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Losses in pendular suspensions due to centrifugal coupling

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Abstract. We present an analysis of the centrifugal coupling of a simple pendulum to a dissipative support. We show that such a coupling leads to an amplitude dependent quality factor. For amplitudes which could be present in laser interferometer gravitational wave detector suspensions, this mechanism could limit the quality factor of the test mass suspension significantly to 10^{10} and should be considered in the design of advanced LIGO type detectors.

Keywords. Gravitational wave detectors; seismic isolation; Q-factor.

PACS Nos 04·30; 46·30

1. Introduction

Large scale laser interferometric gravitational wave detectors are being planned [1–5] and constructed [6]. Such detectors require extremely low amplitudes of vibrational noise in their mirror suspensions. The chief sources of noise are seismic vibrations and thermal noise. The former can, in principle, be reduced to arbitrarily low levels by suitable filter design. Thermal noise however, is generated internally by the acoustic losses in the mirror and by the losses in the mirror suspension. Internal mirror losses generally give rise to a thermal noise peak in the kHz range (assuming a suitable shape for the mirror). Pendulum losses give rise to a noise amplitude which, in the frequency range ~ 1–100 Hz, is generally expected to dominate the noise of a large scale detector.

In figure 1 we show the predicted thermal noise of a 100 kg, 1 Hz pendulum for a range of Q-factors. There would clearly be a great advantage in using pendula with Q-factors $> 10^{10}$ (see [7]). Losses in such pendula are normally considered to arise from elastic losses in the flexure or hinge from which the pendulum is supported. The elastic losses can be substantially reduced, due to the fact that the majority of the *elastic* stored energy is in the gravitational potential energy. Practical pendulum designs can use thin foil flexures in which typically only one part in 10^4 of the elastic stored energy is in the lossy foil. It has been shown by Saulson [8] that under these circumstances the pendulum Q-factor is given by,

$$Q = Q_0 \frac{K_g}{K_s}, \tag{1.1}$$

where K_g/K_s is the ratio of the elastic spring constants and Q_0 is the intrinsic Q of the foil. Since materials such as niobium have an intrinsic $Q \sim 10^6$ at room temperature,

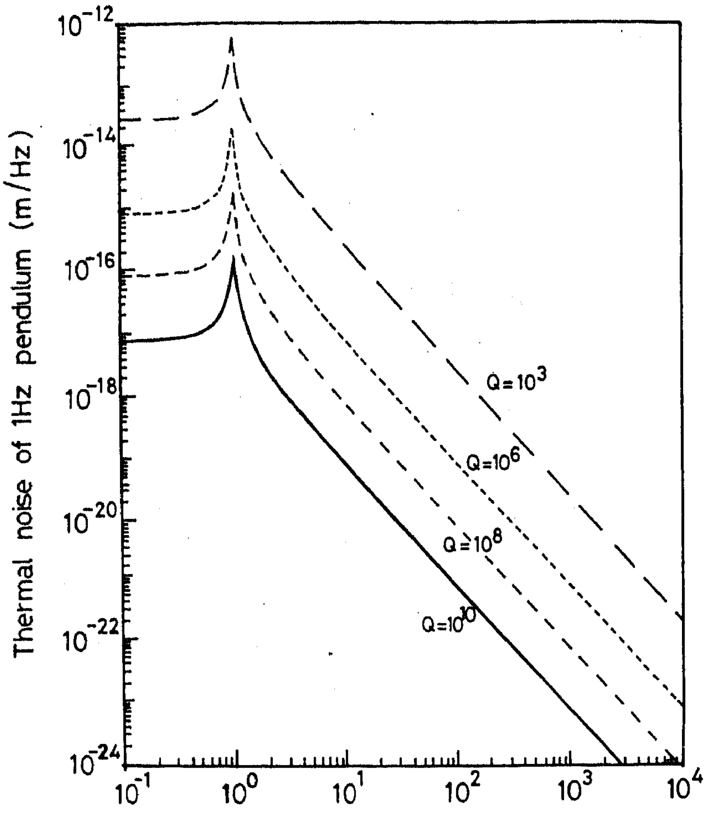

Figure 1. The frequency dependence of the thermal noise predicted for a 1 Hz pendulum for various Q-factors. A pendulum with $Q = 10^{10}$ would allow a strain sensitivity of about $10^{-23}/\sqrt{\mathrm{Hz}}$ at 35 Hz in a 1 km laser interferometer gravitational wave detector.

pendulum Q-factors limited by linear losses in the suspension exceeding 10^{10} are possible [9].

However, even if the elastic loss was reduced to nearly zero using suitable materials and configurations, losses in practice can remain due to the coupling of the pendulum to its support structure. There are two ways in which this coupling can arise:

(a) One is a simple linear horizontal coupling of the suspension point to its support structure. This can be modelled by conventional linear analysis and is accounted for in most isolation system designs. These recoil losses are difficult to overcome. However, Hough *et al* [10] and Braginsky [11] have demonstrated Q factors $\sim 10^7$ and 10^8 respectively. The thermal noise associated with recoil losses is filtered by the suspension. Thus a significant level of recoil losses is tolerable as shown by Saulson [8].
(b) The second is a loss which arises from the centrifugal (vertical) coupling of the pendulum to vertical losses in the support structure. This is a non-linear problem, surprisingly difficult to solve. Here we present the solution to this problem, and give examples of the limiting Q-factor in various situations.

The dominant and relevant effect of a high quality factor is to reduce the thermal noise of the pendulum suspension. The dependence of the thermal noise spectral density on the quality factor has been derived earlier [1]. The thermal noise is given in terms of the quantity $\tilde{h}^2$ where the *tilde* denotes the Fourier transform of the quantity below it. The quantity h is the metric perturbation of the gravitational wave one is trying to detect. The thermal noise is given in terms of this metric perturbation. Thus,

$$\tilde{h}^2_{\text{thermal}} \sim \frac{16kT\omega_p}{mQ_{\text{eff}}\omega^4 l^2}. \tag{1.2}$$

Here k is the Boltzmann constant, T the absolute temperature, l the length of the arm of the laser interferometric detector, Q_{eff} the effective quality factor, ω_p the pendulum frequency and ω the frequency of the thermal noise. The $\tilde{h}$ falls off as $Q_{\text{eff}}^{-(1/2)}$ and therefore a higher Q_{eff} has the effect of reducing the thermal noise. If we observe the noise profiles for burst or continuous wave sources [1] of laser interferometric gravitational wave detectors, the thermal noise is present at the lower end of the band of detectable frequencies. Increasing the Q_{eff} will push this thermal curve 'downwards' reducing the noise at lower frequencies near the seismic cut-off. This will have the effect of increasing the signal to noise ratios for sources which emit gravitational waves predominantly at lower frequencies. For example coalescing binaries which radiate more power at lower frequencies will have their signal to noise enhanced if the quality factor is boosted.

2. The equations of motion

We consider here a simple model of a seismic isolator which consists of a pendulum attached to a spring. The spring motion has a dissipative element in it while the dissipation in the pendular motion is neglected. We set up the classical equations for the system which turn out to be non-linearly coupled. We then solve this system of equations numerically and find that under the assumption that the amplitudes of the motion are small, analytic approximations are possible and an analytic but approximate solution can be derived. We finally compute an effective quality factor for the pendular motion which now depends on amplitude (or time) because the motion is not damped according to the usual exponential law.

Consider the system as shown in figure 2. In terms of the two dynamical variables, x the extension of the spring beyond its normal length and θ the angular displacement of the pendulum, the Lagrangian $\mathscr{L}$ for the system is given by,

$$\mathscr{L} = \frac{1}{2}M(\dot{x}^2 - \omega_s^2 x^2 + l^2\dot{\theta}^2) - Ml\sin\theta\dot{x}\dot{\theta} + Mg(x + l\cos\theta), \tag{2.1}$$

where M is the mass of the bob of the pendulum, l the length of the pendulum, ω_s the natural frequency of the spring and g the acceleration due to gravity. We assume that the damping of the spring is proportional to the velocity which means that the damping force Q_x is of the form,

$$Q_x = -2\frac{M}{\tau}\dot{x}, \tag{2.2}$$

where τ is the damping time constant. We assume that the damping for the θ motion can be neglected ($Q_\theta = 0$). In terms of the dimensionless variables,

$$\alpha = \frac{\omega_p}{\omega_s}, \quad z = \frac{x}{l} - \alpha^2, \quad Q = \frac{1}{2}\omega_s\tau, \quad T = \omega_s t, \tag{2.3}$$

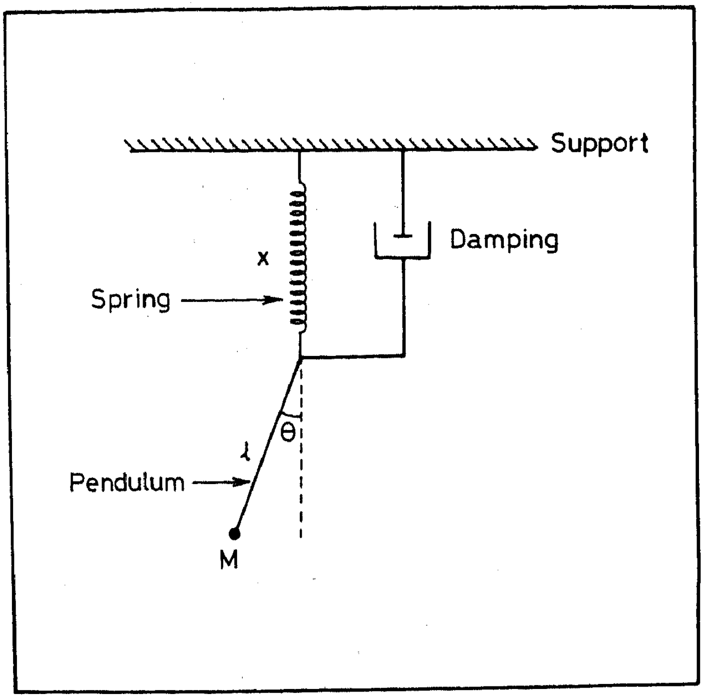

Figure 2. A schematic diagram of a simple model for a seismic isolator is shown which consists of a pendulum attached to a spring. The spring motion is damped with the damping force proportional to the velocity.

where, $\omega_p = (g/l)^{1/2}$, the Euler–Lagrange equations can be reduced to the following equations of motion, for the two dynamical variables z and θ:

$$\ddot{z} + \frac{1}{Q}\dot{z} + z = \theta\ddot{\theta} + \dot{\theta}^2, \tag{2.4a}$$

$$\ddot{\theta} + \alpha^2\theta = \theta\ddot{z}, \tag{2.4b}$$

where the dot is the time derivative with respect to the dimensionless time T. *The aim is to compute the quality factor Q_{eff} of the θ motion.*

The equations are too complex to solve in full generality. They are nonlinearly coupled and do not possess any exact solutions. There are basically only two parameters, namely, α and Q. All the other quantities have been thrown into the background by resorting to dimensionless units. However, if the actual values are known for these parameters it is possible to obtain approximate analytic solutions. We may guess approximate analytic solutions by first solving the equations numerically.

3. The solutions

3.1 *General features*

Many of the qualitative features of the solutions can be seen by examining the nature of the equations themselves, but in order to get a quantitative idea of the timescales of oscillations and damping we resorted to numerical computations. Although the parameters used in the numerical solutions did not correspond to actual situations, they had the advantage of providing us with useful guidelines in assuming the form of the analytic solution. We experimented with several values for the initial parameters (for instance, $z_0 = \dot{z}_0 = \dot{\theta}_0 = 0, \theta_0 = 0{\cdot}05$, where the subscript '0' denotes the values of the variables at the initial time t_0) and came up with the following salient features:

(i) The θ motion is oscillatory with period α and the amplitude of the oscillations is damped over a much larger timescale (this is for a sensible choice of the parameters). We have taken the computation far enough up to $T = 10^4$ so that the damping is appreciable.
(ii) For lower values of α and θ_0 the damping is milder and is not easily seen. For such values we have taken the computation till $T \sim 10^6$. However the results are not qualitatively different.
(iii) The z-motion consists of two superimposed oscillations:

(a) The cycles are at a frequency 2α. This is due to the quadratic nonlinearity, namely, θ^2 forcing the z motion.
(b) On these oscillations is superimposed a transient of about unit frequency which is damped at the rate $e^{-T/2Q}$. At late times the transients die out and the spring oscillates with the frequency 2α.

At late times both the θ and z oscillations are slowly damped. The damping is not exponential but slower and is quantified in the next section. It is remarkable that in the regime of interest, analytic solutions to this system of equations are possible. The above mentioned features are observed in the analytic solution.

3.2 *The transients and the particular solution*

Since the θ displacement is small, we start with a trial solution

$$\theta(T) = \theta_1 \cos(\alpha T), \tag{3.1}$$

so that at $T = 0$ the θ displacement is maximum, namely, $\theta_1 = \theta_0$. θ_1 here is assumed to be constant although it happens to be a slowly varying function of time on the time scale of the damping rate. For a few cycles this assumption is alright during which little decay in the amplitude occurs. The oscillation time scale is of course of the order of α^{-1}. This solution assumes that $\theta\ddot{z} \sim 0$ which is justified later. We further proceed to compute the right hand side of the z equation from equation (3.1). Thus,

$$\ddot{z} + \frac{1}{Q}\dot{z} + z = -\alpha^2 \theta_1^2 \cos 2\alpha T, \tag{3.2}$$

which means that the z motion is forced at twice the frequency of the θ oscillation, which is at the second harmonic of the θ oscillation. The total solution for $z(t)$ is obtained as a superposition of the transient solution and the forced solution. Thus,

$$z(t) = z_{\text{transient}}(t) + z_{\text{forced}}(t), \tag{3.3}$$

where,

$$z_{\text{transient}} = e^{-T/2Q}(A_1 \cos \beta T + A_2 \sin \beta T), \tag{3.4}$$

where, $\beta = (1 - (1/4Q^2))^{1/2}$, and A_1 and A_2 are to be determined from initial conditions imposed on the full solution. The forced solution is,

$$z_{\text{forced}} = z_1 \cos(2\alpha T + \Phi_1), \tag{3.5}$$

where,

$$z_1 = -\alpha^2\theta_1^2/A, \quad \tan\Phi_1 = \frac{2\alpha}{Q(1-4\alpha^2)}, \quad A = [(1-4\alpha^2)^2 + 4\alpha^2/Q^2]^{1/2}. \tag{3.6}$$

Again here the behaviour of z_1 is analogous to θ_1. The transient solution dies out in the timescale Q^{-1} so that at late times ($T \gg Q^{-1}$) only the forced solution survives.

The trial solution for $\theta(T)$ as given in equation (3.1) can be justified in the following way. We observe that if we choose $\alpha \sim 1$ or less and $Q \sim 1$ to 100 then $z \sim \alpha^2\theta_1^2$. Since the z motion is also sinusoidal with frequency $2\alpha, \ddot{z} \sim \alpha^4\theta_0^2$. Thus, the term $\theta\ddot{z}$ is of the order of $\alpha^4\theta_0^3$ which is very small. Therefore the assumption of neglecting this term in our trial solution for θ is not unjustified.

3.3 *Damped motion*

The motion is damped since the system loses energy because of the dissipation in the spring. We analyse the damping from energy considerations. We need to evaluate first the total average energy of the system and then relate it to the rate of loss of average energy. Since the dissipative element is in the spring, the system loses energy *only* through the z-motion. However, the θ-motion will also be damped because of the coupling. The instantaneous energy is the Hamiltonian of the system and is given in dimensionless units by,

$$E = \frac{1}{2}\dot{z}^2 + \frac{1}{2}\dot{\theta}^2 - \theta\dot{\theta}\dot{z} + \frac{1}{2}z^2 + \frac{1}{2}\alpha^2\theta^2. \tag{3.7}$$

We now substitute the late time solutions,

$$z = z_1(T)\cos(2\alpha T + \Phi_1), \quad \theta = \theta_1(T)\cos\alpha T, \tag{3.8}$$

where θ_1 and z_1 are 'slowly' varying functions on the timescale of α^{-1} in equation (3.7) and compute the average energy per cycle $\langle E\rangle$ of the oscillations. The calculation leads to,

$$\langle E\rangle = \left(\alpha^6\theta_1^4 - \frac{1}{4}\alpha^4\theta_1^4 + 2\alpha^6\theta_1^4\right)A^{-2} + \frac{1}{2}\alpha^2\theta_1^2. \tag{3.9}$$

Since θ_1 has been assumed to be small, the higher order terms in θ_1 may be neglected and to the lowest order in θ_1, we have, $\langle E\rangle \sim \frac{1}{2}\alpha^2\theta_1^2$. The rate of loss of energy is computed by first differentiating the expression for the Hamiltonian and then using the equations of motion. We then have,

$$\frac{\mathrm{d}}{\mathrm{d}T}E = -\frac{1}{Q}\dot{z}^2. \tag{3.10}$$

The average loss of energy per unit time can then be computed from equations (3.6) and (3.8) and averaging the trigonometric functions over unit time. Thus,

$$\frac{\mathrm{d}}{\mathrm{d}T}\langle E\rangle = -\frac{2\alpha^6\theta_1^4}{A^2 Q}. \tag{3.11}$$

The milder damping rate, i.e. milder than the exponential rate, is evident from the above analysis and can be seen in the following way:

The dominant contribution to the energy comes from the θ motion and is quadratic in θ_1. On the other hand, the rate of energy loss is quadratic in z_1 since the damping, which is proportional to the velocity, is in this vertical mode. But since the θ mode is coupled quadratically to the z mode the rate of energy decay depends on the *fourth* power of θ_1. This is a much milder decay rate than the one compared to the exponential case, where the energy of the system as well as the energy decay rate, scale quadratically in the amplitude. Also since the rate of the decay of energy is governed by the amplitude of the oscillations in the above manner, the damping is less for lower amplitudes. This was one of our observations in subsection 3.1.

From (3.9) and (3.11) we get a differential equation for the decay of the average energy, or equivalently for the amplitude θ_1:

$$\frac{\mathrm{d}}{\mathrm{d}T}\theta_1^2 = -\frac{4\alpha^4\theta_1^4}{A^2 Q}. \tag{3.12}$$

Solving this equation with the initial conditions $T=0$, $\theta=\theta_0$, we have,

$$\theta_1(T) = \frac{\theta_0}{B(T)}, \tag{3.13a}$$

where,

$$B(T) = (1+\varepsilon T)^{1/2}, \quad \varepsilon = \frac{4\alpha^4\theta_0^2}{A^2 Q}.$$

The decay of the z-motion is obtained from equation (3.6). Thus,

$$z_1(T) = -\frac{\alpha^2\theta_0^2}{AB^2(T)}. \tag{3.13b}$$

Therefore we notice that the z motion is damped at a faster rate than the θ motion. This behaviour remains true for various values of α, Q and θ_0 as has been verified on the computer and cross checked with the above formulae. But if α and θ are very small then enormous amount of time is required on the computer to produce appreciable damping as can be seen from the foregoing discussion. The analytic solution then is useful in predicting the damping profile.

4. The effective Q for the pendular motion

Consider a damped simple harmonic oscillator with natural frequency ω_0 and damping time constant τ. Then the time dependence of the amplitude is $\sim e^{-t/\tau \pm i\omega_0 t}$. The quality factor Q is then given by the formula: $Q = \frac{1}{2}\omega_0\tau$. In terms of the dimensionless time T the effective quality factor Q_{eff} for general damped motion, which is not necessarily an exponential decay, is given by,

$$Q_{\mathrm{eff}} = \frac{1}{2}\alpha\left[-\frac{\mathrm{d}}{\mathrm{d}T}(\ln\theta_1(T))\right]^{-1}. \tag{4.1}$$

Using equation (3.13a), the above expression leads to the result,

$$Q_{\mathrm{eff}} = \alpha\frac{(1+\varepsilon T)}{\varepsilon}. \tag{4.2}$$

Therefore in this case the quality factor is a function of time which is to be expected since the damping is not exponential. We observe that, the Q_{eff} increases with time. At early times, i.e. when $\varepsilon T \ll 1$ the Q_{eff} is given by,

$$Q_{\text{eff}} \sim \frac{\alpha}{\varepsilon} = \frac{A^2 Q}{4\alpha^3 \theta_0^2}. \tag{4.3}$$

We observe that at early times Q_{eff} is a constant and hence in this regime the system behaves like a normal damped harmonic oscillator i.e. the amplitude θ_1 decays exponentially with time $\theta_1(T) \sim e^{-\alpha T/2Q_{\text{eff}}}$. However when $T \sim \varepsilon^{-1}$ the decay of the amplitude is slower than the exponential rate. Figure 3 compares the decay of the amplitudes of the two oscillators:

(a) exponentially damped,
(b) pendular, which corresponds to the system under consideration.

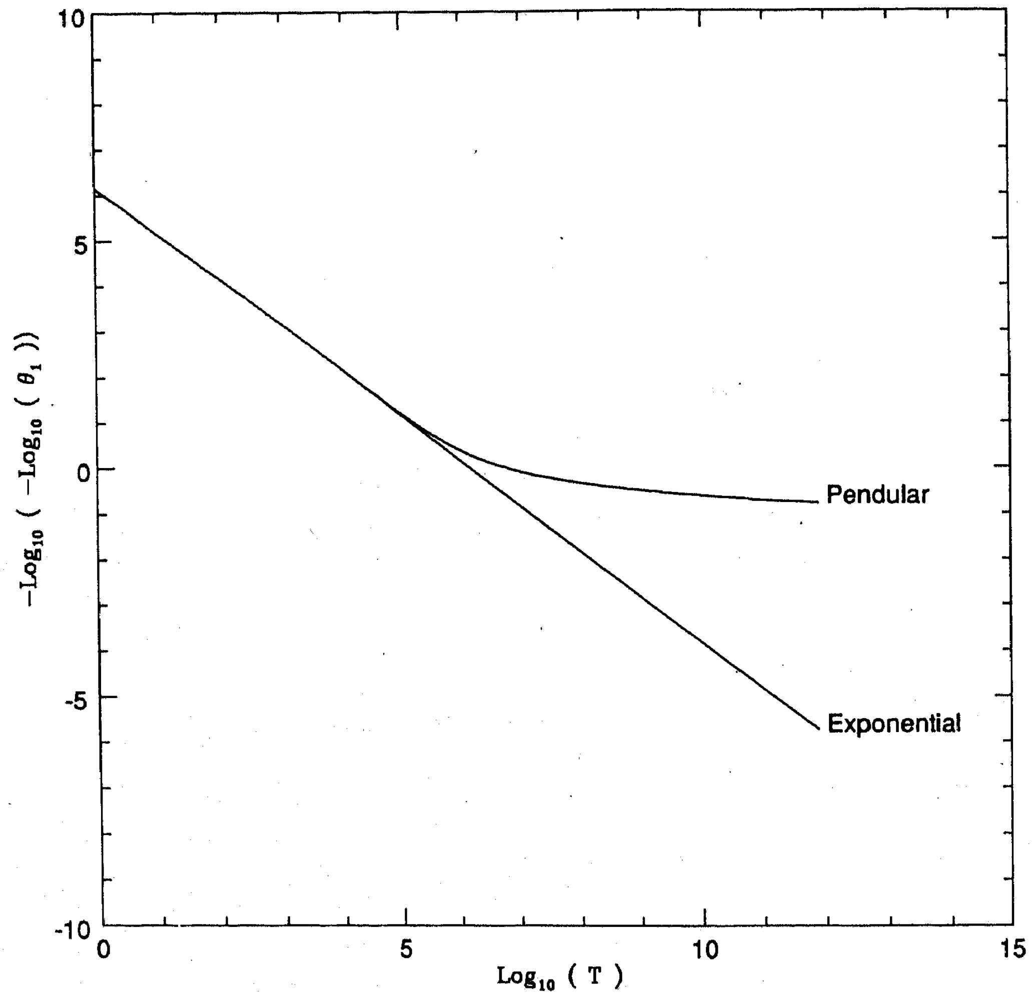

Figure 3. The figure depicts the decay of the amplitudes with time for the standard exponential case and for the model considered here (pendular). The parameters have the following values $\alpha = 0{\cdot}3$, $Q = 2$ and $\theta_0 = 0{\cdot}01$. (a) *exponential*: The amplitude is damped exponentially with time. This appears as a straight line with slope $= -1$ in the logarithmic scale. (b) *pendular*: In the model considered the amplitude decays slower than in the standard case (a).

In figure, $\alpha = 0{\cdot}3$, $Q = 2$ and $\theta_0 = 0{\cdot}01$ which corresponds to $\varepsilon \sim 10^{-6}$. It is convenient to use logarithmic scales for depicting the behaviour. We plot $-\log_{10}(-\log_{10}\theta_1)$ verses $\log_{10} T$. The usual case of the exponentially damped oscillator appears as a straight line with slope $= -1$. This curve is labelled as *exponential*. The intercept on the vertical axis turns out to be $\log_{10}Q_{\text{eff}} - \log_{10}(\frac{1}{2}\alpha\log_{10}e) - \log_{10}(-\log_{10}\theta_0)$ which increases with Q_{eff}. We observe that when $T \sim \varepsilon^{-1} \sim 10^6$ the pendular curve departs from the exponential and the decay is slower. Although the parameters used here are not realised in actual situations, they have the advantage of bringing out the difference in the behaviours of the two types of damping.

For the case of the spring constant $k \sim 10^7\,\text{kg}\cdot\text{s}^{-2}$, $M \sim 10^3\,\text{kg}$, $l \sim 1\,\text{m}$ and Q say 10, we have the following values for the relevant parameters: $\omega_s \sim 100$, $\omega_p \sim 3$, $\alpha \sim 0{\cdot}03$ and $A \sim 1$. From the equations (3.13) and (4.2) we get an amplitude dependent quality factor for the pendular motion. Thus,

$$Q_{\text{eff}} = \frac{A^2 Q}{4\alpha^3} \times \frac{1}{\theta_1^2}. \tag{4.4}$$

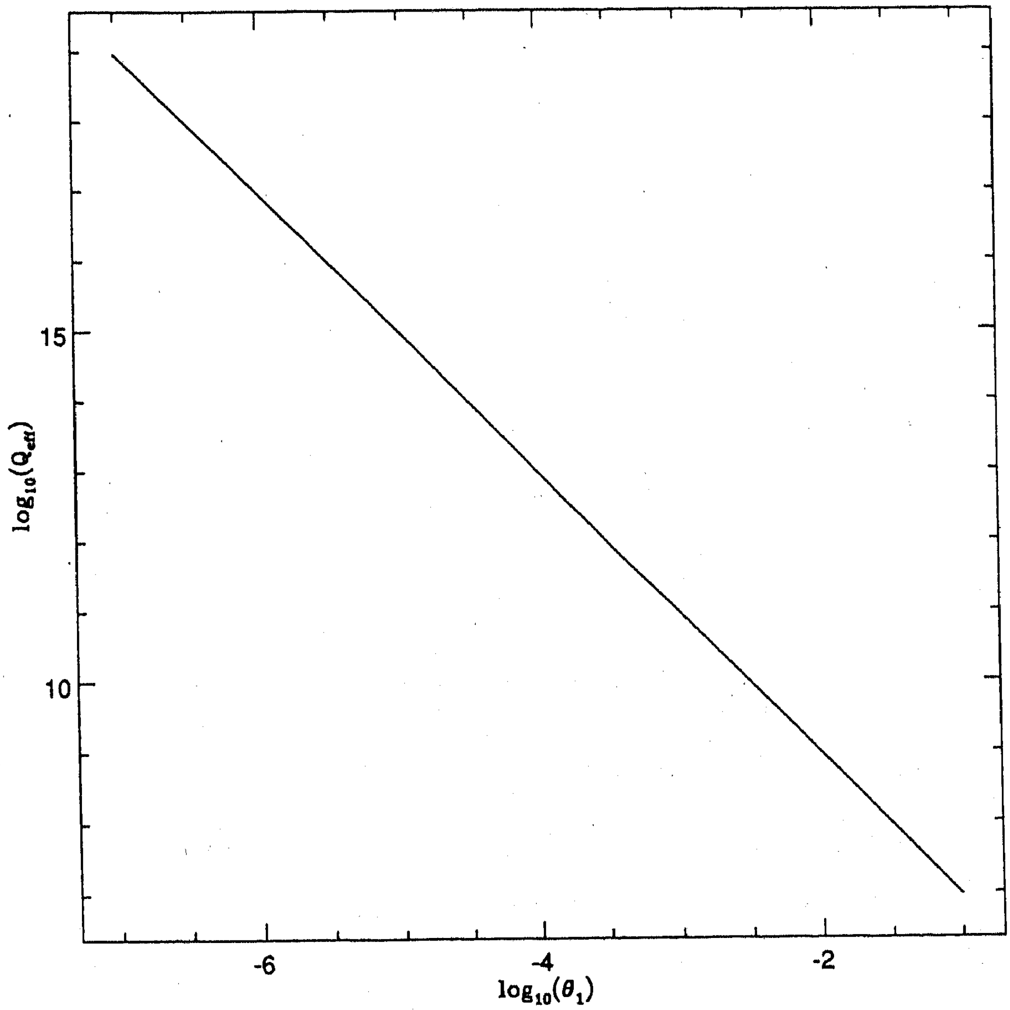

Figure 4. The figure shows that the quality factor Q_{eff} is amplitude dependent for the model considered here. The Q_{eff} is plotted versus the amplitude θ_1 in a logarithmic scale. We find that the Q_{eff} increases as the inverse square of the amplitude. In this figure the relevant parameters have the values $Q = 10$, $\alpha = 0{\cdot}03$ and hence the constant $A^2Q/4\alpha^3 \sim 10^5$.

Figure 4 displays this behaviour in which $\log_{10} Q_{\rm eff}$ is plotted against $\log_{10} \theta_1$. For the values of the parameters mentioned above, the constant $A^2 Q/4\alpha^3 \sim 10^5$. The tendency is for the $Q_{\rm eff}$ to increase with the decrease of amplitude. Therefore a value of $Q_{\rm eff} \sim 10^{10}$ is not impossible under the circumstances.

5. Conclusions

We have shown that the coupling of a pendulum to a lossy support structure can create an amplitude dependent Q-factor and this can significantly degrade the Q of an intrinsically high Q pendulum if care is not taken in the design. Since metal and rubber vibration isolator elements have intrinsically low Q, these can contribute particularly large amplitude dependent losses. The coupling of noise into such a suspension will occur through *parametric amplification* type processes more familiar in optical and radio frequency parametric amplifiers. Some interferometer designs have proposed a suspension point servo which uses a secondary interferometer to lock together the pendulum suspension points. Such a suspension does not eliminate seismic noise, but forces it into common mode so that there is no differential motion. In such a situation residual seismic amplitudes could be large enough to degrade the suspension Q through the mechanism discussed here.

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References

[1] J Hough *et al*, Proposal for a Joint German–British Interferometric Gravitational Wave Detector, *MPQ* 147, (1989)
[2] A Brillet and A Giazotto *et al*, VIRGO Final Conceptual Design, INFN, Pisa, (1992)
[3] Australian International Gravitational Research Centre Proposal, The University of Western Australia, (1990)
[4] D G Blair, D McClelland and H Bachor, Gravitational Waves and the AIGO proposal, *Australian and New Zealand Physicist*, **29**, 4, (1992)
[5] H Bachor, D G Blair, J Ferreirinho, D McClelland, R J Sandeman, P Veitch and A Wrightson; Should Australia have a gravity telescope? *Asia-Pacific Physics News*, **5**, (1990)
[6] A Abramovici, W E Althouse, R W P Drever, Y Gürsel, S Kawamura, F J Raab, D Shoemaker, L Sievers, R E Spero, K S Thorne, R E Vogt, R Weiss, S E Whitcomb and M E Zucker. LIGO: The laser interferometer gravitational wave observatory, *Science*, **256**, 325 (1992)
[7] Ju Li, D G Blair and M Notcutt, Ultra high Q pendulum suspensions for gravitational wave detectors. (To be published 1992)
[8] P Saulson, *Phys. Rev.* **D42**, 2437 (1990)
[9] D G Blair, L Ju and M Notcutt, *Rev. Sci. Instrum.* **64**, 1899 (1993)
[10] W Martin, Ph. D. thesis, University of Glasgow, U.K., (1978)
[11] V B Braginsky, V F Mitrofanof and O A Oklirimenko, *Phys. Lett.* **A175**, 82 (1993)