Equivariant spectral triples on the quantum $SU(2)$ group

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Abstract

We characterize all equivariant odd spectral triples for the quantum $SU(2)$ group acting on its L_2-space and having a nontrivial Chern character. It is shown that the dimension of an equivariant spectral triple is at least three, and given any element of the K-homology group of $SU_q(2)$, there is an equivariant odd spectral triple of dimension 3 inducing that element. The method employed to get equivariant spectral triples in the quantum case is then used for classical $SU(2)$, and we prove that for $p < 4$, there does not exist any equivariant spectral triple with nontrivial K-homology class and dimension p acting on the L_2-space.

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1 Introduction

Study of quantum groups originated in the early eighties in the work of Fadeev, Sklyanin & Takhtajan in the context of quantum inverse scattering theory, and was given a more definitive shape by Drinfeld. It picked up momentum during the mid eighties, and connections were established with various other areas in mathematics. They were first studied in the topological setting independently by Woronowicz ([11]) and Vaksman & Soibelman ([10]), who treated the q-deformation of the $SU(2)$ group. Woronowicz then went on to characterize the family of compact quantum groups and studied their representation theory.

Noncommutative geometry was introduced around the same time by Alain Connes, drawing inspiration mainly from the work of Atiyah and Kasparov. While their classical counterparts are very intimately connected to each other, there has so far been very little work on the connection and relationship between the two notions of quantum groups and noncommutative geometry. One of the first questions that one would like to settle, for example, is that given a compact quantum group, does it admit a Dirac operator that is equivariant under its own

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(co-)action. There has been some work on this theme ([1], [5]), but none of these resolve the question satisfactorily. Present article is a modest attempt towards answering this.

The most well-known example of a compact quantum group is the q-deformation of the $SU(2)$-group, which has been studied very thoroughly for real values of the deformation parameter q by Woronowicz in [11]. It has a natural (co-)action on itself, so that it can be thaught of as an $SU_q(2)$-homogeneous space. We investigate geometries on this homogeneous space equivariant under the (co-)action of $SU_q(2)$. The earliest work on noncommutative geometry on $SU_q(2)$ is probably the paper by Masuda and Watanabe ([7]), but their treatment was more from the point of view of noncommutative topology; they do not talk about the Dirac operator which is of fundamental importance in Connes' theory, and which captures topological as well as geometric information about the space concerned. More recently, Bibikov & Kulish ([1]) and Goswami ([5]) made attempts to get an equivariant 'Dirac' operator on $SU_q(2)$, but none of them could accomplish it satisfactorily within the framework of Connes' theory, which is what we plan to do in the present article. We restrict ourselves to odd spectral triples. We characterize all equivariant odd spectral triples on the L_2 space of the haar state. In particular, we show the existence of a 3-summable equivariant spectral triple. It is also shown that an equivariant spectral triple can not be p-summable for $p < 3$. Then we go on to prove that the associated Chern character is nontrivial by computing the pairing between the induced Fredholm module and a generator for $K_1(C(SU_q(2)))$, which is $\mathbb{Z}$. Computation of the pairing along with the results of Rosenberg & Schochet ([9]) shows that the associated Fredholm module is a generator of K^1. One immediate corollary is the universality of equivariant odd spectral triples in the sense that given any odd spectral triple, there is an equivariant one that induces the same element in K^1. In the last section, we show that the spectral triples that we produce is purely a noncommutative phenomena—for classical $SU(2)$, there does not exist any equivariant 3-summable Dirac operator D with nontrivial sign acting on the L_2-space.

Except in section 5, where we treat the classical case, we will assume q to be *a real parameter lying in the interval* $(0, 1)$.

2 Preliminaries

To fix notation, let us give here a very brief description of the quantum $SU(2)$ group. The C^*-algebra of continuous functions on $SU_q(2)$, to be denoted by $\mathcal{A}$, is the C^*-algebra generated by two elements α and β satisfying the following relations:

$$\begin{aligned} \alpha^*\alpha + \beta^*\beta = I, &\qquad \alpha\alpha^* + q^2\beta\beta^* = I, \\ \alpha\beta - q\beta\alpha = 0, &\qquad \alpha\beta^* - q\beta^*\alpha = 0, \\ \beta^*\beta &= \beta\beta^*. \end{aligned}$$

We will denote by $\mathcal{A}_f$ the dense $*$-subalgebra of $\mathcal{A}$ generated by α and β. Group structure is given by the coproduct $\Delta : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$ defined by

$$\begin{aligned} \Delta(\alpha) &= \alpha \otimes \alpha - q\beta^* \otimes \beta, \\ \Delta(\beta) &= \beta \otimes \alpha + \alpha^* \otimes \beta. \end{aligned}$$

For two continuous linear functionals ρ_1 and ρ_2 on $\mathcal{A}$, one defines their convolution product by: $\rho_1 * \rho_2(a) = (\rho_1 \otimes \rho_2)\Delta(a)$. It is known ([11]) that $\mathcal{A}$ admits a faithful state h, called the Haar state, that satisfies

$$h * \rho(a) = h(a)\rho(I) = \rho * h(a)$$

for all continuous linear functionals ρ and all $a \in \mathcal{A}$. We will denote by $\mathcal{H}$ the GNS space associated with this state.

The representation theory of $SU_q(2)$ is strikingly similar to its classical counterpart. In particular, for each $n \in \{0, \frac{1}{2}, 1, \ldots\}$, there is a unique irreducible unitary representation $t^{(n)}$ of dimension $2n+1$. Denote by $t^{(n)}_{ij}$ the ijth entry of $t^{(n)}$. These are all elements of $\mathcal{A}_f$ and they form an orthogonal basis for $\mathcal{H}$. Denote by $e^{(n)}_{ij}$ the normalized $t^{(n)}_{ij}$'s, so that $\{e^{(n)}_{ij} : n = 0, \frac{1}{2}, 1, \ldots, i, j = -n, -n+1, \ldots, n\}$ is an orthonormal basis.

Remark 2.1 One has to be a little careful here, because, unlike in the classical case, the choice of matrix entries does affect the orthogonality relations. Therefore one has to specify the matrix entries one is working with. In our case, $t^{(n)}_{ij}$'s are the same as in Klimyk & Schmuedgen (page 74, [6]).

We will use the symbol ν to denote the number $1/2$ throughout this article, just to make some expressions occupy less space. Using formulas for Clebsch-Gordon coefficients, and the orthogonality relations (page 80–81 and equation (57), page 115 in [6]), one can write down the actions of α, β and β^* on $\mathcal{H}$ explicitly as follows:

$$\alpha : e^{(n)}_{ij} \mapsto a_+(n,i,j)e^{(n+\nu)}_{i-\nu,j-\nu} + a_-(n,i,j)e^{(n-\nu)}_{i-\nu,j-\nu}, \tag{2.1}$$

$$\beta : e^{(n)}_{ij} \mapsto b_+(n,i,j)e^{(n+\nu)}_{i+\nu,j-\nu} + b_-(n,i,j)e^{(n-\nu)}_{i+\nu,j-\nu}, \tag{2.2}$$

$$\beta^* : e^{(n)}_{ij} \mapsto b^+_+(n,i,j)e^{(n+\nu)}_{i-\nu,j+\nu} + b^+_-(n,i,j)e^{(n-\nu)}_{i-\nu,j+\nu}, \tag{2.3}$$

where

$$a_+(n,i,j) = \Big(q^{2(n+i)+2(n+j)+2}\frac{(1-q^{2n-2j+2})(1-q^{2n-2i+2})}{(1-q^{4n+2})(1-q^{4n+4})}\Big)^\nu, \tag{2.4}$$

$$a_-(n,i,j) = \Big(\frac{(1-q^{2n+2j})(1-q^{2n+2i})}{(1-q^{4n})(1-q^{4n+2})}\Big)^\nu, \tag{2.5}$$

$$b_+(n,i,j) = -\Big(q^{2(n+j)}\frac{(1-q^{2n-2j+2})(1-q^{2n+2i+2})}{(1-q^{4n+2})(1-q^{4n+4})}\Big)^\nu, \tag{2.6}$$

$$b_-(n,i,j) = \left(q^{2(n+i)}\frac{(1-q^{2n+2j})(1-q^{2n-2i})}{(1-q^{4n})(1-q^{4n+2})}\right)^\nu, \tag{2.7}$$

$$b_+^+(n,i,j) = \left(q^{2(n+i)}\frac{(1-q^{2n+2j+2})(1-q^{2n-2i+2})}{(1-q^{4n+2})(1-q^{4n+4})}\right)^\nu, \tag{2.8}$$

$$b_-^+(n,i,j) = -\left(q^{2(n+j)}\frac{(1-q^{2n-2j})(1-q^{2n+2i})}{(1-q^{4n})(1-q^{4n+2})}\right)^\nu. \tag{2.9}$$

3 Equivariant spectral triples

In this section, we will formulate the notion of equivariance, and investigate the behaviour of D, where D is the Dirac operator of an equivariant spectral triple.

In the classical context of a compact Lie group G, a left invariant differential operator is one that commutes with the left regular representation of G. Now in the case of abelian G, the C^*-algebra generated by the left regular representation is nothing but $C(\widehat{G})$. Therefore we can rephrase the left invariance condition as a commutation condition with $C(\widehat{G})$. For $C(SU_q(2))$, Woronowicz has explicitly described the generators for $C(\widehat{G})$. Therefore, a proper analog of a left invariant Dirac operator would be a Dirac operator commuting with these generators.

Let A_0 and A_1 be the following operators on $\mathcal{H}$:

$$\begin{aligned} A_0 &: e_{ij}^{(n)} \mapsto q^j e_{ij}^{(n)}, \\ A_1 &: e_{ij}^{(n)} \mapsto \begin{cases} 0 & \text{if } j=n, \\ (q^{-2n}+q^{2n+2}-q^{-2j}-q^{2j+2})^\nu e_{ij+1}^{(n)} & \text{if } j<n. \end{cases} \end{aligned}$$

The operators A_0 and A_1 generate the C^*-algebra of continuous functions on the dual of $SU_q(2)$ and thus are the 'generators' of the regular representation of $SU_q(2)$ (For more details, see [8]; A_0 and A_1 are the operators **a** and **n** there). We say that an operator T on $\mathcal{H}$ is **equivariant** if it commutes with A_0, A_1 and A_1^*. It is clear that any equivariant self-adjoint operator with discrete spectrum must be of the form

$$D : e_{ij}^{(n)} \mapsto d(n,i)e_{ij}^{(n)}, \tag{3.1}$$

where $d(n,i)$'s are real. Assume then that D is such an operator. Let us first write down the commutators of D with α and β.

$$\begin{aligned} [D,\alpha]e_{ij}^{(n)} &= a_+(n,i,j)(d(n+\nu,i-\nu)-d(n,i))e_{i-\nu,j-\nu}^{(n+\nu)} \\ &\quad + a_-(n,i,j)(d(n-\nu,i-\nu)-d(n,i))e_{i-\nu,j-\nu}^{(n-\nu)}, \end{aligned} \tag{3.2}$$

$$\begin{aligned} [D,\beta]e_{ij}^{(n)} &= b_+(n,i,j)(d(n+\nu,i+\nu)-d(n,i))e_{i+\nu,j-\nu}^{(n+\nu)} \\ &\quad + b_-(n,i,j)(d(n-\nu,i+\nu)-d(n,i))e_{i+\nu,j-\nu}^{(n-\nu)}. \end{aligned} \tag{3.3}$$

We are now in a position to prove the following.

Proposition 3.1 *Let D be an operator of the form $e_{ij}^{(n)} \mapsto d(n,i)e_{ij}^{(n)}$. Then $[D,a]$ is bounded for all $a \in \mathcal{A}_f$ if and only if $d(n,i)$'s satisfy the following two conditions:*

$$d(n+\nu, i+\nu) - d(n,i) = O(1), \tag{3.4}$$

$$d(n+\nu, i-\nu) - d(n,i) = O(n+i+1). \tag{3.5}$$

Proof: Assume that $[D,a]$ is bounded for all $a \in \mathcal{A}_f$. Then, in particular, $[D,\alpha]$ and $[D,\beta]$ are bounded, so that there is a positive constant C such that

$$\|[D,\alpha]\| \le C, \quad \|[D,\beta]\| \le C.$$

It follows from equations (3.2) and (3.3) that

$$|a_+(n,i,j)(d(n+\nu,i-\nu)-d(n,i))|^2 + |a_-(n,i,j)(d(n-\nu,i-\nu)-d(n,i))|^2 \le C^2, \tag{3.6}$$

$$|b_+(n,i,j)(d(n+\nu,i+\nu)-d(n,i))|^2 + |b_-(n,i,j)(d(n-\nu,i+\nu)-d(n,i))|^2 \le C^2 \tag{3.7}$$

for all n, i and j. From the second inequality above, we get

$$|b_+(n,i,j)(d(n+\nu,i+\nu)-d(n,i))| \le C \quad \forall n,i,j.$$

Now

$$|b_+(n,i,j)| = \left(\frac{q^{2n+2j}-q^{4n+2}}{1-q^{4n+2}}\right)^\nu \left(\frac{1-q^{2n+2i+2}}{1-q^{4n+4}}\right)^\nu.$$

Hence

$$1-q^2 \le \frac{1-q^2}{1-q^{4n+4}} \le \max_j |b_+(n,i,j)|^2 = \frac{1-q^{2n+2i+2}}{1-q^{4n+4}} \le \frac{1}{1-q^4}.$$

Hence $|d(n+\nu,i+\nu)-d(n,i)| \le \frac{C}{(1-q^2)^{1/2}}$ for all n, i, i. e. we have (3.4). We also have from equation (3.6), $|a_+(n,i,j)(d(n+\nu,i-\nu)-d(n,i))| \le C$. But

$$a_+(n,i,j) = q\left(\frac{q^{2n+2j}-q^{4n+2}}{1-q^{4n+2}}\right)^\nu \left(\frac{q^{2n+2i}-q^{4n+2}}{1-q^{4n+4}}\right)^\nu.$$

Hence

$$\max_j |a_+(n,i,j)| = q\left(\frac{q^{2n+2i}-q^{4n+2}}{1-q^{4n+4}}\right)^\nu.$$

Therefore

$$q\left(\frac{q^{2n+2i}-q^{4n+2}}{1-q^{4n+4}}\right)^\nu |d(n+\nu,i-\nu)-d(n,i)| \le C \quad \forall n,i.$$

Consequently, $q^{n+i}|d(n+\nu,i-\nu)-d(n,i)| \le q^{-1}C\frac{1}{(1-q^2)^\nu}$, i. e.

$$|d(n+\nu,i-\nu)-d(n,i)| = O(q^{-n-i}). \tag{3.8}$$

Let us next write the difference $d(n+\nu, i-\nu) - d(n,i)$ as follows:

$$\begin{aligned}\sum_{r=0}^{n+i-1} &(d(n+\nu-r\nu, i-\nu-r\nu) - d(n+\nu-(r+1)\nu, i-\nu-(r+1)\nu)) \\ &- \sum_{r=0}^{n+i-1} (d(n-r\nu, i-r\nu) - d(n-(r+1)\nu, i-(r+1)\nu)) \\ &+ d(n+\nu-(n+i)\nu, i-\nu-(n+i)\nu) - d(n-(n+i)\nu, i-(n+i)\nu)\end{aligned}$$

Using this expression together with (3.8) for the case $n+i=0$ and (3.4), we get (3.5).

Next assume that the $d(n,i)$'s satisfy the conditions (3.4) and (3.5). We will show that $[D,\alpha]$ and $[D,\beta]$ are bounded, which in turn will ensure that $[D,a]$ is bounded for all $a \in \mathcal{A}_f$. It follows from (3.4) and (3.5) that there is a positive constant $C > 0$ such that

$$|d(n+\nu, i+\nu) - d(n,i)| \leq C, \quad q^{n+i}|d(n+\nu, i-\nu) - d(n,i)| \leq C.$$

It follows from the above two inequalities that

$$\begin{aligned}|a_+(n,i,j)(d(n+\nu, i-\nu) - d(n,i))| &\leq C(1-q^4)^{-1/2}, \\ |a_-(n,i,j)(d(n-\nu, i-\nu) - d(n,i))| &\leq Cq^{-1}(1-q^2)^{-1/2}.\end{aligned}$$

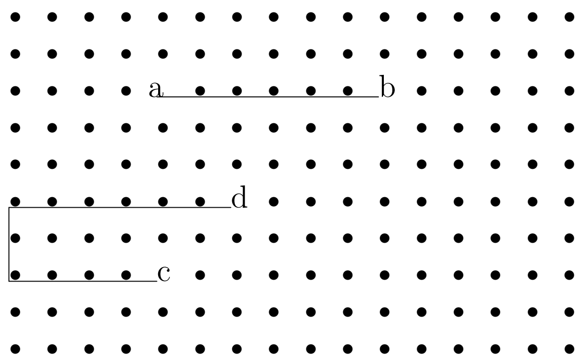
We now conclude from (3.2) that $[D,\alpha]$ is bounded. Proof of boundedness of $[D,\beta]$ is similar. □

Next, we exploit the condition that D must have compact resolvent. It is straightforward to see that a necessary and sufficient condition for an operator D of the form $e_{ij}^{(n)} \mapsto d(n,i)e_{ij}^{(n)}$ to have compact resolvent is that if we write the $d(n,i)$'s in a single sequence, it should not have any limit point other than ∞ or $-\infty$. As we shall see below, in presence of (3.4) and (3.5), we can say much more about the $d(n,i)$'s. In particular, we can extract information about the sign of D also.

Proposition 3.2 *Let D be an operator of the form $e_{ij}^{(n)} \mapsto d(n,i)e_{ij}^{(n)}$ such that $d(n,i)$'s satisfy conditions (3.4) and (3.5) and D has compact resolvent. Then*

1. *For each $k \in \mathbb{N}$, there exists an $r_k \in \mathbb{N}$, $r_k \geq k$ such that $d(n, n-k)$ are of the same sign for all $n \geq r_k$.*
2. *There exists an $r \in \mathbb{N}$ such that for all $k \geq r$ and for all n, $d(n, n-k)$ are of the same sign.* (3.9)

Proof: In the following diagram, each dot stands for a $d(n,i)$, the dot at the i^{th} row and j^{th} column representing $d(\frac{i+j}{2}, \frac{j-i}{2})$ (here i and j range from 0 onwards).



There are two restrictions imposed on these numbers, given by equations (3.4) and (3.5). Equation (3.4) says that: (i) the difference of two consecutive numbers along any row is bounded by a fixed constant, and (3.5) says that: (ii) the difference of two consecutive numbers along the j^{th} column is $O(j+1)$. Suppose C is a big enough constant which works for both (i) and (ii).

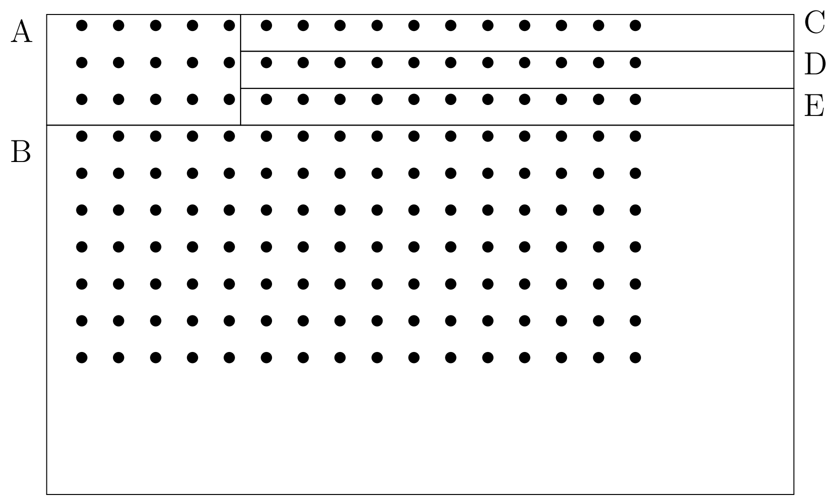
Now suppose a and b are two elements in the same row. Connect them with a path as in the diagram. If a and b are of opposite sign, then because of restriction (i) above, there has to be some dot between a and b for which the corresponding $d(n,i)$ lies in $[-C,C]$. Therefore, if the signs of the $d(n,i)$'s change infinitely often along a row, one can produce infinitely many $d(n,i)$'s in the interval $[-C,C]$. But this will prevent D from having a compact resolvent. This proves part 1.

For part 2, employ a similar argument, this time connecting two dots, say c and d, by a path as shown in the diagram, and observing that the difference between any two consecutive numbers along the path is bounded by C. $\square$

Let m and n be two nonnegative integers. Let

$$\begin{aligned}
F(m,n) &= \left\{ d\left(\frac{j+i}{2}, \frac{j-i}{2}\right) : 0 \leq i \leq m, 0 \leq j \leq n \right\}, \\
S(m,n,r) &= \left\{ d\left(\frac{j+r}{2}, \frac{j-r}{2}\right) : j > n \right\}, \quad 0 \leq r \leq m, \\
T(m) &= \left\{ d\left(\frac{j+i}{2}, \frac{j-i}{2}\right) : i > m, j \geq 0 \right\}.
\end{aligned}$$

In the following diagram, for example, A is $F(2,4)$, B is $T(2)$, and C, D and E are $S(2,4,0)$, $S(2,4,1)$ and $S(2,4,2)$ respectively.



What the last proposition says is the following. There exist big enough integers m and n such that in each of the sets $T(m)$, $S(m,n,0),\ldots,S(m,n,m)$, all elements are of the same sign, i. e. each of the sets $T(m)$, $S(m,n,0),\ldots,S(m,n,m)$ is contained in either $\mathbb{R}_+$ or $-\mathbb{R}_+$.

Remark 3.3 One can extend the argument in the proof of the last proposition a little further and prove that if D is as in the previous proposition, then

given any nonnegative real N, there exist positive integers m and n such that each of the sets $T(m)$, $S(m,n,0),\ldots,S(m,n,m)$ is contained in either $\{x \in \mathbb{R} : x > N\}$ or $\{x \in \mathbb{R} : x < -N\}$. (3.10)

Theorem 3.4 *An operator D on $L_2(h)$ gives rise to an equivariant spectral triple if and only if it is of the form $e_{ij}^{(n)} \mapsto d(n,i)e_{ij}^{(n)}$, where $d(n,i)$'s are real and satisfy conditions (3.4), (3.5) and (3.10).*

Proof: It is enough to prove that if the $d(n,i)$'s obey condition (3.10), then D has compact resolvent. But this is clear, because (3.10) implies that for any real number $N > 0$, the interval $[-N,N]$ contains only a finite number of the $d(n,i)$'s. □

It is clear then that up to a compact perturbation, D will have nontrivial sign if and only if the following condition holds:

1. there exist positive integers m and n such that in each of the sets $T(m)$, $S(m,n,0),\ldots,S(m,n,m)$, all elements are of the same sign, and
2. there are two sets in this collection whose elements are of opposite sign.

(3.11)

A natural question to ask now is whether there does indeed exist any D with nontrivial sign satisfying (3.4) and (3.5). It is easy to see that the operator D determined by the family

$d(n,i)$, where

$$d(n,i)=\begin{cases}2n+1 & \text{if } n\neq i,\\ -(2n+1) & \text{if } n=i,\end{cases}\tag{3.12}$$

satisfy all the requirements in propositions 3.1 and 3.2. In fact, one can easily see that $D^{-3}\in\mathcal{L}^{(1,\infty)}$, where $\mathcal{L}^{(1,\infty)}$ stands for the ideal of Dixmier traceable operators. Thus we have the following.

Theorem 3.5 *$SU_q(2)$ admits an equivariant odd 3-summable spectral triple.*

The classical $SU(2)$ has (both topological as well as metric) dimension 3. For $SU_q(2)$, however, the topological dimension turns out to be 1, as can be seen from the following short exact sequence

$$0\longrightarrow\mathcal{K}\otimes C(S^1)\longrightarrow\mathcal{A}\longrightarrow C(S^1)\longrightarrow 0,$$

where $\mathcal{K}$ denotes the algebra of compact operators. The next theorem tells us that as far as metric dimension is concerned, it behaves more like its classical counterpart; in fact along with the previous theorem, it says that the metric dimension of $SU_q(2)$ is 3.

Theorem 3.6 *Let $(\mathcal{A},\mathcal{H},D)$ be an equivariant odd spectral triple. Then D can not be p-summable for $p<3$.*

Proof: Conditions (3.4) and (3.5) impose the following growth restriction on the $d(n,i)$'s:

$$\max_i|d(n,i)|=O(n).\tag{3.13}$$

The conclusion of the theorem follows easily from this. □

The next proposition gives a nice property of the operator D, namely, it says that the derivative of any nonconstant function is nonzero.

Proposition 3.7 *Let D be given by (3.12). Then for $a\in\mathcal{A}_f$, $[D,a]=0$ if and only if a is a scalar.*

Proof: Take $a=\sum_{(i,j,k)\in F}c_{ijk}\alpha_i\beta^j\beta^{*k}$, where F is a finite subset of $\mathbb{Z}\times\mathbb{N}\times\mathbb{N}$ and all the c_{ijk}'s are nonzero (here α_i is α^i for $i\geq 0$ and $(\alpha^*)^{-i}$ for $i<0$). We will show that $[D,a]\neq 0$.

Let $m=\max\{|i|+j+k:(i,j,k)\in F\}$. Let (r,s,t) be a point of F such that $|r|+s+t=m$. Write $p=\frac{1}{2}(s-t-r)$, $p'=\frac{1}{2}(t-s-r)$. Then it is easy to see that

$$\begin{aligned}&\langle e_{pp'}^{(n+m/2)},[D,a]e_{00}^{(n)}\rangle\\ &=\ \langle e_{pp'}^{(n+m/2)},[D,c_{rst}\alpha_r\beta^s\beta^{*t}]e_{00}^{(n)}\rangle\\ &=\ c_{rst}\prod_{i=1}^{t}b_+^+\left(n+\frac{i-1}{2},-\frac{i-1}{2},\frac{i-1}{2}\right)\prod_{i=t+1}^{t+s}b_+\left(n+\frac{i-1}{2},-t+\frac{i-1}{2},t-\frac{i-1}{2}\right)\end{aligned}$$

$$\times \prod_{i=s+t+1}^{m} a_+^{\#}\left(n+\frac{i-1}{2}, p+\operatorname{sign}(r)\frac{m-i}{2}, p'+\operatorname{sign}(r)\frac{m-i}{2}\right)$$
$$\times \Big(d(n+m/2,p)-d(n,0)\Big),$$

where $a_+^{\#}$ stands for a_+ or a_+^+ depending on the sign of r. The right hand side above is clearly nonzero because of our choice of D. □

The above proposition says, in particular, that the Dirac operator given by (3.12) is really a Dirac operator for the full tangent bundle rather than that of some lower dimensional subbundle.

4 Nontriviality of the Chern character

In this section we will examine the D given by the family (3.12) in more detail and see that the nontriviality in sign does indeed result in nontriviality at the Fredholm level. For this, we will compute the pairing between $\operatorname{sign} D$ and a generator of $K_1(\mathcal{A})$. Let u denote the element $\chi_{\{1\}}(\beta^*\beta)(\beta-I)+I$ of $\mathcal{A}$, where, for a normal operator T, $\chi_F(T)$ denotes the spectral projection of T corresponding to a subset F of the spectrum. It can be shown that this is a generator of $K_1(\mathcal{A})$. What we will do is the following. We will choose an invertible element γ in $\mathcal{A}_f$ that is close enough to u so that γ and u are the same in $K_1(\mathcal{A})$. We then compute the pairing between $\operatorname{sign} D$ and this γ.

Theorem 4.1 *The Chern character of the spectral triple $(\mathcal{A}_f, \mathcal{H}, D)$ is nontrivial.*

Remark 4.2 Goswami ([5]) gives an example of an equivariant operator D acting on $\mathcal{H}\otimes\mathbb{C}^2$, and having bounded commutators with the algebra elements. But this D ($|D|$ in his notation) is positive, hence has trivial pairing with K-theory. Now in most cases it is possible to find a self-adjoint operator with compact resolvent and bounded commutators with the algebra elements just by looking at the elements affiliated to the commutant of the algebra represented on a Hilbert space. It is to avoid this kind of trivialities that the nontrivial pairing is a very crucial requirement. In the present case, as the theorem above shows, the Dirac operator defined by (3.12) does have a nontrivial pairing with K-theory.

Before we begin the proof of the theorem, let us observe from equations (2.2) and (2.3) that the action of $\beta\beta^*$ on $\mathcal{H}$ is given by

$$(\beta\beta^*)(e_{ij}^{(n)}) = \sum_{\epsilon=-1}^{1} k_\epsilon(n,i,j)e_{ij}^{(n+\epsilon)}, \tag{4.1}$$

where

$$k_1(n,i,j) = -\Big(q^{4n+2i+2j+2}\frac{1-q^{2n+2j+2}}{1-q^{4n+2}}\frac{1-q^{2n-2i+2}}{1-q^{4n+4}}\frac{1-q^{2n-2j+2}}{1-q^{4n+4}}\frac{1-q^{2n+2i+2}}{1-q^{4n+6}}\Big)^{\nu}, \tag{4.2}$$

$$k_0(n,i,j) = q^{2(n+j)}\frac{(1-q^{2n-2j})(1-q^{2n+2i})}{(1-q^{4n})(1-q^{4n+2})} + q^{2(n+i)}\frac{(1-q^{2n+2j+2})(1-q^{2n-2i+2})}{(1-q^{4n+2})(1-q^{4n+4})}, \tag{4.3}$$

$$k_{-1}(n,i,j) = -\Big(q^{4n+2i+2j-2}\frac{(1-q^{2n-2j})(1-q^{2n+2i})(1-q^{2n+2j})(1-q^{2n-2i})}{(1-q^{4n-2})(1-q^{4n})(1-q^{4n})(1-q^{4n+2})}\Big)^{\nu}. \tag{4.4}$$

Proof of theorem 4.1 : Choose $r \in \mathbb{N}$ such that $q^{2r} < \frac{1}{2} < q^{2r-2}$. Define $\gamma_r = (\beta^*\beta)^r(\beta - I) + I$. By our choice of r, we have

$$\begin{aligned} \|\gamma_r - u\| &\leq \|(\beta^*\beta)^r - \chi_{\{1\}}(\beta^*\beta)\| \cdot \|\beta - I\| \\ &\leq 2q^{2r} < 1. \end{aligned}$$

Hence γ_r and u are the same in $K_1(\mathcal{A})$. Therefore it is enough for our purpose if we can show that the pairing between $\operatorname{sign} D$ and γ_r is nontrivial. Denote by P_k the projection onto the space spanned by $\{e_{n-k,j}^{(n)} : n, j\}$. Then $\operatorname{sign} D = I - 2P_0$. Therefore we now want to compute the index of the operator $P_0\gamma_r P_0$ thaught of as an operator on $P_0\mathcal{H}$.

It follows from (4.1) that

$$(\beta\beta^*)^r(e_{ij}^{(n)}) = \sum_{\epsilon_t\in\{-1,0,1\}}\Big(\prod_{t=1}^{r} k_{\epsilon_t}(n+\sum_{s=1}^{t-1}\epsilon_s, i, j)\Big)e_{ij}^{(n+\sum_1^r \epsilon_s)}. \tag{4.5}$$

Since β is normal, we have

$$\begin{aligned} \gamma_r e_{ij}^{(n)} = & \sum_{\epsilon_t\in\{-1,0,1\}}\Big(\prod_{t=1}^{r} k_{\epsilon_t}(n+\sum_{s=1}^{t-1}\epsilon_s, i, j)\Big)\Big(b_+(n+\sum_1^r \epsilon_s, i, j)e_{i+\nu,j-\nu}^{(n+\sum_1^r\epsilon_s+\nu)} \\ & \qquad + b_-(n+\sum_1^r \epsilon_s, i, j)e_{i+\nu,j-\nu}^{(n+\sum_1^r\epsilon_s-\nu)}\Big) \\ & - \sum_{\epsilon_t\in\{-1,0,1\}}\Big(\prod_{t=1}^{r} k_{\epsilon_t}(n+\sum_{s=1}^{t-1}\epsilon_s, i, j)\Big)e_{ij}^{(n+\sum_1^r\epsilon_s)} + e_{ij}^{(n)}. \end{aligned} \tag{4.6}$$

Consequently,

$$\begin{aligned} \gamma_r e_{nj}^{(n)} = & \sum_{\epsilon_t\in\{-1,0,1\}}\Big(\prod_{t=1}^{r} k_{\epsilon_t}(n+\sum_{s=1}^{t-1}\epsilon_s, n, j)\Big)\Big(b_+(n+\sum_1^r \epsilon_s, n, j)e_{n+\nu,j-\nu}^{(n+\sum_1^r\epsilon_s+\nu)} \\ & \qquad + b_-(n+\sum_1^r \epsilon_s, n, j)e_{n+\nu,j-\nu}^{(n+\sum_1^r\epsilon_s-\nu)}\Big) \\ & - \sum_{\epsilon_t\in\{-1,0,1\}}\Big(\prod_{t=1}^{r} k_{\epsilon_t}(n+\sum_{s=1}^{t-1}\epsilon_s, n, j)\Big)e_{nj}^{(n+\sum_1^r\epsilon_s)} + e_{nj}^{(n)}. \end{aligned}$$

When we cut this off by P_0, we get

$$
\begin{aligned}
P_0\gamma_r e_{nj}^{(n)} &= \sum_{\sum \epsilon_t=0} \Big(\prod_{t=1}^{r} k_{\epsilon_t}(n+\sum_{s=1}^{t-1}\epsilon_s, n, j)\Big) b_+(n,n,j) e_{n+\nu,j-\nu}^{(n+\nu)} \\
&\quad + \sum_{\sum \epsilon_t=1} \Big(\prod_{t=1}^{r} k_{\epsilon_t}(n+\sum_{s=1}^{t-1}\epsilon_s, n, j)\Big) b_-(n+1,n,j) e_{n+\nu,j-\nu}^{(n+\nu)} \\
&\quad - \sum_{\sum \epsilon_t=0} \Big(\prod_{t=1}^{r} k_{\epsilon_t}(n+\sum_{s=1}^{t-1}\epsilon_s, n, j)\Big) e_{nj}^{(n)} + e_{nj}^{(n)}.
\end{aligned}
$$

A closer look at the quantities k_ϵ and $b_\pm$ tells us that if we do the calculations modulo compact operators, which we can because we want to compute the index, we find that there is no contribution from the second term, while in the case of the first and the third term, contribution comes from only the coefficient where the product $\prod_{t=1}^{r} k_{\epsilon_t}(n+\epsilon_1+\ldots+\epsilon_{t-1}, n, j)$ consists solely of k_0's, i. e. when each $\epsilon_t = 0$. A further examination of the terms k_0 and b_+ then yield the following:

$$
\begin{aligned}
P_0\gamma_r P_0 e_{nj}^{(n)} &= k_0(n,n,j)^r b_+(n,n,j) e_{n+\nu,j-\nu}^{(n+\nu)} + (1-k_0(n,n,j)^r) e_{nj}^{(n)} \\
&= -q^{2rn+2rj}(1-q^{2n-2j})^r q^{n+j}(1-q^{2n-2j+2})^{1/2} e_{n+\nu,j-\nu}^{(n+\nu)} \\
&\qquad + \left(1-q^{2rn+2rj}(1-q^{2n-2j})^r\right) e_{nj}^{(n)},
\end{aligned}
$$

and

$$
\begin{aligned}
P_0\gamma_r^* P_0 e_{nj}^{(n)} &= -q^{2rn+2rj}(1-q^{2n-2j-2})^r q^{n+j}(1-q^{2n-2j})^{1/2} e_{n-\nu,j+\nu}^{(n-\nu)} \\
&\qquad + \left(1-q^{2rn+2rj}(1-q^{2n-2j})^r\right) e_{nj}^{(n)}
\end{aligned}
$$

From these, one can easily show that the index of $P_0\gamma_r P_0$ is -1. Since P_0 is the eigenspace corresponding to the eigenvalue -1 of $\operatorname{sign} D$, the value of the K-homology–K-theory pairing $\langle [u], [(\mathcal{A}, \mathcal{H}, D)]\rangle$ coming from Kasparov product of K^1 and K_1 is $-\operatorname{index} P_0\gamma_r P_0$, which is nonzero. □

Remark 4.3 Strictly speaking, it is not essential to introduce the element u as a generator for $K_1(\mathcal{A})$. It is enough if one computes the pairing between $\operatorname{sign} D$ and a suitable γ_r and show that it is nontrivial. But the introduction of u makes the choice of γ_r's and hence the proof above more transparent.

It follows from proposition 3.2 that for the purposes of computing the index pairing, sign of any equivariant D must be of the form $I - 2P$ where $P = \sum_{k\in F} P_k$, F being a finite subset of $\mathbb{N}$ (the actual P would be a compact perturbation of this). Conversely, given a P of this form, it is easy to produce a D satisfying the conditions in proposition 3.2 for which $\operatorname{sign} D = I - 2P$. One could, for example, take the D given by $d(n,i)$'s, where

$$d(n,i) = \begin{cases} -(2n+1) & \text{if } n-i \in F, \\ 2n+1 & \text{otherwise.} \end{cases} \tag{4.7}$$

We are now in a position to prove the following.

Proposition 4.4 *Given any $m \in \mathbb{Z}$, there exists an equivariant spectral triple D acting on $\mathcal{H}$ such that $\langle \gamma_r, [(\mathcal{A}, \mathcal{H}, D)] \rangle = m$, where $\langle , \cdot, \cdot \rangle : K_1(\mathcal{A}) \times K^1(\mathcal{A}) \to \mathbb{Z}$ denotes the map coming from the Kasparov product.*

Proof: It is enough to prove the statement for m positive. Let D be an equivariant Dirac operator whose sign is $I - 2P$ where $P = \sum_{k \in F} P_k$, F being a subset of size m of $\mathbb{N}$. In order to compute the pairing $\langle \gamma_r, [(\mathcal{A}, \mathcal{H}, D)] \rangle$, we must first have a look at $P_{k+l}\gamma_r P_k$.

We get from equation (4.6)

$$\begin{aligned} \gamma_r e^{(n)}_{n-k,j} &= \sum_{\epsilon_t \in \{-1,0,1\}} \Big(\prod_{t=1}^{r} k_{\epsilon_t}(n + \sum_{s=1}^{t-1} \epsilon_s, n-k, j) \Big) \\ &\quad \times \Big(b_+(n + \sum_1^r \epsilon_s, n-k, j) e^{(n+\sum_1^r \epsilon_s + \nu)}_{n-k+\nu, j-\nu} + b_-(n + \sum_1^r \epsilon_s, n-k, j) e^{(n+\sum_1^r \epsilon_s - \nu)}_{n-k+\nu, j-\nu} \Big) \\ &\quad - \sum_{\epsilon_t \in \{-1,0,1\}} \Big(\prod_{t=1}^{r} k_{\epsilon_t}(n + \sum_{s=1}^{t-1} \epsilon_s, n-k, j) \Big) e^{(n+\sum_1^r \epsilon_s)}_{n-k,j} + e^{(n)}_{n-k,j}. \end{aligned}$$

and consequently,

$$\begin{aligned} P_{k+l}\gamma_r e^{(n)}_{n-k,j} &= \sum_{\sum \epsilon_t = l} \Big(\prod_{t=1}^{r} k_{\epsilon_t}(n + \sum_{s=1}^{t-1} \epsilon_s, n-k, j) \Big) b_+(n+l, n-k, j) e^{(n+l+\nu)}_{n-k+\nu, j-\nu} \\ &\quad + \sum_{\sum \epsilon_t = l+1} \Big(\prod_{t=1}^{r} k_{\epsilon_t}(n + \sum_{s=1}^{t-1} \epsilon_s, n-k, j) \Big) b_-(n+l+1, n-k, j) e^{(n+l+\nu)}_{n-k+\nu, j-\nu} \\ &\quad - \sum_{\sum \epsilon_t = l} \Big(\prod_{t=1}^{r} k_{\epsilon_t}(n + \sum_{s=1}^{t-1} \epsilon_s, n-k, j) \Big) e^{(n+l)}_{n-k,j} + \delta_{l0} e^{(n)}_{n-k,j}. \end{aligned}$$

Now because of the nature of the quantities k_ϵ and $b_\pm$, we see that for index calculations, none of the terms contribute anything for $l \neq 0$, while for $l = 0$, the first, third and the fourth term survive, with coefficient in the first term being $k_0(n, n-k, j)^r b_+(n, n-k, j)$ and that in the third being $(1 - k_0(n, n-k, j)^r)$. It follows from here that

$$\operatorname{index} P_k \gamma_r P_k = -1,$$

and $P_{k+l}\gamma_r P_k$ is compact for $l \neq 0$. Therefore the pairing between $\operatorname{sign} D$ and γ_r produces m. □

An immediate corollary of the above proposition and theorem 1.17 in [9] is the following universality property of equivariant spectral triples.

Corollary 4.5 *Given any odd spectral triple $(\mathcal{A}, \mathcal{K}, D)$, there is an equivariant triple $(\mathcal{A}, \mathcal{H}, D')$ inducing the same element in $K^1(\mathcal{A})$.*

Finally, we have the following characterization theorem for equivariant Dirac operators.

Theorem 4.6 *$(\mathcal{A}, \mathcal{H}, D)$ is an equivariant odd spectral triple with nontrivial Chern character if and only if D is given by (3.1) and the $d(n,i)$'s obey conditions (3.4), (3.5), (3.10) and (3.11).*

Proof: If D is of the form $e_{ij}^{(n)} \mapsto d(n,i) e_{ij}^{(n)}$, where $d(n,i)$'s are real and satisfy conditions (3.4), (3.5), (3.10) and (3.11), then proposition 3.1 says $[D, a]$ is bounded and nontriviality of Chern character follows from arguments of proposition 4.4. Conversely, if D is equivariant, then by propositions 3.1, 3.2 and remark 3.3, we have (3.4), (3.5) and (3.10). Since D has nontrivial Chern character, it has nontrivial sign so that we have (3.11). $\square$

5 The case $q = 1$

It would be interesting at this point to see what happens in the case $q = 1$, i. e. for the classical $SU(2)$. In particular, if the operator D given by (3.12) yields anything in that case.

The representation of $C(SU(2))$ on $L_2(SU(2))$ is given by

$$\begin{aligned} \alpha : e_{ij}^{(n)} &\mapsto a_+(n,i,j) e_{i-\nu,j-\nu}^{(n+\nu)} + a_-(n,i,j) e_{i-\nu,j-\nu}^{(n-\nu)}, \\ \beta : e_{ij}^{(n)} &\mapsto b_+(n,i,j) e_{i+\nu,j-\nu}^{(n+\nu)} + b_-(n,i,j) e_{i+\nu,j-\nu}^{(n-\nu)}, \end{aligned}$$

where

$$a_+(n,i,j) = \Big(\frac{(n-j+1)(n-i+1)}{(2n+1)(2n+2)}\Big)^\nu, \tag{5.1}$$

$$a_-(n,i,j) = \Big(\frac{(n+j)(n+i)}{2n(2n+1)}\Big)^\nu, \tag{5.2}$$

$$b_+(n,i,j) = -\Big(\frac{(n-j+1)(n+i+1)}{(2n+1)(2n+2)}\Big)^\nu, \tag{5.3}$$

$$b_-(n,i,j) = \Big(\frac{(n+j)(n-i)}{2n(2n+1)}\Big)^\nu, \tag{5.4}$$

Remark 5.1 From these, it is immediate that the operator D given by (3.12) in this case does not have bounded commutators. One should, however, note that the associated Kasparov module has a nontrivial K-homology class, as can be seen by directly computing the pairing of $\operatorname{sign} D$ with the fundamental unitary $\begin{pmatrix} \alpha & -\beta^* \\ \beta & \alpha^* \end{pmatrix}$.

Observe that the representation of (the complexification of) $\mathfrak{su}(2)$ on $L_2(SU(2))$ is given by

$$\begin{aligned} he^{(n)}_{ij} &= (n-2j)e^{(n)}_{ij}, \\ ee^{(n)}_{ij} &= j(n-2j+1)e^{(n)}_{i,j-1}, \\ fe^{(n)}_{ij} &= e^{(n)}_{i,j+1}, \end{aligned}$$

where h, e and f obey

$$[h,e]=2e, \quad [h,f]=-2f, \quad [e,f]=h.$$

Therefore, in this case also any equivariant self-adjoint operator with discrete spectrum must be of the form

$$D : e^{(n)}_{ij} \mapsto d(n,i)e^{(n)}_{ij}. \tag{5.5}$$

Commutators of this operator with α and β are once again given by (3.2) and (3.3), where $a_\pm$ and $b_\pm$ are now given by equations (5.1)–(5.4).

Lemma 5.2 *Suppose D is an operator on $L_2(SU(2))$ given by (5.5) and having bounded commutators with α and β. Assume that except for finitely many n's, the set $\{d(n,i) : i = -n, -n+1, \ldots, n\}$ contains elements of both signs, then D can not be p-summable for $p < 4$.*

Proof: Conditions for boundedness of the commutators give us

$$|d(n+\nu, i+\nu) - d(n,i)| = O\Big(\Big(\frac{2n+2}{n+i+1}\Big)^{\nu}\Big), \tag{5.6}$$

$$|d(n+\nu, i-\nu) - d(n,i)| = O\Big(\Big(\frac{2n+2}{n-i+1}\Big)^{\nu}\Big). \tag{5.7}$$

Clearly, then, there is a $K > 0$ such that

$$\begin{aligned} |d(n+\nu, i+\nu) - d(n,i)| &\le K\sqrt{n}, \\ |d(n+\nu, i-\nu) - d(n,i)| &\le K\sqrt{n}. \end{aligned}$$

By assumption, there is an i such that $d(n,i)$ and $d(n,i+1)$ are of opposite signs. Therefore, either the pair $d(n-\nu, i+\nu)$ and $d(n,i)$ or the pair $d(n-\nu, i+\nu)$ and $d(n,i+1)$ must be of opposite sign, so that their difference is really the sum of their absolute values. Because of the above inequalities, either $|d(n,i)| \le K\sqrt{n}$ or $|d(n,i+1)| \le K\sqrt{n}$. Thus for all but finitely many n's, there is an i such that $|d(n,i)| \le K\sqrt{n}$. Now it is a routine exercise in operator theory to show that D can not be p-summable for $p < 4$. □

Proposition 5.3 *Suppose D be an operator on $L_2(SU(2))$ given by (5.5), having nontrivial sign and having bounded commutators with α and β. Then except for finitely many n's, the set $\{d(n,i) : i = -n, -n+1, \ldots, n\}$ contains elements of both signs.*

Note that we call sign D trivial if it is I or $-I$ up to a compact perturbation.

Proof: Observe from (5.6) and (5.7) that if we restrict ourselves to the region $i \geq 0$, then

$$|d(n+\nu, i+\nu) - d(n,i)| = O(1), \tag{5.8}$$

and if we restrict to $i \leq 0$, then

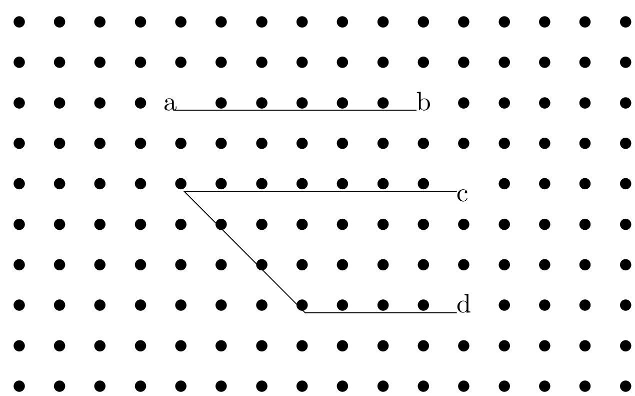
$$|d(n+\nu, i-\nu) - d(n,i)| = O(1). \tag{5.9}$$

Also, it is not too difficult to see that

$$|d(n+1, 0) - d(n,0)| = O(1). \tag{5.10}$$

Suppose $C > 0$ is a constant that works for (5.6)–(5.10).

For the region $i \geq 0$, using arguments similar to that employed in the proof of proposition 3.2 but connecting two elements c and d lying on two different rows by a path as shown in the diagram here and using (5.8) and (5.10), one can show that



1. for any given row, signs of all the $d(n,i)$'s are eventually the same,
2. there exists an integer $K > 0$ such that $(K+1)^{\text{th}}$ row onwards, all the $d(n,i)$'s have the same sign in the region $i \geq 0$.

Similar reasoning tells us that for the region $i \leq 0$,

3. for any given column, signs of all the $d(n,i)$'s are eventually the same,
4. there exists an integer $K' > 0$ such that $(K'+1)^{\text{th}}$ column onwards, all the $d(n,i)$'s have the same sign in the region $i \leq 0$.

Since the intersection of the regions $i \leq 0$ and $i \geq 0$ is nonempty, it follows that

5. if we leave out the first K rows and first K' columns, all the remaining $d(n,i)$'s have the same sign.

Let R_{mk} be the set of $d(n,i)$'s in the k^{th} row lying on the $(m+1)^{\text{th}}$ column onwards, C_{mk} be the set of $d(n,i)$'s in the k^{th} column lying on the $(m+1)^{\text{th}}$ row onwards and T_m be the set of $d(n,i)$'s at the ij^{th} position, where $i \geq m+1$ and $j \geq m+1$. In other words, let

$$\begin{aligned} R_{mk} &= \Big\{d(n, n-k) : n \geq \frac{m+k}{2}\Big\}, \\ C_{mk} &= \Big\{d(n, k-n) : n \geq \frac{m+k}{2}\Big\}, \\ T_m &= \Big\{d(n,i) : n \geq m, -n+m \leq i \leq n-m\Big\}. \end{aligned}$$

From the observations (1–5) above, we conclude that there is a big enough integer m such that the sets $R_{m0}, \ldots, R_{mm}$, $C_{m0}, \ldots, C_{mm}$, and T_m are all contained in either $\mathbb{R}_+$ or $-\mathbb{R}_+$, and there are at least two sets in this collection whose elements are of opposite signs. This immediately tells us that for all $n \geq m$, the set $\{d(n,i) : i = -n, -n+1, \ldots, n\}$ has both positive as well as negative elements. $\square$

Combining lemma 5.2 and proposition 5.3, we now get the following theorem.

Theorem 5.4 *Suppose $(C(SU(2)), L_2(SU(2)), D)$ is an equivariant spectral triple, and assume that D has nontrivial sign. Then D can not be p-summable for $p < 4$.*

The following example illustrates that the bound obtained in the above theorem on the summability of D is the best possible.

Let D be given by the following $d(n,i)$'s

$$d(n,i) = \begin{cases} -[\sqrt{2n}] & \text{if } i = n, \\ 2[\sqrt{2n}] & \text{if } i = n-1, \\ 3[\sqrt{2n}] & \text{if } i = n-2, \\ \cdots & \cdots \\ k_{2n}[\sqrt{2n}] & \text{if } i = n - k_{2n} + 1, \\ 2n & \text{if } i \leq n - k_{2n}, \end{cases} \tag{5.11}$$

where $k_{2n}+1$ is the least integer greater than or equal to $\frac{2n}{[\sqrt{2n}]}$. Then for the operator $|D|$, following are the eigenvalues along with their multiplicities on the n^{th} chunk, i. e. on span$\{e_{ij}^{(n)} : i,j = -n, -n+1, \ldots, n\}$:

eigenvalue	multiplicity
$[\sqrt{2n}]$	$2n+1$
$2[\sqrt{2n}]$	$2n+1$
$3[\sqrt{2n}]$	$2n+1$
$\cdots$	$\cdots$
$k_{2n}[\sqrt{2n}]$	$2n+1$
$2n$	$(2n+1)(2n-k_{2n}+1)$

Therefore each integer $n \in \mathbb{N}$ is an eigenvalue for $|D|$, and the multiplicity m_n of n is given by

$$m_n = (n+1)(n+1-k_n) + \sum_{r|n}(\frac{n^2}{r^2}+1) + \sum_{r|n}(\frac{n^2}{r^2}+2) + \ldots + \sum_{r|n}(\frac{n^2}{r^2}+2n+1). \quad (5.12)$$

It follows from this that $m_n = O(n^3)$, so that D is at most 4-summable. But D has nontrivial sign, and therefore by the theorem above, it can not be p-summable for $p < 4$. Hence D is 4-summable.

Since the sign of this operator coincides with the sign of the operator given by (3.12), it follows from remark 5.1 that the K-homology class of this D is nontrivial.

Remark 5.5 *The analysis in this section shows that if one restricts oneself to $L_2(SU(2))$, it is not possible to get an equivariant Dirac operator with the right summability and having a nontrivial K-homology class at the same time. Thus the classical Dirac operator for $SU(2)$, which resides on $L_2(SU(2)) \otimes \mathbb{C}^2$ is in some sense the minimal one.*

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