c_0(x,t,s)$ and

$$\begin{aligned} v(x,t) &= v_0(x - p_1 t) + t f^*(p_1) \\ &= v_0(x - p_1 t) + s f^*(p_1) + (t - s) f^*(p_1) \\ &\ge v(y,s) + (t - s) f^*(p_1) \\ &\ge W(x,t,s). \end{aligned}$$

Case 5. Let $\xi \in c_b(x,t)$.

Let $y = \xi(s)$ and define $\alpha = \xi|_{[s,t]}$. Then $\alpha \in c(x,t,s)$ and if $t-s-x/p_1 \geq 0$, then

$$\begin{aligned} v(x,t) &= v_0(\xi(0)) + \frac{x}{p_1} f^*(p_1) + \left(t - \frac{x}{p_1}\right) g^*(p_3) \\ &= v_0(\xi(0)) + \frac{x}{p_1} f^*(p_1) + \left(t - s - \frac{x}{p_1}\right) g^*(p_3) + s g^*(p_3) \\ &\geq v(y,s) + \frac{x}{p_1} f^*(p_1) + \left(t - s - \frac{x}{p_1}\right) g^*(p_3) \\ &\geq W(x,t,s)\,. \end{aligned}$$

If $t - s - x/p_1 < 0$, then

$$\begin{aligned} v(x,t) &= v_0(\xi(0)) + \frac{x}{p_1} f^*(p_1) + \left(t - \frac{x}{p_1}\right) g^*(p_3) \\ &= v_0(\xi(0)) + \frac{y}{p_1} f^*(p_1) + \left(s - \frac{y}{p_1}\right) g^*(p_3) + \frac{x-y}{p_1} f^*(p_1) \\ &\geq v(y,s) + (t-s) f^*(p_1) \\ &\geq W(x,t,s)\,. \end{aligned}$$

Case 6. $\xi \in c_r(x,t)$.

Let $\alpha = \xi|_{[s,t]}$, then $\alpha \in c(x,t,s)$. If $s \in [t_1, t]$, then

$$\begin{aligned} v(x,t) &= v_0(\xi(0)) + \frac{x}{p_1} f^*(p_1) + (t_1 - t_2)\min(f^*(0), g^*(0)) + t_2 f^*(p_3) \\ &= v_0(\xi(0)) + \frac{y}{p_1} f^*(p_1) + (t_1 - t_2)\min(f^*(0), g^*(0)) + t_2 f^*(p_3) \\ &\quad + (t-s) f^*(p_1) \\ &\geq v(y,s) + (t-s) f^*(p_1) \\ &\geq W(x,t,s)\,. \end{aligned}$$

If $s \in (t_1, t_2]$, then

$$\begin{aligned} v(x,t) &= v_0(\xi(0)) + (s - t_2)\min(f^*(0), g^*(0)) + t_2 f^*(p_3) \\ &\qquad + (t_1 - s)\min(f^*(0), g^*(0)) + \frac{x}{p_1} f^*(p_1) \\ &\geq v(y,s) + \frac{x}{p_1} f^*(p_1) + (t_1 - s)\min(f^*(0), g^*(0)) \\ &\geq W(x,t,s)\,. \end{aligned}$$

If $s \in [0, t_1]$, then

$$v(x,t) = v_0(\xi(0)) + \frac{x}{p_1} f^*(p_1) + (t_1 - t_2)\min(f^*(0), g^*(0)) + (t_2 - s) f^*(p_3)$$

$$
\begin{aligned}
&+ sf^*(p_3)\\
&\geq v(y,s) + \frac{x}{p_1}f^*(p_1) + (t_1 - t_2)\min(f^*(0), g^*(0)) + (t_2 - s)f^*(p_3)\\
&\geq W(x,t,s)\,.
\end{aligned}
$$

Combining (4.8) and the above estimates to obtain $v(x,t) = W(x,t,s)$. Similarly for $x \leq 0$ and this proves the lemma. $\square$

Now the following corollary is the immediate consequence of Lemma 4.4 which we state without proof.

Corollary 4.5. *Let $0 \leq s < t$, $\xi \in ch(x,t)$ and $\alpha = \xi|_{[s,t]}$, $\beta = \xi|_{[0,s]}$. Then $\alpha \in ch(x,t,s)$ and $\beta \in ch(\xi(s),s)$.*

Lemma 4.6. *v is a uniformly Lipschitz continuous function.*

Proof. In view of Lemma (4.3), it is enough to prove that $t \mapsto v(x,t)$ is a Lipschitz continuous function with Lipschitz constant depend only on v_0, f^* and g^*. Without loss of generality we can assume that $x \geq 0$ and $0 \leq s \leq t$. Let $\xi = (\xi_1, \xi_2, \xi_3) \in ch(x,t)$ with $(p_1, p_2, p_3) = (\dot{\xi}_1, \dot{\xi}_2, \dot{\xi}_3)$ defined on $0 = t_3 \leq t_2 \leq t_1 \leq t_0 = t$. From corollary (4.5), $\alpha|_{[s,t]} \in ch(x,t,s)$. Let $\tilde{L}$ denote a constant depending only on Lipschitz constant of $x \mapsto v(x,t)$, f^*, g^*. From Lemma 4.2, we have,

(i) if $\alpha \in c_0(x,t,s)$ then

$$
\begin{aligned}
v(x,t) - v(x,s) &\geq v(\alpha(s),s) + (t-s)f^*(p_1) - v(x,s)\\
&\geq -\tilde{L}\{|\alpha(s) - x| + (t-s)|\\
&\geq -2\tilde{L}(t-s)\,.
\end{aligned}
$$

(ii) if $\alpha \in c_b(x,t,s)$, then $x/p_1 \leq t-s$ and $|\alpha(s) - x| = |p_3(x/p_1 - (t-s)) - x| \leq (x/p_1)(p_3 + p_1) + (t-s) \leq \tilde{L}(t-s)$. Hence

$$
\begin{aligned}
v(x,t) - v(x,s) &\geq v(\alpha(s),s) + \frac{x}{p_1}f^*(p_1) + (t - s - x/p_1)\, g^*(p_3)\\
&\quad - v(x,s)\\
&\geq -\tilde{L}|\xi(s) - x| - (|f^*(p_1)| + |g^*(p_3)|)(t-s)\\
&\geq -\tilde{L}(t-s)\,.
\end{aligned}
$$

(iii) if $\alpha \in c_r(x,t,s)$, then $x/p_1 \leq t-s$, $t_1 - t_2 \leq t-s$, hence

$$
\begin{aligned}
v(x,t) - v(x,s) &= v(\alpha(s),s) + \frac{x}{p_1}f^*(p_1) + (t_1 - t_2)\min\left(f^*(0), g^*(0)\right)\\
&\quad + (t_2 - s)g^*(p_3) - v(x,s)\\
&\geq -\tilde{L}\left\{\frac{|\alpha(s) - x|}{(t-s)} + |f^*(0)| + |g^*(0)| + |g^*(p_3)|\right\}(t-s)\\
&\geq -\tilde{L}(t-s),
\end{aligned}
$$

since $|\alpha(s) - x| \leq |p_3|(t_2 - s)| + p_1(x/p_1) \leq \tilde{L}(t-s)\,.$

Also from Lemma 4.4, $v(x,t) - v(x,s) \leq (t-s)f^*(0)$. Combining all the above estimates to obtain $|v(x,t) - v(x,s)| \leq \tilde{L}|t-s|$. This proves the lemma. $\square$

Analysis of characteristic curves $ch(x,t)$**.** Since the admissible curves consists of three types of curves. Hence we need to understand the behaviour of the characteristic curves in a systematic way.

Definition 4.7. Let $\alpha \in c(x_1,t_1)$ and $\beta \in c(x_2,t_2)$. α and β is said to intersect properly if there exist a $\theta_0 \in (0, \min(t_1,t_2)]$ such that $\alpha(\theta_0) = \beta(\theta_0)$, $\dot{\alpha}(\theta_0) \neq \dot{\beta}(\theta_0)$ and $\alpha(\theta_0) \neq 0$.

Lemma 4.8. *Let* $\alpha \in ch(x_1,t_1)$ *and* $\beta \in ch(x_2,t_2)$*. Then* α *and* β *do not interset properly.*

Proof. Suppose α, β intersect properly. Then there exist a $\theta_0 \in (0, \min(t_1, t_2)]$ such that $\alpha(\theta_0) = \beta(\theta_0) \neq 0$, $\dot{\alpha}(\theta_0) \neq \dot{\beta}(\theta_0)$. Without loss of generality we can assume that $x = \alpha(\theta_0) > 0$ and $t_1 \leq t_2$. Then we can find $s_2 < \theta_0 < s_1$ such that α and β are positive line segments in $[s_2, \min(t_1,s_1)]$ and $(s_2, s_1]$ respectively. Let $\lambda = (\beta(s_1) - \alpha(s_2))/(s_1 - s_2)$, $\gamma(\theta) = \beta(s_1) + \lambda(\theta - s_1)$ for $\theta \in [s_2, s_1]$. Then $\gamma \in c_0(\beta(s_1), s_1, s_2)$ and from strict convexity of f^* and Corollary 4.5

$$\begin{aligned} v(\beta(s_1), s_1) &\leq v(\gamma(s_2), s_2) + (s_1 - s_2) f^*(\lambda) \\ &< v(\alpha(s_2), s_2) + (\theta_0 - s_1) f^*(\dot{\alpha}) + (s_1 - \theta_0) f^*(\dot{\beta}) \\ &= v(x, \theta_0) + (s_1 - \theta_0) f^*(\dot{\beta}) = v(\beta(s_1), s_1), \end{aligned}$$

which is a contradiction. This proves the lemma. $\square$

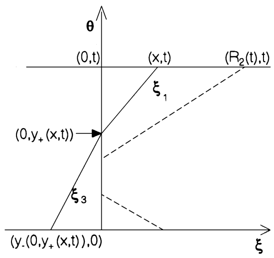

Fig.4.3

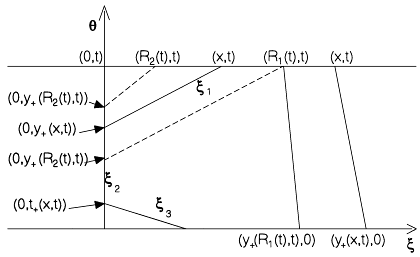

Fig.4.4

Let $t > 0$ and define

$$R_1(t) = \inf\{x; x \geq 0, ch(x,t) \subset c_0(x,t)\}, \tag{4.9}$$

$$R_2(t) = \begin{cases} \inf\{x; 0 \leq x \leq R_1(t), \quad ch(x,t) \cap c_r(x,t) \neq \phi\}, \\ R_1(t) \qquad \text{if the above set is empty.} \end{cases} \tag{4.10}$$

$$L_1(t) = \sup\{x; x \leq 0, ch(x,t) \subset c_0(x,t)\}, \tag{4.11}$$

$$L_2(t) = \begin{cases} \sup\{x : L_1(t) \le x \le 0, ch(x,t) \cap c_r(x,t) \neq \phi\}, \\ L_1(t) \qquad \text{if the above set is empty.} \end{cases} \tag{4.12}$$

Then we have the following

Lemma 4.9. *For $i = 1, 2$, let R_i and L_i are as above. Let*

$$M = \sup\{|\dot{\xi}|_\infty : \xi \in ch(x,t),\ (x,t) \in \mathbb{R} \times \mathbb{R}_+\}.$$

Then

(i) *$R_1(t) \le Mt$, $L_1(t) \ge -Mt$. For $x \in (R_1(t), \infty) \cup (-\infty, L_1(t))$, $ch(x,t) \cap (c_r(x,t) \cup c_b(x,t)) = \phi$. Furthermore if $f^*(0) \le g^*(0)$ then for $x > 0, ch(x,t) \cap c_r(x,t) = \phi$. If $f^*(0) \ge g^*(0)$, then for $x < 0$, $ch(x,t) \cap c_r(x,t) = \phi$.*

(ii) *If $x \in [R_2(t), R_1(t)) \cup (L_1(t), L_2(t)]$, then $ch(x,t) \cap c_r(x,t) \neq \phi$.*

(iii) *If $x \in (0, R_2(t)) \cup (L_2(t), 0)$, then $ch(x,t) \cap ((c_0(x,t) \cup c_r(x,t)) = \phi$.*

(iv) *$R_i, L_i (i = 1, 2)$ are Lipschitz continuous functions.*

(v) *Let $\xi = (\xi_1, \xi_2, \xi_3) \in c(x,t)$ defined on the partition $0 = t_3 \le t_2 \le t_1 \le t$. Define*

$$y_+(x,t) = \begin{cases} \inf\{\xi(0); & \xi \in ch(x,t), & x > R_1(t)\}, \\ \sup\{\xi(0); & \xi \in ch(x,t), & x = R_1(t)\}, \\ \inf\{t_1; & \xi \in ch(x,t), & 0 \le x < R_1(t)\}, \end{cases} \tag{4.13}$$

$$t_+(x,t) = \inf\{t_2 : \quad \xi \in ch(x,t),\ R_2(t) \le x \le R_1(t)\}, \tag{4.14}$$

$$y_-(x,t) = \begin{cases} \sup\{\xi(0); & \xi \in ch(x,t); x < L_1(t)\}, \\ \inf\{\xi(0); & \xi \in ch(x,t); x = L_1(t)\}, \\ \inf\{t_1; & \xi \in ch(x,t),\ 0 \le x < R_1(t)\}, \end{cases} \tag{4.15}$$

$$t_-(x,t) = \inf\{t_2; \quad \xi \in ch(x,t), \quad L_1(t) \le x \le L_2(t)\}. \tag{4.16}$$

Then $x \mapsto y_+(x,t)$ is non decreasing in $[R_1(t), \infty)$, non increasing in $(0, R_1(t))$ and $x \mapsto y_-(x,t)$ is non decreasing in $(0, \infty)$. Also $t \mapsto y_+(R_1(t), t)$ is non decreasing and $t \to y_-(L_1(t), t)$ is non increasing in $(0, \infty)$. Furthermore,

$$\begin{aligned} \lim_{x \to R_1(t)+} y_+(x,t) = y_+(R_1(t), t), \lim_{x \to L_1(t)-} y_-(x,t) = y_-(L_1(t), t)\,. \\ t_+(x,t) = t_+(R_1(t), t), \quad t_-(x,t) = t_-(L_1(t), t). \end{aligned} \tag{4.17}$$

(vi) *For almost every x, $x \mapsto v(x,t)$ is differentiable and at the points of differentiability $x \neq 0$,*

(4.18)
$$v_x(x,t) = \begin{cases} f^{*'}\left(\dfrac{x-y_+(x,t)}{t}\right) & \text{if} \quad x > R_1(t), \\ f^{*'}\left(\dfrac{x}{t-y_+(x,t)}\right) & \text{if} \quad 0 < x \le R_1(t), \\ g^{*'}\left(\dfrac{x-y_-(x,t)}{t}\right) & \text{if} \quad x < L_1(t), \\ g^{*'}\left(\dfrac{x}{t-y_-(x,t)}\right) & \text{if} \quad L_1(t) \le x < 0\,. \end{cases}$$

Furthermore the following limits exist.

(4.19)
$$v_x(0+,t) = \lim_{x\to 0^+} v_x(x,t), \quad v_x(0-,t) = \lim_{x\to 0^-} v_x(x,t)\,.$$

Proof. It is enough to prove the Lemma for $x \ge 0$. Similar arguments follows for $x \le 0$. From Lemma 4.2, $M < \infty$. Let $\xi \in ch(x,t)$ and $x/t \ge M$. Then $\xi_1(0) = x - \dot{\xi}_1 t \ge 0$ and hence $\xi \in c_0(x,t)$. This implis that $R_1(t) \le Mt$. By compactness of $ch(x,t)$, there exist a $\xi_0 \in ch(R_1(t),t) \cap c_0(R_1(t),t)$. Hence if for some $x > R_1(t)$, $\xi \in ch(x,t) \cap (c_b(x,t) \cup c_r(x,t))$, then ξ_0 and ξ must necessarily intersect properly which contradicts Lemma 4.8. Hence $ch(x,t) \cap ((c_b(x,t) \cup c_r(x,t)) = \phi$. Let $f^*(0) \le g^*(0)$ and $\xi \in ch(x,t) \cap c_r(x,t)$ defined on the partition $0 = t_3 \le t_2 \le t_1 \le t_0 = t$ with $\dot{\xi} = (p_1,p_2,p_3)$. Let $\lambda = (x + p_3 t_2)/t$ and $\alpha(\theta) = x + \lambda(\theta - t)$, $\theta \in [0,t]$. Then $\alpha \in c_0(x,t)$ and from strict convexity of f^*

$$\begin{aligned} v(x,t) &\le v_0(\alpha(0)) + t f^*(\lambda) \\ &< v_0(\xi(0)) + \frac{x}{p} f^*(p) + (t_1 - t_2) f^*(0) + t_2 f^*(p_3) \\ &= v(x,t)\,, \end{aligned}$$

which is a contradiction. This proves (i).

Let $R_2(t) < R_1(t)$, then there exist $\alpha = (\alpha_1,\alpha_2,\alpha_3) \in ch(R_2(t),t) \cap c_r(R_2(t),t)$ defined on the partition $0 \le t_3 \le t_2 \le t_1 \le t_0 = t$. We claim that if $x > R_2(t)$ and $ch(x,t) \cap c_b(x,t) \ne \phi$, then $ch(x,t) \cap c_r(x,t) \ne \phi$. For let $\xi = (\xi_1,\phi,\xi_3) \in ch(x,t) \cap c_b(x,t)$ with $s_1 = t - x/\dot{\xi}_1$. From Lemma 4.8, ξ and α do not intersect properly and hence $t_2 \le s_1 \le t_1$. Define $\beta \in c_r(x,t)$ by $\beta = \xi$ in $[s_1,t]$ and $\beta = \alpha$ in $[0,s_1]$. Then from Corollary (4.5)

$$\begin{aligned} v(x,t) &\le v_0(\beta(0)) + \frac{x}{\dot{\beta}_1} f^*(\dot{\beta}_1) + (s_1 - t_2)\min(f^*(0),g^*(0)) + t_2 f^*(\dot{\beta}_3) \\ &= v_0(\alpha(0)) + (s_1 - t_2)\min(f^*(0),g^*(0)) + t_2 f^*(\dot{\alpha}_3) + \frac{x}{\dot{\xi}_1} f^*(\dot{\xi}_1) \\ &= v(0,s_1) + \frac{x}{\dot{\xi}_1} f^*(\dot{\xi}_1) \\ &= v(x,t) \end{aligned}$$

and hence $\beta \in ch(x,t)$. This proves the claim. Now (ii) follows from the above claim, since for $x \in [R_2(t), R_1(t))$, $ch(x,t) \cap ((c_r(x,t) \cap c_b(x,t) \ne \phi$. Now (iii) follows from (i) and the definition of $R_2(t)$.

Let $0 \le s < t$ and $\xi \in ch(R_1(t),t) \cap c_0(R_1(t),t), \eta = (\eta_1,\eta_2,\eta_3) \in ch(R_1(t),t) \setminus c_0(R_1(t),t)$ defined on the partition $0 = t_3 \le t_2 \le t_1 \le t_0 = t$. From Lemma 4.8, if $t_1 \le s \le t$, then $R_1(t) + \dot{\eta}_1(s-t) = \eta(s) \le R_1(s) \le \xi(s) = R_1(t) + \dot{\xi}_1(s-t)$. Hence $|R_1(t) - R_1(s)| \le M(t-s)$. Suppose $s \le t_1 = t - R_1(t)/\dot{\eta}_1$, then $R_1(t) \le \dot{\eta}_1(t-s) \le M(t-s)$ and $0 \le R_1(s) \le R_1(t) + \dot{\xi}_1(s-t) \le 2M(t-s)$. This implies $|R_1(t) - R_1(s)| \le 2M(t-s)$. Let $R_2(t) = R_1(t)$, then with ξ and η as above, if $t_1 \le s \le t$, then $R_2(t) + \dot{\eta}_1(s-t) = \eta(s) \le R_2(s) \le \xi(s) = R_2(t) + \dot{\xi}_1(s-t)$, this implies that $|R_2(t) - R_2(s)| \le M(t-s)$. If $x \le t$, then $R_2(t) \le M(t-s)$ and $R_2(s) \le R_2(t) + \dot{\xi}_1(s-t) \le 2M(t-s)$. Suppose $R_2(t) < R_1(t)$, then choose $\alpha \in c_r(R_2(t),t)$ and $\beta \in c_b(R_2(t),t)$. Let $t - R_2(t)/\dot{\alpha}_1 \le s \le t$, then $\beta(s) \le R_2(s) \le \alpha(s)$ and hence $|R_2(t) - R_2(s)| \le M(t-s)$. If $s \le t - R_2(t)/\dot{\alpha}_1$, then $R_2(t) \le \dot{\alpha}_1(t-s) \le M(t-s)$ and $R_2(s) = 0$ if $s \in [t_2,t_1]$ and if $s \le t_2$, $\quad R_2(s) \le \alpha(s) = \dot{\alpha}_3(s-t) \le M(t-s)$. This proves (iv).

From Lemma 4.8, no two characteristics intersects properly. Hence from (i) through (iii), it follows immediately that $x \mapsto y_+(x,t)$ is non increasing in $[0, R_1(t))$ and $x \mapsto y_+(x,t)$, $t \mapsto y_+(R_1(t),t)$ are non decreasing functions in $[R_1(t),\infty)$ and $(0,\infty)$ respectively. From the definition it follows that $\lim_{x\to R_1(t)+} y_+(x,t) = y_+(R_1(t),t)$. Let $R_2(t) \le x < R_1(t)$ and $\alpha \in ch(R_1(t),t)$ such that $t_2 = t_+(R_1(t),t)$. Let $\beta \in ch(x,t) \cap c_r(x,t)$ with $t_2 = t_+(x,t)$. From Lemma 4.8, $t - x/\dot{\beta}_1 \ge t - R_1(t)/\dot{\alpha}_1$. Hence if $t_+(x,t) > t_+(R_1(t),t)$ then from Corollary 4.5, $\gamma \in c_r(x,t)$ where γ is defined by $\gamma|_{[0,t_+(R_1(t),t)]} = \alpha$ and $\gamma|_{[t_+(R_1(t),t),t]} = \beta$. This implies that $t_+(x,t) \le t_+(R_1(t),t)$ which is a contradiction. If $t_+(x,t) < t_+(R_1(t),t)$, let $\gamma|_{[0,t_+(x,t)]} = \alpha$, $\gamma|_{[t_+(x,t),t]} = \beta$, then $\gamma \in ch(R_1(t),t) \cap c_r(R_1(t),t)$ and hence $t_+(R_1(t),t) \le t_+(x,t) < t_+(R_1(t),t)$ which is a contradiction. This proves (v).

$\xi = (\xi_1,\xi_2,\xi_3) \in ch(x,t)$ with $\dot{\xi} = (p_1,p_2,p_3)$ defined on $0 = t_3 \le t_2 \le t_1 \le t_0 = t$. Since $\xi \in ch(x,t)$ and hence $(\partial/\partial p_1)\rho_\xi(x,t) = 0$ if $p_1 > 0$ or $x > 0, (\partial/\partial t_2)\rho_\xi(x,t) = 0$ if $t_2 > 0$ and $(\partial/\partial p_3)\rho_\xi(x,t) = 0$ if $p_3 \ne 0$. Expanding these equations to obtain:
If $p_1 \ne 0$ or $x > 0$.

$$v_0'(x - p_1 t) = f^{*'}(p_1) \qquad \text{if} \quad \xi \in c_0(x,t), \tag{4.20}$$

$$\min(f^*(0), g^*(0)) = f^*(p_1) - p_1 f^{*'}(p_1) \qquad \text{if} \quad \xi \in c_r(x,t), \tag{4.21}$$

$$p_3 v_0'\left(p_3\left(\frac{x}{p_1} - t\right)\right) = p_1 f^{*'}(p_1) - f^*(p_1) + g^*(p_3) \qquad \text{if} \quad \xi \in c_b(x,t)\,. \tag{4.22}$$

If $p_3 \ne 0$,

$$v_0'(-p_3 t_2) = f^{*'}(p_3) \qquad \text{if} \quad \xi \in c_r(x,t), \tag{4.23}$$

$$v_0'\left(p_3\left(\frac{x}{p_1} - t\right)\right) = g^{*'}(p_3) \qquad \text{if} \quad \xi \in c_b(x,t). \tag{4.24}$$

If $t_2 \neq 0$,

$$\min(f^*(0), g^*(0)) = f^*(p_3) - p_3 v_0'(-p_3 t_2) = f^*(p_3) - p_3 f^{*'}(p_3). \tag{4.25}$$

Now from the compactness of $ch(x,t)$, we can choose a $\eta = (\eta_1, \eta_2, \eta_3) \in ch(x,t)$ with $\dot{\eta} = (q_1, q_2, q_3)$ satisfying the following:

$$\begin{aligned} \eta(0) &= y_+(x,t) && \text{if} \quad x \geq R_1(t), \\ t - \frac{x}{q_1} &= y_+(x,t) && \text{if} \quad 0 \leq x < R_1(t), \\ t_+(x,t) &= t_+(R_1(t), t) && \text{if} \quad x \in [R_2(t), R_1(t)), \\ q_3\left(\frac{x}{q_1} - t\right) &= y_-\left(0, t - \frac{x}{q_1}\right) && \text{if} \quad x \in (0, R_2(t)). \end{aligned} \tag{4.26}$$

From Lemma 4.8, it follows easily that for $x \in (0, R_2(t))$ $x \mapsto q_3(x/q_1 - t)$ is a non increasing function. Hence $x \mapsto y_-(0, t - x/q_1)$ is a non increasing function. Hence $y_+(x,t), y_-(0, t - x/q_1)$ are differentiable almost every x and therefore q_1, q_3, t_+ are differentiable almost every x. Hence from (4.20) to (4.26) at the points x of differentiability of q_1, q_3, t_+,

(4.27)

$$\begin{aligned} \frac{\partial v}{\partial x}(x,t) &= \left(\frac{\partial}{\partial x} + \frac{\partial q_1}{\partial x}\frac{\partial}{\partial q_1} + \frac{\partial t_+}{\partial x}\frac{\partial}{\partial t_+} + \frac{\partial q_3}{\partial x}\frac{\partial}{\partial q_3}\right)\rho_\eta(x,t) \\ &= \frac{\partial}{\partial x}\rho_\eta(x,t) \\ &= \begin{cases} v_0'(x - q_1 t) & \text{if} \quad x > R_1(t), \\ \dfrac{f^*(q_1) - \min(f^*(0), g^*(0))}{q_1} & \text{if} \quad x \in (R_2(t), R_1(t)), \\ \dfrac{q_3 v_0'\left(q_3\left(\frac{x}{q_1} - t\right)\right) + f^*(q_1) - g^*(q_3)}{q_1} & \text{if} \quad x \in (0, R_2(t)), \end{cases} \\ &= f^{*'}(q_1) \\ &= \begin{cases} f^{*'}\left(\dfrac{x - y_+(x,t)}{t}\right) & \text{if} \quad x > R_1(t), \\ f^{*'}\left(\dfrac{x}{t - y_+(x,t)}\right) & \text{if} \quad x \in [0, R_1(t)]. \end{cases} \end{aligned}$$

If $R_1(t) = 0$, then by monotonicity of $y_+(x,t)$, $\lim_{x \to 0^+} v_x(x,t)$ exist. Since $f(f^{*'}(p)) = pf^{*'}(p) - f^*(p)$ hence for $x \in (R_2(t),\ R_1(t))$ and from (4.21), $f(f^*(q_1)) = -\min(f^*(0), g^*(0))$. Now $f'(f^{*'}(q_1)) = q_1 > 0$ and hence from the above equation q_1 is constant. Therefore if $R_2(t) = 0$, then $\lim_{x \to 0^+} v_x(x,t) = f^{*'}(q_1)$. Let $R_2(t) > 0$, then from (4.22) and (4.24), $f(f^{*'}(q_1)) = g(g^{*'}(q_3))$ and $(f'(f^{*'}(q_1)) = q_1 > 0$. Hence $f^{*'}(q_1) = f^{-1}(g(g^{*'}(q_3)))$. From (4.26),

$q_3 = (-y_-(0, t - x/q_1))/(y_+(x,t))$ and hence $\lim_{x\to 0+} q_3$ exist. This implies that $\lim_{x\to 0+} v_x(x,t) = \lim_{x\to 0+} f^{*'}(q_1) = \lim_{x\to 0+} f^{-1}(g(g^{*'}(q_3)))$ exist. This proves the lemma. □

Lemma 4.10. *For almost every t*

$$f(v_x(0+,t)) = g(v_x(0-,t)), \tag{4.28}$$

and one of the following inequality holds:

$$\begin{aligned} &f'(v_x(0+,t)) \geq 0, && g'(v_x(0-,t)) \geq 0,\\ &f'(v_x(0+,t)) \leq 0, && g'(v_x(0-,t)) \geq 0,\\ &f'(v_x(0+,t)) \leq 0, && g'(v_x(0-,t)) \leq 0. \end{aligned} \tag{4.29}$$

Proof. Proof is lengthy and we divide into several steps. Without loss of generality we can assume that

$$f^*(0) \geq g^*(0). \tag{4.30}$$

Hence from (i) of Lemma 4.9, $L_1(t) = L_2(t)$ and $ch(x,t) \cap c_r(x,t) = \phi$ for $x < 0$.

Step 1. Lemma holds in $U = \{t;\ R_2(t) > 0\}$.

From (iv) of Lemma (4.9), U is an open set and hence $U = \bigcup_{i=1}^{\infty} I_i$ where I_i is an open interval. Since no two characteristics intersects properly and $R_2 > 0$ in U, hence for all $s \in U$, $ch(z,s) \subset c_0(z,s)$ for $z < 0$. This implies that $L_1 = 0$ on U and $y_-(0,s)$ is continuous for almost every $s \in U$. Let t be a point of continuity of $y_-(0,\cdot), x \in (0, R_2(t))$ and $\eta \in ch(x,t)$ with $\eta = (q_1, q_2, q_3)$ satisfying (4.26). From (4.22), (4.24) and (4.27), $f(v_x(x,t)) = f(f^{*'}(q_1)) = g(g^{*'}(q_3))$. From (4.18), $g(v_x(0-,t)) = g(g^{*'}(-y_-(0,t)/t))$. As $x \to 0, t - x/q_1 \to t$ and hence from (4.26), $\lim_{x\to 0+} q_3 = -y_-(0,t)/t$. This implies that $f(v_x(0+,t)) = \lim_{x\to 0+} g(g^{*'}(q_3)) = g(g^{*'}(-y_-(0,t)/t)) = g(v_x(0-,t))$. Furthermore $f'(v_x(0+,t)) = \lim_{x\to 0+} q_1 \geq 0$ and $g'(v_x(0-,t)) = \lim_{x\to 0+} q_3 \geq 0$. This proves Step (1).

Step 2. Lemma holds in $S = (0,\infty) \setminus U$.

In order to prove this we define two auxiliary function as follows:

$$\begin{aligned} v_+(x,t) &= \inf\{\rho_\xi(x,t); x \geq 0, \xi \in c_0(x,t) \cup c_r(x,t)\}.\\ v_-(x,t) &= \inf\{\rho_\xi(x,t); x \leq 0, \xi \in c_0(x,t) \cup c_b(x,t)\}. \end{aligned}$$

Using the same arguments as in previous lemmas, $v_\pm$ satisfies the following properties:

(1) Since $f^*(0) \geq g^*(0)$ it follows from (i) of Lemma 4.9, $v_-(x,t) = v(x,t)$ for $x < 0$. Furthermore $t \in S$ implies that $v(x,t) = v_+(x,t)$ for $x \geq 0$ and $v(x,t) = v_-(x,t)$ for $x \leq 0$ and $S = \{t; v_+(0,t) = v_-(0,t)\}$.

(2) $v_\pm$ are Lipschitz continuous functions

(3) For $x \le 0$, there exist $\eta(x,t)$ such that

$$\eta(x,t) = (\eta_1(x,t), \phi, \eta_3(x,t)) \in c(x,t) \quad \text{with } \dot{\eta} = \frac{\partial \eta}{\partial \theta} = (p_1(x,t), \phi, p_3(x,t))$$

minimizing $v_-(x,t)$ and satisfying

(i) $x \mapsto \dot{\eta}(x,t)$ is differentiable almost every $x < 0$,

(ii) $t \mapsto \dot{\eta}(0,t)$ is of bounded variation,

(iii) at differentiable points $x < 0$,

$$\frac{\partial v_-}{\partial x}(x,t) = g^{*'}(p_1(x,t)),$$
$$f(f^{*'}(p_3(x,t))) = g(g^{*'}(p_1(x,t))) \qquad \text{if} \quad p_3(x,t) \text{ exist.}$$

(iv) For almost every t,

$$\frac{\partial v_-}{\partial t}(0,t) = -g(g^{*'}(p_1(0,t))).$$

(v) If $p_3(x,t)$ exist, then $p_1(x,t) < 0$ and $p_3(x,t) < 0$.

(4) For $x \ge 0$, there exist $\xi(x,t) = (\xi_1(x,t), \xi_2(x,t), \xi_3(x,t)) \in c(x,t)$ with $\dot{\xi}(x,t) = d\xi/d\theta = (q_1(x,t), q_2(x,t), q_3(x,t))$ defined on the partition $0 = t_3 \le t_2(x,t) \le t_1(x,t) \le t_0 = t$ which minimizes $v_+(x,t)$ and satisfies the following properties:

(i) $x \mapsto \dot{\xi}(x,t)$ is differentiable almost every $x > 0$.

(ii) $t \mapsto \dot{\xi}(0,t)$ is of bounded variation.

(iii) at differentiable points $x < 0$,

$$\frac{\partial v_+}{\partial x}(x,t) = f(f^{*'}(q_1(x,t))),$$
$$f(f^{*'}(q_1(x,t))) = f(f^{*'}(q_3(x,t))) \qquad \text{if} \quad q_3(x,t) \quad \text{exist,}$$

(iv) for almost every t

$$\frac{\partial v_+}{\partial t}(0,t) = -\begin{cases} f(f^{*'}(q_1(0,t))) & \text{if} \quad q_1(0,t) > 0, \\ f(f^{*'}(q_3(0,t))) & \text{if} \quad \xi_1(0,t) = \phi\,. \end{cases}$$

(v) $q_1(0,t) \ge 0$.

Except (iv) of (3) and (4) every other statements follows exactly as earlier and the proof of this is given below. Suppose for x close to 0, $\eta(x,t) \in c_b(x,t)$, since no two different characteristics intersect properly and hence for $s < t$ and close to $t, \eta(0,s) \in c_b(0,t)$. Hence $v_-(0,s) = v_0(-sp_3(0,s)) + sf^{*'}(p_3(0,s))$. Since $(\partial/\partial p_3)\rho_\eta = 0$ implies that $v_0'(-sp_3(0,s)) = f^{*'}(p_3(0,s))$ and from (iii) of (3) we have, $p_1(0,t) \le 0$, $p_3(0,t) \le 0$,

$$\begin{aligned}\frac{\partial v_-}{\partial t}(0,t) &= \frac{\partial}{\partial s} v_-(0,s)|_{s=t} \\ &= -p_3(0,t)v_0'(-tp_3(0,t)) + f^*(p_3(0,t))\end{aligned}$$

$$
\begin{aligned}
&= -(p_3(0,t)f^{*'}(p_3(0,t)) - f^*(p_3(0,t))) \\
&= -f(f^{*'}(p_3(0,t))) \\
&= g(g^{*'}(p_1(0,t)))\,.
\end{aligned}
$$

Suppose for all $x, \eta(x,t) \in c_0(x,t)$, then $p_1(0,t) \geq 0$ and at points of differentiability of $p_1(0,t)$ and $v_-(0,t)$

$$
\begin{aligned}
\frac{\partial v_-}{\partial t}(0,t) &= \frac{\partial}{\partial t}(v_0(-tp_1(0,t)) + tg^*(p_1(0,t)) \\
&= -(p_1(0,t)g^{*'}(p_1(0,t)) - g^*(p_1(0,t))) \\
&= -g(g^{*'}(p_1(0,t)))\,.
\end{aligned}
$$

This proves (iv) and (v) of (3). Similarly (iv) and (v) of (4) also follows.

Now for almost every $t \in S$, $(\partial/\partial t)(v_+(0,t) - v_-(0,t)) = 0$ and $v_+(x,t) = v(x,t)$ if $x \geq 0$ and $v_-(x,t) = v(x,t)$ if $x \leq 0$. Hence form (iii), (iv) of (3) and (4) $g(v_x(0-,t)) = \lim_{x\to 0^-} g((\partial v_-/\partial x)(x,t)) = g(g^{*'}(p_1(0,t))) = -(\partial v_-/\partial t)(0,t) = -(\partial v_+/\partial_t)(0,t) = f(f^{*'}(q_1(0,t))) = \lim_{x\to 0^+} f((\partial v/\partial x)(x,t)) = f(v_x(0+,t))$ provided $q_1(0,t) > 0$.

If $\xi_1(0,t) = \phi$, then by convexity of g^* and $v_+(0,t) = v_-(0,t)$ gives $p_3(0,t) = q_3(0,t)$ and this gives again $f(v_x(0+,t)) = g(v_x(0-,t))$. This proves (4.28). Now from (v) of (3), (4) gives (4.29). This proves the lemma. ☐

Lemma 4.11. *For $x \neq 0$, Let $v_x(x,t)$ be defined as in* (4.18) *and $M = \sup_{x,t}\{|\dot{\xi}|_\infty;\ \xi \in ch(x,t)\}$. Then for $z > 0$,*

(4.31)

$$
f'(v_x(x+z,t)) - f'(v_x(x,t)) \leq \begin{cases} \dfrac{Mz}{z+x} & \text{if } \ 0 < x < x+z < R_1(t), \\ \dfrac{z}{t} & \text{if } \ x > 0 \ \text{ and not in the above range of values.} \end{cases}
$$

(4.32)

$$
g'(v_x(x,t)) - g'(v_x(x-z,t)) \leq \begin{cases} \dfrac{Mz}{z+|x|} & \text{if } \ L_1(t) < x < x-z < 0, \\ \dfrac{z}{t} & \text{if } \ x < 0 \ \text{ and not in the above range of the values.} \end{cases}
$$

Proof. It is enough to prove for $x > 0$. Let $\eta(x) = (\eta_1(x), \eta_2(x), \eta_3(x)) \in ch(x,t)$ with $\dot{\eta} = d\eta/d\theta = (q_1(x), q_2(x), q_3(x))$ satisfying (4.26).

Case (i). $y_+(x,t)$ is a non decreasing function and hence

$$
\begin{aligned}
f'(v_x(x+z,t)) - f'(v_x(x,t)) &= \frac{x+z-y_+(x+z,t)}{t} - \frac{x-y_+(x,t)}{t} \\
&\leq \frac{x+z-y_+(x+z,t)}{t} - \frac{x-y_+(x+z,t)}{t} \\
&= \frac{z}{t}\,.
\end{aligned}
$$

Case (ii). $0 < x < R_1(t),\ x + z \geq R_1(t)$.
In this case $q_1(x) \geq x/t$ and hence

$$\begin{aligned} f'(v_x(x+z,t)) - f'(v_x(x,t)) &= \frac{x+z-y_+(x+z,t)}{t} - q_1(x) \\ &\leq \frac{x+z}{t} - \frac{x}{t} \\ &= \frac{z}{t}. \end{aligned}$$

Case (iii). $0 < x < x + z < R_1(t)$.
Since $x \mapsto y_+(x,t)$ is a non increasing function in $(0, R_1(t))$ and hence

$$\begin{aligned} f'(v_x(x+z,t)) - f'(v_x(x,t)) &= \frac{x+z}{t-y_+(x+z,t)} - \frac{x}{t-y_+(x,t)} \\ &\leq \frac{x+z}{t-y_+(x+z,t)} - \frac{x}{t-y_+(x+z,t)} \\ &= \frac{z}{t-y_+(x+z,t)} \\ &= \frac{q_1(x+z)z}{x+z} \\ &\leq \frac{Mz}{x+z}\,. \end{aligned}$$

This proves the lemma. □

Proof of Theorem 3.1. From Lemma 4.6, v is a Lipschitz continuous function with $\lim_{t\to 0} v(x,t) = v_0(x)$. In order to prove (3.5) we have to show that v is a sub and supersolution.

Sub solution. Let $x_0 > 0, t_0 > 0, \varphi \in C^1(\mathbb{R} \times \mathbb{R}_+)$ such that $v - \varphi$ has a local maximum at (x_0, t_0) with $v(x_0,t_0) = \varphi(x_0,t_0)$. Let $0 < \varepsilon < x_0$ such that if $B(\varepsilon) = \{(x,s); |x - x_0| \leq \varepsilon,\ t_0 - \varepsilon \leq s \leq t_0\}$, then $B(\varepsilon) \subset \{(x,t);\ x > 0, t > 0\}$ and $v - \varphi < 0$ in $B(\varepsilon)$. Let $t_0 - \varepsilon < s < t_0$, $|p| < \varepsilon/(t_0 - s)$ and $\gamma_p(\theta) = x_0 + p(\theta - t_0)$ for $\theta \in [s, t_0]$. Then by the choice of p, $\gamma_p \in c_0(x_0, t_0, s)$ and hence from Lemma 4.4

$$\begin{aligned} (t_0 - s)f^*(p) &\geq v(x_0,t_0) - v(x_0 + p(s-t_0), s) \\ &\geq \varphi(x_0,t_0) - \varphi(x_0 + p(s-t_0), s) \\ &= \varphi(x_0,t_0) - \varphi(x_0,s) - p\varphi_x(x_0,s)(t_0 - s) + \circ(t_0 - s)\,. \end{aligned}$$

Dividing by $(t_0 - s)$ and letting $s \to t_0$ to obtain $f^*(p) \geq \varphi_t(x_0,t_0) - p\varphi_x(x_0,t_0)$. Hence $(\varphi_t + f(\varphi_x))(x_0,t_0) = \sup_p\{\varphi_t + p\varphi_x - f^*(p)\}(x_0,t_0) \leq 0$. Similarly if $x_0 < 0$. Hence v is a sub solution.

Super solution. Let $x_0 > 0,\ t_0 > 0, \varphi \in C^1(\mathbb{R} \times \mathbb{R}_+)$ such that $v(x_0,t_0) = \varphi(x_0,t_0)$, $v - \varphi$ has a local minimum at (x_0,t_0). Let $M = \{|\dot{\xi}|_\infty; \xi \in ch(x_0,t_0,s)$, $0 \leq s < t_0\}$. Then from Corollary 4.5 and Lemma 4.2, $M < \infty$ independently of (x_0,t_0). Let $0 < \varepsilon < x_0/M$ such that $B(\varepsilon) = \{(x,s);\ |x - x_0| \leq \varepsilon, t_0 - \varepsilon \leq$

$s \le t_0\} \subset \{(x,t); x>0, t>0\}$ and $v-\varphi \ge 0$ in $B(\varepsilon)$. Let $t_0-\varepsilon<s<t_0$ and $\xi_s \in ch(x_0,t_0,s)$. By the choice of ε and M, $\xi_s \in c_0(x_0,t_0,s)$ and $(\xi_s(\theta),\theta) \in B(\varepsilon)$ for all $\theta \in [s,t_0]$. Let for a subsequence $s \to t_0$, $\dot{\xi}_s \to p_1$. Since $v \ge \varphi$ in $B(\varepsilon)$, it follows that

$$\begin{aligned}(t_0-s)f^*(\dot{\xi}_s) &= v(x_0,t_0)-v(x_0+\dot{\xi}_s(s-t_0),s)\\ &\le \varphi(x_0,t_0)-\varphi(x_0+\dot{\xi}_s(s-t_0),s)\\ &= \varphi(x_0,t_0)-\varphi(x_0,s)+\dot{\xi}_s\varphi_x(x_0,s)(t_0-s)+\circ(t_0-s)\,.\end{aligned}$$

Dividing by (t_0-s) and letting $s \to t_0$ to obtain

$$\begin{aligned}0 \le (\varphi_t+p_1\varphi_x-f^*(p_1))(x_0,t_0) &\le \varphi_t+\sup_p\{p\varphi_x-f^*(p)\}(x_0,t_0)\\ &= (\varphi_t+f(\varphi_x))(x_0,t_0)\,.\end{aligned}$$

Similarly for $x \le 0$. (ii), (3.6) and (3.7) follows from Lemma 4.10 and (3.8), (3.9) follows from Lemma 4.11. This proves the theorem. ☐

Proof of Theorem 3.2. Let v be as in Theorem 3.4 and $u=\partial v/\partial x$ and defined as in (4.18). Then from Lemmas 4.10 and 4.11, u satisfies the entropy conditions (E_i) and (E_b) and Rankin Hugonoit condition (1.4) at $x=0$.

Let $\varphi \in C_0^\infty(\mathbb{R}_+ \times \overline{\mathbb{R}}_+)$. Since v is a Lipschitz continuous function, and v satisfies (3.5), it follows that

$$\begin{aligned}0 &= \int_0^\infty\int_0^\infty (v_t+f(v_x))\varphi_x dx\,dt\\ &= \int_0^\infty v(x,0)\varphi_x(x,0)dx-\int_0^\infty\int_0^\infty(v\varphi_{xt}-f(v_x)\varphi_x)dx\,dt\\ &= -\int_0^\infty u_0(x)\varphi(x,0)dx+\int_0^\infty\int_0^\infty(v_x\varphi_t+f(v_x)\varphi_x)dx\,dt\\ &= -\int_0^\infty u_0(x)\varphi(x,0)dx+\int_0^\infty\int_0^\infty(u\varphi_t+f(u)\varphi_x)dx\,dt\,.\end{aligned}$$

Similarly if $x<0$ to obtain for all $\varphi \in C_0^\infty(\mathbb{R}_- \times \overline{\mathbb{R}}_+)$

$$0=-\int_{-\infty}^0 u_0(x)\varphi(x,0)dx+\int_{-\infty}^0\int_{-\infty}^0(u\varphi_t+g(u)\varphi_x)dx\,dt\,.$$

This implies u is a weak solution of (1.3). (3.10) and (3.11) follows from (v) and (vi) of Lemma 4.9 and uniqueness follows from Theorem (2.2). This proves Theorem 3.2. ☐

Proof of Remark 3.3. It is enough to prove (1). Choose a σ such that $g^{*'}(\sigma)=0$. Then $g^*(\sigma)=g^*(g'(0))=-g(0)=0$. Since $g'(0)\ge 0$ and hence

$\sigma \geq 0$. This imples that $v(0,t) \leq v_0(-\sigma t) + tg^*(\sigma) \leq |v_0|_\infty$. Also by Taylor formula for any p there exist a ξ such that

$$\begin{aligned} g^*(p) &= g^*(p - \sigma + \sigma) \\ &= g^*(\sigma) + g^{*'}(\sigma)(p-\sigma) + g^{*''}(\xi)(p-\sigma)^2 \\ &\geq \frac{1}{c}(p-\sigma)^2 . \end{aligned}$$

Since $f^*(0) \geq g^*(0)$ and hence $c_r(x,t) = \phi$ for any $x \leq 0$. Hence if $t \in U_+$, then

$$\begin{aligned} v(0,t) &= v_0(y_-(0,t)) + tg^*\left(-\frac{y_-(0,t)}{t}\right) \\ &\geq -|v_0|_\infty + \frac{t}{c}\left(\frac{y_-(0,t)}{t} + \sigma\right)^2 . \end{aligned}$$

On the other hand since $g^*(\sigma) = 0$,

$$v(0,t) \leq v_0(-\sigma t) + tg^*(\sigma) \leq |v_0|_\infty .$$

Hence combining the above two inequalities gives

$$\left(\frac{y_-(0,t)}{t} + \sigma\right)^2 \leq \frac{2|v_0|_\infty c}{t} .$$

From Lemma 4.2, there exist a $M > 0$ such that $|(y_-(0,t))/t| \leq M$. Let $L = |g^{*'}(-y_-(0,t)/t)|_\infty$. Since $g(0) = 0$ and hence there exist a constant $M(L) > 0$ such that $|g(g^{*'}(-y_-(0,t)/t))| \leq M(L)|g^{*'}(-y_-(0,t)/t)|$. Hence from (1.3)

$$\begin{aligned} |f(u(0+,t))| &= |g(u(0-,t)| \\ &= \left|g\left(g^{*'}\left(\frac{-y_-(0,t)}{t}\right)\right)\right| \\ &\leq M(L)\left|g^{*'}\left(\frac{-y_-(0,t)}{t} - \sigma + \sigma\right)\right| \\ &= M(L)\left|g^{*'}(\sigma) + g^{*''}(\xi)\left(\frac{-y_-(0,t)}{t} - \sigma\right)\right| \\ &\leq cM(L)\left|\frac{y_-(0,t)}{t} + \sigma\right| \\ &\leq \frac{cM(L)(2|v_0|_\infty c)^{1/2}}{t^{1/2}} . \end{aligned}$$

This proves the remark. □

5. Solution of the Riemann Problem

Let

$$u(x,0) = u_0(x) = \begin{cases} u_r & \text{if} \quad x > 0, \\ u_l & \text{if} \quad x < 0. \end{cases} \tag{5.1}$$

Equation (1.1) with the initial data (5.1) is called Riemann Problem. To obtain the solution of the Riemann Problem, consider the initial data

$$v(x,0) = v_0(x) = \begin{cases} u_r\, x & \text{if} \quad x \geq 0, \\ u_l\, x & \text{if} \quad x \leq 0, \end{cases} \tag{5.2}$$

for the Hamilton-Jacobi equation (3.5). By using the formula (3.4), obtain the solution $v(x,t)$. Now the solution to the Riemann problem is given by $u(x,t) = \partial v/\partial x$. For completeness we provide the solution $v(x,t)$ and $u(x,t)$ in the following Corollary without proof. Proof is a direct calculation from formula (3.4).

Corollary 5.1. *Let $v_0(x)$ be given by (5.2). Then solutions $v(x,t)$ of (3.5) and $u(x,t)$ of (1.3), (1.4) are given as follows:*
(1) *Suppose $f^*(0) \geq g^*(0)$, then*

Case (i) $g'(u_l) \leq 0$ *and* $f'(u_r) \geq 0$:
Let u_+ be such that $f(u_+) = \min g(u)$ and $f'(u_+) \geq 0$. Since f is convex u_+ is unique.
For $x \leq 0$,

$$\begin{aligned} v(x,t) &= \begin{cases} u_l\, x - tg(u_l) & \text{if} \quad x \leq tg'(u_l), \\ tg^*\left(\dfrac{x}{t}\right) & \text{if} \quad tg'(u_l) \leq x \leq 0, \end{cases} \\ u(x,t) &= \begin{cases} u_l & \text{if} \quad x \leq tg'(u_l), \\ (g')^{-1}\left(\dfrac{x}{t}\right) & \text{if} \quad tg'(u_l) \leq x \leq 0. \end{cases} \end{aligned} \tag{5.3}$$

For $x \geq 0$ and $u_+ > u_r$,

$$\begin{aligned} v(x,t) &= \begin{cases} u_+\, x - tf(u_+) & \text{if} \quad 0 \leq x < st, \\ u_r\, x - tf(u_r) & \text{if} \quad x > st, \end{cases} \\ u(x,t) &= \begin{cases} u_+ & \text{if} \quad 0 \leq x < st, \\ u_r & \text{if} \quad x > st, \end{cases} \end{aligned} \tag{5.4}$$

where $s = (f(u_+) - f(u_r))/(u_+ - u_r)$.

For $x \geq 0$ *and* $u_+ \leq u_r$,

(5.5)
$$v(x,t) = \begin{cases} u_+ x - tf(u_+) & \text{if} \quad 0 \leq x \leq tf'(u_+), \\ tf^*\left(\dfrac{x}{t}\right) & \text{if} \quad tf'(u_+) \leq x \leq tf'(u_r), \\ u_r x - tf(u_r) & \text{if} \quad x \geq tf'(u_r), \end{cases}$$
$$u(x,t) = \begin{cases} u_+ & \text{if} \quad 0 \leq x \leq tf'(u_+), \\ (f')^{-1}\left(\dfrac{x}{t}\right) & \text{if} \quad tf'(u_+) \leq x \leq tf'(u_r), \\ u_r & \text{if} \quad x \geq tf'(u_r). \end{cases}$$

Case (ii) $g'(u_l) \leq 0$ *and* $f'(u_r) \leq 0$:

Let u_+ *be such that* $f(u_+) = \min g(u)$ *and* $f'(u_+) \geq 0$. *Since* f *is convex, such a* u_+ *is unique and* $u_+ \geq u_r$.

a) *Suppose* $f(u_+) > f(u_r)$:

For $x \leq 0$,

(5.6)
$$v(x,t) = \begin{cases} u_l x - tg(u_l) & \text{if} \quad x \leq tg'(u_l), \\ tg^*\left(\dfrac{x}{t}\right) & \text{if} \quad tg'(u_l) \leq x \leq 0, \end{cases}$$
$$u(x,t) = \begin{cases} u_l & \text{if} \quad x \leq tg'(u_l), \\ (g')^{-1}\left(\dfrac{x}{t}\right) & \text{if} \quad tg'(u_l) \leq x \leq 0. \end{cases}$$

For $x \geq 0$,

(5.7)
$$v(x,t) = \begin{cases} u_+ x - tf(u_+) & \text{if} \quad 0 \leq x < st, \\ u_r x - tf(u_r) & \text{if} \quad x > st, \end{cases}$$
$$u(x,t) = \begin{cases} u_+ & \text{if} \quad 0 \leq x < st, \\ u_r & \text{if} \quad x > st. \end{cases}$$

where $s = (f(u_+) - f(u_r))/(u_+ - u_r)$.

b) *Suppose* $f(u_+) \leq f(u_r)$:

In this case there exists u_- *such that* $g(u_-) = f(u_r)$ *and* $g'(u_-) \leq 0$. *Since* g *is convex such a* u_- *is unique.*

For $x \leq 0$ *and* $u_- < u_l$,

(5.8)
$$v(x,t) = \begin{cases} u_- x - tg(u_-) & \text{if} \quad x > st, \\ u_l x - tg(u_l) & \text{if} \quad x < st, \end{cases}$$
$$u(x,t) = \begin{cases} u_- & \text{if} \quad x > st, \\ u_l & \text{if} \quad x < st. \end{cases}$$

where $s = (g(u_-) - g(u_l))/(u_- - u_l)$.

For $x \leq 0$ *and* $u_- \geq u_l$,

$$
(5.9)\qquad
\begin{aligned}
v(x,t) &= \begin{cases} u_-\,x - tg(u_-) & \text{if} \quad x \geq tg'(u_-), \\ tg^*\left(\dfrac{x}{t}\right) & \text{if} \quad tg'(u_l) \leq x \leq tg'(u_-), \\ u_l\,x - tg(u_l) & \text{if} \quad x \leq tg'(u_l), \end{cases} \\
u(x,t) &= \begin{cases} u_- & \text{if} \quad x \geq tg'(u_-), \\ (g')^{-1}\left(\dfrac{x}{t}\right) & \text{if} \quad tg'(u_l) \leq x \leq tg'(u_-), \\ u_l & \text{if} \quad x \leq tg'(u_l). \end{cases}
\end{aligned}
$$

For $x \geq 0$,

$$
(5.10)\qquad
\begin{aligned}
v(x,t) &= u_r\,x - tf(u_r), \\
u(x,t) &= u_r.
\end{aligned}
$$

Case (iii) $g'(u_l) \geq 0$ *and* $f'(u_r) \geq 0$:

Let u_+ *be such that* $f(u_+) = g(u_l)$ *and* $f'(u_+) \geq 0$. *Since* f *is convex such a* u_+ *is unique.*

For $x \leq 0$,

$$
(5.11)\qquad
\begin{aligned}
v(x,t) &= u_l\,x - tg(u_l), \\
u(x,t) &= u_l.
\end{aligned}
$$

For $x \geq 0$ *and* $u_+ > u_r$,

$$
(5.12)\qquad
\begin{aligned}
v(x,t) &= \begin{cases} u_+\,x - tf(u_+) & \text{if} \quad 0 \leq x < st, \\ u_r\,x - tf(u_r) & \text{if} \quad x > st, \end{cases} \\
u(x,t) &= \begin{cases} u_+ & \text{if} \quad 0 \leq x < st, \\ u_r & \text{if} \quad x > st, \end{cases}
\end{aligned}
$$

where $s = (f(u_+) - f(u_r))/(u_+ - u_r)$.

For $x \geq 0$ *and* $u_+ \leq u_r$,

$$
(5.13)\qquad
\begin{aligned}
v(x,t) &= \begin{cases} u_+\,x - tf(u_+) & \text{if} \quad 0 \leq x \leq tf'(u_+), \\ tf^*\left(\dfrac{x}{t}\right) & \text{if} \quad tf'(u_+) \leq x \leq tf'(u_r), \\ u_r\,x - tf(u_r) & \text{if} \quad x \geq tf'(u_r), \end{cases} \\
u(x,t) &= \begin{cases} u_+ & \text{if} \quad 0 \leq x \leq tf'(u_+), \\ (f')^{-1}\left(\dfrac{x}{t}\right) & \text{if} \quad tf'(u_+) \leq x \leq tf'(u_r), \\ u_r & \text{if} \quad x \geq tf'(u_r). \end{cases}
\end{aligned}
$$

Case (iv) $g'(u_l) \geq 0$ *and* $f'(u_r) \leq 0$:

a) *Suppose* $g(u_l) \geq f(u_r)$:

Let u_+ *be such that* $f(u_+) = g(u_l)$ *and* $f'(u_+) \geq 0$. *Since* f *is convex such a* u_+ *is unique and* $u_+ \geq u_r$.

For $x \leq 0$,

$$v(x,t) = u_l\, x - tg(u_l), \quad u(x,t) = u_l. \tag{5.14}$$

For $x \geq 0$,

$$\begin{aligned} v(x,t) &= \begin{cases} u_+\, x - tf(u_+) & \text{if} \quad 0 \leq x < st, \\ u_r\, x - tf(u_r) & \text{if} \quad x > st, \end{cases} \\ u(x,t) &= \begin{cases} u_+ & \text{if} \quad 0 \leq x < st, \\ u_r & \text{if} \quad x > st, \end{cases} \end{aligned} \tag{5.15}$$

where $s = (f(u_+) - f(u_r))/(u_+ - u_r)$.

b) *Suppose* $g(u_l) \leq f(u_r)$:

In this case there exists u_- *such that* $g(u_-) = f(u_r)$ *and* $g'(u_-) \leq 0$. *Since* g *is convex such a* u_- *is unique and* $u_- < u_l$.

For $x \leq 0$,

$$\begin{aligned} v(x,t) &= \begin{cases} u_-\, x - tg(u_-) & \text{if} \quad x > st, \\ u_l\, x - tg(u_l) & \text{if} \quad x < st, \end{cases} \\ u(x,t) &= \begin{cases} u_- & \text{if} \quad x > st, \\ u_l & \text{if} \quad x < st, \end{cases} \end{aligned} \tag{5.16}$$

where $s = (g(u_-) - g(u_l))/(u_- - u_l)$.

For $x \geq 0$,

$$v(x,t) = u_r\, x - tf(u_r), \quad u(x,t) = u_r. \tag{5.17}$$

(2) *Suppose* $f^*(0) \leq g^*(0)$, *then*

Case (i) $g'(u_l) \leq 0$ *and* $f'(u_r) \geq 0$:

Let u_- *be such that* $g(u_-) = \min f(u)$ *and* $g'(u_-) \leq 0$. *Since* g *is convex such a* u_- *is unique.*

For $x \leq 0$ *and* $u_- \geq u_l$,

$$\begin{aligned} v(x,t) &= \begin{cases} u_l\, x - tg(u_l) & \text{if} \quad x \leq tg'(u_l), \\ tg^*\left(\dfrac{x}{t}\right) & \text{if} \quad tg'(u_l) \leq x \leq tg'(u_-), \\ u_-\, x - tg(u_l) & \text{if} \quad tg'(u_-) \leq x \leq 0, \end{cases} \\ u(x,t) &= \begin{cases} u_l & \text{if} \quad x \leq tg'(u_l), \\ (g')^{-1}\left(\dfrac{x}{t}\right) & \text{if} \quad tg'(u_l) \leq x \leq tg'(u_-), \\ u_-\, x - tg(u_l) & \text{if} \quad tg'(u_-) \leq x \leq 0. \end{cases} \end{aligned} \tag{5.18}$$

For $x \leq 0$ and $u_- < u_l$,

$$(5.19)\qquad \begin{aligned} v(x,t) &= \begin{cases} u_-\,x - tg(u_-) & \text{if} \quad 0 \geq x > st, \\ u_l\,x - tg(u_l) & \text{if} \quad x < st, \end{cases} \\ u(x,t) &= \begin{cases} u_- & \text{if} \quad 0 \geq x > st, \\ u_l & \text{if} \quad x < st, \end{cases} \end{aligned}$$

where $s = (f(u_-) - f(u_l))/(u_- - u_l)$.

For $x \geq 0$,

$$(5.20)\qquad \begin{aligned} v(x,t) &= \begin{cases} tf^*\left(\dfrac{x}{t}\right) & \text{if} \quad 0 \leq x \leq tf'(u_r), \\ u_r\,x - tf(u_r) & \text{if} \quad x \geq tf'(u_r), \end{cases} \\ u(x,t) &= \begin{cases} (f')^{-1}\left(\dfrac{x}{t}\right) & \text{if} \quad 0 \leq x \leq tf'(u_r), \\ u_r & \text{if} \quad x \geq tf'(u_r). \end{cases} \end{aligned}$$

Case (ii) $g'(u_l) \leq 0$ *and* $f'(u_r) \leq 0$:

Let u_- be such that $g(u_-) = f(u_r)$ and $g'(u_-) \leq 0$. Since g is convex such a u_- is unique.

For $x \leq 0$ and $u_- < u_l$,

$$(5.21)\qquad \begin{aligned} v(x,t) &= \begin{cases} u_-\,x - tg(u_-) & \text{if} \quad x > st, \\ u_l\,x - tg(u_l) & \text{if} \quad x < st, \end{cases} \\ u(x,t) &= \begin{cases} u_- & \text{if} \quad x > st, \\ u_l & \text{if} \quad x < st, \end{cases} \end{aligned}$$

where $s = (g(u_-) - g(u_l))/(u_- - u_l)$.

For $x \leq 0$ and $u_- \geq u_l$,

$$(5.22)\qquad \begin{aligned} v(x,t) &= \begin{cases} u_-\,x - tg(u_-) & \text{if} \quad x \geq tg'(u_-), \\ tg^*\left(\dfrac{x}{t}\right) & \text{if} \quad tg'(u_l) \leq x \leq tg'(u_-), \\ u_l\,x - tg(u_l) & \text{if} \quad x \leq tg'(u_l), \end{cases} \\ u(x,t) &= \begin{cases} u_- & \text{if} \quad x \geq tg'(u_-), \\ (g')^{-1}\left(\dfrac{x}{t}\right) & \text{if} \quad tg'(u_l) \leq x \leq tg'(u_-), \\ u_l\,x - tg(u_l) & \text{if} \quad x \leq tg'(u_l). \end{cases} \end{aligned}$$

For $x \geq 0$,

$$(5.23)\qquad \begin{aligned} v(x,t) &= u_r\,x - tf(u_r), \\ u(x,t) &= u_r. \end{aligned}$$

Case (iii) $g'(u_l) \geq 0$ *and* $f'(u_r) \geq 0$:

a) *Suppose* $g(u_l) \leq \min f(u)$:

Let u_- *be such that* $g(u_-) = \min f(u)$ *and* $g'(u_-) \leq 0$. *Since* g *is convex such a* u_- *is unique and* $u_- \leq u_l$.

For $x \leq 0$,

$$
\begin{aligned}
v(x,t) &= \begin{cases} u_- \, x - tg(u_-) & \text{if} \quad x > st, \\ u_l \, x - tg(u_l) & \text{if} \quad x < st, \end{cases} \\
u(x,t) &= \begin{cases} u_- & \text{if} \quad x > st, \\ u_l & \text{if} \quad x < st, \end{cases}
\end{aligned} \tag{5.24}
$$

where $s = (g(u_l) - g(u_-))/(u_l - u_-)$.

For $x \geq 0$,

$$
\begin{aligned}
v(x,t) &= \begin{cases} tf^*\left(\frac{x}{t}\right) & \text{if} \quad x \leq tf'(u_r), \\ u_r \, x - tf(u_r) & \text{if} \quad x > tf'(u_r), \end{cases} \\
u(x,t) &= \begin{cases} (f')^{-1}\left(\frac{x}{t}\right) & \text{if} \quad x \leq tf'(u_r), \\ u_r & \text{if} \quad x > tf'(u_r). \end{cases}
\end{aligned} \tag{5.25}
$$

b) *Suppose* $g(u_l) \geq \min f(u)$:

In this case there exists u_+ *such that* $f(u_+) = g(u_l)$ *and* $f'(u_+) \geq 0$. *Since* f *is convex such a* u_+ *is unique.*

For $x \leq 0$,

$$
\begin{aligned}
v(x,t) &= u_l \, x - tg(u_l), \\
u(x,t) &= u_l.
\end{aligned} \tag{5.26}
$$

For $x \geq 0$ *and* $u_+ > u_r$,

$$
\begin{aligned}
v(x,t) &= \begin{cases} u_+ \, x - tf(u_+) & \text{if} \quad 0 \leq x < st, \\ u_r \, x - tf(u_r) & \text{if} \quad x > st, \end{cases} \\
u(x,t) &= \begin{cases} u_+ & \text{if} \quad 0 \leq x < st, \\ u_r & \text{if} \quad x > st, \end{cases}
\end{aligned} \tag{5.27}
$$

where $s = (f(u_+) - f(u_r))/(u_+ - u_r)$.

For $x \geq 0$ *and* $u_+ < u_r$,

$$
\begin{aligned}
v(x,t) &= \begin{cases} u_+ \, x - tf(u_+) & \text{if} \quad 0 \leq x \leq tf'(u_+), \\ tf^*\left(\frac{x}{t}\right) & \text{if} \quad tf'(u_+) \leq x \leq tf'(u_r), \\ u_r \, x - tf(u_r) & \text{if} \quad x \geq tf'(u_r), \end{cases} \\
u(x,t) &= \begin{cases} u_+ & \text{if} \quad 0 \leq x \leq tf'(u_+), \\ (f')^{-1}\left(\frac{x}{t}\right) & \text{if} \quad tf'(u_+) \leq x \leq tf'(u_r), \\ u_r & \text{if} \quad x \geq tf'(u_r). \end{cases}
\end{aligned} \tag{5.28}
$$

Case (iv) $g'(u_l) \geq 0$ *and* $f'(u_r) \leq 0$:

a) *Suppose* $g(u_l) \geq f(u_r)$:

Let u_+ *be such that* $f(u_+) = g(u_l)$ *and* $f'(u_+) \geq 0$. *Since* f *is convex such a* u_+ *is unique and* $u_+ \geq u_r$.

For $x \leq 0$,

$$\begin{aligned} v(x,t) &= u_l\, x - tg(u_l), \\ u(x,t) &= u_l. \end{aligned} \tag{5.29}$$

For $x \geq 0$,

$$\begin{aligned} v(x,t) &= \begin{cases} u_+\, x - tf(u_+) & \text{if } \; 0 \leq x < st, \\ u_r\, x - tf(u_r) & \text{if } \; x > st, \end{cases} \\ u(x,t) &= \begin{cases} u_+ & \text{if } \; 0 \leq x < st, \\ u_r & \text{if } \; x > st, \end{cases} \end{aligned} \tag{5.30}$$

where $s = (f(u_+) - f(u_r))/(u_+ - u_r)$.

b) *Suppose* $g(u_l) \leq f(u_r)$:

In this case there exists u_- *such that* $g(u_-) = f(u_r)$ *and* $g'(u_-) \leq 0$. *Since* g *is convex such a* u_- *is unique and* $u_- \leq u_l$.

For $x \leq 0$,

$$\begin{aligned} v(x,t) &= \begin{cases} u_-\, x - tg(u_-) & \text{if } \; x > st, \\ u_l\, x - tg(u_l) & \text{if } \; x < st, \end{cases} \\ u(x,t) &= \begin{cases} u_- & \text{if } \; x > st, \\ u_l & \text{if } \; x < st, \end{cases} \end{aligned} \tag{5.31}$$

where $s = (g(u_-) - g(u_l))/(u_- - u_l)$.

For $x \geq 0$,

$$\begin{aligned} v(x,t) &= u_r\, x - tf(u_r), \\ u(x,t) &= u_r. \end{aligned} \tag{5.32}$$

This completes the corollary.

If $f = g$, then two shocks cannot cross each other. On the other hand if $f \neq g$, then in general the line $x = 0$ allows a shock pass through it. The following example illustrates this phenomena.

Example 5.2. Let $f(u) = u^2/2$, $g(u) = (u^2+1)/2$, and $u_0(x) = \chi_{(-\infty,-1)}(x)$.

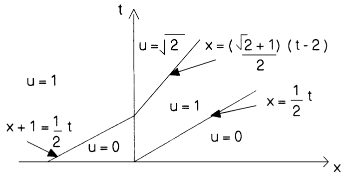

Fig 5.1. Shock passing through the line x = 0

Then the solution u of (1.1) and (1.3) (Fig. 5.1) is given by

$$u(x,t) = \begin{cases} 1 & \text{if} \quad x+1 \le \frac{1}{2}t, \quad x \le 0 \quad \text{or} \quad \frac{(\sqrt{2}+1)}{2}(t-2) \le x \le \frac{1}{2}t, \ x \ge 0\,, \\ 0 & \text{if} \quad x+1 > \frac{1}{2}t, \quad x \le 0 \quad \text{or} \ x > \frac{1}{2}t\,, \\ \sqrt{2} & \text{if} \quad 0 \le x \le \left(\frac{\sqrt{2}+1}{2}\right)(t-2), \ t \ge 2\,. \end{cases}$$

Example 5.3. In this example we show that solution constructed by Diehl [1], [2] differs from us. Let f, g satisfies the hypothesis (H).

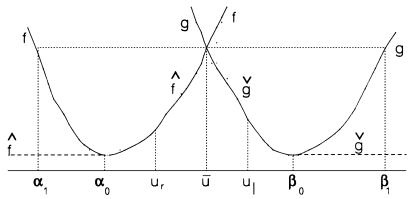

Fig.5.2

Let $f(\alpha_0) = \min f = g(\beta_0) = \min g$. Assume that $\alpha_0 < \beta_0$ and let $\overline{u} \in [\alpha_0, \beta_0]$ be the unique point where $f(\overline{u}) = g(\overline{u})$ with $f'(\overline{u}) > 0$ and $g'(\overline{u}) < 0$. Let $\alpha_0 < u_r < \overline{u} < u_l < \beta_0$, $\nu = f(\overline{u}) = g(\overline{u})$, $\alpha_1 < \overline{u} < \beta_1$ such that $f(\alpha_1) = g(\beta_1) = \nu$ (see Fig. 5.2),

$$\hat{f}(u, u_r) = \begin{cases} \min\limits_{\theta \in [u, u_r]} f(\theta) & \text{if} \quad u \le u_r \\ \max\limits_{\theta \in [u_r, u]} f(\theta) & \text{if} \quad u \ge u_r \end{cases} = \begin{cases} f(u) & \text{if} \quad u \ge \alpha_0, \\ f(\alpha_0) & \text{if} \quad u \le \alpha_0, \end{cases}$$

$$\breve{g}(u, u_l) = \begin{cases} \max\limits_{\theta \in (u, u_l]} g(\theta) & \text{if} \quad u \le u_l \\ \min\limits_{\theta \in [u_l, u]} g(\theta) & \text{if} \quad u \ge u_l \end{cases} = \begin{cases} g(u) & \text{if} \quad u \le \beta_0, \\ g(\beta_0) & \text{if} \quad u \ge \beta_0, \end{cases}$$

$$P(f, u_r) = \{u_r\} \cup \Big\{u < u_r; \quad \hat{f}(u+\varepsilon, u_r) > \hat{f}(u, u_r), \quad \forall \varepsilon > 0\Big\}$$
$$\cup \Big\{u > u_r; \quad \hat{f}(u-\varepsilon, u_r) < \hat{f}(u, u_r) \quad \forall \varepsilon > 0\Big\}$$

$$
\begin{aligned}
&= [\alpha_0, \infty),\\
N(g, u_l) &= \{u_l\} \cup \{u < u_l; \quad \breve{g}(u+\varepsilon, u_l) < \breve{g}(u, u_l) \quad \forall \varepsilon > 0\}\\
&\quad \cup \{u > u_l; \quad \breve{g}(u-\varepsilon_1, u_l) > \breve{g}(u, u_l) \quad \forall \varepsilon < 0\}\\
&= (-\infty, \beta_0],\\
\overline{U} &= \left\{u \in \mathbb{R}^1; \quad \hat{f}(u, u_r) = \breve{g}(u, u_l)\right\} = \{\overline{u}\},\\
\Gamma(u_r, u_l) &= \{(\alpha, \beta); \quad f(\alpha) = g(\beta) = \nu\}\\
&= \{(\alpha_1, \overline{u}),\ (\overline{u}, \beta_1),\ (\overline{u}, \overline{u})\},
\end{aligned}
$$

$s_1 = (g(u_l) - g(\overline{u}))/(u_l - \overline{u})$, $s_2 = (f(\overline{u}) - f(u_r))/(\overline{u} - u_r)$. Then the solution constructed by Diehl [1], [2] is given by

$$
u(x,t) = \begin{cases} u_l & \text{if} \quad x \le s_1 t,\\ \overline{u} & \text{if} \quad s_1 t < x < 0,\\ \overline{u} & \text{if} \quad 0 < x < s_2 t,\\ u_r & \text{if} \quad s_2 t < x\,. \end{cases}
$$

The solution from our method (see Corollary 5.1, Case (i) with $f^*(0) = g^*(0) = -f(\alpha_0)$) is given by

$$
u(x,t) = \begin{cases} u_l & \text{if} \quad x \le t g'(u_l),\\ (g')^{-1}\left(\dfrac{x}{t}\right) & \text{if} \quad t g'(u_l) \le x < 0,\\ (f')^{-1}\left(\dfrac{x}{t}\right) & \text{if} \quad 0 < x \le t f'(u_r),\\ u_r & \text{if} \quad x \ge t f'(u_r). \end{cases}
$$

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