MAPPING CLASS GROUPS AND INTERPOLATING COMPLEXES: RANK

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Abstract. A family of interpolating graphs $\mathcal{C}(S,\xi)$ of complexity ξ is constructed for a surface S and $-2 \leq \xi \leq \xi(S)$. For $\xi = -2, -1, \xi(S) - 1$ these specialize to graphs quasi-isometric to the marking graph, the pants graph and the curve graph respectively. We generalize the notion of a hierarchy and Theorems of Brock-Farb and Behrstock-Minsky to show that the rank of $\mathcal{C}(S,\xi)$ is r_ξ, the largest number of disjoint copies of subsurfaces of complexity greater than ξ that may be embedded in S. The interpolating graphs $\mathcal{C}(S,\xi)$ interpolate between the pants graph and the curve graph.

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Contents

1. Introduction

1.1. Motivation and Statement of Results. Starting with Masur-Minsky's result that the curve complex is hyperbolic [14], the coarse geometry of mapping class groups has attracted much attention. A motivating scholium is the following.

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The mapping class group behaves like a non-uniform rank one lattice away from the peripheral subgroups and like a higher rank non-uniform lattice at the peripheral subgroups.

Hyperbolicity of the curve complex is an instance of rank one behavior. Intersection patterns of peripheral subgroups and resultant structures similar to the Tits complex illustrate higher rank behavior. In this paper, we investigate further this interbreeding of rank one and higher rank behavior.

There are two pieces of motivation behind this paper:

Motivation 1: Higher rank lattices admit a whole family of compactifications, for instance the Borel-Serre, Reduced Borel-Serre and toroidal compactifications. These are built from configurations of parabolic subgroups. (See Borel-Ji [3] for instance.) Of particular relevance to this paper is the fact that the Furstenberg (or maximal) boundary is obtained as a quotient space of the Tits boundary by identifying certain Weyl chambers at infinity to points. **A coarse geometric analog of such a topological quotienting operation is "coning"** (see below). This intuitive idea will play an important role in the construction of interpolating graphs.

Moduli spaces too admit such compactifications, of which the Deligne-Mumford compactification is probably the most well-known. If we look at the universal cover of the compactified moduli space, we find an intersection pattern of boundary strata. This is encoded in the Curve Complex $\mathcal{C}(S)$ of a surface S (discovered by Harvey [10] from the above-mentioned analogy with non-uniform lattices of higher rank). Another such simplicial complex (originally discovered by Hatcher and Thurston [11]) is the pants complex $\mathcal{P}(S)$. Recently, Brock [4] has shown that the pants complex is quasi-isometric to Teichmuller space equipped with the Weil-Petersson metric. The first aim of this paper is to describe a collection of simplicial complexes interpolating (in a sense to be made precise) between the curve-complex $\mathcal{C}(S)$ and the pants complex $\mathcal{P}(S)$.

Motivation 2: Masur and Minsky develop in [15] a detailed combinatorial structure of *hierarchies* to get a handle on quasigeodesics in $MCG(S)$. We develop in this paper a related hierarchy of spaces where the bottom level is given by the curve complex and the top level by the marking complex (quasi-isometric to the the mapping class group). Thus in a sense again, we describe a collection of simplicial complexes interpolating between the curve-complex $\mathcal{C}(S)$ and the marking complex $\mathcal{M}(S)$.

The basic point of this paper is to introduce the notion of interpolating graphs and make the observation that existing notions of hierarchies, etc. have a natural generalization to this situation.

Quasi-isometric Models First off, we describe quasi-isometric models for these interpolating complexes as they are easy to define. Let $\Gamma(S)$ be a Cayley graph of the mapping class group of a surface S of genus g with n punctures. Let $\xi(S) = 3g - 3 + n = \xi_0$ denote the complexity of the surface S. Note that $\Gamma(S)$ depends on the choice of generators and is therefore well-defined up to quasi-isometry.

We now describe a variety of graphs associated with $\Gamma(S)$. Fix a ξ with $-2 \leq \xi \leq \xi_0 - 1$.

Consider all essential (i.e. π_1-injective) subsurfaces S of complexity less than or equal to ξ. These fall into finitely many orbits under the action of the mapping class group, $S_1, \cdots S_k$, say. Then we may assume (changing the generating set if necessary) that the Cayley graph $\Gamma_{MCG(S_i)} \subset \Gamma(S)$. Here $\Gamma_{MCG(S_i)}$ denotes the Cayley graph of the subgroup of $MCG(S)$ stabilizing S_i (Dehn twists on boundaries allowed). We may further assume for convenience that the S_i's are maximal (i.e. no S_i is a proper subset of another S_j.) We define $\Gamma(S, \xi)$ to be the graph obtained from $\Gamma(S)$ by coning (a la Farb [7]) cosets of sub-mapping class groups $MCG(S_i)$, i.e. by introducing a vertex v_{gH} for each coset gH of the mapping class group $H = MCG(S_i)$ for each of the above S_i's and joining it by an edge of length $\frac{1}{2}$ to every element of $gH \subset \Gamma(S)$. The main theorem of this paper determines the rank of $\Gamma(S, \xi)$. In the next subsection, we shall give a more intrinsic (geometric) model $\mathcal{C}(S, \xi)$ quasi-isometric to $\Gamma(S, \xi)$ with the additional restriction that it is defined *naturally.* These shall be termed the *interpolating graphs* or *complexity ξ graphs* for the surface S. Note that $\xi = -1$ gives a graph quasi-isometric to the pants graph and $\xi = -2$ cones off the trivial sub-mapping class groups corresponding to mapping class groups of disks, yielding therefore the mapping class group (or equivalently the marking graph).

Let r_ξ denote the maximum number of disjoint subsurfaces of complexity $(\xi + 1)$ that can be embedded in S. Then the main theorem of this paper states:

Theorem 2.12 : *The rank of the interpolating graph $\mathcal{C}(S, \xi)$ of complexity ξ (> 0) or its quasi-isometric model $\Gamma(S, \xi)$ is $r_\xi(S)$.*

Once we set up the framework, the proof is a re-working of that due to Brock-Farb [5] and Behrstock-Minsky [2]. In the final subsection of this paper, we shall draw a conjectural picture of the interconnections between hierarchies, rank and interpolating graphs. One interesting fallout is the following.

Conjecture: Rank one implies Hyperbolic
If for $\xi > 0$, $r_\xi(S) = 1$ is $\mathcal{C}(S, \xi)$ a hyperbolic metric space?

1.2. **Complexes Associated to the Mapping Class Group.** In this subsection, we describe the motivating examples: the curve-graph, the pants graph and the marking graph. We then proceed to give the promised description of interpolating (or complexity ξ) graphs.

Curve graph
Case 1: $\xi(S) \geq 2$
The *curve graph* of S, denoted $\mathcal{C}(S)$ is a graph with
1) vertices corresponding to nontrivial homotopy classes of non-peripheral, simple closed curves on S
2) edges corresponding to pairs of (homotopically distinct) simple closed curves which can be realized disjointly on S.

Case 2: $\xi(S) = 1$
Then either $g = n = 1$ (S is a one-holed torus) or $g = 0, n = 4$ (S is a 4-holed sphere). The vertex set is as in Case 1 above. The edge set consists of pairs of curves which realize the minimal possible intersection on S (1 for the one-holed torus, 2 for the 4-holed sphere).

Case 3: $\xi(S) = 0$
In this case, S is the 3-holed sphere with empty curve-complex as the vertex set is empty.

Case 4: $\xi(S) = -1$
In this case, S is the 2-holed sphere, i.e. an annulus. Fix a point in each boundary component.
1) Vertices are (homotopy classes of) arcs connecting the give boundary points up to homotopy rel endpoints.
2)Edges are pairs of non-intersecting arcs.

Pants Graph
The *pants graph* $\mathcal{P}(S)$ of S is a graph with

(1) Vertices consisting of pants decompositions of S.
(2) Edges consist of pants decompositions that agree on all but one curve, and further, those curves differ by an edge in the curve

complex of the complexity one subsurface (complementary to the rest of the curves) in which they lie.

Marking Graph
We consider pants decompositions μ of S. (We may identify μ with a maximal simplex in $\mathcal{C}(S)$). The set of underlying curves shall be denoted *base* (μ). The transversals τ_μ of μ consist of one curve each for each component of base (μ), intersecting it transversely, non-trivially and minimally (i.e. with minimum non-zero number of intersections. This number is one or two). A pair (base $(\mu), \tau_\mu$) of pants decompositions and transversals shall be referred to as a *marking*.

A marking (base $(\mu), \tau_\mu$) is *clean, complete* if,
1) for each $\gamma \in$ *base* (μ), the transversal curve t_γ to γ is disjoint from the rest of *base* (μ).
2) Each pair (γ, t_γ) fills a non-annular surface W satisfying $\xi(W) = 1$ and for which $d_{\mathcal{C}(W)}(\gamma, t) = 1$.

The *marking graph* or *marking complex*, $\mathcal{M}(S)$ is defined as follows. Vertices correspond to clean complete markings. The edges of $\mathcal{M}(S)$ are of two types (See Masur-Minsky [15] and also Behrstock-Minsky [2].):
Twist: Replace a transversal curve by another obtained by performing a full Dehn twist (resp. half-twist) along the associated base curve, if the transversal curve intersects the base curve in one (resp. two) points.
Flip: Exchange the roles of a base curve and its transversal curve. Perform surgery if necessary to reinstate disjointness. (Note that after switching base and transversal, the disjointness requirement on the transversals may be violated. However, Masur-Minsky show in [15] that one can surger the new transversal to obtain one that does satisfy the disjointness requirement. They further show that only a finite (uniformly bounded) number of flip moves are possible.)

Lemma 1.1. Masur-Minsky [15] *$\mathcal{M}(S)$ is quasi-isometric to the mapping class group of S.*

Remark 1.2. [15] *Masur and Minsky note that the pants graph is exactly what remains of the marking complex when annuli (and hence transverse curves) are ignored.*

Interpolating Graphs

We are now in a position to define the (natural or) geometric models that are the main object of study in this paper.

Definition 1.3. *An* interpolating graph *or* complexity ξ graph $\mathcal{C}(S,\xi)$ *consists of the following.*
1) vertices are pants decompositions (or maximal simplices in the curve complex)
2) edges are of two types:
a) edges of the pants graph of S
b) additional edges connecting pairs of pants decompositions agreeing on the complement of a (connected) subsurface of complexity less than or equal to ξ

Note that edges of type (2b) above *include* edges of type (2a) for $\xi \geq 1$. However, in order to include the pants graph as a starting point, we have mentioned edges of type (2a) separately.

As in [15], the same definitions apply to essential (possibly disconnected) subsurfaces of S. For a disconnected surface $W = \sqcup_{i=1}^n W_i$, $\mathcal{C}(W,\xi) = \prod_{i=1}^n \mathcal{C}(W_i,\xi)$.

Remark 1.4. We note that the interpolating complexes $\mathcal{C}(S,\xi)$ may be regarded as (quasi-isometric models) unifying the 3 types of complexes given above:
1) $\xi = \xi_0 - 1$: (denoting $\xi(S)$ by ξ_0.) Subsurfaces where moves are considered are (arbitrary) proper subsurfaces. This gives a model quasi-isometric to the curve-graph.
To see this, note that all proper pants subgraphs (i.e. pants graphs of all proper essential subsurfaces) have diameter one in $\mathcal{C}(S,\xi)$ here. Recall (Remark 1.2) that the pants graph is exactly what remains of the marking complex when annuli (and hence transverse curves) are ignored, or equivalently, when all moves on annuli are at distance one from each other. Thus $\mathcal{C}(S,\xi)$ is quasi-isometric to what one gets from the marking complex by first forgetting annuli and then all subsurfaces of complexity ξ. This is equivalent to coning cosets of all proper mapping class subgroups (i.e. mapping class groups of all proper essential subsurfaces) of the mapping class group of S, which is quasi-isometric to the curve complex $\mathcal{C}(S)$ of S.
An explicit quasi-isometry from $\mathcal{C}(S)$ to $\mathcal{C}(S,\xi)$ can be set up by sending any curve $\zeta \in \mathcal{C}(S)$ to some (any) pants decomposition containing ζ. There is a bounded amount of ambiguity in this as the set of pants decomposition containing ζ has diameter one in $\mathcal{C}(S,\xi)$.
2) $\xi = -1$: This coincides exactly with the definition of the pants graph. There are no edges of type (2b) of Definition 1.3 (vacuously). We mention this case separately to underscore the point that the pants graph is what one gets when annuli (and hence transverse curves) are

ignored.
3) $\xi = -2$: Here, moves are restricted to surfaces of complexity -2, i.e. disks. To correctly interpret this case, the curve graph of the annulus has to be resurrected and re-instated over the pants graph in order to get the marking graph. More precisely, if we want to incorporate the marking graph in our collection, we should define our interpolating complex starting with the marking complex and then adjoin edges for moves occurring in subsurfaces of complexity $\leq \xi$. i.e. we should redefine $\mathcal{C}(S,\xi)$ as a graph whose vertex set consists of elements of the marking graph, and edges between markings coinciding on the complement of a subsurface of complexity $\leq \xi$. (This takes care of the marking complex; but in all subsequent discussions, it would make the exposition more awkward. Once we reach the pants graph, the marking is forgotten in any case - hence the definition here. We include the marking graph here largely for completeness and to indicate that all the known graphs associated to the mapping class group are included in this discussion.

Remark 1.5. *Given the definition above, it is easy to see that (for some, hence any, finite generating set)* $\Gamma(S,\xi)$ *is quasi-isometric to* $\mathcal{C}(S,\xi)$.

In particular, (up to quasi-isometry) coning Dehn twists gives the pants complex and coning all proper sub-mapping class groups the curve complex (see [14] *). Not coning anything (or coning only the trivial sub-mapping class group for disks) gives the marking complex.*

Remark 1.6. *A useful heuristic is:*
The complexity ξ *graph is what remains of the marking complex when surfaces of complexity* $\leq \xi$ *are ignored.*

1.3. **Projections, Hierarchies, Distance Formulae.** In this subsection, we summarize some of the foundational work of Masur-Minsky [14] [15], followed by more recent work of Behrstock-Minsky [2]. An essential tool for the next section is Theorem 1.11 giving a distance formula for interpolating complexes.

Theorem 1.7. Masur-Minsky [14] *For any surface* S*, the complex of curves* $\mathcal{C}(S)$ *is an infinite diameter* δ*-hyperbolic space (as long as it is non-empty).*

Definition 1.8. Masur-Minsky [15] *Given a subsurface* $W \subset S$*, a* subsurface projection *is a map* $\pi_W : \mathcal{C}(S) \to 2^{\mathcal{C}(W)}$ *defined as follows.*
Case 1: W *is not an annulus.*
Given any curve $\gamma \in \mathcal{C}(S)$ *intersecting* Y *essentially, we define* $\pi_W(\gamma)$

to be the collection of curves (vertices) in $\mathcal{C}(W)$ obtained by surgering the essential arcs of $\gamma \cap W$ along ∂W to obtain simple closed curves in W.
Case 2: W is an annulus.
Here, the curve-graph is assumed to be that for the compactified cover $\widetilde{W}$ of S corresponding to the subgroup $\pi_1(W)$. Note that $\widetilde{W}$ can be identified with the annulus. If γ intersects W transversely and essentially, we lift γ to an arc crossing the annulus $\widetilde{W}$ and let this be $\pi_W(\gamma)$. If γ is a core curve of W or fails to intersect it, we let $\pi_W(\gamma) = \emptyset$.

$d_{\mathcal{C}(W)}(\mu, \nu)$ will be used as a short form for $d_{\mathcal{C}(W)}(\pi_W(\mu), \pi_W(\nu))$.

Next, for any $\mu \in \mathcal{C}(S, \xi)$ and any non-annular $W \subseteq S$ the above projection map induces $\pi_W : \mathcal{C}(S, \xi) \to 2^{\mathcal{C}(W)}$. This map is simply the union over $\gamma \in \textbf{base}(\mu)$ of the usual projections $\pi_W(\gamma)$. As in the case of curve complex projections, we write $d_{\mathcal{C}(W)}(\mu, \nu)$ for $d_{\mathcal{C}(W)}(\pi_{(W)}(\mu), \pi_{(W)}(\nu))$. The distance in the interpolating graph of complexity ξ shall be denoted by d_ξ.

Hierarchies

As summarized in [2], hierarchy paths are quasigeodesics in $\mathcal{M}(S)$ with constants depending only on the topological type of S such that

(1) any two points $\mu, \nu \in \mathcal{M}(S)$ are connected by at least one hierarchy path γ. The base geodesic of γ in $\mathcal{C}(S)$ is β, i.e. β is a geodesic in $\mathcal{C}(S)$ joining **base** (μ) to **base** (ν).

(2) There is a monotonic map $v : \gamma \to \beta$, such that $v(\gamma_n)$ is a vertex in **base** (γ_n) for every γ_n in γ.

(3) Subsurfaces of S which "separate" μ from ν in a significant way must play a role in the hierarchy paths from μ to ν in the following sense:
There exists a constant $M_2 = M_2(S)$ such that, if W is an essential subsurface of S and $d_{\mathcal{C}(W)}(\mu, \nu) > M_2$, then for any hierarchy path γ connecting μ to ν, there exists a marking γ_n in γ with $[\partial W] \subset$ **base** (γ_n), where $[\partial W]$ denotes the free homotopy class of the multicurve represented by the boundary ∂W of W. Furthermore there exists a vertex v in the geodesic β shadowed by γ such that $W \subset S \setminus v$. This property follows directly from Lemma 6.2 of [15].

Complexity ξ Partial Hierarchies

The notion of a *partial hierarchy* was introduced by Masur and Minsky in [15], Sec 4.2, as a hierarchy minus the restriction that every component domain must support a tight geodesic. Analogously we may define a *complexity ξ partial hierarchy* as a collection H of tight geodesics in S satisfying the following:

(1) There is a main geodesic g_H with domain S and initial and terminal markings $I(H)$, $T(H)$ respectively.
(2) If $b, f \in H$ and $Y \subset S$ is a subsurface *of complexity greater than* ξ such that Y is backward (resp. forward) subordinate to b (resp. f), then H contains a unique tight geodesic k such that $D(k) = Y$ and k is backward (resp. forward) subordinate to b (resp. f). *This is the only point of difference with a hierarchy.*
(3) For every geodesic k in H other than g_H, there are $b, f \in H$ such that k is backward (resp. forward) subordinate to b (resp. f).

As in the case of hierarchies, any two points $\mu, \nu \in \mathcal{C}(S, \xi)$ are connected by at least one hierarchy path γ. Also, one can define $d_{\mathcal{C}(W,\xi)}(\mu, \nu)$ as a generalization of $d_{\mathcal{C}(W)}(\mu, \nu)$ of [15] by taking projections of $\mathcal{C}(W, \xi)$ to $\mathcal{C}(W, \xi)$. Then Lemma 6.2 of [15] generalizes to: There exists a constant $M_2 = M_2(S)$ such that, if W is an essential subsurface of S and $d_{\mathcal{C}(W,\xi)}(\mu, \nu) > M_2$ (in particular $\xi(W) > \xi$), then for any complexity ξ hierarchy path γ connecting μ to ν, there exists a marking γ_n in γ with $[\partial W] \subset$ **base** (γ_n). Furthermore there exists a vertex v in the geodesic β shadowed by γ such that $W \subset S \setminus v$. Further complexity ξ hierarchy paths are quasigeodesics in $\mathcal{C}(W, \xi)$. Similar generalizations hold for Lemma 6.6 (Common Links) and Lemma 6.7 (Slice Comparison) of [15].

Distance Formulae

Masur–Minsky prove the following distance formula for distances in the marking complex:

Theorem 1.9. Masur–Minsky [15] *If $\mu, \nu \in \mathcal{M}(S)$, then there exists a constant $K(S)$, depending only on S, such that for each $K > K(S)$ there exists $a \geq 1$ and $b \geq 0$ for which:*

$$d_{\mathcal{M}(S)}(\mu, \nu) \approx_{a,b} \sum_{W \subseteq S} Tf_K d_{\mathcal{C}(W)}(\pi_Y(\mu), \pi_Y(\nu))$$

Here the *threshold function* $Tf_K N$ is defined to be N if $N > K$ and 0 else. Also we write $f \approx_{a,b} g$ if $\frac{1}{a} f - b \leq g \leq af + b$.

In the final section of [14], Masur and Minsky indicate generalizations of their main results to the pants complexes. This is explicated in [2], where, Behrstock and Minsky show that this yields the following formula for distances in the pants complex. They use Remark 1.2 to forget projections to annuli.

Theorem 1.10. Behrstock-Minsky [2] *If $\mu, \nu \in \mathcal{P}(S)$, then there exists a constant $K(S)$, depending only on the topological type of S, such that for each $K > K(S)$ there exists $a \geq 1$ and $b \geq 0$ for which:*

$$d_{\mathcal{P}(S)}(\mu, \nu) \approx_{a,b} \sum_{\mathrm{non-annular\ Y \subseteq S}} Tf_K d_{\mathcal{C}(Y)}(\pi_Y(\mu), \pi_Y(\nu))$$

Exactly analogously, we state the generalization of Theorem 1.10 above to interpolating graphs $\mathcal{C}(S, \xi)$. Here, we disregard subsurfaces of complexity $\leq \xi$ as per Remark 1.6. The proof follows that of Theorems 1.9 and 1.10 above.

Theorem 1.11. *If $\mu, \nu \in \mathcal{C}(S, \xi)$, there exists a constant $K(S)$, depending only on the topology of S, such that for each $K > K(S)$ there exists $a \geq 1$ and $b \geq 0$ for which:*

$$d_\xi(\mu, \nu) \approx_{a,b} \sum_{W \subseteq S, \xi(W) > \xi} Tf_K d_{\mathcal{C}(W)}(\pi_Y(\mu), \pi_Y(\nu))$$

2. Rank of $\mathcal{C}(S, \xi)$

2.1. **Lower Bound on Rank of $\mathcal{C}(S, \xi)$.** In this section, we generalize a result of Brock-Farb [5] to the case of interpolating graphs. The proof is virtually an exact replica modulo two extra ingredients. First, the observation in Remark 1.4 that $\mathcal{C}(W, \xi)$ is quasi-isometric to the curve complex $\mathcal{C}(W)$ if $\xi(W) = \xi + 1$. The other ingredient is Theorem 1.11 which generalizes Theorem 1.9 and Theorem 1.10.

Recall that a *quasiflat* in a metric space X is a quasi-isometric embedding of Euclidean n-space in X; also, (Gromov [9] Section 6.2) that the *rank* of a metric space X is the maximal dimension n of a *quasi-flat* in X.

As in [5], we shall say that S *decomposes* into essential subsurfaces $R_1, \ldots, R_k$ if each R_j is essential and if $R_1, \ldots, R_k$ may be modified by an isotopy so that they are pairwise disjoint and $S - R_1 \sqcup \ldots \sqcup R_k$ is a collection of open annular neighborhoods of simple closed curves on S, each isotopic to a boundary component of some R_j.

Fix $\xi \in \mathbb{N}$. Let $r_\xi(S)$ denote the maximum number k such that S decomposes into essential subsurfaces $R_1, \ldots, R_k, T$ such that each R_j, $j = 1, \ldots, k$ has $\xi(R_j) = \xi + 1$ and T is either empty or has $\xi(T) \le \xi$.

Theorem 2.1. *The graph $\mathcal{C}(S, \xi)$ contains a quasi-flat of dimension $r_\xi(S)$.*

Proof: The proof is essentially a reworking in the present context of complexity ξ graphs of Theorem 4.2 of [5] by Brock and Farb. By definition of $r_\xi = r$(say), the surface S decomposes into subsurfaces

$$R_1, \ldots, R_{r(S)}, T$$

so that $\xi(R_j) = \xi + 1$ for each j and either T is empty or $\xi(T) \le \xi$.

We now construct a quasi-isometric embedding of the Cayley graph for $\mathbb{Z}^r$ with the standard generators into the complex $\mathcal{C}(S, \xi)$.

Let $\{c_j\}$ be a pants decomposition of R_j and γ a pants decomposition of T. Along with the core curves of the open annuli in $S - R_1 \sqcup \ldots \sqcup R_r \sqcup T$, the curves $\mathbf{base}(c_j)$ and $\mathbf{base}(\gamma)$ form a pants decomposition of S.

We let $g_j \colon \mathbb{Z} \to \mathcal{C}(R_j, \xi)$ be a (bi-infinite) geodesic so that $g_j(0) = c_j$. Since $\mathcal{C}(R_j, \xi)$ is quasi-isometric to $\mathcal{C}(R_j)$ (Remark 1.4) we might as well assume that $g_j \colon \mathbb{Z} \to \mathcal{C}(R_j)$ is a quasigeodesic (though strictly speaking, we should compose g_j with the quasi-isometry between $\mathcal{C}(R_j, \xi)$ and $\mathcal{C}(R_j)$). Note that the quasi-isometry constants depend only on R_j and hence only on the topology of S. Further, we identify $\mathcal{C}(R_j, \xi)$ and $\mathcal{C}(R_j)$ via these uniform quasi-isometries. For the rest of this proof, we shall assume that $\mathcal{C}(R_j, \xi) = \mathcal{C}(R_j)$ rather than just being quasi-isometric to it.

We define the embedding $Q \colon \mathbb{Z}^r \to \mathcal{C}(S, \xi)$ as:

$$Q(k_1, \ldots, k_r) = (g_1(k_1), \ldots, g_r(k_r))$$

.

Let $\vec{k} = (k_1, \ldots, k_n)$ and $\vec{l} = (l_1, \ldots, l_n)$. Since elementary moves in the pants graph and hence in $\mathcal{C}(S, \xi)$ along g_j can be made independently in each R_j, we have

$$d_\xi(Q(\vec{k}), Q(\vec{l})) \le \sum_{j=1}^{n} |l_j - k_j| = d_{\mathbb{Z}^r}(\vec{k}, \vec{l})$$

which shows that Q is 1-Lipschitz.

Given R_j, the subsurface projection $\pi_{R_j}(Q(\vec{k}))$ to R_j simply picks out the curve $g_j(k_j)$ so we have

$$\pi_{R_j}(Q(\vec{k})) = g_j(k_j).$$

Thus, the projection distance

$$d_{R_j}(Q(\vec{k}), Q(\vec{l})) = d_{R_j}(g_j(k_j), g_j(l_j)) = |k_j - l_j|$$

since we are identifying $\mathcal{C}(R_j, \xi)$ and $\mathcal{C}(R_j)$.

By Theorem 1.11, there exists $K_0 = K_0(S)$ so that for all $K \geq K_0$ there exist constants a and b so that if we let $P_{\vec{k}} = Q(\vec{k})$ and $P_{\vec{l}} = Q(\vec{l})$ then we have the inequality

$$\sum_{\substack{Y \subseteq S\, \xi(Y) > \xi \\ d_{\mathcal{C}(Y)}(\pi_Y(P_{\vec{k}}), \pi_Y(P_{\vec{l}})) > M}} d_Y(P_{\vec{k}}, P_{\vec{l}}) \leq a\, d_\xi(P_{\vec{k}}, P_{\vec{l}}) + b.$$

But the left-hand-side of the inequality is bounded below by

$$\max_j |k_j - l_j| \geq \frac{\sum_j |k_j - l_j|}{r}.$$

Thus, Q is a quasi-isometric embedding. □

2.2. **Upper Bound on Rank of $\mathcal{C}(S, \xi)$.** In this section, we generalize a recent result of Behrstock-Minsky [2] to the context of interpolating graphs. The proof is again virtually an exact replica. As we shall be following [2] closely, we shall indicate only the steps in the argument and the necessary modifications.

Step 1: Coarse Product Regions in $\mathcal{C}(S, \xi)$

First, we describe the geometry of the set of pants decompositions in $\mathcal{C}(S, \xi)$ containing a prescribed set of base curves. Equivalently, in the coned-off mapping class group $\Gamma(S, \xi)$, such a set corresponds to the coned off coset of the stabilizer of a simplex in the complex of curves. These regions coarsely decompose as products as in [2].

Let Δ be a multicurve in S. Partition S into subsurfaces isotopic to complementary components of Δ. Let $\sigma(\Delta)$ denote the subsurface obtained by discarding the components homeomorphic to $S_{0,3}$. This is called the *partition* of Δ.

Theorem 1.11 gives the following generalization of Lemma 2.1 of Behrstock-Minsky [2].

Lemma 2.2. *Let $\mathcal{Q}(\Delta) \subset \mathcal{C}(S, \xi)$ denote the set of pants decompositions whose bases contain Δ. Then, sending each (family of base curves of a) pants decomposition to the restrictions to elements of $\sigma(\Delta)$ we obtain a quasi-isometric identification*

$$\mathcal{Q}(\Delta) \approx \prod_{U \in \sigma(\Delta), \xi(U) > \xi} \mathcal{C}(U, \xi)$$

with uniform constants.

Step 2: Ultralimits of $\mathcal{Q}(\Delta)$

We refer the reader to [2] Section 1.4 for the necessary background on ultralimits and asymptotic cones. $\mathcal{C}^\omega(S,\xi)$ will denote an asymptotic cone of $\mathcal{C}(S,\xi)$ and μ_0 a preferred base-point. Let $\{s_n\}_n \in \mathbb{N}$ be a monotonic sequence of positive numbers such that $s_n \to \infty$.

Definition 2.3. *For a sequence $\boldsymbol{\Delta} = \{\Delta_n\}$ such that $\lim_\omega \frac{1}{s_n} d_\xi(\mu_0, \mathcal{Q}(\Delta_n)) < \infty$, define $\mathcal{Q}^\omega(\boldsymbol{\Delta}) \subset \mathcal{C}^\omega(S,\xi)$ to be the ultralimit of $\mathcal{Q}(\Delta_n)$, with metrics rescaled by $1/s_n$.*

Also, since the topological type of $\sigma(\Delta_n)$ is ω-a.e. constant, we may define $\sigma(\boldsymbol{\Delta})$ to be the ω-limit of $\sigma(\Delta_n)$'s. Note also that the complexities $\xi(U_{nj})$ are ω-a.e. constant for components U_{nj} of U_n.

Then Lemma 2.2 and the fact that ultralimits commute with finite products gives the following generalization of Equation 2.2 of [2]:

Lemma 2.4. *There is a uniform bi-Lipschitz identification*

$$\mathcal{Q}^\omega(\boldsymbol{\Delta}) \cong \prod_{\boldsymbol{U} \in \sigma(\boldsymbol{\Delta}), \xi(\boldsymbol{U}) > \xi} \mathcal{C}^\omega(\boldsymbol{U},\xi).$$

Step 3: $\mathbb{R}$-trees and Product Regions in Asymptotic Cones

The following definition is adapted from Behrstock [1] and Behrstock-Minsky [2].

Definition 2.5. *Let $\boldsymbol{W} = (W_n)$ be a sequence of connected subsurfaces (considered mod ω) and $\boldsymbol{x} \in \mathcal{C}^\omega(\boldsymbol{W},\xi)$, Then $F_{\boldsymbol{W},\boldsymbol{x},\xi} \subset \mathcal{C}^\omega(\boldsymbol{W},\xi)$ is defined as:*

$$F_{\boldsymbol{W},\boldsymbol{x},\xi} = \{\boldsymbol{y} \in \mathcal{C}^\omega(\boldsymbol{W},\xi) : d_{\mathcal{C}^\omega(\boldsymbol{W},\xi)}(\boldsymbol{x},\boldsymbol{y}) = 0 \textit{ for all proper subsets } \boldsymbol{U} \subset \boldsymbol{W}\}.$$

The next theorem is a version of a theorem due to Behrstock [1] adapted to our context of interpolation graphs.

Theorem 2.6. *Let $\boldsymbol{W} = (W_n)$ be a sequence of connected proper subsurfaces of S, and $\boldsymbol{x} \in \mathcal{C}^\omega(\boldsymbol{W},\xi)$. Any two points $\boldsymbol{y}, \boldsymbol{z} \in F_{\boldsymbol{W},\boldsymbol{x},\xi}$ are connected by a unique embedded path in $\mathcal{C}^\omega(\boldsymbol{W},\xi)$, and this path lies in $F_{\boldsymbol{W},\boldsymbol{x}}$. In particular, $F_{\boldsymbol{W},\boldsymbol{x},\xi}$ is an $\mathbb{R}$-tree.*

Next, for $\boldsymbol{W}$ and $\boldsymbol{x}$ as above, *separating product regions* in $\mathcal{C}^\omega(\boldsymbol{W},\xi)$, denoted $P_{\boldsymbol{W},\boldsymbol{x},\xi}$, are subsets of $\mathcal{Q}^\omega(\partial\boldsymbol{W})$ defined as follows:
In the bi-Lipschitz product structure on $\mathcal{Q}^\omega(\partial\boldsymbol{W})$ (Lemma 2.4), $\boldsymbol{W}$ is a member of $\sigma(\partial\boldsymbol{W})$. Therefore, $\mathcal{C}^\omega(\boldsymbol{W},\xi)$ appears as a factor. Define

$P_{\boldsymbol{W},\boldsymbol{x},\xi}$ to be the subset of $\mathcal{Q}^\omega(\partial\boldsymbol{W},\xi)$ consisting of points whose co-ordinate in the $\mathcal{C}^\omega(\boldsymbol{W},\xi)$ factor lies in $F_{\boldsymbol{W},\boldsymbol{x},\xi}$. The following Lemma generalizes Lemma 3.3 of [2].

Lemma 2.7. *There exists a bi-Lipschitz identification of $P_{\boldsymbol{W},\boldsymbol{x},\xi}$ with*

$$F_{\boldsymbol{W},\boldsymbol{x},\xi} \times \mathcal{C}^\omega(\boldsymbol{W}^c,\xi).$$

Step 4: Global Projection Maps

The following Theorem generalizes Theorem 3.5 of [2] and gives a global projection map for $F_{\boldsymbol{W},\boldsymbol{x},\xi}$:

Theorem 2.8. *Given $\boldsymbol{x} \in \mathcal{C}^\omega(\boldsymbol{W},\xi)$, there is a continuous map*

$$\Phi = \Phi_{\boldsymbol{W},\boldsymbol{x},\xi} \colon \mathcal{C}^\omega(S,\xi) \to F_{\boldsymbol{W},\boldsymbol{x},\xi}$$

with these properties:

1. *Φ restricted to $P_{\boldsymbol{W},\boldsymbol{x},\xi}$ is projection to the first factor in the product structure $P_{\boldsymbol{W},\boldsymbol{x},\xi} \cong F_{\boldsymbol{W},\boldsymbol{x},\xi} \times \mathcal{C}^\omega(\boldsymbol{W}^c,\xi)$.*
2. *Φ is locally constant in the complement of $P_{\boldsymbol{W},\boldsymbol{x},\xi}$.*

Step 5: Separating Sets

The sets $P_{\boldsymbol{W},\boldsymbol{x},\xi} \cong F_{\boldsymbol{W},\boldsymbol{x},\xi} \times \mathcal{C}^\omega(\boldsymbol{W}^c,\xi)$ give rise to separating sets in $\mathcal{C}^\omega(S,\xi)$ as in Theorem 3.6 of [2].

Theorem 2.9. *There is a family $\mathcal{L}$ of closed subsets of $\mathcal{C}^\omega(S,\xi)$ such that any two points in $\mathcal{C}^\omega(S,\xi)$ are separated by some $L \in \mathcal{L}$. Moreover each $L \in \mathcal{L}$ is isometric to $\mathcal{C}^\omega(Z,\xi)$, where Z is some proper essential (not necessarily connected) subsurface of S, with $r(Z) < r(S)$.*

Step 6: Inductive Dimension

Using the separating sets, we complete the argument as in Theorem 4.1 of [2].

Let ind, Ind and *dim* denote small inductive dimension, ind, large inductive dimension, and covering (or topological) dimension respectively.

Let $\widehat{\mathrm{ind}}(X)$ denote the supremum of $\mathrm{ind}(X')$ over all locally-compact subsets $X' \subset X$; similarly $\widehat{\mathrm{Ind}}$ and $\widehat{\dim}$. Then using Theorem 2.9 above the argument by Behrstock and Minsky yields

Theorem 2.10. $\widehat{\mathrm{ind}}(\mathcal{C}^\omega(S,\xi)) = \widehat{\mathrm{Ind}}(\mathcal{C}^\omega(S,\xi)) = \widehat{\dim}(\mathcal{C}^\omega(S,\xi)) =$ $r_\xi(S)$.

Since a quasiflat of dimension r would yield a dimension r locally compact subset of $\mathcal{C}^\omega(S,\xi)$, Theorem 2.10 immediately gives:

Corollary 2.11. *If the graph $\mathcal{C}(S,\xi)$ contains a quasi-flat of dimension r, then $r \leq r_\xi(S)$.*

Combining Theorem 2.1 and Corollary 2.11 we obtain the main theorem of this paper:

Theorem 2.12. *The rank of the interpolating graph $\mathcal{C}(S,\xi)$ of complexity ξ is $r_\xi(S)$ for $\xi > 0$.*

2.3. **Problems.** A number of (hopefully interesting) issues arise from the notion of interpolating graphs of complexity ξ.

Conjecture 1: Rank one implies Hyperbolic
If for $\xi > 0$, $r_\xi(S) = 1$ is $\mathcal{C}(S,\xi)$ a hyperbolic metric space?

This is motivated by

(1) the observation (Remark 1.4) that $\mathcal{C}(S,\xi(S)-1)$ is quasi-isometric to the curve graph and that in this case, $r_{\xi(S)-1}$ is clearly one (all of S is embedded in S by the identity map).
(2) Next, for $\xi = -2$, $r_\xi = 1$ means that the maximum number of disjoint homotopically distinct annuli (connected subsurface of complexity $(-2+1=1)$) that S admits is precisely one. Then $\xi(S) = 1$ and it is precisely these cases that have hyperbolic marking complex $\mathcal{C}(S,-2)$.
(3) $\xi = -1$ corresponds to the pants graph. This case is special and here the appropriate hypothesis would be $r_0(S) = 1$ because $S_{0,3}$ has trivial curve complex; hence in order to get an infinite diameter curve complex, we have to step up $\xi = -1$ by 2 and then calculate the maximum number of disjoint embedded subsurfaces of complexity ≥ 1. Brock and Farb [5] show that the pants graph is hyperbolic iff $\xi(S) = 2$ and each of the two possibilities (5-holed sphere and two-holed torus) admit exactly one (disjoint) subsurface of complexity 1.
(4) $\xi = 1$, $\xi(S) = 3$. Recent work of Brock and Masur [6] is closely related to this case. Brock and Masur show that the pants graph $P(S)$ for $\xi(S) = 3$ is strongly hyperbolic relative to certain sets $X_\gamma \subset P(S)$ consisting of pants decompositions that contain γ. This is equivalent to coning off the (products of) pants subcomplexes of subsurfaces Σ where $\xi(\Sigma) = 1$.

Thus the above Conjecture would serve to unify all the above cases. A similar conjecture has been formulated by Saul Schleimer in [18]

Problem 2 Is the automorphism group of the interpolating graph commensurable with the mapping class group $MCG(S)$?

This question is a special case of Ivanov's metaconjecture [12] that every object naturally associated to a surface S and having a suciently rich structure has $MCG(S)$ as its groups of automorphisms.

One other piece of motivation is Margalit's result [13] reducing the automorphism group of the pants graph to that of the curve complex C(S).

Problem 3 There is a hierarchy

$$\mathcal{C}(S,-2) \to \mathcal{C}(S,-1) \to \mathcal{C}(S,1) \cdots C(S,\xi) \to C(S,\xi+1) \to \cdots \to C(S,\xi(S)-1)$$

where the first term is the marking graph, the second the pants graph, the last the curve graph.

The map $C(S,\xi) \to C(S,\xi+1)$ is given by coning a collection of subsets corresponding to curve graphs of subsurfaces of complexity $\xi+1$. Thus the last term is hyperbolic and the preimages of points at each stage are hyperbolic by Masur-Minsky's Theorem 1.7. This raises the hope that the hierarchy paths constructed by Masur and Minsky in [15] may alternately be inductively constructed in a bottom-up approach from the curve complex. Further, at each stage we should obtain a hierarchy path in the interpolating graph $\mathcal{C}(S,\xi)$.

Problem 4 (independently due to Yair Minsky [16]. See also Question 10, due to Wise and Behrstock in the Geometric Group Theory Problems wiki, Section on *Relative Hyperbolicity*.) Finally, there ought to be a general geometric structure lying between strong and weak relative hyperbolicity of which the mapping class group is a special case. Let us call this putative structure *graded relative hyperbolicity* (terminology independently due to Yair Minsky [16]) . A possible definition would be the existence of a sequence

$$X_n \to X_{n-1} \to \cdots X_1$$

of spaces and maps where at the ith stage one cones off a collection $\mathcal{C}_i$ of (uniformly) quasiconvex hyperbolic subsets of X_i. Further, we demand that X_1 be hyperbolic.

A toy example is given by $X = X_n = \Gamma_G$ the Cayley graph of a hyperbolic group G. H is assumed to be a quasiconvex subgroup of height n (see Gitik-Mitra-Rips-Sageev [8] for instance). In passing from X_i to X_{i-1}, we cone all cosets of $\cap_{j=1\cdots i-1} g_j H g_j^{-1}$ for essentially distinct cosets $g_j H$. In this particular case, the role of hierarchy paths might be taken by *electro-ambient quasigeodesics* introduced by the author in [17].

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