Jack polynomials, generalized binomial coefficients and polynomial solutions of the generalized Laplace's equation

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Abstract

We discuss the symmetric homogeneous polynomial solutions of the generalized Laplace's equation which arises in the context of the Calogero-Sutherland model on a line. The solutions are expressed as linear combinations of Jack polynomials and the constraints on the coefficients of expansion are derived. These constraints involve generalized binomial coefficients defined through Jack polynomials. Generalized binomial coefficients for partitions of k upto $k = 6$ are tabulated.

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In recent years, a considerable progress has been made in finding the exact eigen functions of Calogero-Sutherland type models [1-6] both through operator as well as analytical methods [7-15] . This has largely become possible through a better understanding of the mathematical properties of the Jack polynomials $J_\kappa(x_1, \cdots, x_N; \alpha)$ [15-18], which are homogeneous symmetric polynomials of degree k in the N variables $x \equiv x_1, \cdots, x_N$. They are labelled by partitions κ of k and depend on a parameter α and, among other things, are eigenfunctions of the operator

$$D_2 = \sum_i x_i^2 \frac{\partial^2}{\partial x_i^2} + \frac{2}{\alpha} \sum_{i<j} \frac{1}{(x_i - x_j)} \left(x_i^2 \frac{\partial}{\partial x_i} - x_j^2 \frac{\partial}{\partial x_j} \right) \quad , \tag{1}$$

corresponding to the eigenvalue

$$E_\lambda = \sum_i \left[\kappa_i(\kappa_i - 1) - \frac{2}{\alpha}(i-1)) \right] + \frac{2}{\alpha} k(N-1) \quad , \tag{2}$$

and of the Euler operator

$$E_1 = \sum_i x_i \frac{\partial}{\partial x_i} \quad , \tag{3}$$

corresponding to the eigenvalue k. By repeatedly calculating the commutator of the operator D_2 with the operator

$$E_0 = \sum_i \frac{\partial}{\partial x_i} \quad , \tag{4}$$

one can construct two other useful operators

$$D_1 = \frac{1}{2}[E_0, D_2] = \sum_i x_i \frac{\partial^2}{\partial x_i^2} + \frac{2}{\alpha} \sum_{i<j} \frac{1}{(x_i - x_j)} \left(x_i \frac{\partial}{\partial x_i} - x_j \frac{\partial}{\partial x_j} \right) \quad , \tag{5}$$

$$D_0 = [E_0, D_1] = \sum_i \frac{\partial^2}{\partial x_i^2} + \frac{2}{\alpha} \sum_{i<j} \frac{1}{(x_i - x_j)} \left(\frac{\partial}{\partial x_i} - \frac{\partial}{\partial x_j} \right) \quad . \tag{6}$$

The operator D_0 is referred to as the generalized Laplacian operator. In order to compute the action of E_0 on the Jack polynomials and hence of D_1 and D_0, it proves convenient to introduce the notion of generalised binomial coefficients [19] associated with partitions as follows

$$\mathcal{J}_\kappa(x+1) = \sum_{s=o}^{k} \sum_{\sigma} \begin{pmatrix} \kappa \\ \sigma \end{pmatrix}_\alpha \mathcal{J}_\sigma(x) \quad , \tag{7}$$

where

$$\mathcal{J}_\kappa(x) = \frac{J_\kappa(x)}{J_\kappa(1)} \quad , \tag{8}$$

and the sum over σ in (7) denotes sum over all partitions of s. The action of the operator E_0 on Jack polynomials can now be written as

$$E_0 \mathcal{J}_\kappa(x+1) = \sum_{i}^{N} \begin{pmatrix} \kappa \\ \kappa_{(i)} \end{pmatrix}_\alpha \mathcal{J}_{\kappa_{(i)}}(x) \quad , \tag{9}$$

where $\kappa_{(i)}$ is the partition obtained by decreasing the i^{th} part of κ by 1 provided the result is an admissible partition.

By making the replacements

$$x^2 \frac{d^2}{dx^2} \rightarrow D_2 \; ; \; x \frac{d^2}{dx^2} \rightarrow D_1 \; ; \; \frac{d^2}{dx^2} \rightarrow D_0 \; ; \; x \frac{d}{dx} \rightarrow E_1 \; ; \; \frac{d}{dx} \rightarrow E_0 \; , \tag{10}$$

in the appropriate second order differential equations obeyed by the classical polynomials, new one-parameter family of multivariate special functions have been defined [20] and their properties have been investigated in detail [13]. (These multivariate special functions subsume those defined through zonal harmonics [21-23] as a special case corresponding to $\alpha = 2$) In this context, $\mathcal{J}_\kappa(x)$ plays the same role as x^k does in the single variable case. These multivariate generalised special functions appear as a factor in the exact eigenfunctions of Calogero-Sutherland type models. Thus, it is now known that the eigenfunction of Calogero-Sutherland model on a line involving quadratic and inversely quadratic potentials can be expressed as a product of (a) a Jastrow factor (b) a Gaussian and (c) a generalised Hermite polynomial [12,13]. However, as shown by Calogero [1] and Perelemov [5,6], the exact eigenfunctions of this model can also be expressed as a product of (a) a Jastrow factor (b) a Gaussian (c) a Laguerre polynomial and (d) symmetric polynomial solutions of the generalised Laplace's equation

$$D_0 P_k = 0 \quad . \tag{11}$$

(This equation also turns up in the conext of the Calogero-Sutherland model involving only the inversely quadratic potentials). Finding symmetric polynomial solutions of this equation is our main concern here. In principle, one could expand P_k in terms of any basis in the space of symmetric polynomials in N variables and work out the constraints on the coefficients of expansion arising from the requirement that P_k satisfy the generalised Laplace's equation. In the past, monomial symmetric functions and power symmetric functions have been used for this purpose. However, no explicit expression for the constraints on the coefficients of expansion could be given because the generalised Laplacian operator has no simple action on these functions. In this letter we suggest that the most appropriate basis functions for this purpose are the Jack polynomials $\mathcal{J}_\kappa(x)$ as can be seen from the following considerations.

From the known action of D_2 and E_0 on $\mathcal{J}_\kappa(x)$ one finds that the actions of $D_1 = 1/2\ [E_0, D_2]$ and $D_0 = [E_0, D_1]$ on $\mathcal{J}_\kappa(x)$ are given by

$$D_1 \mathcal{J}_\kappa(x) = \sum_i^N \begin{pmatrix} \kappa \\ \kappa_{(i)} \end{pmatrix}_\alpha (\kappa_i - 1 + \frac{1}{\alpha}(N-i)) \mathcal{J}_{\kappa_{(i)}}(x) \quad , \tag{12}$$

$$\begin{aligned} D_0 \mathcal{J}_\kappa(x) = & \sum_i^N \begin{pmatrix} \kappa \\ \kappa_{(i)} \end{pmatrix}_\alpha \begin{pmatrix} \kappa_{(i)} \\ \kappa_{(i,i)} \end{pmatrix}_\alpha \mathcal{J}_{\kappa_{(i,i)}}(x) \\ + & \sum_{i<j}^N \left[\begin{pmatrix} \kappa \\ \kappa_{(i)} \end{pmatrix}_\alpha \begin{pmatrix} \kappa_{(i)} \\ \kappa_{(i,j)} \end{pmatrix}_\alpha - \begin{pmatrix} \kappa \\ \kappa_{(j)} \end{pmatrix}_\alpha \begin{pmatrix} \kappa_{(j)} \\ \kappa_{(i,j)} \end{pmatrix}_\alpha \right] ((\kappa_i - \kappa_j) - \frac{1}{\alpha}(i-j)) \mathcal{J}_{\kappa_{(i,j)}}(x) \ . \end{aligned} \tag{13}$$

Here $\kappa_{(i,j)}$ denotes the sequence obtained by decreasing the i^{th} and j^{th} part of the partition κ by 1 provided the result of this operation is an admissible partition. Armed with the knowledge of the action of the generalised Laplace's operator on $\mathcal{J}_\kappa(x)$, we now expand P_k in terms of these as follows

$$P_k(x) = \sum_\kappa a_\kappa \mathcal{J}_\kappa(x) \quad , \tag{14}$$

wkere the sum on the rhs runs over all partitions of k. Substituting this expansion in (11), using (13), and equating the coefficients of the $\mathcal{J}$'s to zero we obtain the following constraint on the a_κ's

$$\sum_{i<j}^{N} a_{\kappa^{(i,j)}} \left[\begin{pmatrix} \kappa^{(i,j)} \\ \kappa^{(j)} \end{pmatrix}_\alpha \begin{pmatrix} \kappa^{(j)} \\ \kappa \end{pmatrix}_\alpha - \begin{pmatrix} \kappa^{(i,j)} \\ \kappa^{(i)} \end{pmatrix}_\alpha \begin{pmatrix} \kappa^{(i)} \\ \kappa \end{pmatrix}_\alpha \right] ((\kappa_i^{(i,j)} - \kappa_j^{(i,j)}) - \frac{1}{\alpha}(i-j))$$
$$+ \sum_{i}^{N} a_{\kappa^{(i,i)}} \begin{pmatrix} \kappa^{(i,i)} \\ \kappa_{(i)} \end{pmatrix}_\alpha \begin{pmatrix} \kappa_{(i)} \\ \kappa \end{pmatrix}_\alpha = 0 \quad . \tag{15}$$

Here $\kappa^{(i)}$ ($\kappa^{(i,j)}$) denote partitions obtained by increasing the i^{th} part (i^{th} and j^{th} parts) of κ by one. Using these constraints in (14) one obtains the linearly independent symmetric homogeneous polynomial solutions of the generalized Laplace's equation.

The method outlined above for finding the linearly independent solution of the generalized Laplace's equation requires the knowledge of the generalized binomial coefficients. Some general properties of these which follow from the duality and the interpolational properties of Jack polynomials are listed below

(a) The duality of Jack polynomials i.e. the algebraic homomorphism $\mathcal{J}_\kappa(x;\alpha) \to \mathcal{J}_{\kappa'}(x;1/\alpha)$ where κ' denotes the partition conjugate to κ, implies that

$$\begin{pmatrix} \kappa \\ \sigma \end{pmatrix}_\alpha = \begin{pmatrix} \kappa' \\ \sigma' \end{pmatrix}_{1/\alpha} \quad . \tag{16}$$

(b) For $\alpha = \infty,\ 2\ ,\ 1\ ,\ 0$ Jack polynomials reduce to monomial symmetric functions, zonal harmonics, Schur functions and the elementary symmetric functions

$$\mathcal{J}_\kappa(x;\infty) = \mathcal{M}_\kappa(x)\ ;\ \mathcal{J}_\kappa(x;2) = \mathcal{C}_\kappa(x)\ ;\ \mathcal{J}_\kappa(x;1) = \mathcal{S}_\kappa(x)\ ;\ \mathcal{J}_\kappa(x;0) = \mathcal{E}_{\kappa'}(x)\ . \tag{17}$$

The generalised binomial coefficients for the elementary symmetric functions are easy to calculate and one finds that

$$\begin{pmatrix} \kappa \\ \sigma \end{pmatrix}_0 = \sum_{distinct\ perm.\ of\ \sigma'_1,\cdots,\sigma'_N} \begin{pmatrix} \kappa'_1 \\ \sigma'_1 \end{pmatrix} \cdots \begin{pmatrix} \kappa'_N \\ \sigma'_N \end{pmatrix} \quad , \tag{18}$$

and hence on using the duality in (16)

$$\begin{pmatrix} \kappa \\ \sigma \end{pmatrix}_{\infty} = \sum_{distinct\ perm.\ of\ \sigma_1,\cdots,\sigma_N} \begin{pmatrix} \kappa_1 \\ \sigma_1 \end{pmatrix} \cdots \begin{pmatrix} \kappa_N \\ \sigma_N \end{pmatrix} \quad . \tag{19}$$

This appears to be the most natural extension of the notion of the binomial coefficients for integers to those for partitions. If we denote by $\{P\sigma\}$ the sequences obtained by distinct permutations of the entries in the partition σ then it follows from (19) that

$$\begin{pmatrix} \kappa \\ \sigma \end{pmatrix}_{\infty} = 0 \quad , \tag{20}$$

if all elements of the set $\{P\sigma\}$ are such that the sequences obtained by subtracting the sequence κ from them contain only non negative integers. From (19) it also follows that if κ and σ are both partitions of k then

$$\begin{pmatrix} \kappa \\ \sigma \end{pmatrix}_{\infty} = \delta_{\kappa,\sigma} \quad . \tag{21}$$

We have explicitly computed the generalized binomial coefficients for partitions of k upto $k = 5$ from the knowledge of the expressions for the Jack polynomials in terms of monomial symmetric functions given in ref 24. The results are summarized in the Tables $1 - 5$. These calculations indicate that

(i) the properties (20) and (21) hold true for generalized binomial coefficients for arbitrary α.

(ii) the generalized binomial coefficients are rational functions of α.

(iii) If κ is a partition of k and σ denote the partitions of s then

$$\sum_{\sigma} \begin{pmatrix} \kappa \\ \sigma \end{pmatrix}_{\alpha} = \begin{pmatrix} k \\ s \end{pmatrix} \quad . \tag{22}$$

Using (ii) and (iii) and the values of the generalised binomial coefficients for $\alpha = 2$ [25,26] , and those for $\alpha = \infty$ and $\alpha = 0$ we have computed the table for $k = 6$. The results are displayed in Table 6. In a recent work Okounkov and Olshanski [26] have unravelled the general structure of the generalised binomial coefficients.

The results given above can be used for constructing the linearly independent solutions P_k of the generalized Laplaces's equation. As an illustration, explicit expressions for the constraints on the the coefficients of expansion for $k = 3, 4$ are given below

[1] $k = 3$

$$3\alpha a_3 + \alpha(\alpha - 1)a_{21} - 3a_{1^3} = 0 \quad . \tag{23}$$

[2] $k = 4$

$$6\alpha(1+\alpha)a_4 + (\alpha - 1)a_{31} + 2\alpha(2+\alpha)a_{2^2} - \alpha(3+\alpha)a_{21^2} = 0 \quad , \tag{24}$$

$$\alpha(1+3\alpha)a_{31} - 2(1+2\alpha)a_{2^2} + \alpha(\alpha - 1)a_{21^2} = 0 \quad . \tag{25}$$

To conclude, we have suggested a convenient basis for constructing the linearly independent solutions of the generalized Laplace's equation which arises in the context of Calogero-Sutherland models. The generalized binomial coefficients required for this purpose are constructed for partitions of k for $k \leq 6$.

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Table 1 : Generalized binomial coefficients for k = 1

		σ	
	k=1	(0)	(1)
κ	(1)	1	1

Table 2 : Generalized binomial coefficients for k=2

			σ		
	k=1	(0)	(1)	(2)	(1^2)
κ	(2)	1	2	1	0
	(1^2)	1	2	0	1

Table 3 : Generalized binomial coefficients for k=3

				σ			
k=3	(0)	(1)	(2)	(1^2)	(3)	(2,1))	(1^3)
(3)	1	3	3	0	1	0	0
κ (2,1)	1	3	$\frac{2+\alpha}{1+\alpha}$	$\frac{1+2\alpha}{1+\alpha}$	0	1	0
(1^3)	1	3	0	3	0	0	1

Table 4a : Generalized binomial coefficients for k=4

				σ			
k=4	(0)	(1)	(2)	(1^2)	(3)	(2,1)	(1^3)
(4)	1	4	6	0	4	0	0
(3,1)	1	4	$\frac{5+3\alpha}{1+\alpha}$	$\frac{1+3\alpha}{1+\alpha}$	$\frac{2+2\alpha}{1+2\alpha}$	$\frac{2+6\alpha}{1+2\alpha}$	0
κ (2^2)	1	4	$\frac{4+2\alpha}{1+\alpha}$	$\frac{2+4\alpha}{1+\alpha}$	0	4	0
$(2,1^2)$	1	4	$\frac{3+\alpha}{1+\alpha}$	$\frac{3+5\alpha}{1+\alpha}$	0	$\frac{6+2\alpha}{2+\alpha}$	$\frac{2+2\alpha}{2+\alpha}$
(1^4)	1	4	0	6	0	0	4

Table 4b : Generalized binomial coefficients for k=4

		σ				
	k=4	(4)	(3,1)	(2^2)	$(2,1^2)$	(1^4)
	(4)	1	0	0	0	0
	(3,1)	0	1	0	0	0
κ	(2^2)	0	0	1	0	0
	$(2,1^2)$	0	0	0	1	0
	(1^4)	0	0	0	0	1

Table 5a : Generalized binomial coefficients for k=5

	σ						
k=5	(0)	(1)	(2)	(1^2)	(3)	(2,1)	(1^3)
(5)	1	5	10	0	10	0	0
(4,1)	1	5	$\frac{9+6\alpha}{1+\alpha}$	$\frac{1+4\alpha}{1+\alpha}$	$\frac{7+8\alpha}{1+2\alpha}$	$\frac{3+12\alpha}{1+2\alpha}$	0
(3,2)	1	5	$\frac{8+4\alpha}{1+\alpha}$	$\frac{2+6\alpha}{1+\alpha}$	$\frac{4+2\alpha}{1+2\alpha}$	$\frac{6+18\alpha}{1+2\alpha}$	0
κ $(3,1^2)$	1	5	$\frac{7+3\alpha}{1+\alpha}$	$\frac{3+7\alpha}{1+\alpha}$	$\frac{3+2\alpha}{1+2\alpha}$	$\frac{12(1+3\alpha+\alpha^2)}{(1+2\alpha)(2+\alpha)}$	$\frac{2+3\alpha}{2+\alpha}$
$(2^2,1)$	1	5	$\frac{6+2\alpha}{1+\alpha}$	$\frac{4+8\alpha}{1+\alpha}$	0	$\frac{18+6\alpha}{2+\alpha}$	$\frac{2+4\alpha}{2+\alpha}$
$(2,1^3)$	1	5	$\frac{4+\alpha}{1+\alpha}$	$\frac{6+9\alpha}{1+\alpha}$	0	$\frac{12+3\alpha}{2+\alpha}$	$\frac{8+7\alpha}{2+\alpha}$
(1^5)	1	5	0	10	0	0	10

Table 5b : Generalized binomial coefficients for k=5

	σ				
k=5	(4)	(3,1)	(2^2)	$(2,1^2)$	(1^4)
(5)	5	0	0	0	0
(4,1)	$\frac{2+3\alpha}{1+3\alpha}$	$\frac{3+12\alpha}{1+3\alpha}$	0	0	0
(3,2)	0	$\frac{4+2\alpha}{1+\alpha}$	$\frac{1+3\alpha}{1+\alpha}$	0	0
κ $(3,1^2)$	0	$\frac{3+2\alpha}{1+\alpha}$	0	$\frac{2+3\alpha}{1+\alpha}$	0
$(2^2,1)$	0	0	$\frac{3+\alpha}{1+\alpha}$	$\frac{2+4\alpha}{1+\alpha}$	0
$(2,1^3)$	0	0	0	$\frac{12+3\alpha}{3+\alpha}$	$\frac{3+2\alpha}{3+\alpha}$
(1^5)	0	0	0	0	5

Table 5c : Generalized binomial coefficients for k=5

				σ			
k=5	(5)	(4,1)	(3,2)	$(3,1^2)$	$(2^2,1)$	$(2,1^3)$	(1^5)
(5)	1	0	0	0	0	0	0
(4,1)	0	1	0	0	0	0	0
(3,2)	0	0	1	0	0	0	0
κ $(3,1^2)$	0	0	0	1	0	0	0
$(2^2,1)$	0	0	0	0	1	0	0
$(2,1^3)$	0	0	0	0	0	1	0
(1^5)	0	0	0	0	0	0	1

Table 6a : Generalized binomial coefficients for k=6

k=6	σ (0)	(1)	(2)	(1^2)	(3)	(2,1)	(1^3)
(6)	1	6	15	0	20	0	0
(5,1)	1	6	$\frac{14+10\alpha}{1+\alpha}$	$\frac{1+5\alpha}{1+\alpha}$	$\frac{16+20\alpha}{1+2\alpha}$	$\frac{4+20\alpha}{1+2\alpha}$	0
(4,2)	1	6	$\frac{13+7\alpha}{1+\alpha}$	$\frac{2+8\alpha}{1+\alpha}$	$\frac{12+8\alpha}{1+2\alpha}$	$\frac{8+32\alpha}{1+2\alpha}$	0
$(4,1^2)$	1	6	$\frac{12+6\alpha}{1+\alpha}$	$\frac{3+9\alpha}{1+\alpha}$	$\frac{10+8\alpha}{1+2\alpha}$	$\frac{6(3+11\alpha+4\alpha^2)}{(1+2\alpha)(2+\alpha)}$	$\frac{2+4\alpha}{2+\alpha}$
(3^2)	1	6	$\frac{12+6\alpha}{1+\alpha}$	$\frac{3+9\alpha}{1+\alpha}$	$\frac{8+4\alpha}{1+2\alpha}$	$\frac{12+36\alpha}{1+2\alpha}$	0
κ (3,2,1)	1	6	$\frac{11+4\alpha}{1+\alpha}$	$\frac{4+11\alpha}{1+\alpha}$	$\frac{(3+2\alpha)(2+\alpha)}{(1+\alpha)(1+2\alpha)}$	$\frac{26+83\alpha+26\alpha^2}{(2+\alpha)(1+2\alpha)}$	$\frac{(1+2\alpha)(2+3\alpha)}{(1+\alpha)(1+2\alpha)}$
$(3,1^3)$	1	6	$\frac{9+3\alpha}{1+\alpha}$	$\frac{6+12\alpha}{1+\alpha}$	$\frac{4+2\alpha}{1+2\alpha}$	$\frac{6(4+11\alpha+3\alpha^2)}{(2+\alpha)(1+2\alpha)}$	$\frac{8+10\alpha}{2+\alpha}$
(2^3)	1	6	$\frac{9+3\alpha}{1+\alpha}$	$\frac{6+12\alpha}{1+\alpha}$	0	$\frac{36+12\alpha}{2+\alpha}$	$\frac{4+8\alpha}{2+\alpha}$
$(2^2,1^2)$	1	6	$\frac{8+2\alpha}{1+\alpha}$	$\frac{7+13\alpha}{1+\alpha}$	0	$\frac{32+8\alpha}{2+\alpha}$	$\frac{8+12\alpha}{2+\alpha}$
$(2,1^4)$	1	6	$\frac{5+\alpha}{1+\alpha}$	$\frac{10+14\alpha}{1+\alpha}$	0	$\frac{20+4\alpha}{2+\alpha}$	$\frac{20+16\alpha}{2+\alpha}$
(1^6)	1	6	0	15	0	0	20

Table 6b : Generalized binomial coefficients for k=6

	σ				
k=6	(4)	(3,1)	(2^2)	$(2,1^2)$	(1^4)
(6)	15	0	0	0	0
(5,1)	$\frac{9+15\alpha}{1+3\alpha}$	$\frac{6+30\alpha}{1+3\alpha}$	0	0	0
(4,2)	$\frac{2(2+3\alpha)(1+\alpha)}{(1+2\alpha)(1+3\alpha)}$	$\frac{2(5+3\alpha)(1+4\alpha)}{(1+\alpha)(1+3\alpha)}$	$\frac{(1+3\alpha)(1+4\alpha)}{(1+\alpha)(1+2\alpha)}$	0	0
$(4,1^2)$	$\frac{3+3\alpha}{1+3\alpha}$	$\frac{3(3+13\alpha+8\alpha^2)}{(1+\alpha)(1+3\alpha)}$	0	$\frac{3+6\alpha}{1+\alpha}$	0
(3^2)	0	$\frac{12+6\alpha}{1+\alpha}$	$\frac{3+9\alpha}{1+\alpha}$	0	0
κ (3,2,1)	0	$\frac{3(3+2\alpha)(2+\alpha)}{2(1+\alpha)^2}$	$\frac{3(1+3\alpha+\alpha^2)}{(1+\alpha)^2}$	$\frac{3(2+3\alpha)(1+2\alpha)}{2(1+\alpha)^2}$	0
$(3,1^3)$	0	$\frac{6+3\alpha}{1+\alpha}$	0	$\frac{3(8+13\alpha+3\alpha^2)}{(1+\alpha)(3+\alpha)}$	$\frac{3+3\alpha}{3+\alpha}$
(2^3)	0	0	$\frac{9+3\alpha}{1+\alpha}$	$\frac{6+12\alpha}{1+\alpha}$	0
$(2^2,1^2)$	0	0	$\frac{(3+\alpha)(4+\alpha)}{(1+\alpha)(2+\alpha)}$	$\frac{2(3+5\alpha)(4+\alpha)}{(1+\alpha)(3+\alpha)}$	$\frac{2(3+2\alpha)(1+\alpha)}{(2+\alpha)(3+\alpha)}$
$(2,1^4)$	0	0	0	$\frac{30+6\alpha}{3+\alpha}$	$\frac{15+9\alpha}{3+\alpha}$
(1^6)	0	0	0	0	15

Table 6c : Generalized binomial coefficients for k=6

k=6	σ (5)	(4,1)	(3,2)	$(3,1^2)$	$(2^2,1)$	$(2,1^3)$	(1^5)
(6)	6	0	0	0	0	0	0
(5,1)	$\frac{2+4\alpha}{1+4\alpha}$	$\frac{4+20\alpha}{1+4\alpha}$	0	0	0	0	0
(4,2)	0	$\frac{4+4\alpha}{1+2\alpha}$	$\frac{2+8\alpha}{1+2\alpha}$	0	0	0	0
$(4,1^2)$	0	$\frac{6+6\alpha}{2+3\alpha}$	0	$\frac{6+12\alpha}{2+3\alpha}$	0	0	0
(3^2)	0	0	6	0	0	0	0
κ (3,2,1)	0	0	$\frac{(3+2\alpha)(2+\alpha)}{2(1+\alpha)^2}$	$\frac{(1+2\alpha)(2+\alpha)}{(1+\alpha)^2}$	$\frac{(2+3\alpha)(1+2\alpha)}{2(1+\alpha)^2}$	0	0
$(3,1^3)$	0	0	0	$\frac{12+6\alpha}{3+2\alpha}$	0	$\frac{6+6\alpha}{3+2\alpha}$	0
(2^3)	0	0	0	0	6	0	0
$(2^2,1^2)$	0	0	0	$\frac{8+2\alpha}{2+\alpha}$	$\frac{4+4\alpha}{2+\alpha}$	0	0
$(2,1^4)$	0	0	0	0	0	$\frac{20+4\alpha}{4+\alpha}$	$\frac{4+2\alpha}{4+\alpha}$
(1^6)	0	0	0	0	0	0	6

Table 6d : Generalized binomial coefficients for k=6

k=6	(6)	(5,1)	(4,2)	$(4,1^2)$	(3^2)	(3,2,1)	$(3,1^3)$	(2^3)	$(2^2,1^2)$	$(2,1^4)$	(1^6)
	σ										
(6)	1	0	0	0	0	0	0	0	0	0	0
(5,1)	0	1	0	0	0	0	0	0	0	0	0
(4,2)	0	0	1	0	0	0	0	0	0	0	0
$(4,1^2)$	0	0	0	1	0	0	0	0	0	0	0
(3^2)	0	0	0	0	1	0	0	0	0	0	0
κ (3,2,1)	0	0	0	0	0	1	0	0	0	0	0
$(3,1^3)$	0	0	0	0	0	0	1	0	0	0	0
(2^3)	0	0	0	0	0	0	0	1	0	0	0
$(2^2,1^2)$	0	0	0	0	0	0	0	0	1	0	0
$(2,1^4)$	0	0	0	0	0	0	0	0	0	1	0
(1^6)	0	0	0	0	0	0	0	0	0	0	1