More on a Problem of Zarankiewicz

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Abstract. We show tight necessary and sufficient conditions on the sizes of small bipartite graphs whose union is a larger bipartite graph that has no large bipartite independent set. Our main result is a common generalization of two classical results in graph theory: the theorem of Kővári, Sós and Turán on the minimum number of edges in a bipartite graph that has no large independent set, and the theorem of Hansel (also Katona and Szemerédi and Krichevskii) on the sum of the sizes of bipartite graphs that can be used to construct a graph (non-necessarily bipartite) that has no large independent set. Our results unify the underlying combinatorial principles developed in the proof of tight lower bounds for depth-two superconcentrators.

1 Introduction

Consider a bipartite graph $G = (V, W, E)$, where $|V|, |W| = n$. Suppose every k element subset $S \subseteq V$ is connected to every k element subset $T \subseteq W$ by at least one edge. How many edges must such a graph have? This is the celebrated Zarankiewicz problem.

Definition 1 (Bipartite independent set). *A bipartite independent set of size $k \times k$ in a bipartite graph $G = (V, W, E)$ is a pair of subsets $S \subseteq V$ and $T \subseteq W$ of size k each such that there is no edge connecting S and T, i.e., $(S \times T) \cap E = \emptyset$.*

The Zarankiewicz problem asks for the minimum number of edges in a bipartite graph that does not have any bipartite independent set of size $k \times k$. We may think of an edge as a complete bipartite graph where each side of the bipartition is just a singleton. This motivates the following generalization where we consider bipartite graphs as formed by putting together not just edges, but, more generally, small complete bipartite graphs.

Definition 2. *A bipartite graph $G = (V, W, E)$ is said to be the union of complete bipartite graphs $G_i = (V_i, W_i, E_i = V_i \times W_i)$, $(i = 1, 2, \ldots, r)$ if each $V_i \subseteq V$, each $W_i \subseteq W$, and $E = E_1 \cup \cdots \cup E_r$.*

Definition 3. *We say that a sequence of positive integers $(n_1, n_2, \ldots, n_r)$ is (n, k)-strong if there is a bipartite graph $G = (V, W, E)$ that is a union of graphs $G_i = (V_i, W_i, E_i = V_i \times W_i)$, $i = 1, 2, \ldots, r$, such that*

- *$|V|, |W| = n$;*
- *$|V_i| = |W_i| = n_i$;*
- *G has no bipartite independent set of size $k \times k$.*

What conditions must the n_i's satisfy for $(n_1, n_2, \ldots, n_r)$ to be (n, k)-strong? Note that the Zarankiewicz problem is a special case of this question where each n_i is 1 and $\sum_i n_i$ corresponds to the number edges in the final graph G.

Remark. The Zarankiewicz problem is more commonly posed in the following form: What is the maximum number of edges in a bipartite graph with no $k \times k$ bipartite *clique*. By interchanging edges and non-edges, we can ask for the maximum number of non-edges (equivalently the minimum number of edges) such that there is no $k \times k$ bipartite *independent set*. This complementary form is more convenient for our purposes.

The Kővári, Sós and Turán bound

The following classical theorem gives a lower bound on the number of edges in that have no large independent set.

Theorem 1 (Kővári, Sós and Turán [6]; see, e.g., [2], Page 301, Lemma 2.1.). *If G does not have an independent set of size $k \times k$, then*

$$n \binom{n - \overline{d}}{k} \binom{n}{k}^{-1} \leq k - 1,$$

where $\overline{d}$ is the average degree of G.

The above theorem implies that

$$n \leq (k - 1) \binom{n - \overline{d}}{k}^{-1} \binom{n}{k}$$

$$\leq (k - 1) \left(\frac{n - k + 1}{n - \overline{d} - k + 1} \right)^k$$

$$= (k - 1) \left(1 + \frac{\overline{d}}{n - \overline{d} - k + 1} \right)^k$$

$$\leq (k - 1) \exp \left(\frac{\overline{d} k}{n - \overline{d} - k + 1} \right),$$

which yields,

$$\overline{d} \geq \frac{(n - k + 1) \log(n/(k - 1))}{k + \log(n/(k - 1))}.$$

In this paper, we will mainly be interested in $k \in [n^{1/10}, n^{9/10}]$, in which case we obtain

$$|E(G)| = n\overline{d} = \Omega \left(\frac{n^2}{k} \log n \right).$$

For the problem under consideration, this immediately gives the necessary condition

$$\sum_{i=1}^{r} n_i^2 = \Omega \left(\frac{n^2}{k} \log n \right). \tag{1}$$

It will be convenient to normalize n_i and define $\alpha_i = \frac{n_i}{n/k}$. With this notation, the inequality above can be restated as follows.

$$\sum_{i=1}^{r} \alpha_i^2 = \Omega(k \log n). \tag{2}$$

The Hansel bound

The same question can also be asked in the context of general graphs. In that case, we have the classical theorem.

Theorem 2 (Hansel [4], Katona and Szemerédi [5]). *Suppose it is possible to place one copy each of K_{n_i,n_i}, $i = 1, 2, \ldots, r$, in a vertex set of size n such that the resulting graph has no independent set of size k. Then,*

$$\sum_{i=1}^{r} n_i \geq n \log \left(\frac{n}{k-1} \right).$$

Although this result pertains to general graphs and is not directly applicable to the bipartite graph setting, it can be used (details omitted as we will use this bound only to motivate our results, not to derive them) to derive a necessary condition for bipartite graphs as well. In particular, normalizing n_i by setting $n_i = \alpha_i \frac{n}{k}$ as before, one can obtain the necessary condition

$$\sum_{i=1}^{r} \alpha_i = \Omega(k \log n). \tag{3}$$

Note that neither of the two bounds above strictly dominates the other: if all α_i are small (say $\ll 1$), then the first condition derived from the Kővári, Sós and Turán bound is stronger, wheras if all α_i are large ($\gg 1$), then the condition derived from the Hansel bound is stronger.

In our applications, we will meet situations where the α_i's will not be confined to one or the other regime. To get optimal results, one must, therefore, devise a condition appropriate for the entire range of values for the α_i's. Towards this goal, we start by trying to guess the form of this general inequality by asking a dual question: what is a sufficient condition on n_i's (equivalently α_i's) for $(n_1, n_2, \ldots, n_r)$ to be (n, k)-strong? We derive the following.

Theorem 3 (Sufficient condition). *Suppose $k \in [n^{1/10}, n^{9/10}]$, and let $\alpha_i \in [n^{-1/100}, n^{1/100}]$, $i = 1, 2 \ldots, r$. Then, there is a constant $A > 0$ such that if*

$$\sum_{i:\alpha_i \leq 1} \alpha_i^2 + \sum_{i:\alpha_i > 1} \alpha_i \geq Ak \log n,$$

then $(n_1, n_2, \ldots, n_r)$ is (n, k)-strong, where $n_i = \alpha_i(n/k)$.

Using this insight, we then show that this sufficient condition is indeed also necessary (with slightly different constants).

Theorem 4 (Necessary condition). *Suppose $k \in [n^{1/10}, n^{9/10}]$, and let $\alpha_i \in [n^{-1/100}, n^{1/100}]$, $i = 1, 2 \ldots, r$. Then, there is a constant $B > 0$ such that if $(n_1, n_2, \ldots, n_r)$ is (n, k)-strong where $n_i = \alpha_i(n/k)$, then*

$$\sum_{i:\alpha_i \leq 1} \alpha_i^2 + \sum_{i:\alpha_i > 1} \alpha_i \geq Bk \log n.$$

Our proof of Theorem 4 uses a refinement of the ideas used in Radhakrishnan and Ta-Shma [7]. In a later section, we also show that our inequality leads to a modular proof of their tight lower bound on the size of depth-two superconcentrators.

A tradeoff result for depth-two superconcentrators was shown by Dutta and Radhakrishnan [3]. Their main argument leads one to consider situations where the small bipartite graphs used to build the bigger one are not symmetric, instead of being of the form K_{n_i,n_i}, they are of the form K_{m_i,n_i} (with perhaps $m_i \neq n_i$).

Definition 4. *We say that a sequence of pairs of positive integers $((m_1, n_1), (m_2, n_2), \ldots, (m_r, n_r))$ is (n, k)-strong if there is a bipartite graph $G = (V, W, E)$ that is a union of graphs $G_i = (V_i, W_i, E_i = V_i \times W_i)$, $i = 1, 2, \ldots, r$, such that*

- *$|V|, |W| = n$;*
- *$|V_i| = m_i$ and $|W_i| = n_i$.*
- *G has no bipartite independent set of size $k \times k$.*

We refine our arguments and provide tight necessary and sufficient conditions for this asymmetric setting as well. From the necessary condition for the asymmetric setting, we derive the tradeoff result shown earlier by Dutta and Radhakrishnan [3]. Our main results are the following.

Theorem 5 (Sufficient condition: asymmetric case). *Suppose $k \in [n^{1/10}, n^{9/10}]$, and let $\alpha_i, \beta_i \in [n^{1/100}, n^{-1/100}]$, $i = 1, 2 \ldots, r$. Then, there is a constant $C > 0$ such that if*

$$\sum_{i \in X} \alpha_i \beta_i + \sum_{i \in \{1,2,\ldots,r\} \setminus X} (\alpha_i + \beta_i) \mathsf{H}(p_i) \geq Ck \log n$$

for every $X \subseteq \{1, 2, \ldots, r\}$, where $p_i = \frac{\alpha_i}{\alpha_i + \beta_i}$ and $\mathsf{H}(p_i) = -p_i \log(p_i) - (1 - p_i) \log(1 - p_i)$ is the binary entropy function, then the sequence $((m_1, n_1), (m_2, n_2), \ldots, (m_r, n_r))$ is (n, k)-strong where $m_i = \alpha_i(n/k)$, $n_i = \beta_i(n/k)$.

Theorem 6 (Necessary condition: asymmetric case). *Suppose $k \in [n^{1/10}, n^{9/10}]$, and let $\alpha_i, \beta_i \in [n^{-1/100}, n^{1/100}]$, $i = 1, 2 \ldots, r$. Then, there is a constant $D > 0$ such that if the sequence $((m_1, n_1), (m_2, n_2), \ldots, (m_r, n_r))$ is (n, k)-strong where $m_i = \alpha_i(n/k)$ and $n_i = \beta_i(n/k)$, then*

$$\sum_{i \in X} \alpha_i \beta_i + \sum_{i \in \{1,2,\ldots,r\} \setminus X} (\alpha_i + \beta_i) \mathsf{H}(p_i) \geq Dk \log n$$

for every $X \subseteq \{1, 2, \ldots, r\}$, where $p_i = \frac{\alpha_i}{\alpha_i + \beta_i}$ and $\mathsf{H}(p_i) = -p_i \log(p_i) - (1 - p_i) \log(1 - p_i)$.

Organization

In Section 2, we first derive the sufficient condition (Theorem 3) and then prove the necessary condition (Theorem 4) for the symmetric setting. We then present generalizations of these results to the asymmetric setting in Section 3. In Section 4, we derive the depth-two superconcentrator results.

2 Building bipartite graph from smaller symmetric bipartite graphs

2.1 Sufficient condition: Theorem 3

Let us consider a probabilistic construction of a bipartite graph $G = (V, W, E)$ where, given $n_1, n_2, \ldots, n_r$ such that $\alpha_i = \frac{n_i}{n/k} \in [n^{-1/100}, n^{1/100}]$, we place an independently drawn random copy G_i of K_{n_i, n_i} between V and W. In other words, G is the union of $G_1, G_2, \ldots, G_r$ where $G_i = (V_i, W_i, E_i = V_i \times W_i)$ and V_i, W_i are uniformly chosen random n_i element subsets of V and W respectively. Fix a potential independent set (S, T) of size $k \times k$. Then, as shown below,

$$\Pr[E(H_i) \cap S \times T = \emptyset] \leq 1 - (1 - \exp(-\alpha_i))^2. \tag{4}$$

Thus, since the graphs G_i's are chosen independently, the probability that (S,T) is independent in G is

$$p = \Pr[(S,T) \text{ is independent in } G] \le \prod_{i=1}^{r}(1-(1-\exp(-\alpha_i))^2).$$

By the union bound, if $p\binom{n}{k}^2 < 1$ then there is bipartite graph built by putting together one copy each of K_{n_i,n_i} that avoids all independent sets of size $k \times k$. The interesting aspect of this calculation is in the form of the expression $p_i = 1-(1-\exp(-\alpha_i))^2$. We will show below that

$$p_i \le \begin{cases} \exp\left(-\frac{\alpha_i^2}{3}\right) & \text{if } \alpha_i \le 1 \\ \exp(-(1-\ln 2)\alpha_i) & \text{if } \alpha_i > 1 \end{cases} \tag{5}$$

This immediately gives us our first result, the sufficient condition stated as Theorem 3 in the introduction.

Proof of (4): Recall that $G_i = (V_i, W_i, V_i \times W_i)$, where V_i and W_i are uniformly chosen random subsets of V and W of size n_i each.

$$\Pr[E(H_i) \cap S \times T = \emptyset] = \Pr[V_i \cap S = \emptyset \vee W_i \cap T = \emptyset] = 1-(1-\Pr[V_i \cap S = \emptyset])(1-\Pr[W_i \cap T = \emptyset]).$$

Then, (4) follows from this because

$$\Pr[V_i \cap S = \emptyset], \Pr[W_i \cap T = \emptyset] = \binom{n-k}{n_i}\binom{n}{n_i}^{-1}$$
$$\le \left(\frac{n-k}{n}\right)^{n_i}$$
$$\le \exp\left(-\frac{kn_i}{n}\right)$$
$$= \exp(-\alpha_i).$$

Proof of (5): We have

$$p_i = 1-(1-\exp(-\alpha_i))^2 = \exp(-\alpha_i)(2-\exp(-\alpha_i)).$$

If $\alpha_i \le 1$, then we have

$$2-\exp(-\alpha_i) \le 1+\alpha_i - \frac{\alpha_i^2}{2} + \frac{\alpha_i^3}{6} \le \exp\left(\alpha_i - \frac{\alpha_i^2}{3}\right).$$

Thus,

$$p_i = \exp(-\alpha_i)(2-\exp(-\alpha_i)) \le \exp\left(-\frac{\alpha_i^2}{3}\right).$$

If $\alpha_i > 1$, then we have

$$p_i = \exp(-\alpha_i)(2-\exp(-\alpha_i)) \le 2\exp(-\alpha_i) = \exp(-\alpha_i + \ln 2) \le \exp(-(1-\ln 2)\alpha_i).$$

We might ask if this sufficient condition is also necessary. As noted in the introduction, the Kővári, Sós and Turán bound (Inequality 2) explains the first term in the above sufficient condition, and the Hansel bound (Inequality 3) explains the second term. We thus have explanations for both the terms in the LHS of the sufficient condition, using two classical theorems of graph theory. However, neither of them implies in full generality that the sufficient condition derived above is necessary.

2.2 Necessary condition: Theorem 4

We show that the sufficient condition derived in Theorem 3 is also necessary upto constants.

Let $k \in [n^{\frac{1}{10}}, n^{\frac{9}{10}}]$ and $\alpha_i \in [n^{-1/100}, n^{1/100}]$, $i = 1, 2, \ldots, r$. Suppose we are given a bipartite graph $G = (V, W, E)$ which is the union of complete bipartite graphs $G_1, G_2, \ldots, G_r$ and has no bipartite independent set of size $k \times k$, where $G_i = (V_i, W_i, E_i = V_i \times W_i)$ with $|V_i| = |W_i| = n_i = \alpha_i(n/k)$. We want to show that for some constant $B > 0$,

$$\sum_{i:\alpha_i \leq 1} \alpha_i^2 + \sum_{i:\alpha_i > 1} \alpha_i \geq Bk \log n.$$

We will present the argument for the case when $k = \sqrt{n}$; the proof for other k is similar, and focussing on this k will keep the notation and the constants simple. We will show that if the second term in the LHS of the above inequality is small, say,

$$\mathsf{SecondTerm} = \sum_{i:\alpha_i > 1} \alpha_i \leq \frac{1}{100} k \log n,$$

then the first term must be large, i.e.,

$$\mathsf{FirstTerm} = \sum_{i:\alpha_i \leq 1} \alpha_i^2 \geq \frac{1}{100} k \log n.$$

Assume $\mathsf{SecondTerm} \leq \frac{1}{100} k \log n$. Let us call a G_i for which $\alpha_i > 1$ as *large* and a G_i for which $\alpha_i \leq 1$ as *small*. We start as in [7] by deleting one of the sides of each large G_i independently and uniformly at random from the vertex set of G. For a vertex $v \in V$, let d_v be number of large G_i's such that $v \in V_i$. The probability that v survives at the end of the random deletion is precisely 2^{-d_v}. Now,

$$\sum_v d_v = \sum_{i:\alpha_i > 1} n_i \leq \frac{1}{100} n \log n,$$

where the inequality follows from our assumption that $\mathsf{SecondTerm} \leq \frac{1}{100} k \log n$. That is, the average value of d_v is $\frac{1}{100} \log n$, and by Markov's inequality, at least half of the vertices have their d_v's at most $d = \frac{1}{50} \log n$. We focus on a set V' of $n/2$ such vertices, and if they survive the first deletion, we delete them again with probability $1 - 2^{-(d-d_v)}$, so that every one of these $n/2$ vertices in V' survives with probability exactly $2^{-d} = n^{-1/50}$. Let X be the vertices of V' that survive. Similarly, we define $W' \subseteq W$, and let $Y \subseteq W'$ be the vertices that survive.

Claim. With probability $1 - o(1)$, $|X|, |Y| \geq \frac{n}{4} 2^{-d}$.

The claim can be proved as follows. For $v \in V'$, let I_v be the indicator variable for the event that v survives. Then, $\Pr[I_v = 1] = 2^{-d} = n^{-1/50}$ for all $v \in V'$. Furthermore, I_v and $I_{v'}$ are dependent precisely if there is a common large G_i such that both $v, v' \in V_i$. Thus, any one I_v is dependent on at most $\Delta = d_v \times \max\{n_i : \alpha_i > 1\} \leq (1/50) \log n \, n^{1/100}(n/k) = (1/50) n^{51/100} \log n$ such events (recall $k = \sqrt{n}$). We thus have (see Alon-Spencer [1])

$$\mathbb{E}[|X|] = \sum_{v \in V'} I_v = \frac{n}{2} 2^{-d} = \frac{1}{2} n^{49/50};$$
$$\mathsf{Var}[|X|] \leq E[|X|]\Delta.$$

By Chebyshev's inequality, the probability that $|X|$ is less than $\frac{\mathbb{E}[|X|]}{2}$ is at most

$$\frac{4\mathsf{Var}[X]}{\mathbb{E}[|X|]^2} \leq \frac{4\Delta}{\mathbb{E}[|X|]} = o(1).$$

A similar calculation can be done for $|Y|$. (End of Claim.)

The crucial consequence of our random deletion process is that no large G_i has any edge between X and Y. Since G does not have any independent set (S, T) of size $k \times k$, the small G_i's must provide the necessary edges to avoid such independent sets between X and Y. Consider an edge (v, w) of a small G_i. The probability that this edge survives in $X \times Y$ is precisely the probability of the event $I_v \wedge I_w$. Note that the two events I_v and I_w are either independent (when v and w do not belong to a common large G_i), or they are mutually exclusive. Thus, the expected number of edges supplied between X and Y by small G_i's is at most

$$\sum_{i:\alpha_i \leq 1} \alpha_i^2 (n/k)^2 2^{-2d} = \mathsf{FirstTerm} \times (n/k)^2 2^{-2d},$$

and by Markov's inequality, with probability $1/2$ it is at most twice its expectation. Using the Claim above we conclude that the following three events happen simultaneously: (a) $|X| \geq \frac{n}{4} 2^{-d}$, (b) $|Y| \geq \frac{n}{4} 2^{-d}$, (c) the number of edges conecting X and Y is at most $\mathsf{FirstTerm} \times (n/k)^2 2^{-2d}$. Using (1), this number of edges must be at least $\frac{1}{3} \frac{49}{50} \frac{(n2^{-d})^2}{16k} \log n$. (Note that $\frac{1}{3}$ suffices as the constant in (1) for the case $|X|, |Y| \geq \frac{n^{49/50}}{4}$ and $k = \sqrt{n}$.) Comparing the upper and lower bounds on the number of edges thus established, we obtain the required inequality

$$\mathsf{FirstTerm} \geq \frac{1}{100} k \log n.$$

3 Building bipartite graph from smaller asymmetric bipartite graphs

3.1 Sufficient condition: Theorem 5

3.2 Necessary condition: Theorem 6

Let $k \in [n^{\frac{1}{10}}, n^{\frac{9}{10}}]$ and $\alpha_i, \beta_i \in [n^{-\frac{1}{100}}, n^{\frac{1}{100}}]$, $i = 1, 2, \ldots, r$. Suppose we are given a bipartite graph $G = (V, W, E)$ which is the union of complete bipartite graphs $G_1, G_2, \ldots, G_r$ and has no bipartite independent set of size $k \times k$, where $G_i = (V_i, W_i, E_i = V_i \times W_i)$ with $|V_i| = m_i = \alpha_i(n/k)$ and $|W_i| = n_i = \beta_i(n/k)$. As stated in Theorem 6, we let $p_i = \frac{\alpha_i}{\alpha_i + \beta_i}$ and $\mathsf{H}(p_i) = -p_i \log(p_i) - (1-p_i)\log(1-p_i)$. We wish to show that there is a constant $D > 0$ such that

$$\sum_{i \in X} \alpha_i \beta_i + \sum_{i \in \{1,2,\ldots,r\} \setminus X} (\alpha_i + \beta_i)\mathsf{H}(p_i) \geq Dk \log n$$

for every $X \subseteq \{1, 2, \ldots, r\}$.

The proof is similar to the proof of Thereom 4 and again we present the argument for the case when $k = \sqrt{n}$. Fix a subset $X \subseteq \{1, 2, \ldots, r\}$. Our plan is to assume that the second term in the LHS of the above inequality is small,

$$\mathsf{SecondTerm} = \sum_{i \in \{1,2,\ldots,r\} \setminus X} (\alpha_i + \beta_i)\mathsf{H}(p_i) \leq \frac{1}{100} k \log n,$$

and from this conclude that the first term must be large,

$$\mathsf{First\,Term} = \sum_{i \in X} \alpha_i \beta_i \geq \frac{1}{100} k \log n.$$

Assume $\mathsf{Second\,Term} \leq \frac{1}{100} k \log n$. We will call a graph G_i whose index $i \in X$ as *marked*, and a graph G_i whose index $i \notin X$ as *unmarked*. As before, we will delete one of the sides of each unmarked G_i independently at random from the vertex set of G. However, since this time there are different number of vertices on the two sides of G_i, we need to be more careful and choose the deletion probabilities cleverly. We do so as follows. For every unmarked G_i independently, we delete all the vertices in W_i with probability p_i and all its vertices in V_i with the remaining probability $1 - p_i$.

For a vertex $v \in V$, let S_v be the set of $i \notin X$ such that $v \in V_i$. Define d_v to be the quantity $\sum_{i \in S_v} \log(1/p_i)$. The probability that v survives the random deletion process is 2^{-d_v}. Using the fact that $p_i = \frac{\alpha_i}{\alpha_i + \beta_i}$ and plugging the expression for $\mathsf{H}(p_i)$ in the assumption that $\mathsf{Second\,Term} \leq \frac{1}{100} k \log n$, we get

$$\sum_{i \notin X} \left(\alpha_i \log(1/p_i) + \beta_i \log(1/(1 - p_i)) \right) \leq \frac{1}{100} k \log n.$$

Multiplying both sides by (n/k), this implies

$$\sum_{i \notin X} m_i \log(1/p_i) \leq \frac{1}{100} n \log n, \tag{6}$$

and

$$\sum_{i \notin X} n_i \log(1/(1 - p_i)) \leq \frac{1}{100} n \log n. \tag{7}$$

Since

$$\sum_{v \in V} d_v = \sum_{i \notin X} m_i \log(1/p_i),$$

the average value of d_v is at most $\frac{1}{100} \log n$, and by Markov's inequality, at least $3n/4$ vertices $v \in V$ have their d_v at most $d = \frac{1}{25} \log n$. Moreover, since $\alpha_i, \beta_i \in [n^{-1/100}, n^{1/100}]$, we have

$$p_i \leq \frac{n^{1/100}}{n^{1/100} + n^{-1/100}},$$

and thus

$$\begin{aligned}
\frac{1}{p_i} &\geq \frac{n^{1/100} + n^{-1/100}}{n^{1/100}} \\
&= \frac{1}{1 - \frac{n^{-1/100}}{n^{1/100} + n^{-1/100}}} \\
&\geq \exp\left(\frac{n^{-1/100}}{n^{1/100} + n^{-1/100}} \right) \\
&\geq \exp\left(\frac{1}{2} n^{-1/50} \right).
\end{aligned}$$

The above implies $\log(1/p_i) \geq \frac{1}{2}n^{-1/50}$, which combined with (6) yields

$$\sum_{i \notin X} m_i \leq \frac{1}{50}n^{51/50}\log n.$$

Since

$$\sum_{v \in V}|S_v| = \sum_{i \notin X} m_i,$$

the average value of $|S_v|$ is at most $\frac{1}{50}n^{1/50}\log n$, and again by Markov's inequality, at least $3n/4$ vertices $v \in V$ satisfies $|S_v| \leq d' = \frac{4}{50}n^{1/50}\log n$.

We focus on a set V' of $n/2$ vertices $v \in V$ such that $d_v \leq d$ and $|S_v| \leq d'$. If any vertex $v \in V'$ survives the first deletion, we delete it further with probability $1 - 2^{-(d-d_v)}$, so that the survival probability of each vertex in V' is exactly $2^{-d} = n^{-1/25}$. Let X be the set of vertices in V' that survive. Similarly, we define $W' \subseteq W$, and let Y be the set of vertices in W' that survive.

Claim. With probability $1 - o(1)$, $|X|, |Y| \geq \frac{n}{4}2^{-d}$.

The proof of the claim is exactly like the previous time. For $v \in V'$, we let I_v be the indicator variable for the event that v survives in X. Then, $\Pr[I_v = 1] = 2^{-d} = n^{-1/25}$ for all $v \in V'$. Furthermore, I_v and $I_{v'}$ are dependent precisely if there is a common unmarked G_i such that both $v, v' \in V_i$. Thus, any one I_v is dependent on at most $\Delta = |S_v| \times \max\{m_i : i \notin X\} \leq (4/50)n^{1/50}\log n n^{1/100}(n/k) = (4/50)n^{53/100}\log n$ such events (recall $k = \sqrt{n}$). Now we have

$$\mathbb{E}[|X|] = \sum_{v \in V'} I_v = \frac{n}{2}2^{-d} = \frac{1}{2}n^{24/25};$$
$$\mathsf{Var}[|X|] \leq E[|X|]\Delta.$$

By Chebyshev's inequality, the probability that $|X|$ is less than $\frac{\mathbb{E}[|X|]}{2}$ is at most

$$\frac{4\mathsf{Var}[X]}{\mathbb{E}[|X|]^2} \leq \frac{4\Delta}{\mathbb{E}[|X|]} = o(1).$$

A similar calculation can be done for $|Y|$. (End of Claim.)

Since no unmarked G_i has any edge between X and Y, the marked G_i's must provide enough edges to avoid all independent sets of size $k \times k$ between X and Y. As in the proof of Theorem 4, we can argue that an edge of a marked G_i survives in $X \times Y$ with probability at most 2^{-2d}. Thus the expected number of edges supplied between X and Y by marked G_i's is at most

$$\sum_{i \in X} m_i n_i 2^{-2d} = \sum_{i \in X}\alpha_i \beta_i (n/k)^2 2^{-2d} = \mathsf{FirstTerm} \times (n/k)^2 2^{-2d},$$

and by Markov's inequality with probability $1/2$ it is at most twice its expectation. Thus the event where both X and Y are of size at least $\frac{n}{4}2^{-d}$ and the number of edges connecting them is at most $\mathsf{FirstTerm} \times (n/k)^2 2^{-2d}$ occurs with positive probability. From (1), this number of edges must be at least $\frac{1}{3}\frac{24}{25}\frac{(n2^{-d})^2}{16k}\log n$. (Note that $\frac{1}{3}$ suffices as the constant in (1) when $|X|, |Y| \geq \frac{n^{24/25}}{4}$ and $k = \sqrt{n}$.) Thus we get

$$\mathsf{FirstTerm} \geq \frac{1}{100}k\log n.$$

4   Depth-two superconcentrators

Consider a graph $G = (V, M, W, E)$ with three sets of vertices V, M and W, where $|V|, |W| = n$, and all edges in E go from V to M or M to W. Such a graph is called a depth-two n-superconcentrator if for every $k \in \{1, 2, \dots, n\}$ and every pair of subsets $S \subseteq V$ and $T \subseteq W$, each of size k, there are k vertex disjoint paths from S to T.

We reprove two known lower bounds for depth-two superconcentrators: the first one is a lower bound on the number of edges (Theorem 7) shown in [7] which we reprove here using Theorem 4; the second one is a tradeoff result between the number of edges going from V to M and the number of them going from M to W (Theorem 8) shown in [3] which we reprove using Theorem 6.

Theorem 7 (Radhakrishnan and Ta-Shma [7]). *If the graph $G = (V, M, W, E)$ is a depth-two n-superconcentrator, then $|E(G)| = \Omega(n \frac{(\log n)^2}{\log \log n})$.*

Proof. This proof is similar to the one used in [7], but the use of Theorem 4 makes the calculations modular. Assume that the number of edges in a depth-two n-superconcentrator G is at most $(B/100)n\frac{(\log n)^2}{\log \log n}$, where B is the constant in Theorem 4. By increasing the number of edges by a factor at most two, we assume that each vertex in M has the same number of edges coming from V and going to W. For a vertex $v \in M$, let $\deg(v)$ denote the number of edges that come from V to v (equivalently the number of edges that go from v to W). Let For $k \in [n^{1/4}, n^{3/4}]$, define

$$\mathsf{High}(k) = \{v \in M : \deg(v) \geq \frac{n}{k}(\log n)^2\};$$

$$\mathsf{Medium}(k) = \{v \in M : \frac{n}{k}(\log n)^{-2} \leq \deg(v) < \frac{n}{k}(\log n)^2\};$$

$$\mathsf{Low}(k) = \{v \in M : \deg(v) < \frac{n}{k}(\log n)^{-2}\}.$$

Claim. For each $k \in [n^{1/4}, n^{3/4}]$, the number of edges incident on $\mathsf{Medium}(k)$ is at least $\frac{B}{2}n \log n$.

Fix a $k \in [n^{1/4}, n^{3/4}]$. First observe that $|\mathsf{High}(k)| < k$, for otherwise, the number of edges in G would already exceed $n(\log n)^2$, contradicting our assumption. Thus, every pair of subsets $S \subseteq V$ and $T \subseteq W$ of size k each has a common neighbour in $\mathsf{Medium}(k) \cup \mathsf{Low}(k)$. We are now in a position to move to the setting of Theorem 4. For each vertex $v \in \mathsf{Medium}(k) \cup \mathsf{Low}(k)$, consider the complete bipartite graph between its in-neighbours in V and out-neighbours in W. The analysis above implies that the union of these graphs is a bipartite graph between V and W that has no independent set of size $k \times k$. For $v \in \mathsf{Medium}(k) \cup \mathsf{Low}(k)$, let $\alpha_v = \frac{\deg(v)}{n/k}$. Using Theorem 4, it follows that

$$\sum_{v \in \mathsf{Medium}(k) \cup \mathsf{Low}(k):\alpha_v \leq 1} \alpha_v^2 + \sum_{v \in \mathsf{Medium}(k) \cup \mathsf{Low}(k):\alpha_v > 1} \alpha_v \geq Bk \log n. \tag{8}$$

For $\alpha_v \leq 1$, $\alpha_v^2 \leq \alpha_v$ and thus we can replace α_v^2 by α_v when $(\log n)^{-2} \leq \alpha_v \leq 1$ and conclude

$$\sum_{v \in \mathsf{Low}(k)} \alpha_v^2 + \sum_{v \in \mathsf{Medium}(k)} \alpha_v \geq Bk \log n. \tag{9}$$

One of the two terms in the LHS is at least half the RHS. If it is the first term then noting that $\alpha_v < (\log n)^{-2}$ for all $v \in \mathsf{Low}(k)$, we obtain

$$\sum_{v \in \mathsf{Low}} \deg(v) = (n/k) \sum_{v \in \mathsf{Low}} \alpha_v$$

$$\geq (n/k)(\log n)^2 \sum_{v \in \mathsf{Low}} \alpha_v^2$$

$$\geq \frac{B}{2} n (\log n)^3,$$

Since the left hand side is precisely the number of edges entering $\mathsf{Low}(k)$, this contradicts our assumption that G has few edges. So, it must be that the second term in the LHS of (9) is at least $\frac{B}{2} k \log n$. Then, the number of edges incident on $\mathsf{Medium}(k)$ is

$$\sum_{v \in \mathsf{Medium}} \deg(v) = (n/k) \sum_{v \in \mathsf{Medium}} \alpha_v \geq \frac{B}{2} n \log n.$$

This completes the proof of the claim.

Now, consider values of k of the form $n^{1/4}(\log n)^{4i}$ in the range $[n^{1/4}, n^{3/4}]$. Note that there are at least $(\frac{1}{10}) \log n / \log \log n$ such values of k and the sets $\mathsf{Medium}(k)$ for these values of k are disjoint. By the claim above, each such $\mathsf{Medium}(k)$ has at least $\frac{B}{2} n \log n$ edges incident on it, that is G has a total of at least $\frac{B}{20} n \frac{(\log n)^2}{\log \log n}$ edges, again contradicting our assumption.

Theorem 8 (Dutta and Radhakrishnan [3]). *If the graph $G = (V, M, W, E)$ is a depth-two n-superconcentrator with average degree of nodes in V and W being a and b respectively and $a \leq b$, then*

$$a \log \left(\frac{a+b}{a} \right) \log b = \Omega(\log^2 n).$$

Proof. We may assume that $b > \log n$, otherwise the total number of edges in G is at most $2n \log n$ which contradicts Theorem 7 proved earlier. We may also assume that $b < (\log n)^{\frac{1}{10}}$, otherwise the theorem can be easily seen to be true. For a vertex $v \in M$, let $\deg_V(v)$ denote the number of edges that come from V to v and $\deg_W(v)$ denote the number of edges that go from v to W. We will assume that the ratio $\frac{\deg_V(v)}{\deg_W(v)}$ is equal to $\frac{a}{b}$ for each vertex $v \in M$. This is without loss of generality as we can make the ratio $\frac{\deg_V(v)}{\deg_W(v)}$ equal to $\frac{a}{b}$ by increasing the number of edges from V to v (if $\frac{\deg_V(v)}{\deg_W(v)}$ is smaller) or increasing the number of edges from v to W (if the ratio is larger), and this process does not increase the number of edges between V and M or between M and W more than by a factor two. (We ignore the rounding issues as they are not important.)

For $k \in [n^{1/4}, n^{3/4}]$, define

$$\mathsf{High}(k) = \{v \in M : \deg(v) \geq \frac{n}{k} b^2\};$$

$$\mathsf{Medium}(k) = \{v \in M : \frac{n}{k} b^{-2} \leq \deg(v) < \frac{n}{k} b^2\};$$

$$\mathsf{Low}(k) = \{v \in M : \deg(v) < \frac{n}{k} b^{-2}\}.$$

We consider values of k of the form $n^{1/4} b^{4i}$ in the range $[n^{1/4}, n^{3/4}]$. There are at least $L = \frac{\log n}{10 \log b}$ such values of k and the sets $\mathsf{Medium}(k)$ for these values of k are disjoint. Thus out of these values of k, we can find one, say k_0, such that the number of edges from V to $\mathsf{Medium}(k_0)$ is at most $\frac{an}{L}$.

We observe that $|\mathsf{High}(k_0)| < k_0$, otherwise the number of edges between M and W would be at least $b^2 n > bn$ which is a contradition. Thus every pair of subsets $S \subseteq V$ and $T \subseteq W$ of size k each has a common neighbour in $\mathsf{Medium}(k_0) \cup \mathsf{Low}(k_0)$. For each vertex $v \in \mathsf{Medium}(k_0) \cup \mathsf{Low}(k_0)$, consider the complete bipartite graph between the in-neighbours and out-neighbours of v. The union of these graphs is a bipartite graph between V and W that has no independent set of size $k \times k$. For $v \in \mathsf{Medium}(k_0) \cup \mathsf{Low}(k_0)$, let $\alpha_v = \frac{\deg_V(v)}{n/k_0}$ and $\beta_v = \frac{\deg_W(v)}{n/k_0}$. It follows from Theorem 6 that

$$\sum_{v \in \mathsf{Low}(k_0)} \alpha_v \beta_v + \sum_{v \in \mathsf{Medium}(k_0)} (\alpha_v + \beta_v) \mathsf{H}(\frac{\alpha_v}{\alpha_v + \beta_v}) \geq D k_0 \log n. \tag{10}$$

where D is the constant from Theorem 6. One of the two terms in the LHS is at least half the RHS. If it is the first term, noting that $\beta_v < b^{-2}$ for all $v \in \mathsf{Low}(k_0)$, we obtain

$$\sum_{v \in \mathsf{Low}(k_0)} \deg_V(v) = (n/k_0) \sum_{v \in \mathsf{Low}(k_0)} \alpha_v$$

$$\geq (n/k_0) b^2 \sum_{v \in \mathsf{Low}(k_0)} \alpha_v \beta_v$$

$$\geq \frac{D}{2} n \log n b^2$$

$$\geq \frac{D}{2} n (\log n)^3$$

Since the left hand side is precisely the number of edges entering $\mathsf{Low}(k_0)$, we get $a \geq \frac{D}{2}(\log n)^3$ which proves the theorem. If the second term in the LHS of (10) is at least $\frac{D}{2} k_0 \log n$, we get

$$\sum_{v \in \mathsf{Medium}(k_0)} (\alpha_v + \beta_v) \mathsf{H}(\frac{\alpha_v}{\alpha_v + \beta_v}) \geq \frac{D}{2} k_0 \log n.$$

Simplifying we get

$$\sum_{v \in \mathsf{Medium}(k_0)} \left(\alpha_v \log \left(\frac{\alpha_v + \beta_v}{\alpha_v} \right) + \beta_v \log \left(\frac{\alpha_v + \beta_v}{\beta_v} \right) \right) \geq \frac{D}{2} k_0 \log n.$$

We know that

$$\left(\frac{\alpha_v + \beta_v}{\beta_v} \right)^{\beta_v} = \left(1 + \frac{\alpha_v}{\beta_v} \right)^{\beta_v} \leq \exp(\alpha_v),$$

which means

$$\beta_v \log \left(\frac{\alpha_v + \beta_v}{\beta_v} \right) \leq \frac{\alpha}{\ln 2}.$$

Noting $\frac{\alpha_v + \beta_v}{\alpha_v} = \frac{a+b}{a}$, we have

$$\sum_{v \in \mathsf{Medium}(k_0)} \alpha_v \log \left(\frac{a+b}{a} + \frac{1}{\ln 2} \right) \geq \frac{D}{2} k_0 \log n.$$

Since $a \leq b$, $\frac{a+b}{a} \geq 1$ and we conclude

$$\sum_{v \in \mathsf{Medium}(k_0)} \alpha_v \log \left(\frac{a+b}{a} \right) = \Omega(k_0 \log n).$$

The number of edges from V to $\mathsf{Medium}(k_0)$ is precisely

$$\sum_{v \in \mathsf{Medium}(k_0)} \deg_V(v) = (n/k_0) \sum_{v \in \mathsf{Medium}(k_0)} \alpha_v.$$

But we know that there are at most $\frac{an}{L}$ edges from V to $\mathsf{Medium}(k_0)$. Thus

$$\frac{ak_0}{L} \log\left(\frac{a+b}{a}\right) = \Omega(k_0 \log n),$$

which implies

$$a \log\left(\frac{a+b}{a}\right) \log b = \Omega(\log^2 n).$$

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