Spectral triples and associated Connes-de Rham complex for the quantum $SU(2)$ and the quantum sphere

Partha Sarathi Chakraborty and Arupkumar Pal

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Abstract

In this article, we construct spectral triples for the C^*-algebra of continuous functions on the quantum $SU(2)$ group and the quantum sphere. There has been various approaches towards building a calculus on quantum spaces, but there seems to be very few instances of computations outlined in chapter 6, [5]. We give detailed computations of the associated Connes-de Rham complex and the space of L_2-forms.

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1 Introduction

Given a noncommutative space, there is no general method for constructing a spectral triple on it. Even though there are general results asserting the existence of enough unbounded Kasparov modules ([1]), in concrete examples, it is often difficult to carry out this prescription. In [2], the authors characterized all spectral triples for the C^*-algebra $\mathcal{A}$ of continuous functions on $SU_q(2)$ represented on its L_2-space, assuming equivariance under the (co-)action of the group itself. In the present article, we take the more standard representation of $\mathcal{A}$ on $\mathcal{H} = L_2(\mathbb{N}) \otimes L_2(\mathbb{Z})$ (see (1.3) below), and impose equivariance condition under the action of the group $S^1 \times S^1$. Employing similar technques as in [2], we arrive at a spectral triple of dimension 2. One advantage of this triple is that it is relatively easy to compute the associated Connes-de Rham complex, which we give in section 3. This complex is supported on $\{0, 1\}$, and thus captures the topological dimension, which can be seen to be 1 from the following well-known exact sequence

$$0 \longrightarrow \mathcal{K} \otimes C(S^1) \longrightarrow \mathcal{A} \longrightarrow C(S^1) \longrightarrow 0. \tag{1.1}$$

The complex of square integrable forms were introduced by Frolich et. al in [8]. We also present calculations of these L_2-forms for this spectral triple.

In the last section, we briefly indicate how to carry out a similar construction of a spectral triple and the associated calculus for the quantum sphere S^2_{qc}.

Let us start with a brief description of the C^*-algebra of continuous functions on the quantum $SU(2)$, to be denoted by $\mathcal{A}$. This is the canonical C^*-algebra generated by two elements α and β satisfying the following relations:

$$\alpha^*\alpha + \beta^*\beta = I,\ \alpha\alpha^* + q^2\beta\beta^* = I,\ \alpha\beta - q\beta\alpha = 0,\ \alpha\beta^* - q\beta^*\alpha = 0,\ \beta^*\beta = \beta\beta^*. \tag{1.2}$$

The C^*-algebra $\mathcal{A}$ can be described more concretely as follows. Let $\{e_i\}_{i\geq 0}$ and $\{e_i\}_{i\in\mathbb{Z}}$ be the canonical orthonormal bases for $L_2(\mathbb{N})$ and $L_2(\mathbb{Z})$ respectively. We denote by the same symbol N the operator $e_k \mapsto ke_k$, $k \geq 0$, on $L_2(\mathbb{N})$ and $e_k \mapsto ke_k$, $k \in \mathbb{Z}$, on $L_2(\mathbb{Z})$. Similarly, denote by the same symbol ℓ the operator $e_k \mapsto e_{k-1}$, $k \geq 1$, $e_0 \mapsto 0$ on $L_2(\mathbb{N})$ and the operator $e_k \mapsto e_{k-1}$, $k \in \mathbb{Z}$ on $L_2(\mathbb{Z})$. Now take $\mathcal{H}$ to be the Hilbert space $L_2(\mathbb{N}) \otimes L_2(\mathbb{Z})$, and define π to be the following representation of $\mathcal{A}$ on $\mathcal{H}$:

$$\pi(\alpha) = \ell\sqrt{I - q^{2N}} \otimes I, \quad \pi(\beta) = q^N \otimes \ell. \tag{1.3}$$

Then π is a faithful representation of $\mathcal{A}$, so that one can identify $\mathcal{A}$ with the C^*-subalgebra of $\mathcal{L}(\mathcal{H})$ generated by $\pi(\alpha)$ and $\pi(\beta)$. Image of π contains $\mathcal{K} \otimes C(S^1)$ as an ideal with $C(S^1)$ as the quotient algebra, that is we have a useful short exact sequence

$$0 \longrightarrow \mathcal{K} \otimes C(S^1) \stackrel{i}{\longrightarrow} \mathcal{A} \stackrel{\sigma}{\longrightarrow} C(S^1) \longrightarrow 0. \tag{1.4}$$

We will denote by $\mathcal{A}_f$ the *-subalgebra of $\mathcal{A}$ generated by α and β. Let

$$\alpha_i\beta^j\beta^{*k} := \begin{cases} \alpha^i\beta^j\beta^{*k} & \text{if } i \geq 0, \\ (\alpha^*)^{-i}\beta^j\beta^{*k} & \text{if } i < 0. \end{cases}$$

Then $\{\alpha_i\beta^j\beta^{*k} : i \in \mathbb{Z}, j, k \in \mathbb{N}\}$ is a basis for $\mathcal{A}_f$. The Haar state h on $\mathcal{A}$ is given by,

$$h : a \mapsto (1 - q^2)\sum_{i=0}^{\infty} q^{2i}\langle e_{i0}, ae_{i0}\rangle.$$

Remark 1.1 The representation π admits a nice interpretation. Let M be a compact topological manifold and E, a Hermitian vector bundle on M. Let $\Gamma(M, E)$ be the space of continuous sections. Then $\Gamma(M, E)$ is a finitely generated projective $C(M)$ module. Define an inner product on $\Gamma(M, E)$ as

$$\langle s_1, s_2\rangle := \int (s_1(m), s_2(m))_m d\nu(m),$$

where ν is a smooth measure on M and $(\cdot,\cdot)_m$ is the inner product on the fibre on m. Let $\mathcal{H}_E$ be the Hilbert space completion of $\Gamma(M, E)$. Then we have a natural representation of $C(M)$ in $\mathcal{L}(\mathcal{H}_E)$. The same program can be carried out in the noncommutative context also. Let $\mathcal{B}$ be a C^*-algebra and E a Hilbert $\mathcal{B}$-module with its $\mathcal{B}$ valued inner product $\langle\cdot,\cdot\rangle_{\mathcal{B}}$. Let τ be a state on $\mathcal{B}$. Consider the inner product on E given by $\langle e_1, e_2\rangle = \tau(\langle e_1, e_2\rangle_{\mathcal{B}})$. If we denote by $\mathcal{H}_E$ the Hilbert space completion of E, then we get a natural representation of $\mathcal{B}$ in $\mathcal{L}(\mathcal{H}_E)$. Now in the context of $SU_q(2)$, let $p = |e_0\rangle\langle e_0| \otimes I \in \mathcal{A}$. Then it is easy to verify that

$\mathcal{H}_E = l^2(\mathbb{N}) \otimes l^2(\mathbb{Z})$ for $E = \mathcal{A}p$ with its natural left Hilbert $\mathcal{A}$-module structure. Moreover, the associated representation is nothing but the representation of $\mathcal{A}$ described above. Also, viewed this way, one can think of the representation of $\mathcal{A}$ on $L_2(h)$ given in [2] as being a countable direct sum of representations each of which look like π (just think of $\mathcal{A}$ as $\oplus \mathcal{A}p_i$ where $p_i = |e_i\rangle\langle e_i| \otimes I$).

2 $S^1 \times S^1$-equivariant spectral triples

The group $G = S^1 \times S^1$ has the following action on $\mathcal{A}$:

$$\tau_{z,w} : \begin{cases} \alpha \mapsto z\alpha \\ \beta \mapsto w\beta \end{cases} \tag{2.1}$$

Let U be the following representation of G on $\mathcal{H}$: $U_{z,w} = z^N \otimes w^N$. Then for any $a \in \mathcal{A}$, one has $\pi(\tau_{z,w}(a)) = U^*_{z,w}\pi(a)U_{z,w}$, i.e. the action τ is implemented through this representation U of G. A self-adjoint operator with discrete spectrum equivariant under this G-action must be of the form

$$D : e_{ij} \mapsto d_{ij}e_{ij}. \tag{2.2}$$

It is easy to see that if D is such an operator, then $[D, \alpha]$ and $[D, \beta]$ are given by

$$[D, \alpha]e_{ij} = (d_{i-1,j} - d_{ij})\sqrt{1 - q^{2i}}\, e_{i-1,j}, \tag{2.3}$$

$$[D, \beta]e_{ij} = (d_{i,j-1} - d_{ij})q^i\, e_{i,j-1}. \tag{2.4}$$

Employing arguments very similar to those used in the proofs of propositions 3.1 and 3.2 in [2], we now get the following results.

Proposition 2.1 *Let D be an operator of the form $e_{ij} \mapsto d_{ij}e_{ij}$. Then $[D, a]$ is bounded for all $a \in \mathcal{A}_f$ if and only if d_{ij}'s satisfy the following two conditions:*

$$|d_{i-1,j} - d_{ij}| = O(1), \tag{2.5}$$

$$|d_{i,j-1} - d_{ij}| = O(i + 1). \tag{2.6}$$

Corollary 2.2 *Let $(\mathcal{A}_f, \mathcal{H}, D)$ be a spectral triple equivariant under the action of $S^1 \times S^1$. Then D can not be p-summable if $p < 2$.*

Proof: This is a consequence of the following growth restriction on the d_{ij}'s:

$$d_{ij} = O(i + |j| + 1), \tag{2.7}$$

which follows from the last proposition. □

That there indeed exists a spectral triple that is 2-summable is easy to see, by just taking D to be the operator

$$D = N \otimes S + I \otimes N, \tag{2.8}$$

where $S = \sum_{\substack{i\geq 0\\ j\geq 0}} |e_{ij}\rangle\langle e_{ij}| - \sum_{\substack{i\geq 0\\ j<0}} |e_{ij}\rangle\langle e_{ij}|$.

Remark 2.3 The obstruction element given by Voiculescu ([12]) turns out to be zero for the ideals $\mathcal{L}^{(p,\infty)}$ where $p<2$ (note that by proposition 1.7, [12], it is enough to look at positive finite-rank contractions from the commutant $U(G)'$ in order to calculate this obstruction). Though one can not conclude anything definite from this, it is possible that by dropping the condition of $S^1\times S^1$-equivariance, Dirac operators of lower summability might be achievable.

Proposition 2.4 *Let D be as in the previous proposition. Assume that D has compact resolvent. Then up to a compact perturbation, we have*

$$\left.\begin{array}{l}\text{1. For each } j\in\mathbb{Z}, \text{ all the } d_{ij}\text{'s are of the same sign,}\\ \text{2. there is a big enough integer } M \text{ such that}\\ \quad\text{(a) all the } d_{ij}\text{'s for } j\geq M \text{ are of the same sign,}\\ \quad\text{(b) all the } d_{ij}\text{'s for } j\leq -M \text{ are of the same sign.}\end{array}\right\} \tag{2.9}$$

Proof: Again, the proof is very similar to the proof of proposition 3.2 in [2], and hence is omitted. □

This proposition says in particular that if $(\mathcal{A},\mathcal{H},D)$ is a G-equivariant Fredholm module, then upto a compact perturbation, $P=\frac{I+\text{sign}\, D}{2}$ must be one of the following, where E is some finite subset of $\{-M+1,-M+2,\ldots,M-1\}$:

$$P_1=\sum_{\substack{i\geq 0\\ j\leq -M}}|e_{ij}\rangle\langle e_{ij}|+\sum_{\substack{i\geq 0\\ j\in E}}|e_{ij}\rangle\langle e_{ij}|,\qquad P_2=\sum_{\substack{i\geq 0\\ j\geq M}}|e_{ij}\rangle\langle e_{ij}|+\sum_{\substack{i\geq 0\\ j\in E}}|e_{ij}\rangle\langle e_{ij}|,$$
$$P_3=\sum_{\substack{i\geq 0\\ j\in E}}|e_{ij}\rangle\langle e_{ij}|,\qquad P_4=\sum_{\substack{i\geq 0\\ j\in E^c}}|e_{ij}\rangle\langle e_{ij}|.$$

We will prove below that the D given by (2.8) is in some sense the unique nontrivial Dirac operator for the representation π of $\mathcal{A}$.

Theorem 2.5 *Let D' be an G-equivariant Dirac operator. Then the Kasparov module associated with D' is either trivial or is same as the one associated with D or $-D$.*

Proof: Let $u=\chi_{\{0\}}(\beta^*\beta)(\beta-I)+I$. First, observe that $\langle[u],(\mathcal{A},\mathcal{H},D)\rangle=\text{index}\, SuS=1$. Since the K-groups for $SU_q(2)$ are free abelain, by the results of Rosenberg & Schochet ([10]), it is now enough to show that $\langle[u],(\mathcal{A},\mathcal{H},D')\rangle$ is either 0 or ±1 if $P':=\frac{I+\text{sign}\, D'}{2}$ is one of the P_i's above. Since $\langle[u],(\mathcal{A},\mathcal{H},D')\rangle=\text{index}\, P'uP'$, direct calculation now tells us that if P' is P_3 or P_4, the above pairing would be zero; it would be -1 if $P'=P_1$, and it is 1 and if $P'=P_2$. □

The canonical unitary $\begin{pmatrix}\alpha & -q\beta^*\\ \beta & \alpha^*\end{pmatrix}$ that comes in the definition of $SU_q(2)$ has non-trivial K-theory class (see the remark following theorem 5, [6]). One can verify that by computing its pairing with $D\otimes I$ (acting on $\mathcal{H}\otimes\mathbb{C}^2$).

The following proposition can be derived as a corollary to proposition 4.3, [2]. But the proof presented there was just by computing pairings between appropriate elements and does not give an insight as to why it is true. We give a different proof here that sheds light on this.

Proposition 2.6 *Given any* $m \in K^1(SU_q(2)) = \mathbb{Z}$*, there exists a Kasparov module* $(L_2(h), F)$ *which induces this element.*

Proof: Using remark 1.1, one could look at $L_2(h)$ as $\oplus \mathcal{A}p_i$. Representation of $\mathcal{A}$ by left multiplications in each piece looks like π. Now given m in $\mathbb{Z}$, one has to pick m copies of π, and define F to be $(\operatorname{sign} m)S$ on each of these pieces and I on others. Then $(L_2(h), F)$ would be the required module. $\square$

3 Connes-de Rham complex

Let $\Omega^\bullet(\mathcal{A}_f) = \oplus_n \Omega^n(\mathcal{A}_f)$ be the universal graded differential algebra over $\mathcal{A}_f$, i.e. $\Omega^n(\mathcal{A}_f) = \operatorname{span}\{a_0(\delta a_1)\ldots(\delta a_n) : a_i \in \mathcal{A}_f, \delta(ab) = a(\delta b) + (\delta a)b\}$. The universal differential algebra is not very interesting from the cohomological point of view. Interesting cohomologies are obtained from the representations of the algebra. For the spectral triple $(\mathcal{A}_f, \mathcal{H}, D)$, one has the standard Connes-de Rham complex of noncommutative exterior forms $\Omega_D^\bullet(\mathcal{A}_f)$, given by

$$\Omega_D^\bullet(\mathcal{A}) := \Omega^\bullet(\mathcal{A})/(\mathfrak{K} + \delta\mathfrak{K}) \cong \pi(\Omega^\bullet(\mathcal{A}))/\pi(\delta\mathfrak{K}).$$

where $\mathfrak{K} = \oplus_{p\geq 0}\mathfrak{K}_p$ is the two sided ideal of $\Omega^\bullet(\mathcal{A})$ given by $\mathfrak{K}_p = \{\omega \in \Omega^p(\mathcal{A}) : \pi(\omega) = 0\}$. But often, the explicit computation of this complex is rather difficult. What we will do is the following. We will compute the complex obtained from the representation $\theta\circ\pi : \Omega^\bullet(\mathcal{A}) \to \mathcal{Q}(\mathcal{H})$ where $\theta : \mathcal{L}(\mathcal{H}) \to \mathcal{Q}(\mathcal{H}) = \mathcal{L}(\mathcal{H})/\mathcal{K}(\mathcal{H})$ is the projection onto the Calkin algebra. More specifically, let $\widetilde{d} : \mathcal{A}_f \to \mathcal{L}(\mathcal{H})$ be given by $\widetilde{d}a = [D, \pi(a)]$. Define $\pi_n : \Omega^n(\mathcal{A}_f) \to \mathcal{L}(\mathcal{H})$ by $\pi_n(a_0(\delta a_1)\ldots(\delta a_n)) = \pi(a_0)(\widetilde{d}a_1)\ldots(\widetilde{d}a_n)$. Define $d = \theta\circ\widetilde{d}$, $\psi_n = \theta\circ\pi_n$, and $\psi := \oplus\psi_n : \oplus\Omega^n \to \mathcal{Q}(\mathcal{H})$. Let $J_n = \ker\psi_n$. Define $\Omega_d^n(\mathcal{A}_f) = \Omega^n(\mathcal{A}_f)/(J_n + \delta J_{n-1})$.

Then $\Omega_d^n(\mathcal{A}_f) = \psi(\Omega^n(\mathcal{A}_f))/\psi(\delta J_n)$. We will compute these cohomologies $\Omega_d^n(\mathcal{A}_f)$. Before entering the computations, it should be stressed here that by computing these rather than the standard complex, we do not lose much. Because, first, since for a compact operator K one has $\operatorname{Tr}_\omega(K|D|^{-2}) = 0$, proposition 5, page 550, [5] concerning the Yang-Mills functional holds in our present case. Second, in the context of the canonical spectral triple associated with a compact Riemannian spin manifold this prescription also gives back the exterior complex.

First, we need the following lemma which will be very useful for the computations.

Lemma 3.1 *Assume* $a, b \in \mathcal{A}_f$ *and* $c \in \mathcal{K}(\mathcal{H})$*. If* $a(I\otimes S) + b = c$*, then* $a = b = 0$*.*

Proof: For a functional ρ on $\mathcal{L}(L_2(\mathbb{N}))$, and $T \in \mathcal{L}(\mathcal{H})$, denote by a_ρ the operator $(\rho\otimes\mathrm{id})T$. Now observe that for any $a \in \mathcal{A}_f$ and any functional ρ,

$$a_\rho\ell = \ell a_\rho. \tag{3.1}$$

Write $P = \frac{1}{2}(I+S)$. It is easy to see that the given condition implies that $(b_\rho - a_\rho) + 2a_\rho P = c_\rho$, which in turn implies that

$$\begin{aligned} (b_\rho - a_\rho)e_i &= c_\rho e_i \quad \forall i < 0, & (3.2)\\ (b_\rho + a_\rho)e_i &= c_\rho e_i \quad \forall i \geq 0. & (3.3) \end{aligned}$$

Now from (3.1) and (3.2), it follows that for any $i, j \in \mathbb{Z}$ and $j < 0$,

$$\begin{aligned} \|(b_\rho - a_\rho)e_i\| &= \|(b_\rho - a_\rho)\ell^{j-i}e_j\| \\ &= \|\ell^{j-i}(b_\rho - a_\rho)e_j\| \\ &= \|(b_\rho - a_\rho)e_j\| \\ &= \|c_\rho e_j\|. \end{aligned}$$

Since c is compact, $\lim_{j\to-\infty} \|c_\rho e_j\| = 0$. Hence $(b_\rho - a_\rho)e_i = 0$ for all i. In other words, $(b_\rho - a_\rho) = 0$. Since this is true for any ρ, we get $a = b$. Using this equality, together with equations (3.1) and (3.3), a similar reasoning yields $a = 0$. □

Lemma 3.2 *Let $\mathcal{I}_\beta$ denote the ideal in $\mathcal{A}_f$ generated by β and β^*. Then for $n \geq 1$, we have*

$$\psi(\Omega^n(\mathcal{A}_f)) = (I \otimes S)^n \mathcal{A}_f + (I \otimes S)^{n+1}\mathcal{I}_\beta. \tag{3.4}$$

Proof: Let us first prove the equality for $n = 1$. Let $Z_k = q^{N+k}(N+k)$, $B_{jk} = \sum_{i=j-k+1}^{j} |e_{i-1}\rangle\langle e_i|$, and

$$C_j = \begin{cases} \sum_{i=0}^{j-1} |e_i\rangle\langle e_{i-1}| & \text{if } j \geq 1,\\ 0 & \text{if } j = 0. \end{cases}$$

It is easy to check that

$$\begin{aligned} [D, \alpha] &= \alpha(-I \otimes S), \\ [D, \beta] &= q^N N \otimes [S, \ell^*] + \beta. & (3.5) \end{aligned}$$

It follows from these that

$$\begin{aligned} [D, \alpha_i\beta^j\beta^{*k}] = &-i(I \otimes S)\alpha_i\beta^j\beta^{*k} + (j-k)\alpha_i\beta^j\beta^{*k} + 2(Z_i \otimes C_j)\alpha_i\beta^{j-1}\beta^{*k} \\ &-2(Z_i \otimes B_{jk})\alpha_i\beta^j\beta^{*k-1}. & (3.6) \end{aligned}$$

Hence $d(\alpha_i\beta^j\beta^{*k}) = -i(I \otimes S)\alpha_i\beta^j\beta^{*k} + (j-k)\alpha_i\beta^j\beta^{*k}$. Thus for any $a \in \mathcal{A}_f$,

$$da = (I \otimes S)b + c, \qquad \text{where } b \in \mathcal{A}_f, \quad c \in \mathcal{I}_\beta. \tag{3.7}$$

Note that for any $a' \in \mathcal{A}_f$, $\psi(a')(I \otimes S) = (I \otimes S)\psi(a')$ in $\mathcal{Q}(\mathcal{H})$. Hence $\psi(a'(\delta a))$ is again of the form $(I \otimes S)b + c$, where $b \in \mathcal{A}_f$, $c \in \mathcal{I}_\beta$, i.e. is a member of $(I \otimes S)\mathcal{A}_f + \mathcal{I}_\beta$. Thus $\psi(\Omega^1(\mathcal{A}_f)) \subseteq (I \otimes S)\mathcal{A}_f + \mathcal{I}_\beta$. For the reverse inclusion, observe that $(I \otimes S) = (1-q^2)^{-1}((d\alpha)\alpha^* + q^2(d\alpha^*)\alpha)$, $\beta = d\beta$ and $\beta^* = -d\beta^*$.

The inductive step follows easily from (3.7). □

Lemma 3.3 *$J_0 = \{0\}$, and for $n \geq 1$, we have*

$$\psi(\delta J_n) = (I \otimes S)^{n+1}\mathcal{A}_f + (I \otimes S)^{n+2}\mathcal{I}_\beta. \tag{3.8}$$

Proof: By lemma 3.1, $\psi : \mathcal{A}_f \to \mathcal{Q}(\mathcal{H})$ is faithful. Hence it follows that $J_0 = \{0\}$.

We will prove here (3.8) by induction. From lemma 3.2, we have $\psi(\delta J_1) \subseteq \psi(\Omega^2(\mathcal{A}_f)) = \mathcal{A}_f + (I \otimes S)\mathcal{I}_\beta$. Let us show that I, $(I \otimes S)\beta$ and $(I \otimes S)\beta^*$ are all members of $\psi(\delta J_1)$.

Choose $\omega \in \Omega^1(\mathcal{A}_f)$ such that $\psi(\omega) = (I \otimes S)$. Let $\omega_k = k\alpha_k\omega - \delta(\alpha_k), k = \pm 1$. Then it follows from (2.3) that $\psi(\omega_k) = k\alpha_k(I \otimes S) - k\alpha_k(I \otimes S) = 0$, so that $\omega_k \in J_1$. $\psi(\delta\omega_k) = \psi(k(\delta\alpha_k)\omega) = k^2\alpha_k = \alpha_k \in \psi(\delta J_1)$, i.e. both α and α^* are in $\psi(\delta J_1)$. It follows from this that $I \in \psi(\delta J_1)$.

Next we show that $(I \otimes S)\beta \in \psi(\delta J_1)$. Take $\omega = \frac{1}{2}(\alpha(\delta\beta) - \delta(\alpha\beta) + q\beta(\delta\alpha))$. Then $\psi(\omega) = 0$ and $\psi(\delta\omega) = (I \otimes S)\alpha\beta$. So $(I \otimes S)\alpha\beta \in \psi(\delta J_1)$. Similarly taking $\omega = \frac{1}{2}(\alpha^*(\delta\beta) - \delta(\alpha^*\beta) + q^{-1}\beta(\delta\alpha^*))$, it follows that $(I \otimes S)\alpha^*\beta \in \psi(\delta J_1)$. These two together imply $(I \otimes S)\beta \in \psi(\delta J_1)$.

A similar argument shows that $(I \otimes S)\beta^*$ is also in $\psi(\delta J_1)$. Thus $\mathcal{A}_f + (I \otimes S)\mathcal{I}_\beta = \psi(\delta J_1)$.

For the inductive step, notice that $\psi(\delta J_n) \subseteq \psi(\Omega^{n+1}(\mathcal{A}_f)) = (I \otimes S)^{n+1}\mathcal{A}_f + (I \otimes S)^{n+2}\mathcal{I}_\beta$. We will show that the following are all elements of $\psi(\delta J_n)$:

$$\begin{array}{lll} (I \otimes S)^{n+1}\alpha, & (I \otimes S)^{n+2}\alpha\beta, & (I \otimes S)^{n+2}\alpha\beta^*, \\ (I \otimes S)^{n+1}\alpha^*, & (I \otimes S)^{n+2}\alpha^*\beta, & (I \otimes S)^{n+2}\alpha^*\beta^*. \end{array}$$

From the right $\mathcal{A}_f$-module structure of $\psi(\delta J_n)$, it will then follow that $(I \otimes S)^{n+1}$, $(I \otimes S)^{n+2}\beta$ and $(I \otimes S)^{n+2}\beta^*$ are in $\psi(\delta J_n)$, giving us the other inclusion.

Choose $\omega \in J_{n-1}$ such that $\psi(\delta\omega) = (I \otimes S)^n$. Take $\omega_k = k\omega(\delta\alpha_k)$, $k = \pm 1$. Then $\omega_k \in J_n$ and $\psi(\delta\omega_k) = (I \otimes S)^{n+1}\alpha_k$. Similarly choosing ω such that $\psi(\delta\omega) = (I \otimes S)^{n+1}\beta$ and ω_k as before, we get $\omega_k \in J_n$ and $\psi(\delta\omega_k) = q^{-k}(I \otimes S)^{n+2}\alpha\beta$. Finally, take ω such that $\psi(\delta\omega) = (I \otimes S)^{n+1}\beta^*$ and ω_k as before to show that $(I \otimes S)^{n+2}\alpha_k\beta^* \in \psi(\delta J_n)$. □

Theorem 3.4

$$\Omega_d^n(\mathcal{A}_f) = \begin{cases} \mathcal{A}_f \oplus \mathcal{I}_\beta & \text{if } n = 1, \\ \{0\} & \text{if } n \geq 2. \end{cases}$$

Proof: Proof follows from lemmas 3.2 and 3.3. □

Remark 3.5 The differential $d : \mathcal{A}_f \to \Omega_d^1(\mathcal{A}_f) = \mathcal{A}_f \oplus \mathcal{I}_\beta$ is given by

$$d(\alpha_i\beta^j\beta^{*k}) = -i\alpha_i\beta^j\beta^{*k} \oplus (j-k)\alpha_i\beta^j\beta^{*k}.$$

4 L^2-complex of Frohlich et. al.

In this section we will compute the complex of square integrable forms for the spectral triple $(\mathcal{A}_f, \mathcal{H}, D)$. For that we begin with similar computations for the spectral triple $(\mathbb{C}[z, z^{-1}], \mathcal{H}_0 = L_2(\mathbb{Z}), D_0 = N)$ associated with the algebra $\mathbb{C}[z, z^{-1}]$. Here we consider the embedding $\pi_0 : \mathbb{C}[z, z^{-1}] \to \mathcal{L}(\mathcal{H})$ that maps z to ℓ.

Lemma 4.1 (i) $\widetilde{\Omega}^n_{D_0}(\mathbb{C}[z,z^{-1}]) = 0,$ *for* $n \geq 2$,
(ii) $\widetilde{\Omega}^1_{D_0}(\mathbb{C}[z,z^{-1}]) = \mathbb{C}[z,z^{-1}]$.

Proof: (i) Let $\omega = \sum \alpha_{n_0,\cdots,n_k} z^{n_0}\delta z^{n_1}\cdots\delta z^{n_k} \in \Omega^k(\mathbb{C}[z,z^{-1}])$, where the sum is a finite one and δ is the universal differential. Then it is easily verified that

$$(\omega,\omega)_{D_0} = \int (\sum n_1\cdots n_k \alpha_{n_0,\cdots,n_k} z^{\sum_0^k n_j})^* (\sum n_1\cdots n_k \alpha_{n_0,\cdots,n_k} z^{\sum_0^k n_j}) dz,$$

where dz is the Lebesgue measure on the circle. Therefore,

$$\begin{aligned} \mathfrak{K}_k(\mathbb{C}[z,z^{-1}]) &:= \{\omega \in \Omega^k(\mathbb{C}[z,z^{-1}]) : \ (\omega,\omega)_{D_0} = 0\} \\ &= \{\sum \alpha_{n_0,\cdots,n_k} z^{n_0}\delta z^{n_1}\cdots\delta z^{n_k} : \sum_{n_0+\cdots+n_k=r} n_1\cdots n_k \alpha_{n_0,\cdots,n_k} = 0, \forall r\}. \end{aligned}$$

Consequently we have,

$$z^{n_0}\delta z^{n_1}\cdots\delta z^{n_k} - n_1\cdots n_k z^{\sum_0^k n_i - k}\delta z\cdots\delta z \in \mathfrak{K}_k(\mathbb{C}[z,z^{-1}]), \tag{4.9}$$

$$\delta z^r \delta z\cdots\delta z - r z^r \delta z\cdots\delta z \in \mathfrak{K}_k(\mathbb{C}[z,z^{-1}]), \tag{4.10}$$

$$z^r\delta z\cdots\delta z - \frac{1}{r+1}\delta z^{r+1}\delta z\cdots\delta z \in \mathfrak{K}_{k-1}(\mathbb{C}[z,z^{-1}]). \tag{4.11}$$

From (4.11) we get $\delta z^r\delta z\cdots\delta z \in \delta\mathfrak{K}_{k-1}(\mathbb{C}[z,z^{-1}])$. Combining this with (4.9) and (4.10) we get,

$$z^{n_0}\delta z^{n_1}\cdots\delta z^{n_k} \in \mathfrak{K}_k(\mathbb{C}[z,z^{-1}]) + \delta\mathfrak{K}_{k-1}(\mathbb{C}[z,z^{-1}]) \text{ for large } n_0.$$

Since $\mathfrak{K}_k(\mathbb{C}[z,z^{-1}]) + \delta\mathfrak{K}_{k-1}(\mathbb{C}[z,z^{-1}])$ is a bimodule we have

$$z^{n_0}\delta z^{n_1}\cdots\delta z^{n_k} \in \mathfrak{K}_k(\mathbb{C}[z,z^{-1}]) + \delta\mathfrak{K}_{k-1}(\mathbb{C}[z,z^{-1}]) \quad \forall\, n_0,\cdots,n_k.$$

This proves (i).
(ii) It suffices to note that

$$z^{n_0}\delta z^{n_1} - n_1 z^{n_0+n_1-1}\delta z \in \mathfrak{K}_1(\mathbb{C}[z,z^{-1}]).$$

The induced $d : \widetilde{\Omega}^0_{D_0}(\mathbb{C}[z,z^{-1}]) \to \mathbb{C}[z,z^{-1}]$ is given by $d(z^n) = nz^n$. □

Now we are in a position to compute the complex of square integrable forms.

Theorem 4.2 (i) $\widetilde{\Omega}^n_D(\mathcal{A}_f) = 0$ *for* $n \geq 2$.
(ii) $\widetilde{\Omega}^n_D(\mathcal{A}_f) = \mathbb{C}[z,z^{-1}]$ *for* $n = 0,1$ *(equality as an* $\mathcal{A}_f$ *bimodule), and the differential* $d : \mathcal{A}_f \to \widetilde{\Omega}^1_D(\mathcal{A}_f)$ *is given by* $d(\alpha_i\beta^j\beta^{*k}) = -iz^i$.

Proof: Note that the homomorphism σ in (1.4) induces a surjective homomorphism denoted by the same symbol from $\mathcal{A}_f$ to $\mathbb{C}[z,z^{-1}]$. We have the following short exact sequence

$$0 \longrightarrow \mathcal{I}_\beta \longrightarrow \mathcal{A}_f \xrightarrow{\sigma} \mathbb{C}[z,z^{-1}] \longrightarrow 0,$$

Let $\sigma_k : \Omega^k(\mathcal{A}_f) \to \Omega(\mathbb{C}[z,z^{-1}])$ be the induced surjective map. One easily verifies that $(\omega,\omega)_D = (\sigma_k(\omega),\sigma_k(\omega))_{D_0}$. Therefore,

$$\mathfrak{K}_k(\mathcal{A}_f) = \{\omega \in \Omega^k(\mathcal{A}_f) : \ (\omega,\omega)_D = 0\} = \sigma_k^{-1}(\mathfrak{K}_k(\mathbb{C}[z,z^{-1}])).$$

We have the following commutative diagram

$$
\begin{array}{ccccccc}
\mathfrak{K}_0 = I_\beta & \longrightarrow & \mathcal{A}_f & \xrightarrow{\sigma} & \mathbb{C}[z,z^{-1}] & \longrightarrow & \pi_0(\mathbb{C}[z,z^{-1}]) \\
 & & \downarrow & & \downarrow & & \downarrow \\
\mathfrak{K}_1(\mathcal{A}_f) & \longrightarrow & \Omega^1(\mathcal{A}_f) & \xrightarrow{\sigma_1} & \Omega^1(\mathbb{C}[z,z^{-1}]) & \longrightarrow & \tilde{\Omega}^1_{D_0}(\mathbb{C}[z,z^{-1}]) \\
 & & \downarrow & & \downarrow & & \downarrow \\
\mathfrak{K}_2(\mathcal{A}_f) & \longrightarrow & \Omega^2(\mathcal{A}_f) & \xrightarrow{\sigma_2} & \Omega^2(\mathbb{C}[z,z^{-1}]) & \longrightarrow & \tilde{\Omega}^2_{D_0}(\mathbb{C}[z,z^{-1}]) \\
 & \cdots & & \cdots & & \cdots & \cdots \\
\mathfrak{K}_n(\mathcal{A}_f) & \longrightarrow & \Omega^n(\mathcal{A}_f) & \xrightarrow{\sigma_n} & \Omega^n(\mathbb{C}[z,z^{-1}]) & \longrightarrow & \tilde{\Omega}^n_{D_0}(\mathbb{C}[z,z^{-1}]).
\end{array}
$$

This along with the previous lemma proves the theorem. We will only illustrate (i).

Let $\omega_n \in \Omega^n(\mathcal{A}_f)$, then by the previous lemma $\sigma_n(\omega_n) = \omega_{1,n} + \delta\omega_{2,n-1}$ where $\omega_{1,n} \in \mathfrak{K}_n(\mathbb{C}[z,z^{-1}]), \omega_{2,n-1} \in \mathfrak{K}_{n-1}(\mathbb{C}[z,z^{-1}])$. Let $\omega'_{1,n} = \sigma_n^{-1}(\omega_{1,n}), \omega'_{2,n-1} = \sigma_{n-1}^{-1}(\omega_{2,n-1})$, then $\sigma_n(\omega_n - \omega'_{1,n} - \delta\omega'_{2,n-1}) = 0$ implying $\omega_n \in \mathfrak{K}_n + \delta\mathfrak{K}_{n-1}$. □

5 Computations for the quantum sphere

In this section we will briefly indicate how to carry out all the earlier constructions for the quantum spheres. We will be sketchy because most of the arguments are very similar to the case of $SU_q(2)$. Quantum sphere was introduced by Podleś in [9]. This is the universal C*-algebra, denoted by $C(S^2_{qc})$, generated by two elements A and B subject to the following relations:

$$
\begin{aligned}
A^* &= A, \qquad & B^*B &= A - A^2 + cI, \\
BA &= q^2 AB, \qquad & BB^* &= q^2 A - q^4 + cI.
\end{aligned}
$$

Here the deformation parameters q and c satisfy $|q| < 1, c > 0$. For later purpose we also note down two irreducible representations whose direct sum is faithful. Let $\mathcal{H}_+ = l^2(\mathbb{N}), \mathcal{H}_- = \mathcal{H}_+$. Define $\pi_\pm(A), \pi_\pm(B) : \mathcal{H}_\pm \to \mathcal{H}_\pm$ by

$$
\begin{aligned}
\pi_\pm(A)(e_n) &= \lambda_\pm q^{2n} e_n \qquad \text{where} \qquad \lambda_\pm = \frac{1}{2} \pm (c + \frac{1}{4})^{1/2}, \\
\pi_\pm(B)(e_n) &= c_\pm(n)^{1/2} e_{n-1} \quad \text{where} \quad c_\pm(n) = \lambda_\pm q^{2n} - (\lambda_\pm q^{2n})^2 + c.
\end{aligned}
$$

Since $\pi = \pi_+ \oplus \pi_-$ is a faithful representation, an immediate corollary follows.

Theorem 5.1 (Sheu [11]) (i) $C(S^2_{qc}) \cong C^*(\mathscr{T}) \oplus_\sigma C^*(\mathscr{T}) := \{(x,y) : \ x,y \in C^*(\mathscr{T}), \sigma(x) = \sigma(y)\}$ *where* $C^*(\mathscr{T})$ *is the Toeplitz algebra and* $\sigma : C^*(\mathscr{T}) \to C(S^1)$ *is the symbol homomorphism.*
(ii) *We have a short exact sequence*

$$
0 \longrightarrow \mathcal{K} \xrightarrow{i} C(S^2_{qc}) \xrightarrow{\alpha} C^*(\mathscr{T}) \longrightarrow 0 \tag{5.12}
$$

Proof: (i) An explicit isomorphism is given by $x \mapsto (\pi_+(x), \pi_-(x))$.
(ii) Define $\alpha((x,y)) = x$ then $\ker\alpha = \mathcal{K}$. □

Corollary 5.2 (i) $K_0(C(S^2_{qc})) = K^0(C(S^2_{qc})) = \mathbb{Z} \oplus \mathbb{Z}$.
(ii) $K_1(C(S^2_{qc})) = K^1(C(S^2_{qc})) = 0$.

Proof: The six term exact sequence associated with (5.12) along with the KK-equivalence of $\mathcal{K}$ and $C^*(\mathscr{T})$ with $\mathbb{C}$ proves the result. □

Proposition 5.3 *Let $\mathcal{A}_{fin}$ be the *-subalgebra of $C(S^2_{qc})$ generated by A and B. Then*

$$\left(\mathcal{A}_{fin},\ \mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-,\ D = \begin{pmatrix} 0 & N \\ N & 0 \end{pmatrix},\ \gamma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right)$$

is an even spectral triple.

Proof: We only have to show that $[D, a]$ is bounded for $a \in \mathcal{A}_{fin}$. For that it is enough to note that
(i) $N\pi_\pm(A),\ \pi_\pm(A)N$ are bounded,
(ii) $n(c_\pm(n)^{1/2} - \sqrt{c})$ is bounded as n becomes large,
(iii) $[N, l] = l$. □

Remark 5.4 This spectral triple has nontrivial Chern character. This can be seen as follows: let $P_0 = i(|e_0\rangle\langle e_0|) \in C(S^2_{qc})$, then applying proposition 4, page 296, [5],we get the index pairing $\langle [P_0], [(\mathcal{A}_{fin}, \mathcal{H}, D, \gamma)]\rangle = -1$, implying nontriviality of the spectral triple.

Now we will briefly indicate the computations of the complex $(\Omega^\bullet_d(\mathcal{A}_{fin}), d)$ introduced at the beginning of section 3.

Theorem 5.5 (i) $\Omega^n_d(\mathcal{A}_{fin}) = 0$ *for* $n \geq 2$.
(ii) $\Omega^1_d(\mathcal{A}_{fin}) = \mathbb{C}[z, z^{-1}]$, *here also equality is as an* $\mathcal{A}_{fin}$ *bimodule.*

Proof: Let π be the associated representation of $\Omega^\bullet(\mathcal{A}_{fin})$ in $\mathcal{L}(\mathcal{H})$. Then straightforward verification gives (i) $[D, A]$ is compact, (ii) $[D, B] = l \otimes \kappa +$ compact, and (iii) $[D, B^*] = -l^* \otimes \kappa +$ compact, where $\kappa = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Therefore, modulo compacts

$$\begin{aligned} \pi(\Omega^{2k+1}(\mathcal{A}_{fin})) &= C^*_{fin}(\mathscr{T}) \otimes \kappa \\ \pi(\Omega^{2k}(\mathcal{A}_{fin})) &= C^*_{fin}(\mathscr{T}) \otimes I_2, \end{aligned}$$

where $C^*_{fin}(\mathscr{T})$ is the *-algebra generated by $\mathscr{T}$. Now for (i), note that

$$\omega_n = B\delta B^* \underbrace{\delta B \cdots \delta B}_{n-2\ times} + B^*\delta B \underbrace{\delta B \cdots \delta B}_{n-2\ times}$$

satisfies (a) $\pi(\omega_n)$ is compact and (b)$\pi(\delta\omega_n) = 2I$ is invertible, hence (i) follows.
For (ii), observe that if $a \in \mathcal{A}_{fin}$ and $\pi(a)$ is compact then Na and aN both compact. Hence, $\Omega^1_d(\mathcal{A}_{fin}) = \pi(\Omega^1(\mathcal{A}_{fin})) = \mathbb{C}[z, z^{-1}]$ because modulo compacts $\mathbb{C}[z, z^{-1}]$ is $C^*(\mathscr{T})$. □

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Partha Sarathi Chakraborty

Indian Statistical Institute, 203, B. T. Road, Calcutta–700 108, INDIA

email: parthasc_r@isical.ac.in

Arupkumar Pal

Indian Statistical Institute, 7, SJSS Marg, New Delhi–110 016, INDIA

email: arup@isid.ac.in