The quantum communication complexity of the pointer chasing problem: the bit version

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Abstract

We consider the two-party quantum communication complexity of the bit version of the pointer chasing problem, originally studied by Klauck, Nayak, Ta-Shma and Zuckerman [KNTZ01]. We show that in any quantum protocol for this problem, the two players must exchange $\Omega(\frac{n}{k^4})$ qubits. This improves the previous best bound of $\Omega(\frac{n}{2^{2^{O(k)}}})$ in [KNTZ01], and comes significantly closer to the best upper bounds known $O(n+k\log n)$ (classical deterministic [PRV01]) and $O(k\log n+\frac{n}{k}(\log^{\lceil k/2\rceil}(n)+\log k))$ (classical randomized [KNTZ01]). Our proof uses a round elimination argument with correlated input generation, making better use of the information theoretic tools than in previous papers.

1 Introduction

We consider the following pointer chasing problem in the two-party communication model [Yao79, Yao93].

> Let V_A and V_B be disjoint sets of size n. Alice is given a function $F_A : V_A \rightarrow V_B$ and player Bob is given a function $F_B : V_B \rightarrow V_A$. Let $F \triangleq F_A \cup F_B$. There is a fixed vertex s in V_B. The players need to exchange messages and determine the least significant bit of $F^{(k+1)}(s)$, where k and s are known to both parties in advance.

If Bob starts the communication, there is a straightforward classical deterministic protocol where one of the players can determine the answer after k messages of $\log n$ bits have been exchanged. It appears much harder, however, to solve the problem efficiently with k messages, when Alice is required to send the first message. We refer to this as the pointer chasing problem P_k.

Background: The pointer chasing problem has been studied in the past to show rounds versus communication tradeoffs in classical communication complexity. Nisan and Wigderson [NW93] showed (following some earlier results of Papadimitriou and Sipser [PS84], and Duris, Galil and Schnitger [DGS87]) that the players must exchange $\Omega(\frac{n}{k} - k\log n)$ bits to solve P_k; their bound was improved by Klauck [Kla00] to $\Omega(\frac{n}{k} + k)$. These lower bounds hold even if randomization is allowed. A deterministic protocol with $O(n + k\log n)$ bits of communication was given by Ponzio, Radhakrishnan and Venkatesh [PRV01], and a classical randomized protocol with $O(k\log n + \frac{n}{k}(\log^{\lceil k/2\rceil}(n) + \log k))$ bits by Klauck, Nayak, Ta-Shma and Zuckerman [KNTZ01]. Thus, the lower and upper bounds are quite close in the the classical setting.

In the quantum communication complexity model, this problem has been studied recently by Klauck, Nayak, Ta-Shma and Zuckerman [KNTZ01], who, using interesting information-theoretic techniques, showed

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a lower bound of $\Omega(\frac{n}{2^{2^{O(k)}}})$. This bound deteriorates rapidly with k, and becomes trivial for $k \geq \log \log n$. We improve this lower bound.

Result: In any bounded error quantum protocol for the pointer chasing problem P_k, Alice and Bob must exchange $\Omega(\frac{n}{k^4})$ qubits.

Our proof technique: The underlying information theoretic tools we use are, in fact, mainly taken from the paper Klauck, Nayak, Ta-Shma and Zuckerman [KNTZ01]. Our proofs use the round elimination method, stated explicitly in the classical communication complexity setting by Miltersen, Nisan, Safra and Wigderson [MNSW98]. This technique was applied in the quantum setting by Klauck *et al.*, who developed several tools, notably the *average encoding theorem* and the *local transition theorem*. Their argument was refined further by Sen and Venkatesh [SV98]. Recently, Jain, Radhakrishnan and Sen [JRS02] showed an optimal $\Omega(n \log^{(k)} n)$ lower bound for the *full version* of the pointer chasing problem, where the players must determine the full description of $F^{(k+1)}(s)$, and not just its least significant bit. This was also obtained by the round elimination argument. In this paper, we adapt this argument to the *bit version* of the problem. For this, we consider a slightly different pointer chasing problem, where the two players are allowed to generate their own inputs and then proceed to compute the answer. To keep this problem non-trivial we must impose some restrictions on the way the players behave. First, we insist that the inputs they generate must be sufficiently rich. Second, the amount of communication before the input is generated, is limited. In previous round elimination arguments, the inputs were supplied to the two players from 'outside'. While this worked well for many problems, for the pointer chasing problem it made things difficult. However, letting the players generate their inputs gives rise to new technical difficulties, because the inputs they generate are not exactly what we want, but only close to it. So, we need to apply a correction step, that converts a protocol whose inputs have a distribution close to the one we desire into one where the inputs are exactly what we want. Overall, we believe, the main contribution of this work is in showing how existing information theoretic tools can be better exploited for round elimination in quantum communication protocols.

1.1 Organization of the rest of the paper

In the next section, we define the pointer chasing problem formally and derive our main result assuming a Round Elimination Lemma. In Section 3, we collect the probabilistic and information theoretic tools that are required for the proof. Finally, in Section 4, we describe the round elimination argument in detail.

2 Lower bound for the pointer chasing problem

In this section, we formally define the problem and our main result assuming a Round Elimination Lemma, which will be proved in later section.

Quantum communication protocols: We consider two party quantum communication protocols as defined by Yao [Yao93]. Let E, F, G be arbitrary finite sets and $f : E \times F \rightarrow G$ be a function. There are two players Alice and Bob, who hold qubits. When the communication game starts, Alice holds $|x\rangle$ where $x \in E$ together with some ancilla qubits in the state $|0\rangle$, and Bob holds $|y\rangle$ where $y \in F$ together with some ancilla qubits in the state $|0\rangle$. Thus the qubits of Alice and Bob are initially in computational basis states, and the initial superposition is simply $|x\rangle_A|0\rangle_A|y\rangle_B|0\rangle_B$. Here the subscripts denote the ownership of the qubits by Alice and Bob. The players take turns to communicate to compute $f(x, y)$. Suppose it is Alice's turn. Alice can make an arbitrary unitary transformation on her qubits and then send one or more qubits to Bob. Sending qubits does not change the overall superposition, but rather changes the ownership of the qubits, allowing Bob to apply his next unitary transformation on his original qubits plus the newly received qubits. At the end of the protocol, the last recipient places the answer in a special register Ans.

Definition 1 (Safe transformation, protocols) *Let $\mathcal{H}$ and $\mathcal{K}$ be a finite-dimensional Hilbert spaces, with bases $(|h\rangle : h \in H)$ and $(|k\rangle : k \in K)$. We say that a unitary transformation U on $\mathcal{H} \otimes \mathcal{K}$* acts safely *on $\mathcal{H}$ if there exist unitary transformations $(U_h : h \in H)$ acting on $\mathcal{K}$ such that for all $h \in H$ and $k \in K$,*

$$U : |h\rangle|k\rangle \mapsto |h\rangle U_h|k\rangle.$$

We say that a protocol acts safely on a register R, if all unitary transformations in the protocol act safely on R, and R is never sent as part of a message. We say that a protocol is safe *if Alice and Bob act safely on their input registers.*

When the inputs are classical, we can always assume that the protocol is safe. This is possible since the inputs to Alice and Bob are in computational basis states. So, the players can make a secure copy of their inputs before beginning the protocol.

2.1 The pointer chasing problem P_k

The input: Alice's input is a function $F_A : V_A \to V_B$. Bob's input is a function $F_B : V_B \to V_A$. V_A and V_B are disjoint sets of size n each. We assume that $n = 2^r$ for some $r \geq 1$.

The golden path: There is a fixed vertex $s \in V_B$. Let $F \stackrel{\Delta}{=} F_A \cup F_B$; let $\mathsf{ans} \stackrel{\Delta}{=} \mathrm{lsb}(F^{(k+1)}(s))$. Here $\mathrm{lsb}(x)$ is the least significant bit of x; we assume that vertices in V_A and V_B have binary encodings of length $\log n$.

The communication: Alice and Bob exchange messages $M_1, \ldots, M_k$, having lengths $c_1 n, \ldots, c_k n$, via a safe quantum protocol in order to determine ans. Alice starts the communication, that is, she sends M_1. The player receiving M_k places a guess for ans in the register Ans. We require that the bit obtained by measuring Ans in the computational basis[1] should be the correct answer (i.e. equal to $\mathrm{lsb}(F^{(k+1)}(s))$ with probability at least $\frac{3}{4}$, for all F_A, F_B.

2.2 The predicate Q_k^A

We will show our lower bound for P_k using an inductive argument. It will be convenient to state our induction hypothesis by means of a predicates Q_k^A and Q_k^B, defined below. Roughly, the induction proceeds as follows. We show that if there is an efficient protocol for P_k, then Q_k^A is true. We then show independently that Q_ℓ^A implies $Q_{\ell-1}^B$ and Q_ℓ^B implies $Q_{\ell-1}^A$, and that Q_0^A and Q_0^B are false. Thus, there is no efficient protocol for P_k.

We now define $Q_k^A(c_1, \ldots, c_k, n_a, n_b, \epsilon)$ for $k \geq 1$. Then, separately, we define Q_0^A. For $k \geq 0$, Q_k^B is the same as Q_k^A, with the roles of Alice and Bob reversed. Consequently, all our statements involving Q_k^A and Q_k^B have two forms, where one is obtained from the other by reversing the roles of Alice and Bob. We will typically state just one of them, and let the reader infer the other.

The predicate $Q_k^A(c_1, \ldots, c_k, n_a, n_b, \epsilon)$ holds if there is a quantum protocol of the following form.

[1]From now on, all measurements are to be performed using the computational basis.

Input generation: Alice and Bob 'generate' most of their inputs themselves. Alice has n input registers $(F_A[u] : u \in V_A)$ and Bob has n input registers $(F_B[v] : v \in V_B)$. There is a fixed vertex $s \in V_B$, that is known to both players. Each of Alice's registers has $\log n$ qubits so that it can hold a description of a vertex in V_B; similarly, each of Bob's registers can hold a description of a vertex in V_A. In addition, Alice and Bob have registers for their 'work' qubits W_A and W_B.

When the protocol starts, Alice's registers are all initialized to 0. On Bob's side, the register $F_B[s]$ starts off with the uniform superposition $|\mu\rangle \triangleq \frac{1}{\sqrt{n}} \sum_{a \in V_A} |a\rangle$; the other registers are all 0.

Alice starts by generating a pure state in $\widetilde{M}_1 M_1$, where $\widetilde{M}_1, M_1$ are each $c_1 n$ qubit registers. Then she applies a unitary transformation U_A on her registers other than M_1 to generate a state in registers F_A and W_A. Alice then sends M_1 to Bob.

Now, Bob generates his input using the message M_1 as follows. He applies a unitary transformation U_B on the registers that he owns at this point:

- M_1, the message registers just received from Alice;
- $F_B[s]$ the register holding the start pointer, which is in the state $|\mu\rangle$ in tensor with the other register;
- $(F_B[b] : b \in V_B - \{s\})$ and the registers W_B holding the work qubits of B, which contain 0.

U_B must operate "safely" on $F_B[s]$. F_B holds the 'generated input' to Bob for the pointer chasing problem, and W_B Bob's 'work qubits'.

We will use F_A, F_B also to refer to the actual states of the respective registers; f_A, f_B will denote the states that would result, were we to measure F_A, F_B. Thus, typically F_A, F_B will be parts of a pure state (the global state of Alice's and Bob's qubits) whereas f_A, f_B will be mixtures of computational basis states.

For our predicate $Q_k^A(c_1, \ldots, c_k, n_a, n_b, \epsilon)$ to hold, this input generation process must satisfy some conditions.

> **Requirement 1(a):** There is a subset $X_A \subseteq V_A$ of size at most n_a such that the variables $(f_A(u) : u \in V_A)$ are independent, and for $u \in V_A - X_A$, $f_A(u)$ is uniformly distributed.
>
> **Requirement 1(b):** There is a subset $X_B \subseteq V_B - \{s\}$ of size at most n_b such that the random variables $(f_B(v) : v \in V_B)$ are independent, and $f_B(v)$ for $v \in V_B - X_B$ is uniformly distributed. Note that $f_B[s]$ is automatically uniformly distributed, because initially $F_B[s]$ contains the uniform superposition, and U_B acts safely on $F_B[s]$.

Communication: After U_A, U_B have been applied, Alice and Bob follow a quantum protocol exchanging further messages $M_2, \ldots, M_k$ of lengths $c_2 n, \ldots, c_k n$. Bob sends the message M_2. The rest of the protocol is required to act safely on registers F_A, F_B. At the end of the protocol, the player who receives M_k places a qubit in a special register Ans. The protocol then terminates.

The probability of error: Once the protocol has terminated, all registers are measured. Let ans denote the value observed in Ans, and let f_A and f_B be the values observed in F_A and F_B; we treat f_A and f_B as functions (from V_A to V_B and V_B to V_A respectively). Let $f \triangleq f_A \cup f_B$. Note that ans and f are random variables.

> **Requirement 2:** $\Pr[\mathsf{ans} = \mathrm{lsb}(f^{(k+1)}(s))] \geq 1 - \epsilon$.

Base case: In $Q_0^A(\epsilon)$, there is no input generation phase or communication. Bob and Alice start as before, with $|\mu\rangle$ in Bob's register $F_B[s]$. Alice produces a guess ans for $\mathrm{lsb}(f_B(s))$, which must be correct with probability at least $1-\epsilon$. Clearly, we have the following base case for our induction.

Proposition 1 *If $Q_0^A(\epsilon)$ is true then $\epsilon \geq \frac{1}{2}$.*

Our goal is to show that if Q_k^A holds, then $c_1 + c_2 + \ldots + c_k = \Omega(k^{-4})$. By the following lemma, this implies a lower bound $\frac{n}{k^4}$ for P_k^A.

Lemma 1 *If there is a safe quantum protocol for P_k^A with $v_0 = s \in V_B$, messages of lengths $c_1 n, \ldots, c_k n$, and worst case error at most $\frac{1}{4}$, then $Q_k^A(c_1, \ldots, c_k, n_A = 0, n_B = 0, \frac{1}{4})$ is true.*

Proof: We are given a safe quantum protocol $\mathcal{P}$ for P_k, where Alice sends the first message M_1. Consider the operation of $\mathcal{P}$ when uniform superpositions are fed for F_A and F_B. Consider the state of Alice just before M_1 is sent to B. This state has two parts.

1. The qubits that Alice keeps with herself, $F_A W_A$, where F_A is $n \log n$ qubits long.
2. The $c_1 n$ qubits that constitute the message M_1.

Let $\widetilde{M_1} M_1$ contain a canonical purification of M_1, where $\widetilde{M_1}$ is $c_1 n$ qubits long. Clearly, it is within Alice's powers to first generate the canonical purification in $\widetilde{M_1} M_1$, and then apply a unitary transformation U_A on $\widetilde{M_1}$ plus some initially zero ancilla qubits in order to generate the correct state of $F_A W_A M_1$. Alice then sends M_1

In our protocol, on Bob's side, $F_B[s]$ already has a uniform superposition in tensor with the rest of Alice's and Bob's qubits. Then, Bob generates the rest of his "input", $F_B[v], v \neq s$ as a uniform superposition in tensor with everything else. The registers W_B are set to $|0\rangle$. At this point, the state of $F_A W_A M_1 F_B W_B$ is exactly the same as it would be in $\mathcal{P}$ after Bob receives the first message. From now on, Alice and Bob operate exactly as in $\mathcal{P}$, which is "safe" on F_A, F_B. The above parameters for Q_k^A can now be verified easily. □

The following lemma is the key to our inductive argument.

Lemma 2 (Round elimination) *(a) For $k \geq 2$, if $Q_k^A(c_1, \ldots, c_k, n_A, n_B, \epsilon)$ holds (with $n_A < n$) then $Q_{k-1}^B(c_1 + c_2, c_3, \ldots, c_k, n_A, n_B + 1, \epsilon')$ holds with $\epsilon' = \left(\frac{n}{n - n_a}\right)\left[\epsilon + 3((2\ln 2)c_1)^{\frac{1}{4}}\right]$.*

(b) If $Q_1^A(c_1, n_A, n_B, \epsilon)$ holds (with $n_A < n$), then $Q_0^A(\epsilon')$ holds, where ϵ' is exactly as in part (a).

The next section is devoted to the proof of this lemma. Now, let us assume this lemma and prove our main lower bound.

Theorem 1 *Suppose $k \leq n^{\frac{1}{4}}$ and $Q_k^A(c_1, \ldots, c_k, 0, 0, \frac{1}{4})$ holds. Then $c_1 + c_2 + \cdots + c_k = \Omega(k^{-4})$.*

Proof: (Sketch) By $k-1$ applications of Part (a) of Lemma 2 (a) and one application of Part (b), we conclude that either $Q_0^A(\epsilon')$ or $Q_0^B(\epsilon')$ holds with $\epsilon' \leq \left(\frac{n}{n-k}\right)^k \left[\frac{1}{4} + 3k((2\ln 2)(c_1 + c_2 + \cdots + c_k))^{\frac{1}{4}}\right]$. Our theorem follows immediately from this and Proposition 1. □

Now, by using Lemma 1, we can derive from this our lower bound for P_k.

Corollary 1 (Main result) *In any protocol for P_k, Alice and Bob must exchange a total of $\Omega(\frac{n}{k^4})$ qubits.*

3 Preliminaries

We now recall some basic definitions and facts from probability and and information theory, which will be useful in proving our main result. For excellent introductions to classical and quantum information theory, see the books by Cover and Thomas [CT91] and Nielsen and Chuang [NC00] respectively.

If A is a quantum system with density matrix ρ, then $S(A) \stackrel{\Delta}{=} S(\rho) \stackrel{\Delta}{=} -\mathrm{Tr}\, \rho \log \rho$ is the *von Neumann entropy* of A. If A, B are two disjoint quantum systems, the *mutual information* of A and B is defined as $I(A : B) \stackrel{\Delta}{=} S(A) + S(B) - S(AB)$.

Fact 1 (see [KNTZ01]) *Let $X_1, \ldots, X_n$ be classical random variables and let M be a quantum encoding of $X \stackrel{\Delta}{=} X_1 \ldots X_n$. Then, $I(X : M) \geq \sum_{i=1}^{n} I(X_i : M)$. Also, if M is n qubits long, then $I(X : M) \leq n$.*

We will be working with various measures of distance between classical and quantum states. For distributions D and D' on a finite set X, their total variational distance is given by $\|D - D'\|_1 = \sum_{x \in X} |D(x) - D'(x)|$. We will use the following elementary fact, which we state without proof.

Fact 2 *Suppose D, D' are two probability distributions on the same finite set X, whose total variation distance is $\|D - D'\|_1 = \delta$. Then, there exists a stochastic matrix $P = (p_{xx'})_{xx' \in X}$, such that $D = PD'$ and $\sum_{x' \in X} P(x', x')D(x') = 1 - \frac{1}{2}\delta$. Let $\mathcal{H}$ be a Hilbert space with basis $(|x\rangle : x \in X)$. Let C be a unitary transformation on $\mathcal{H} \otimes \mathcal{H}$ that maps basis vectors of the form $|x'\rangle|\mathbf{0}\rangle$ (where $\mathbf{0}$ is a special element of X) according to the rule*

$$|x'\rangle|\mathbf{0}\rangle \to |x'\rangle \otimes \sum_{x \in X} \sqrt{p_{xx'}}|x\rangle,$$

and maps other standard basis vectors suitably. Suppose R' and R are registers that can hold states in $\mathcal{H}$, where R' contains a mixture of basis states with distribution D' and R is in the state $|\mathbf{0}\rangle$. Apply C to (R', R), and then measure the registers in the computational basis. Let the resulting random variables (taking values in X) be Z' and Z. Then, Z' has distribution D', Z has distribution D and $\Pr[Z \neq Z'] \leq \frac{1}{2}\delta$. Note, that C acts safely on R'.

The trace norm of a linear operator A is defined as $\|A\|_t \stackrel{\Delta}{=} \mathrm{Tr}\, \sqrt{A^\dagger A}$. The following fundamental theorem (see [AKN01]) shows that the trace distance between two density matrices ρ_1, ρ_2, $\|\rho_1 - \rho_2\|_t$, bounds how well one can distinguish between ρ_1, ρ_2 by a measurement.

Theorem 2 ([AKN01]) *Let ρ_1, ρ_2 be two density matrices on the same Hilbert space. Let $\mathcal{M}$ be a general measurement (i.e. a POVM), and $\mathcal{M}\rho_i$ denote the probability distributions on the (classical) outcomes of $\mathcal{M}$ got by performing measurement $\mathcal{M}$ on ρ_i. Let the ℓ_1 distance between $\mathcal{M}\rho_1$ and $\mathcal{M}\rho_2$ be denoted by $\|\mathcal{M}\rho_1 - \mathcal{M}\rho_2\|_1$. Then*

$$\|\mathcal{M}\rho_1 - \mathcal{M}\rho_2\|_1 \leq \|\rho_1 - \rho_2\|_t$$

We will need the following "average encoding theorem" of Klauck *et al.* [KNTZ01]. Intuitively speaking, it says that if the mutual information between a classical random variable and its quantum encoding is small, then the various quantum "codewords" are close to the "average codeword".

Theorem 3 (Average encoding theorem [KNTZ01]) *Suppose X, Q are two disjoint quantum systems, where X is a classical random variable, which takes value x with probability p_x, and Q is a quantum encoding $x \mapsto \sigma_x$ of X. Let the density matrix of the average encoding be $\sigma \stackrel{\Delta}{=} \sum_x p_x \sigma_x$. Then*

$$\sum_x p_x \|\sigma_x - \sigma\|_t \leq \sqrt{(2 \ln 2) I(X : Q)}$$

We will also need the following "local transition theorem" of Klauck *et al.* [KNTZ01].

Theorem 4 (Local transition, [KNTZ01]) *Let ρ_1, ρ_2 be two mixed states with support in a Hilbert space $\mathcal{H}$, $\mathcal{K}$ any Hilbert space of dimension at least the dimension of $\mathcal{H}$, and $|\phi_i\rangle$ any purifications of ρ_i in $\mathcal{H} \otimes \mathcal{K}$. Then, there is a local unitary transformation U on $\mathcal{K}$ that maps $|\phi_2\rangle$ to $|\phi_2'\rangle \stackrel{\Delta}{=} (I \otimes U)|\phi_2\rangle$ (I is the identity operator on $\mathcal{H}$) such that*

$$\big\| |\phi_1\rangle\langle\phi_1| - |\phi_2'\rangle\langle\phi_2'| \big\|_t \leq 2\sqrt{\|\rho_1 - \rho_2\|_t}$$

4 Round elimination: proof of Lemma 2

We consider Part (a) first. Part (b) follows using similar argument, and we do not describe them explicitly. Suppose $Q_k^A(c_1, c_2, \ldots, c_k, n_A, n_B, \epsilon)$ is true. That is, there is a protocol $\mathcal{P}$ satisfying Requirements 1 and 2 in the definition of Q_k^A. We need to show that there is a protocol that satisfies the requirements for Q_{k-1}^B with parameters stated in Lemma 2 (a).

In what follows, subscripts of pure and mixed states will denote the registers which are in those states. For $u \in V_A$, we use the subscript u instead of $F_A[u]$. Similarly, for $v \in V_B$, we use the subscript v instead of $F_B[v]$. For example, we say that the register $F_B[s]$ is initially in the state $|\mu\rangle_s = \frac{1}{\sqrt{n}} \sum_{u \in V_A} |u\rangle_s$.

Let $|\psi^A\rangle$ be the (pure) state of Alice's registers just before she sends M_1 to Bob. At this point the state of all the registers taken together is the pure state

$$|\psi_{\text{in}}\rangle = |\psi^A\rangle \otimes \frac{1}{\sqrt{n}} \sum_{a \in V_A} |a\rangle_s |\mathbf{0}\rangle_R, \tag{1}$$

where R is the set of registers corresponding to the rest of B's input $(F_B[v] : v \in V_B - \{s\})$, and work qubits W_B. For $a \in V_A$, we may expand $|\psi^A\rangle$ as

$$|\psi^A\rangle = \frac{1}{\sqrt{\ell_a}} \sum_{b \in V_B} |b\rangle_a |\psi^A_{a \to b}\rangle, \tag{2}$$

where $\ell_a = 1$ if $a \in X_A$ and $\ell_a = n$ otherwise. Here, $|\psi^A_{a\to b}\rangle$ is a pure state of Alice's registers $(F_A(v) : v \in V_A - \{a\})$ and W_A. Note that $|\psi^A_{a\to b}\rangle$ is precisely the state of these registers when $F_A[a]$ is measured and found to be in state $|b\rangle$. (If $\Pr[f_A[a] = b] = 0$, then $|\psi^A_{a\to b}\rangle \stackrel{\Delta}{=} 0$.) From (1) and (2), we have

$$|\psi_{\text{in}}\rangle = \frac{1}{\sqrt{n}} \sum_{a \in V_A} \frac{1}{\sqrt{\ell_a}} \sum_{b \in V_B} |b\rangle_a |\psi^A_{a\to b}\rangle |a\rangle_s |\mathbf{0}\rangle_R \tag{3}$$

At this point the first message M_1 is sent to Bob. Let the rest of the protocol starting from this point be $\mathcal{P}'$; that is, in $\mathcal{P}'$ Bob starts by generating his input from M_1 and $F_B[s]$, sends the message M_2 to A, to which Alice responds with M_3, and so on. At the end of $\mathcal{P}'$ we have a register containing the answer which we measure to find ans, and the input registers of Alice and Bob, which when measured yield f_A and f_B.

Let $\epsilon_{a\to b}$ be the probability of error when $\mathcal{P}'$ is run starting from the state $|b\rangle_a |\psi^A_{a\to b}\rangle |a\rangle_s |\mathbf{0}\rangle_R$. Thus, we have

$$\epsilon_{a\to b} = \Pr[\mathsf{ans} \neq \mathrm{lsb}(f^{(k+1)}(s)) \mid f_B[s] = a \text{ and } f_A[a] = b],$$

in the original protocol $\mathcal{P}$ (or in $\mathcal{P}'$, when it is run starting from $|\psi_{\text{in}}\rangle$). In particular, we have

$$\epsilon = \mathop{\mathrm{E}}_{a,b}[\epsilon_{a\to b}] \geq \frac{n - n_a}{n} \mathop{\mathrm{E}}_{a \in_u V_A - X_A, b \in_u V_B}[\epsilon_{a\to b}]. \tag{4}$$

In the first expectation, (a, b) are chosen with the same distribution as $(f_B[s], f_A[f_B[s]])$ of the given protocol $\mathcal{P}$; in the second, they are chosen uniformly from the sets specified.

Overview: We want to eliminate the first message sent by Alice, at the cost of increasing the probability of error slightly, but preserving the total length of the communication. This is based on the following idea (taken from [KNTZ01]). Let $M_{1,a\to b}$ be the state of the registers holding the first message when the entire state of Alice's registers is $\psi^A_{a\to b}$; that is, $M_{1,a\to b}$ is the state of the message registers corresponding to message M_1, when we measure $F_A[a]$ and observe $|b\rangle$ there. Note, that $\psi^A_{a\to b}$ is a purification of $M_{1,a\to b}$. Also, the state of the first message in $\mathcal{P}$, M_1 is the average, taken over the choices of b, of $M_{1,a\to b}$.

Suppose there is an $a \in V_A - X_A$ such that for all b, the message $M_{1,a\to b}$ is independent of b, that is, it is always the fixed sate M^*. Then, we can eliminate the first message. Informally stated, this amounts to restricting ourselves to the subcase of the protocol when Bob's first pointer $F_B[s]$ is fixed at $|a\rangle$, and Bob generates M^* himself, and sends some small advice along with his message M_2, to enable Alice to reproduce the right entanglement between her registers and Bob's. Unfortunately, we will not be able to show that there is an a and an M^* such that $M_{1,a\to b} = M^*$, for all b. Instead, we will show that there is an M^* that will be close to $M_{1,a\to b}$ for typical b. In fact, the message M_1 (which is the average of $M_{1,a\to b}$ as b varies) will be our M^*.

Let $(M_1, \widetilde{M_1})$ be the canonical purification of the first message of the protocol $\mathcal{P}$. Our first goal is to show that if M_1 is close to $M_{1,a\to b}$, then Alice can create a state close to $|\psi^A_{a\to b}\rangle$ from $(M_1, \widetilde{M_1})$ by applying a unitary transformation on $\widetilde{M_1}$. More precisely, suppose $\|M_{1,a\to b} - M_1\|_t \stackrel{\Delta}{=} \delta_{a\to b}$. Then, by the Local Transition Theorem, there is a unitary transformation $U_{a\to b}$ that when applied to $\widetilde{M_1}$ (together with ancilla qubits initialized to zero) takes the pure state $(M_1, \widetilde{M_1})$ to a state $\tilde{\psi}^A_{a\to b}$ such that

$$\left\| |\psi^A_{a\to b}\rangle\langle\psi^A_{a\to b}| - |\tilde{\psi}^A_{a\to b}\rangle\langle\tilde{\psi}^A_{a\to b}| \right\|_t \leq 2\sqrt{\delta_{a\to b}}. \tag{5}$$

In particular, if the protocol $\mathcal{P}'$ is run starting from the state $|\tilde{\psi}^A_{a\to b}\rangle|\mu\rangle_s|\mathbf{0}\rangle_R$ (instead of $|\psi^A_{a\to b}\rangle|\mu\rangle_s|\mathbf{0}\rangle_R$), the probability of error is at most $\epsilon_{a\to b} + 2\sqrt{\delta_{a\to b}}$.

4.1 The protocol $\mathcal{P}_{a\to b}$

Now, we fix $a \in V_A$ and $b \in V_B$ and consider the case when $f_B(s) = a$ and $f_A(a) = b$. We now describe a protocol that functions for this situation (see Figure 1) . This is just an intermediate protocol. Later we will describe how we obtain our final protocol (satisfying the requirements of Q^B_{k-1}) from this. It will be helpful, meanwhile, to keep in mind that in our final protocol, the roles of A and B will be reversed, $F_B[s]$ will be fixed at $|a\rangle$ (we will add s to X_B), a will be our new s, and the state of $F_A[a]$ will not be fixed at $|b\rangle$ but will be the uniform superposition $|\mu\rangle$.

Step 1: Alice generates the canonical purification $(M_1, \widetilde{M_1})$. Alice applies $U_{a\to b}$ to $\widetilde{M_1}$ (plus some ancilla) to produce the state $|\tilde{\psi}^A_{a\to b}\rangle$ in the registers (M_1, F_A, W_A).
Step 2: Alice and Bob proceed according to the protocol $\mathcal{P}'$ starting from the state $|\tilde{\psi}_{a\to b}\rangle = |\tilde{\psi}^A_{a\to b}\rangle|a\rangle_s|\mathbf{0}\rangle_R$, where, as before, R is the set of registers of Bob corresponding to $(F_B[v] : v \in V_B - \{s\})$ and work qubits W_B .

Figure 1: The intermediate protocol $\mathcal{P}_{a\to b}$

Revised Step 1:

- Alice generates the canonical purification $(M_1, \widetilde{M_1})$. Alice applies $U_{a\to b}$ to $\widetilde{M_1}$ (plus some ancilla) to produce the state $|\tilde{\psi}^A_{a\to b}\rangle$ in the registers (M_1, F_A, W_A). Alice sends M_1 to Bob.

- Next, to produce input registers satisfying Requirement 1(a), Alice uses a fresh set of registers $\hat{F}_A$ and sets $\hat{F}_A[a] = |b\rangle$. Next, Alice applies a unitary transformation to registers $(\hat{F}_A[a], F_A, \tilde{F}_A)$ defined by

$$|b\rangle_{\hat{F}[a]}|\psi\rangle_{F_A,\tilde{F}_A} \to |b\rangle_{\hat{F}[a]}C_{a\to b}|\psi\rangle_{F_A,\tilde{F}_A}.$$

Before the application of this the registers $\tilde{F}_A$ are initialized to $|\mathbf{0}\rangle$ (as in the statement of Fact 2). Alice then copies $(\tilde{F}_A[u] : u \in V_A - \{a\})$ into $(\hat{F}_A[u] : u \in V_A - \{a\})$. The input generation for Alice is now complete.

Note that at this point if we measure $(F_A, \hat{F}_A)$, the resulting random variables $(f'_{A,a\to b}, \hat{f}_{A,a\to b})$ have distribution precisely $D'_{a\to b}$ and $D_{a\to b}$. Furthermore, (see Fact 2),

$$\Pr[f'_{A,a\to b} \neq \hat{f}_{A,a\to b}] \leq \frac{1}{2} \cdot 2\sqrt{\delta_{a\to b}}. \tag{6}$$

Step 2: From this point on, Alice and Bob just follow $\mathcal{P}'$ described above. On receiving M_1, Bob generates his input and work qubits by appropriately applying the unitary transformation U_B. He then generates message M_2 and sends it to Alice.

Let $|\phi_{a\to b}\rangle$ denote the state of the entire system just after M_2 is sent to Alice.

After this, Alice and Bob continue as before. In particular, the Alice continues to use her old input register F_A (safely) as before. The registers $\hat{F}$ are not used until the end, when they are measured in order to decide if the answer returned by the protocol is correct.

Figure 2: The revised protocol $\mathcal{P}_{a\to b}$

Remark on the inputs generated: Suppose we measure registers F_A just after $U_{a\to b}$ has been applied in the above protocol. Let $f'_{A,a\to b}$ be the resulting random variable with distribution $D'_{a\to b}$. On the other hand, if we were to measure the same registers in the state $|\psi^A_{a\to b}\rangle$, then the resulting random variable is $f_{A,a\to b}$ whose distribution is D; that is, D is the distribution of f_A conditioned on the event $f_A[a] = b$. Then, it follows from (5) and Theorem 2 that

$$\|D_{a\to b} - D'_{a\to b}\|_1 \leq 2\sqrt{\delta_{a\to b}}. \tag{7}$$

We will want Alice's input registers to satisfy Requirement 1(b). Unfortunately, the distribution D' may not satisfy this requirement automatically, but (7) will help us 'correct' this.

Next consider Bob's input registers. In $\mathcal{P}$, Bob's register $F_B[s]$ contained the uniform superposition μ and he generated the input in the rest of the registers himself form M_1 using the unitary transformation U_B. The input he generated satisfied Requirement 1(b). In $\mathcal{P}_{a\to b}$, Bob applies the same transformation U_B on M_1, but $F_B[s]$ is now $|a\rangle$ and not $|\mu\rangle$. Suppose F_B is measured at this stage resulting in the random variable $f_{B,a\to b} : V_B \to V_A$. Note that $f_{B,a\to b}$ has the same distribution as f_B conditioned on the event $f_B(s) = a$.

Thus,

B1. $f_{B,a\to b}$ is constant on $X_A \cup \{s\}$ (in fact, $f_B[s] = a$), and

B2. the set of random variables $(f_{B,a\to b}[v] : v \in V_B - X_B - \{s\})$ are independent and uniformly distributed over V_A.

Probability of error in $\mathcal{P}_{a\to b}$: By (5) and Theorem 2, the probability of error of $\mathcal{P}_{a\to b}$, which we denote by $\tilde{\epsilon}_{a\to b}$, is at most $\epsilon_{a\to b} + 2\sqrt{\delta_{a\to b}}$.

Correcting Alice's input registers: The random variable $f'_{A,a\to b}$ that results from measuring F_A has a distribution $D'_{a\to b}$ which is close to the desired distribution $D_{a\to b}$ of $f_{A,a\to b}$ (by (7) above). It will be easier to satisfy Requirement 1(b), however, if we could arrange that the distribution of Alice's inputs is exactly $D_{a\to b}$. To do this, we use Fact 2; let $C_{a\to b}$ be the unitary transformation corresponding to $D'_{a\to b}$ and $D_{a\to b}$. We revise the protocol $\mathcal{P}_{a\to b}$ by including this operation (see Figure 2).

Error probability of the revised protocol: At that end of the protocol, we measure all registers and obtain the answer ans, and the inputs $\hat{f}_{A,a\to b}$ and $f_{B,a\to b}$. We also have $f_{A,a\to b}$ corresponding to Alice's old input registers F_A. Let $\hat{f}_{a\to b} = \hat{f}_{A,a\to b} \cup f_{B,a\to b}$ and $f'_{a\to b} = f'_{A,a\to b} \cup f_{B,a\to b}$. This revised protocol makes an error whenever $\mathsf{ans} \neq \mathrm{lsb}\hat{f}^{(k+1)}_{a\to b}(s)$. We then have

$$
\begin{aligned}
\hat{\epsilon}_{a\to b} &\stackrel{\Delta}{=} \Pr[\mathsf{ans} \neq \mathrm{lsb}\hat{f}^{(k+1)}_{a\to b}(s)] \\
&\leq \Pr[\hat{f}_{a\to b} \neq f'_{a\to b}] + \Pr[\mathsf{ans} \neq \mathrm{lsb}f'^{(k+1)}_{a\to b}(s)] \\
&\leq \frac{1}{2}\cdot 2\sqrt{\delta_{a\to b}} + \epsilon_{a\to b} + 2\sqrt{\delta_{a\to b}} \\
&= \epsilon_{a\to b} + 3\sqrt{\delta_{a\to b}}. \qquad (8)
\end{aligned}
$$

4.2 The final protocol: $\mathcal{P}_a$

A small modification now gives us our final protocol, which will satisfy the requirements for Q^{k-1}. We make two changes to the revised version of $\mathcal{P}_{a\to b}$. First, instead of Alice sending M_1 and retaining $\widetilde{M}_1$, now Bob creates the canonical purification $(M_1, \widetilde{M}_1)$ and sends Alice $\widetilde{M}_1$, while retaining M_1. Second, in $\mathcal{P}_{a\to b}$, the register $\hat{F}_A[a]$ is fixed to the value $|b\rangle$. Now, however, Alice starts with $|\mu\rangle$ in $\hat{F}[a]$. With these modifications, Alice's role in the input generation phase of the new protocol is similar to Bob's role in the protocol we started with. The resulting protocol $\mathcal{P}_a$ (see Figure 3) depends on the choice of a. Using an averaging argument we will conclude that there is a choice for $a \in V_A$ so that $\mathcal{P}_a$ satisfies the requirements for Q^B_{k-1} as needed in Lemma 2(a).

The probability of error of $\mathcal{P}_a$: For $a \in V_A - X_A$, let $\hat{\epsilon}_a$ be the probability of error of $\mathcal{P}_a$. Then, by (8), we have

$$
\begin{aligned}
\hat{\epsilon}_a &= \mathop{\mathrm{E}}_{b\in_u V_B}[\hat{\epsilon}_{a\to b}] \qquad (9) \\
&\leq \mathop{\mathrm{E}}_{b\in_u V_B}[\epsilon_{a\to b} + 3\sqrt{\delta_{a\to b}}]. \qquad (10)
\end{aligned}
$$

The new input registers for Alice will be denoted by $\hat{F}_A$. The old input registers will continue to exist, but they will count as work qubits of Alice. Initially, in the register $\hat{F}_A[a]$ we place a uniform superposition $|\mu\rangle$. All other registers are initialized to 0.

Step 1: Bob generates the canonical purification $(M_1, \widetilde{M_1})$ of the first message of $\mathcal{P}$. He sets his register $F_B[s]$ to the state $|a\rangle$, and using the transformation U_B generates his inputs F_B and work qubits W_B. Then, he generates the first message of protocol $\mathcal{P}'$ (this corresponds message M_2 of the $\mathcal{P}$), and sends this message along with $\widetilde{M_1}$ to Alice.

Step 2: (a) One receiving $\widetilde{M_1}$, Alice applies a unitary transform on registers $(\hat{F}_A[a], \widetilde{M_1}, A)$ to generate a state in registers F_A (the old input registers) and W_A (the work qubits of the original protocol). Here, A is a set of ancilla qubits initialized to 0. This unitary transformation acts according to the rule

$$|b\rangle_{\hat{F}[a]}|\theta\rangle_{\widetilde{M_1},A} \mapsto |b\rangle_{\hat{F}[a]}U_{a\to b}|\theta\rangle_{\widetilde{M_1},A}.$$

Note that this transformation is safe on $\hat{F}[a]$.
(b) Since F_A is not in the desired state, Alice applies the correction used in the revised Step 1 of $\mathcal{P}_{a\to b}$. That is, she applies a unitary transformation to registers $(\hat{F}_A[a], F_A, \tilde{F}_A)$ defined by

$$|b\rangle_{\hat{F}[a]}|\psi\rangle_{F_A,\tilde{F}_A} \mapsto |b\rangle_{\hat{F}[a]}C_{a\to b}|\psi\rangle_{F_A,\tilde{F}_A}.$$

Before the application of this the registers $\tilde{F}_A$ are initialized to 0. Alice then copies $(\tilde{F}_A[u] : u \in V_A - \{a\})$ into $(\hat{F}_A[u] : u \in V_A - \{a\})$. For the purpose of satisfying Requirement 1(b), $\hat{F}_A$ are to be treated as A's input register.

The state of entire system at this point is precisely $\frac{1}{\sqrt{n}}\sum_{b\in V_B}|\phi_{a\to b}\rangle$, where $|\phi_{a\to b}\rangle$ is the state at the corresponding point in the revised protocol $\mathcal{P}_{a\to b}$ (see Figure 2). The rest of the protocol operates safely on F_A,$\hat{F}_A$ and F_B. In fact, no unitary transform will now be applied to registers $\hat{F}_A$.

Step 3: Alice resumes the protocol $\mathcal{P}'$. Note that Bob has already executed the first step of $\mathcal{P}'$ and sent the first message (which corresponds to message M_2 of the original protocol). Alice responds to this message as before.

While executing $\mathcal{P}'$, the old input registers F_A are used. The new registers $\hat{F}_A$ are not touched by any unitary transformation from now on. At the end, however, when we try to decide if an error has been made, we will measure all registers, and check if the answer ans' agrees with the answer $\mathsf{ans}(\hat{f}_A, f_B)$, where $\hat{f}_A$ is the random variable obtained by measuring the new input registers $\hat{F}_A$.

Figure 3: The protocol $\mathcal{P}_a$

We need to show that there exists an a such that $\hat{\epsilon}_a$ is small. For this we consider the average of $\hat{\epsilon}_a$ as a is chosen uniformly from $V_A - X_A$:

$$\mathop{\mathrm{E}}_{a \in_u V_A - X_A}[\hat{\epsilon}_a] \;\leq\; \mathop{\mathrm{E}}_{a,b}[\epsilon_{a \to b} + 3\sqrt{\delta_{a \to b}}], \tag{11}$$

where on the right a is chosen uniformly from $V_A - X_A$ and b is chosen independently and uniformly from V_B. (From now on, when we average over a and b, we will assume that they are chosen in this manner.) By (4), we have

$$\mathop{\mathrm{E}}_{a,b}[\epsilon_{a \to b}] \leq \left(\frac{n}{n - n_a}\right)\epsilon. \tag{12}$$

It remains to bound $\mathrm{E}_{a,b}[\sqrt{\delta_{a \to b}}]$. Consider the state obtained by measuring Alice's input registers F_A just before M_1 is sent to Bob in the original protocol. As stated earlier, if the value b is observed for $F_A[a]$, then the state of the message registers will be $M_{1,a \to b}$; also, M_1 is the average of these states, that is, $M_1 = \frac{1}{n}\sum_{b \in V_B} M_{1,a \to b}$.

Claim 1 *For $a \in V_A - X_A$, $\mathrm{E}_b[\delta_{a \to b}] \leq \sqrt{(2\ln 2) I(f_A[a] : M_1)}$.*

Proof: Consider the encoding of elements of V_B given by $b \mapsto M_{1,a \to b}$ by restricting attention the registers $F_A[a]$ and M_1. Our claim now follows from the Average Encoding Theorem (Theorem 3) and the definition of $\delta_{a \to b}$. □

Claim 2 $\mathrm{E}_{a \in_u V_A - X_A}[I(f_A[a] : M_1)] \leq \left(\frac{n}{n - n_a}\right) c_1$.

Proof: Using Fact 1 and (7), we have $c_1 n \geq I(f_A : M_1) \geq \sum_{a \in V_A} I(f_A[a] : M_1) \geq \sum_{a \in V_A - X_A} I(f_A[a] : M_1)$.
□

By combining these two claims, and noting that the square-root function is concave, we obtain

$$\mathop{\mathrm{E}}_{a,b}[\delta_{a \to b}] \;\leq\; \mathop{\mathrm{E}}_{a}[\sqrt{(2\ln 2) I(f_A[a] : M_1)}] \;\leq\; \sqrt{(2\ln 2)\mathop{\mathrm{E}}_{a}[I(f_A[a] : M_1)]} \;\leq\; \sqrt{\left(\frac{n}{n - n_a}\right)(2\ln 2) c_1}. \tag{13}$$

This implies, again because the square root is concave, that

$$\mathop{\mathrm{E}}_{a,b}[\sqrt{\delta_{a \to b}}] \;\leq\; \left[\left(\frac{n}{n - n_a}\right)(2\ln 2) c_1\right]^{\frac{1}{4}}. \tag{14}$$

Now we return to (11), and use (12) and (14) to obtain

$$\mathop{\mathrm{E}}_{a}[\hat{\epsilon}_a] \leq \left(\frac{n}{n - n_a}\right)\left[\epsilon + 3((2\ln 2) c_1)^{\frac{1}{4}}\right].$$

Thus, there exists an $a \in V_A - X_A$ such that

$$\hat{\epsilon}_a \leq \left(\frac{n}{n - n_a}\right)\left[\epsilon + 3((2\ln 2) c_1)^{\frac{1}{4}}\right].$$

Now, it can be verified, the protocol $\mathcal{P}_a$ satisfies the requirements for $Q^B_{k-1}(c_1 + c_2, c_3, \ldots, c_k, n_A, n_B + 1, \hat{\epsilon}_a)$. This shows Part (a) of Lemma 2. Part (b) can be established similarly.

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