Painlevé singularity structure analysis of three component Gross-Pitaevskii type equations

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Abstract

In this paper, we have studied the integrability nature of a system of three coupled Gross-Pitaevskii type nonlinear evolution equations arising in the context of spinor Bose-Einstein condensates by applying the Painlevé singularity structure analysis. We show that only for two sets of parametric choices, corresponding to the known integrable cases, the system passes the Painlevé test.

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I. INTRODUCTION

Integrable multicomponent nonlinear Schrödinger type equations have attracted considerable current interest in soliton research. Much focus has been paid to identify new integrable multicomponent type equations due to their many faceted applications in different fields of science such as nonlinear optics, Bose-Einstein condensates, biophysics, plasma physics, etc.[1−5]. Painlevé singularity structure analysis is one of the powerful tools to isolate and identify integrable dynamical systems[6−10]. This procedure nicely complements other integrability tools like inverse scattering transform (IST), infinite number of involutive integrals of motion, symmetries, Bäcklund transformations, Hirota's bilinearization method, etc., to study the integrability properties of nonlinear systems[1,2]. By applying the Painlevé test for integrability a class of integrable coupled nonlinear Schrödinger (CNLS) type equations, which arise in different physical contexts, has been identified[11−16].

In this connection, the system of CNLS equations in the presence of confining potential becomes the coupled Gross-Pitaevskii (GP) equations, governing the dynamics of two component Bose-Einstein condensates[17−19]. This kind of multicomponent condensates can also be created with the mixture of two different atomic species or by considering the hyperfine spin of atoms in the presence of optical dipole traps[20−22]. The latter entities are the so-called spinor Bose-Einstein condensates (BECs).

Spinor Bose-Einstein condensates of ultra cold atoms can be created by liberating the hyperfine states by means of optical trapping. Two component condensates have been realized in ^{87}Rb (see Ref. 23) and also optically trapped three component condensates were studied in Refs. 24-27. The evolution of the spinor condensate wave functions is governed by the following set of three-coupled nonlinear Schrödinger type equations[28],

$$
\begin{aligned}
i\hbar\psi_{+1,T} &= -\frac{\hbar^2}{2m}\psi_{+1,XX} + (c_0+c_2)(|\psi_{+1}|^2+|\psi_0|^2)\psi_{+1} \\
&\quad +(c_0-c_2)|\psi_{-1}|^2\psi_{+1} + c_2\psi_{-1}^*\psi_0^2, \qquad (1a)\\
i\hbar\psi_{0,T} &= -\frac{\hbar^2}{2m}\psi_{0,XX} + (c_0+c_2)(|\psi_{+1}|^2+|\psi_{-1}|^2)\psi_0 \\
&\quad +c_0|\psi_0|^2\psi_0 + 2c_2\psi_0^*\psi_{+1}\psi_{-1}, \qquad (1b)\\
i\hbar\psi_{-1,T} &= -\frac{\hbar^2}{2m}\psi_{-1,XX} + (c_0+c_2)(|\psi_{-1}|^2+|\psi_0|^2)\psi_{-1} \\
&\quad +(c_0-c_2)|\psi_{+1}|^2\psi_{-1} + c_2\psi_{+1}^*\psi_0^2, \qquad (1c)
\end{aligned}
$$

where $\psi_{\pm1,0}$'s are the wave functions of the three spin components, T is the time and X denotes the spatial co-ordinate. The effective one-dimensional coupling constants c_0 and c_2 representing the mean field and spin exchange interactions, respectively, are given by $c_0 = \frac{g_0+2g_2}{3}$, $c_2 = \frac{g_2-g_0}{3}$, where $g_f = \frac{4\hbar^2 a_f}{ma_\perp^2}\left(\frac{1}{1-C\frac{a_f}{a_\perp}}\right)$, $f = 0,2$. Here a_f's are the s-wave scattering lengths in the total hyperfine spin channel f, $a_\perp$ is the size of the transverse ground state, m is the atomic mass and the constant $C = -\zeta(1/2) \simeq 1.46$, where ζ is the Reimann zeta-function. With the redefinition of $T = \hbar t$, $X = \frac{\hbar}{\sqrt{2m}}x$ and transforming $(\psi_1, \psi_0, \psi_{-1}) \to (\psi_1, \sqrt{2}\psi_0, \psi_{-1})$, we can rewrite Eq. (1) in the standard form as

$$
\begin{aligned}
i\psi_{+1,t} &= -\psi_{+1,xx} + (c_0+c_2)(|\psi_{+1}|^2+2|\psi_0|^2)\psi_{+1} \\
&\quad +(c_0-c_2)|\psi_{-1}|^2\psi_{+1} + 2c_2\psi_{-1}^*\psi_0^2, \qquad (2a)\\
i\psi_{0,t} &= -\psi_{0,xx} + (c_0+c_2)(|\psi_{+1}|^2+|\psi_{-1}|^2)\psi_0 \\
&\quad +2c_0|\psi_0|^2\psi_0 + 2c_2\psi_0^*\psi_{+1}\psi_{-1}, \qquad (2b)\\
i\psi_{-1,t} &= -\psi_{-1,xx} + (c_0+c_2)(|\psi_{-1}|^2+2|\psi_0|^2)\psi_{-1} \\
&\quad +(c_0-c_2)|\psi_{+1}|^2\psi_{-1} + 2c_2\psi_{+1}^*\psi_0^2. \qquad (2c)
\end{aligned}
$$

We refer to Eq. (2) as the three-component GP type equations. The above system of equations has been solved by the IST method and multicomponent bright and dark solitons have been reported for specific choices of c_0 and c_2[29−31]. Now it is of interest to isolate all the possible integrable models arising from Eq. (2) for arbitrary choices of c_0 and c_2, which can be tuned suitably through Feshbach resonance. For this purpose, we perform a Painlevé singularity structure analysis to the above fairly generalized system. It is also expected that besides BECs the analysis will have wider ramifications in nonlinear optics.

This paper is arranged in the following manner. In section II, the three steps involved in the Painlevé singularity structure analysis, namely the leading order analysis of the Laurent expansion in the neighbourhood of a non-characteristic singular manifold, determination of the resonances (that is, the powers at which arbitrary functions can occur in the Laurent expansion) and analysis of the Laurent expansion for sufficient number of arbitrary functions are carried out. It is shown that only for the two specific parametric choices, namely (i) $c_2 = 0$ and (ii) $c_0 = c_2$ the system (2) passes the Painlevé integrability test. The results are analyzed in the final section.

II. PAINLEVÉ SINGULARITY STRUCTURE ANALYSIS

In order to perform the Painlevé singularity structure analysis of Eq. (2) the dependent variables $\psi_{\pm 1}$, ψ_0 and their complex conjugates are denoted as

$$\psi_{+1} = a, \quad \psi^*_{+1} = b, \quad \psi_{-1} = m, \quad \psi^*_{-1} = n, \quad \psi_0 = p, \quad \psi^*_0 = q. \tag{3}$$

Then Eqs. (2) become

$$\begin{aligned}
ia_t &= -a_{xx} + (c_0 + c_2)\,(ab + 2pq)a + (c_0 - c_2)\,mna + 2c_2np^2, &\text{(4a)}\\
-ib_t &= -b_{xx} + (c_0 + c_2)\,(ab + 2pq)b + (c_0 - c_2)\,mnb + 2c_2mq^2, &\text{(4b)}\\
im_t &= -m_{xx} + (c_0 + c_2)\,(mn + 2pq)m + (c_0 - c_2)\,abm + 2c_2bp^2, &\text{(4c)}\\
-in_t &= -n_{xx} + (c_0 + c_2)\,(mn + 2pq)n + (c_0 - c_2)\,abn + 2c_2aq^2, &\text{(4d)}\\
ip_t &= -p_{xx} + 2c_0p^2q + (c_0 + c_2)\,(ab + mn)p + 2c_2qam, &\text{(4e)}\\
-iq_t &= -q_{xx} + 2c_0q^2p + (c_0 + c_2)\,(ab + mn)q + 2c_2pbn. &\text{(4f)}
\end{aligned}$$

The Painlevé singularity structure analysis (of an analytic polynomial differential equation) is carried out by seeking a generalized Laurent expansion[32] for the dependent variables

$$a = \phi^{\alpha} \sum_{j=0} a_j(x,t)\phi^j, \qquad a_0 \neq 0, \tag{5a}$$

$$b = \phi^{\beta} \sum_{j=0} b_j(x,t)\phi^j, \qquad b_0 \neq 0, \tag{5b}$$

$$m = \phi^{\gamma} \sum_{j=0} m_j(x,t)\phi^j, \quad m_0 \neq 0, \tag{5c}$$

$$n = \phi^{\delta} \sum_{j=0} n_j(x,t)\phi^j, \qquad n_0 \neq 0, \tag{5d}$$

$$p = \phi^{\epsilon} \sum_{j=0} p_j(x,t)\phi^j, \qquad p_0 \neq 0, \tag{5e}$$

$$q = \phi^{\omega} \sum_{j=0} q_j(x,t)\phi^j, \qquad q_0 \neq 0, \tag{5f}$$

in the neighbourhood of the non-characteristic singular manifold $\phi(x,t) = 0$, with nonvanishing derivatives $\phi_x(x,t) \neq 0$ and $\phi_t(x,t) \neq 0$.

A. Leading order analysis

The leading order behaviour of the solution is analyzed by assuming the forms

$$a \approx a_0\phi^{\alpha}, \quad b \approx b_0\phi^{\beta}, \quad m \approx m_0\phi^{\gamma}, \quad n \approx n_0\phi^{\delta}, \quad p \approx p_0\phi^{\epsilon}, \quad q \approx q_0\phi^{\omega} \tag{6}$$

for the dependent variables, where α, β, γ, δ, ϵ and ω are integers to be determined. After substituting these forms into Eq. (4) and by balancing the most dominant terms, at the leading order one obtains

$$\alpha = \beta = \gamma = \delta = \epsilon = \omega = -1, \tag{7}$$

with a set of relations

$$p_0^2 = a_0 m_0, \qquad q_0^2 = b_0 n_0, \tag{8a}$$

$$\phi_x^2 = \frac{(c_0 + c_2)}{2}\left(\sqrt{a_0 b_0} + \sqrt{m_0 n_0}\right)^2. \tag{8b}$$

Note that there are six functions a_0, b_0, m_0, n_0, p_0 and q_0 (besides the arbitrary manifold $\phi(x,t)$) and the above three conditions mean that three of them are arbitrary at this stage of the analysis.

B. Resonances

The second step in the singularity structure analysis is to determine the resonances (powers) at which arbitrary functions can enter into the Laurent series (5). To obtain the resonance values, we substitute the following expressions into Eqs. (4)

$$\begin{aligned} a &= a_0\phi^{-1} + \cdots + a_j\phi^{j-1}, \qquad b = b_0\phi^{-1} + \cdots + b_j\phi^{j-1}, \\ m &= m_0\phi^{-1} + \cdots + m_j\phi^{j-1}, \quad n = n_0\phi^{-1} + \cdots + n_j\phi^{j-1}, \\ p &= p_0\phi^{-1} + \cdots + p_j\phi^{j-1}, \qquad q = q_0\phi^{-1} + \cdots + q_j\phi^{j-1}. \end{aligned} \tag{9}$$

and determine the possible values of j. By collecting the coefficients of ϕ^{j-3}, one can obtain a system of six algebraic equations which can be casted as

$$\mathbf{D}\ \mathbf{X}^T = \mathbf{0}, \tag{10a}$$

where the superscript 'T' denotes the transpose of the matrix and the matrices $\mathbf{X}$ and $\mathbf{D}$ are given by

$$\mathbf{X} = \begin{pmatrix} a_j & b_j & m_j & n_j & p_j & q_j \end{pmatrix}, \tag{10b}$$

$$\mathbf{D} = \begin{pmatrix} Q_1 & r_1\ a_0^2 & r_2 n_0 a_0 & r_1 m_0 a_0 & 2r_1 a_0 q_0 + 4c_2 n_0 p_0 & 2r_1 p_0 a_0 \\ r_1\ b_0^2 & Q_1 & r_1 b_0 n_0 & r_2 m_0 b_0 & 2r_1 q_0 b_0 & 2r_1 p_0 b_0 + 4c_2 m_0 q_0 \\ r_2 b_0 m_0 & r_1 a_0 m_0 & Q_2 & r_1 m_0^2 & 2r_1 q_0 m_0 + 4c_2 b_0 p_0 & 2r_1 p_0 m_0 \\ r_1 b_0 n_0 & r_2 a_0 n_0 & r_1 n_0^2 & Q_2 & 2r_1 q_0 n_0 & 2r_1 p_0 n_0 + 4c_2 a_0 q_0 \\ r_1 b_0 p_0 + 2c_2 q_0 m_0 & r_1 a_0 p_0 & r_1 n_0 p_0 + 2c_2 a_0 q_0 & r_1 m_0 p_0 & Q_3 & 2r_1 a_0 m_0 \\ r_1 b_0 q_0 & r_1 a_0 q_0 + 2c_2 p_0 n_0 & r_1 n_0 q_0 & r_1 m_0 q_0 + 2c_2 p_0 b_0 & 2r_1 b_0 n_0 & Q_3 \end{pmatrix}, \tag{10c}$$

in which

$$
\begin{aligned}
r_1 &= c_0 + c_2, \quad r_2 = c_0 - c_2, \\
Q_1 &= -(j-1)(j-2)\phi_x^2 + 2r_1\ (a_0 b_0 + p_0 q_0) + r_2\ m_0 n_0, \\
Q_2 &= -(j-1)(j-2)\phi_x^2 + 2r_1\ (m_0 n_0 + p_0 q_0) + r_2\ a_0 b_0, \\
Q_3 &= -(j-1)(j-2)\phi_x^2 + r_1\ (a_0 b_0 + m_0 n_0) + 4c_0 p_0 q_0
\end{aligned}
$$

and $\mathbf{0}$ is a (6×1) null matrix. By requiring the determinant of the matrix $\mathbf{D}$ to be zero the following resonance equation is obtained.

$$
j^3(j+1)(j-3)^3(j-4)\left(4c_2 - 3j(c_0+c_2) + j^2(c_0+c_2)\right)^2 = 0. \tag{11}
$$

From Eq. (11) the values of j are obtained as

$$
j = -1, 0, 0, 0, 3, 3, 3, 4, N_1, N_1, N_2, N_2, \tag{12a}
$$

where

$$
N_1 = \frac{1}{2}\left(\frac{3(c_0+c_2) + \sqrt{9c_0^2 + 2c_0c_2 - 7c_2^2}}{c_0+c_2}\right), \tag{12b}
$$

$$
N_2 = \frac{1}{2}\left(\frac{3(c_0+c_2) - \sqrt{9c_0^2 + 2c_0c_2 - 7c_2^2}}{c_0+c_2}\right). \tag{12c}
$$

All the resonances should be integers for the system (2) to satisfy the Painlevé property so that movable algebraic branching type critical singular manifolds are avoided. Hence by requiring N_1 and N_2 to be integers we find the following two cases:

$$
\begin{aligned}
\text{Case(i)}: \quad c_0 &= -\left(1 + \frac{4}{m(m-3)}\right) c_2, \quad m = 1, 2, 4, 5, 6, \ldots (m \neq 3) \\
\text{Case(ii)}: \quad c_2 &= 0, \qquad m = 0, 3.
\end{aligned}
$$

Then the integer resonances for both the cases can be written as

$$
j = -1, 0, 0, 0, 3, 3, 3, 4, m, m, 3-m, 3-m, \quad m = 0, 1, 2, \ldots \tag{13}
$$

Note that in the above, $j = -1$ corresponds to the arbitrariness of the non-characteristic manifold $\phi(x,t)$.

Case (i):

In this case for the choices $m = 1$ and $m = 2$, we get $c_0 = c_2 = c$ (say), where c is a real constant, all the resonances are positive integers (except for $j = -1$) and are given below.

$$j = -1, 0, 0, 0, 1, 1, 2, 2, 3, 3, 3, 4. \quad (14)$$

However for $m \geq 4$, the presence of more negative resonances indicates that there may exist only particular solutions with lesser number of arbitrary functions in the Laurent expansion. For example, the choice $m = 4$ corresponding to $c_0 = -2c_2$, has the resonances $j = -1, -1, -1, 0, 0, 0, 3, 3, 3, 4, 4, 4$. This may be an indication that the Laurent expansion (5) does not correspond to a general solution with required number of arbitrary functions, but represents only a particular solution. We feel that these choices with $m \geq 4$ are the candidates for further deeper analysis mathematically on Laurent expansions in the negative powers. We do not pursue this problem further here. However, one can perform for example a study on the modulation instability of system (2) for these choices of $m(\geq 4)$ and look for solitary wave type solutions which could be of specific physical interest. So, hereafter we will consider only the case having resonances (14) with $c_0 = c_2 = c$.

Case (ii):

For the values $m = 0$ and $m = 3$, we require $c_2 = 0$ and the system (2) reduces to a set of three coupled nonlinear Schrödinger equations with resonances $j = -1, 0, 0, 0, 0, 0, 3, 3, 3, 3, 3, 4$, whose integrability and Painlevé analysis have already been studied in detail in Ref. 12. So we will not consider this case any further.

C. Analysis for arbitrary functions

The next step in the Painlevé singularity structure analysis is to show that there exist sufficient number of arbitrary functions in the Laurent expansion (5) which can arise at the resonance values given by (14) without the introduction of movable critical singular manifolds. To prove this, we expand the dependent variables in Eqs. (4) (upto the highest resonance value in (14)) as below:

$$a = \frac{a_0}{\phi} + a_1 + a_2\phi + a_3\phi^2 + a_4\phi^3, \tag{15a}$$

$$b = \frac{b_0}{\phi} + b_1 + b_2\phi + b_3\phi^2 + b_4\phi^3, \tag{15b}$$

$$m = \frac{m_0}{\phi} + m_1 + m_2\phi + m_3\phi^2 + m_4\phi^3, \tag{15c}$$

$$n = \frac{n_0}{\phi} + n_1 + n_2\phi + n_3\phi^2 + n_4\phi^3, \tag{15d}$$

$$p = \frac{p_0}{\phi} + p_1 + p_2\phi + p_3\phi^2 + p_4\phi^3, \tag{15e}$$

$$q = \frac{q_0}{\phi} + q_1 + q_2\phi + q_3\phi^2 + q_4\phi^3, \tag{15f}$$

where a_j, b_j, m_j, n_j, p_j, q_j, $j = 0, 1, \ldots, 4$, are functions of (x, t) to be determined. Then by collecting various powers of ϕ, we explicitly show that there exist sufficient number of arbitrary functions at each index of the resonance values given in (14). As noted above, the resonance at $j = -1$ corresponds to the arbitrariness of the non-characteristic manifold ϕ.

1. Coefficients of ϕ^{-3} :

The set of algebraic equations resulting at this order is

$$\begin{aligned}
-2\phi_x^2 + 2c(a_0b_0 + 2p_0q_0) + 2cn_0p_0^2/a_0 &= 0, \\
-2\phi_x^2 + 2c(a_0b_0 + 2p_0q_0) + 2cm_0q_0^2/b_0 &= 0, \\
-2\phi_x^2 + 2c(m_0n_0 + 2p_0q_0) + 2cb_0p_0^2/m_0 &= 0, \\
-2\phi_x^2 + 2c(m_0n_0 + 2p_0q_0) + 2ca_0q_0^2/n_0 &= 0, \\
-2\phi_x^2 + 2cp_0q_0 + 2c(a_0b_0 + m_0n_0) + 2ca_0q_0m_0/p_0 &= 0, \\
-2\phi_x^2 + 2cp_0q_0 + 2c(a_0b_0 + m_0n_0) + 2cb_0p_0n_0/q_0 &= 0.
\end{aligned} \tag{16a}$$

Solving Eqs. (16a) again results in the already deduced relations (8),

$$p_0^2 = a_0m_0, \qquad q_0^2 = b_0n_0, \tag{16b}$$

$$\phi_x^2 = c\left(\sqrt{a_0b_0} + \sqrt{m_0n_0}\right)^2. \tag{16c}$$

This clearly shows that three out of the six functions (a_0, b_0, m_0, n_0, p_0 and q_0) are arbitrary at the triple resonance $j = 0, 0, 0$.

2. Coefficients of ϕ^{-2} :

At the power ϕ^{-2}, we obtain the following set of algebraic equations expressed in the matrix form,

$$\mathbf{D_1}\ \mathbf{X_1}^T = (-i\phi_t)\ \mathbf{Y_1}^T, \tag{17a}$$

where

$$\mathbf{D_1} = \begin{pmatrix} 4cl_1 & 2ca_0b_0 & 0 & 2cm_0n_0 & 4cl_2 & 4cp_0q_0 \\ 2ca_0b_0 & 4cl_1 & 2cm_0n_0 & 0 & 4cp_0q_0 & 4cl_2 \\ 0 & 2ca_0b_0 & 4cl_2 & 2cm_0n_0 & 4cl_1 & 4cp_0q_0 \\ 2ca_0b_0 & 0 & 2cm_0n_0 & 4cl_2 & 4cp_0q_0 & 4cl_1 \\ 2cl_2 & 2ca_0b_0 & 2cl_2 & 2cm_0n_0 & 2c(l_1+l_2) & 4cp_0q_0 \\ 2ca_0b_0 & 2cl_1 & 2cm_0n_0 & 2cl_2 & 4cp_0q_0 & 2c(l_1+l_2) \end{pmatrix}, \tag{17b}$$

$$\mathbf{X_1} = \begin{pmatrix} \frac{a_1}{a_0} & \frac{b_1}{b_0} & \frac{m_1}{m_0} & \frac{n_1}{n_0} & \frac{p_1}{p_0} & \frac{q_1}{q_0} \end{pmatrix}, \tag{17c}$$

$$\mathbf{Y_1} = \begin{pmatrix} 1 & -1 & 1 & -1 & 1 & -1 \end{pmatrix}. \tag{17d}$$

In the above matrix $\mathbf{D_1}$ we have introduced the quantities l_1 and l_2, which are defined as

$$l_1 = a_0b_0 + p_0q_0 \quad \text{and} \quad l_2 = m_0n_0 + p_0q_0. \tag{17e}$$

In order to make the calculations simpler here and in the subsequent analysis, we use the Kruskal ansatz[7] by assuming the singular manifold function $\phi(x,t)$ in the form $\phi(x,t) = x + \rho(t)$, with ρ an arbitrary analytic function and the a_j, b_j, m_j, n_j, p_j, q_j are functions of t only. One can solve the above six algebraic equations (17) given in the matrix form and obtain

$$\frac{a_1}{a_0} = -\frac{p_0q_0}{a_0b_0}\left(\frac{i\rho_t}{2c\ l_2} + \frac{p_1}{p_0}\right), \tag{18a}$$

$$\frac{b_1}{b_0} = \frac{p_0q_0}{a_0b_0}\left(\frac{i\rho_t}{2c\ l_2} - \frac{q_1}{q_0}\right), \tag{18b}$$

$$\frac{m_1}{m_0} = -\frac{1}{l_2}\left(\frac{i\rho_t}{2c} + l_1\frac{p_1}{p_0}\right), \tag{18c}$$

$$\frac{n_1}{n_0} = \frac{1}{l_2}\left(\frac{i\rho_t}{2c} - l_1\frac{q_1}{q_0}\right). \tag{18d}$$

From equations (18), we observe that the two functions (p_1 and q_1) out of the six functions a_1, b_1, m_1, n_1, p_1 and q_1 are arbitrary. Naturally, these are associated with the double resonance at $j = 1, 1$.

3. Coefficients of ϕ^{-1} :

At this order we obtain

$$\mathbf{D_1}\ \mathbf{X_2}^T = \mathbf{Y_2}^T, \tag{19a}$$

where the matrix $\mathbf{D_1}$ being defined in Eq. (17b) and

$$\mathbf{X_2} = \begin{pmatrix} \frac{a_2}{a_0} & \frac{b_2}{b_0} & \frac{m_2}{m_0} & \frac{n_2}{n_0} & \frac{p_2}{p_0} & \frac{q_2}{q_0} \end{pmatrix}, \tag{19b}$$

$$\mathbf{Y_2} = \begin{pmatrix} y_2^{(1)} & y_2^{(2)} & y_2^{(3)} & y_2^{(4)} & y_2^{(5)} & y_2^{(6)} \end{pmatrix}. \tag{19c}$$

Here the elements of $\mathbf{Y_2}$ are given by

$$\begin{aligned} y_2^{(1)} &= \frac{ia_{0t}}{a_0} - \frac{2c}{a_0}\left(p_1^2 n_0 + a_1^2 b_0\right), \\ &\quad -\frac{4c}{a_0}\left(a_1 p_1 q_0 + a_1 b_1 a_0 + n_1 p_1 p_0 + a_1 q_1 p_0 + p_1 q_1 a_0\right), \end{aligned} \tag{20a}$$

$$\begin{aligned} y_2^{(2)} &= -\frac{ib_{0t}}{b_0} - \frac{2c}{b_0}\left(q_1^2 m_0 + b_1^2 a_0\right) \\ &\quad -\frac{4c}{b_0}\left(b_1 q_1 p_0 + a_1 b_1 b_0 + m_1 q_1 q_0 + b_1 p_1 q_0 + p_1 q_1 b_0\right), \end{aligned} \tag{20b}$$

$$\begin{aligned} y_2^{(3)} &= \frac{im_{0t}}{m_0} - \frac{2c}{m_0}\left(m_1^2 n_0 + p_1^2 b_0\right) \\ &\quad -\frac{4c}{m_0}\left(m_1 q_1 p_0 + m_1 n_1 m_0 + m_1 p_1 q_0 + b_1 p_1 p_0 + p_1 q_1 m_0\right), \end{aligned} \tag{20c}$$

$$\begin{aligned} y_2^{(4)} &= -\frac{in_{0t}}{n_0} - \frac{2c}{n_0}\left(n_1^2 m_0 + q_1^2 a_0\right) \\ &\quad -\frac{4c}{n_0}\left(a_1 q_1 q_0 + m_1 n_1 n_0 + n_1 q_1 p_0 + p_1 q_1 n_0 + n_1 p_1 q_0\right), \end{aligned} \tag{20d}$$

$$\begin{aligned} y_2^{(5)} &= \frac{ip_{0t}}{p_0} - \frac{2c}{p_0}\left(2p_1 q_1 p_0 + a_1 q_1 m_0 + m_1 n_1 p_0 + a_1 b_1 p_0 + p_1^2 q_0\right) \\ &\quad -\frac{2c}{p_0}\left(a_1 p_1 b_0 + b_1 p_1 a_0 + n_1 p_1 m_0 + a_1 m_1 q_0 + m_1 q_1 a_0 + m_1 p_1 n_0\right), \end{aligned} \tag{20e}$$

$$\begin{aligned} y_2^{(6)} &= -\frac{iq_{0t}}{q_0} - \frac{2c}{q_0}\left(2p_1 q_1 q_0 + n_1 q_1 m_0 + m_1 n_1 q_0 + a_1 b_1 q_0 + q_1^2 p_0\right) \\ &\quad -\frac{2c}{q_0}\left(b_1 p_1 n_0 + b_1 n_1 p_0 + a_1 q_1 b_0 + m_1 q_1 n_0 + b_1 q_1 a_0 + n_1 p_1 b_0\right). \end{aligned} \tag{20f}$$

Proceeding further as in the case of $j = 1$ and by incorporating the results of $j = 0$ and $j = 1$, we express the four functions a_2, b_2, m_2 and n_2 in terms of the remaining two unknown functions p_2 and q_2:

$$\frac{a_2}{a_0} = \frac{1}{l_1}\left(\frac{y_2^{(1)} - y_2^{(3)}}{4c} + l_2\,\frac{m_2}{m_0} - (l_2 - l_1)\frac{p_2}{p_0}\right), \tag{21a}$$

$$\frac{b_2}{b_0} = \frac{1}{l_1}\left(\frac{y_2^{(2)} - y_2^{(4)}}{4c} + l_2\,\frac{n_2}{n_0} - (l_2 - l_1)\frac{q_2}{q_0}\right), \tag{21b}$$

where

$$\frac{m_2}{m_0} = \frac{n_2}{n_0} - \frac{l_1}{l_2}\left(\frac{p_2}{p_0} - \frac{q_2}{q_0}\right) + \frac{y_2^{(3)} - y_2^{(4)}}{2c\ l_2} + \frac{a_0 b_0 (y_2^{(1)} - y_2^{(2)} - y_2^{(3)} + y_2^{(4)})}{4\ c l_1 l_2}, \tag{21c}$$

$$\frac{n_2}{n_0} = -\frac{l_1}{l_2}\left(\frac{q_2}{q_0}\right) + \frac{a_0 b_0 (3y_2^{(4)} + y_2^{(2)} - 2y_2^{(1)}) + 2p_0 q_0 (2y_2^{(4)} - y_2^{(3)})}{12\ c l_1 l_2}, \tag{21d}$$

with l_1 and l_2 being defined in Eq. (17e) and $y_2^{(j)}$'s, $j = 1, 2, ..., 6$, are given in Eqs. (20). From the above equations (21), we can easily see that two (p_2 and q_2) out of the six functions a_2, b_2, m_2, n_2, p_2 and q_2 are arbitrary, as required by the existence of arbitrary functions at the double resonance $j = 2, 2$.

4. Zeroth order in ϕ :

Collecting now the coefficients at the zeroth order, that is ϕ^0, we obtain

$$\mathbf{D_3}\ \mathbf{X_3}^T = \mathbf{Y_3}^T, \tag{22a}$$

where the matrix $\mathbf{D_3} = \mathbf{D_1} - 2\mathbf{I}$, $\mathbf{I}$ is a (6×6) identity matrix and

$$\mathbf{X_3} = \begin{pmatrix} \frac{a_3}{a_0} & \frac{b_3}{b_0} & \frac{m_3}{m_0} & \frac{n_3}{n_0} & \frac{p_3}{p_0} & \frac{q_3}{q_0} \end{pmatrix}, \tag{22b}$$

$$\mathbf{Y_3} = \begin{pmatrix} y_3^{(1)} & y_3^{(2)} & y_3^{(3)} & y_3^{(4)} & y_3^{(5)} & y_3^{(6)} \end{pmatrix}. \tag{22c}$$

Here

$$
\begin{aligned}
y_3^{(1)} &= i\left(\frac{a_{1t}+a_2\rho_t}{a_0}\right)-\frac{4c}{a_0}\left(a_2p_1q_0+a_1a_2b_0+a_2b_1a_0+p_2q_1a_0\right.\\
&\quad +n_2p_1p_0+n_1p_2p_0+a_1q_2p_0+a_1p_2q_0+a_1b_2a_0+a_1p_1q_1\\
&\quad \left.+a_2q_1p_0+p_1q_2a_0+p_1p_2n_0\right)-\frac{2c}{a_0}\left(a_1^2b_1+n_1p_1^2\right), && \text{(22d)}\\
y_3^{(2)} &= -i\left(\frac{b_{1t}+b_2\rho_t}{b_0}\right)-\frac{4c}{b_0}\left(b_1p_2q_0+b_2p_1q_0+b_1p_1q_1+p_2q_1b_0\right.\\
&\quad +a_2b_1b_0+p_1q_2b_0+a_1b_2b_0+m_2q_1q_0+m_1q_2q_0+q_1q_2m_0\\
&\quad \left.+b_1b_2a_0+b_1q_2p_0+b_2q_1p_0\right)-\frac{2c}{b_0}\left(a_1b_1^2+m_1q_1^2\right), && \text{(22e)}\\
y_3^{(3)} &= i\left(\frac{m_{1t}+m_2\rho_t}{m_0}\right)-\frac{4c}{m_0}\left(p_1p_2b_0+m_1m_2n_0+m_1n_2m_0+m_2n_1m_0\right.\\
&\quad +m_1p_1q_1+p_2q_1m_0+b_1p_2p_0+b_2p_1p_0+p_1q_2m_0+m_1p_2q_0\\
&\quad \left.+m_2p_1q_0+m_1q_2p_0+m_2q_1p_0\right)-\frac{2c}{m_0}\left(b_1p_1^2+m_1^2n_1\right), && \text{(22f)}\\
y_3^{(4)} &= -i\left(\frac{n_{1t}+n_2\rho_t}{n_0}\right)-\frac{4c}{n_0}\left(n_1p_2q_0+n_2p_1q_0+n_1q_2p_0+a_1q_2q_0\right.\\
&\quad +m_2n_1n_0+n_2q_1p_0+q_1q_2a_0+q_1a_2q_0+m_1n_2n_0+p_1q_2n_0\\
&\quad \left.+p_2q_1n_0+n_1n_2m_0+n_1p_1q_1\right)-\frac{2c}{n_0}\left(a_1q_1^2+n_1^2m_1\right), && \text{(22g)}\\
y_3^{(5)} &= i\left(\frac{p_{1t}+p_2\rho_t}{p_0}\right)-\frac{4c}{p_0}\left(p_1q_2p_0+p_2q_1p_0\right)-\frac{2c}{p_0}\left(p_1p_2q_0+m_2p_1n_0+m_1n_1p_1\right.\\
&\quad +a_1b_1p_1+a_2b_1p_0+m_1n_2p_0+m_2n_1p_0+a_1b_2p_0+b_1p_2a_0+m_1p_2n_0\\
&\quad +n_2p_1m_0+n_1p_2m_0+b_2p_1a_0+a_1p_2b_0+a_2p_1b_0+m_1q_2a_0\\
&\quad \left.+m_2q_1a_0+a_1m_2q_0+a_1m_1q_1+a_2m_1q_0+a_2q_1m_0+a_1q_2m_0\right), && \text{(22h)}\\
y_3^{(6)} &= -i\left(\frac{q_{1t}+q_2\rho_t}{q_0}\right)-\frac{4c}{q_0}\left(p_1q_2q_0+p_2q_1q_0\right)-\frac{2c}{p_0}\left(q_1q_2p_0+m_2q_1n_0+m_1n_1q_1\right.\\
&\quad +a_1b_1q_1+b_1q_2a_0+m_1q_2n_0+n_2q_1m_0+n_1q_2m_0+b_2q_1a_0+m_1n_2q_0\\
&\quad +a_2b_1q_0+m_2n_1q_0+a_1b_2q_0+b_1n_2p_0+a_1q_2b_0+a_2q_1b_0\\
&\quad \left.+b_2n_1p_0+n_2p_1b_0+b_1n_1p_1+n_1p_2b_0+b_2p_1n_0+b_1p_2n_0\right). && \text{(22i)}
\end{aligned}
$$

After a straightforward but lengthy algebra one can solve the above Eqs. (22) and deduce the following three expressions:

$$
\begin{aligned}
\frac{a_3}{a_0} &= \frac{\left((4cl_2-2)\frac{m_3}{m_0}-(4c(l_2-l_1)-2)\frac{p_3}{p_0}+y_3^{(1)}-y_3^{(3)}\right)}{(4cl_1-2)}, && \text{(23a)}\\
\frac{b_3}{b_0} &= \frac{\left((4cl_2-2)\frac{n_3}{n_0}-(4c(l_2-l_1)-2)\frac{q_3}{q_0}+y_3^{(2)}-y_3^{(4)}\right)}{(4cl_1-2)}, && \text{(23b)}
\end{aligned}
$$

where

$$\frac{m_3}{m_0} = -\frac{n_3}{n_0} + \frac{\left(4cl_1\left(\frac{p_3}{p_0}+\frac{q_3}{q_0}\right)-y_3^{(3)}\right)}{2c(l_1-l_2)} + \frac{a_0b_0(y_3^{(2)}-y_3^{(4)})}{2c(l_1-l_2)^2}. \tag{23c}$$

The above expressions (23) indicate the arbitrariness of three functions (n_3, p_3 and q_3) out of the six functions a_3, b_3, m_3, n_3, p_3 and q_3. Thus system (2) with $c_0 = c_2 = c$, satisfies the requirement of the presence of three arbitrary functions corresponding to the triple resonance at $j = 3, 3, 3$.

5. Coefficients of ϕ^1 :

In a similar manner, after a lengthy algebra carried out using *Maple* we have verified that the resulting six algebraic equations at the coefficient of ϕ, which we do not present here for want of space, reduce to five equations with six unknown functions. Thus we observe that there exists one arbitrary function corresponding to the resonance $j = 4$, as required.

Our preceding analysis shows that for the choice $c_0 = c_2 = c$, there exist sufficient number of arbitrary functions at the resonance values given by Eq. (14). One can proceed further to obtain the higher order coefficient functions for all $j > 4$ in terms of the previous coefficients without the introduction of any movable critical singular manifold into the Laurent expansion for the case $c_0 = c_2 = c$. So we conclude that the system of three component Gross-Pitaevskii (GP) type equation (2) passes the Painlevé test only for the two cases (i) $c_2 = 0$ and (ii) $c_0 = c_2 = c$ and is expected to be integrable. Of course, this fact has already been shown for the case $c_2 = 0$ through the Painlevé analysis[12] and the second case $c_0 = c_2 = c$ can be reduced to the (2×2) matrix NLS equation[28] which is integrable through the IST method[33,34].

III. CONCLUSION

In this paper, we have studied the integrability property of the three component Gross-Pitaevskii (GP) type equations arising as the evolution equations for spinor condensates by applying the Painlevé singularity structure analysis. We have identified that only for the following two choices of the effective one dimensional coupling constants, (i) $c_2 = 0$ and (ii) $c_0 = c_2 = c$, the system (2) passes the Painlevé test and possesses Laurent expansion with full complement of arbitrary functions without the introduction of movable critical singular manifolds. The integrability of the first choice ($c_2 = 0$) has been discussed already[12]. For the second choice ($c_0 = c_2 = c$) by applying the IST method the multi-bright solitons under vanishing[28,29] as well as non-vanishing boundary conditions[31] and multi-dark solitons[30] have been obtained by reducing the system (2) to the known IST integrable (2×2) matrix nonlinear Schrödinger equation[33,34]. Our present analysis also shows that the system (2) passes the Painlevé test for integrability when c_0 and c_2 are equal and non-zero, in addition to the choice $c_2 = 0$. Apart from finding the choices for which the system (2) can be integrable, one can also obtain information regarding the Hirota's bilinearization for such cases from the above analysis. This has already been exploited for the case $c_2 = 0$ to obtain multi-soliton solutions[35,36]. Work is now in progress to make a similar analysis for the case $c_0 = c_2 = c$. Also it is of future interest to identify multicomponent integrable systems with higher degree of hyperfine spin ($F > 1$), see for example the coupled evolution equations given in[37], for which also the above type of Painlevé singularity structure analysis can be carried out.

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