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# Magnetization curves of hard superconductor samples with non-zero demagnetization factor

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**Abstract.** We present exact solutions of Bean's critical state model for some sample shapes having non-zero demagnetization factor $N$. Virgin and hysteresis magnetization curves are obtained for samples in the shape of (i) a sphere (ii) a spheroid (iii) a cylinder of circular cross-section with its axis perpendicular to the field and (iv) a cylinder of elliptical cross section with its axis perpendicular to the field. Some interesting features seen in these first solutions for $N \neq 0$ are discussed.

**Keywords.** Hard superconductors; magnetization curves; critical state model; demagnetization factor.

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## 1. Introduction

Bean (1962, 1964) had proposed the critical state model to explain the isothermal magnetization of a zero-field-cooled (ZFC) hard type II superconductor. He had obtained the virgin and hysteresis magnetization curves, under the assumption that the current density $J_C$ is independent of magnetic field, for the two sample geometries of an infinite slab and an infinite cylinder of circular cross-section. In both cases the field is taken to be along the long axis and the demagnetization factor $N$ is thus zero. While Bean's work has been extended to various field-dependent forms $J_C(H)$, this extension has been done only for the above two simplifying geometries (see e.g. Campbell and Evetts 1972; Kim *et al* 1963; Kes *et al* 1973; Chaddah *et al* 1989; Ravi Kumar and Chaddah 1989). Campbell and Evetts (1972) have considered other infinitely long sample shapes (with $N = 0$), and have obtained the magnetization for some of these under the assumption of a field-independent $J_C$.

Experimental samples always have a non-zero demagnetization factor ($N \neq 0$), and attempts to obtain magnetization curves for such samples, within Bean's model, have been continuously made (Campbell and Evetts 1972; Brechna 1973; Wilson 1983; Clem and Kogan 1987). With the assumption of field-independent $J_C$, Campbell and Evetts (1972) have obtained approximate solutions to the virgin magnetization curve, in the limit of zero applied field, for a sphere ($N = 1/3$) and for an infinitely long circular cylinder perpendicular to the field ($N = 1/2$). Wilson (1983), and Clem and Kogan (1987), have *assumed* solutions of Bean's model for the circular cylinder and the sphere respectively. While the former is for arbitrary fields, the latter is for extremely large fields. This unhappy situation contrasts markedly with that for the thermodynamic magnetization of type I and reversible type II (see Fetter and Hohenberg 1969) superconductors where solutions are available for various $N$. The discovery of high $T_C$ superconductors, which have intrinsic pinning because of their small coherence length and are thus always irreversible or "hard", has highlighted the need for solving Bean's critical state model for $N \neq 0$. In this paper we shall present first solutions for $N \neq 0$, under the assumption that $J_C$ is independent of field. As in earlier approximate or assumed solutions, we shall ignore $H_{C1}$ as small. While the inclusion of $H_{C1}$ in the virgin magnetization curve is straightforward (Bean 1962), the same is not true for the hysteresis curve (Chaddah 1988).

In the next section we state Bean's model. We also argue that this model provides the only solution of Maxwell's equations, other than London's equation (which is for a reversible magnetization), for a material with infinite conductivity for $J \leqslant J_C$. In §3 we present a contrast of the virgin magnetization for samples with $N = 0$ and with $N \neq 0$. While in the former shielding currents flow in only one sense, this is not true for the latter. In §4 we obtain the shielding current distribution for samples with $N \neq 0$. In §5 we obtain the virgin magnetization curves for these sample shapes and also the hysteresis curves, assuming field reversal from very large fields. We finally state our conclusions, and discuss possible experiments which would be relevant to this work.

## 2. Isothermal magnetization

### 2.1 *Bean's model*

The basic premise of Bean's model (Bean 1962, 1964) is that when any region of a hard superconductor sample is exposed to a variation in the local magnetic field, a shielding current of magnitude $J_C$ will flow in that region. In this model only the following states of current flow are possible with a given axis of the applied field.

(i) Zero current will flow in those regions that have never felt the magnetic field. For a ZFC sample being subjected to a magnetic field, such regions will constitute the central region of the sample and will shrink in volume as the applied field is increased.
(ii) Shielding currents of magnitude $J_C$ will flow in those regions that have experienced a non-zero $H$ (after zero field cooling). The direction of the shielding current will be perpendicular to the field axis and its sense depends on the sense of the emf that accompanied the last local change of field through Lenz's law.

If we denote the shielding currents by $\mathbf{J}_s$, then Maxwell's equation gives us

$$\operatorname{curl}\mathbf{H} = 0$$

$$\operatorname{curl}\mathbf{B} = \mu_0 \mathbf{J}_s. \tag{1}$$

Bean's model adds to these the condition that only in a region where $|\mathbf{B}|$ has always remained zero after zero-field cooling $|\mathbf{J}_s| = 0$ and in all other regions

$$|\mathbf{J}_s| \equiv J_s = J_C \tag{2}$$

### 2.2 *Macroscopic justification*

We now argue that for a material having $\sigma = \infty$ for $J_s \leqslant J_{C'}$ (and a finite $\sigma$ for $J_s > J_C$) Bean's model presents the only alternative to London's equation. The latter, of course, yields a magnetization curve with no hysteresis and cannot be applied to hard type II superconductors.

In the region $J_s < J_C$ we have $\sigma = \infty$ and this results in (using standard notations)

$$\mathbf{E} = (m/ne^2)\,\partial \mathbf{J}_s/\partial t.$$

When substituted in Maxwell's equation

$$\text{curl}\,\mathbf{E} = -\,\partial \mathbf{B}/\partial t$$

we get

$$\partial \mathbf{B}/\partial t = -(m/ne^2)\text{curl}\,\partial \mathbf{J}_s/\partial t. \tag{3}$$

The virgin magnetization curve of a ZFC sample has the initial conditions $\mathbf{B}(\mathbf{r}, t_0) = 0$, and $\mathbf{J}_s(\mathbf{r}, t_0) = 0$, throughout the sample and (3) yields

$$\mathbf{B}(t) = -(m/ne^2)\text{curl}\,\mathbf{J}_s(t). \tag{4}$$

Equations (1) and (4), which must be satisfied for $J_s < J_C$ result in London's equation.

If however, $J_s > J_C$ then (4) would not be satisfied. And for $J_s > J_C$, $\sigma \neq \infty$ and thus the shielding currents would decay as in a normal metal. This decay must be arrested once $J_s$ reduces to $J_C$ since now $\sigma = \infty$. Thus, to deviate from (4) (or from London's equation) we need a transient $J_s(t) > J_C$ wherever $\tilde{\mathbf{B}} \neq 0$. Once such a transient is set up, $J_s$ cannot decay to a magnitude smaller than $J_C$. We thus get Bean's model as the only macroscopic alternative to London's equation.

### 2.3 *Virgin magnetization of a spherical sample*

In this section we shall attempt to solve the problem of a spherical sample while satisfying (11) and (2).

For a ZFC spherical sample subjected to $\mathbf{H}_A = H_A\mathbf{k}$, the shielding current density will be along $-\mathbf{e}_\varphi$ (Campbell and Evetts 1972; Clem and Kogan 1987) and will have a magnitude $J_C$. These shielding currents will flow in a region (which we call region I) inwards from the surface, up to a certain depth such that $\mathbf{B} = 0$ in the interior region (called region II). Since this interior region has never been exposed to a non-zero $B$, Bean's model states that $J_s = 0$ in region II. Region I is bound externally by the surface of the sphere of radius $R$, and internally by a certain surface described by $r = f(\theta)$. The problem has reduced to obtaining $f(\theta)$ for various $H_A$.

Introducing $\mathbf{M}(= \mathbf{B} - \mu_0\mathbf{H})$, (1) can be rewritten as

$$\text{curl}\,\mathbf{M} = \mu_0\mathbf{J}_s \tag{5}$$

and a solution consistent with Bean's model requires

$$\text{curl}\,\mathbf{M}(\mathbf{r}) = 0 \tag{6}$$

for $\mathbf{r}$ lying in region II. Equation (6) thus provides a necessary condition to be satisfied by $f(\theta)$. To ensure that $f(\theta)$ provides a solution of the problem, we must show that

$$\mathbf{B}(\mathbf{r}) = 0 \tag{7}$$

for $\mathbf{r}$ lying in region II, where $\mathbf{B}$ will be calculated as a superposition of $\mathbf{H}_A$ and the magnetic induction $\mathbf{B}_J$ created by $-J_C\mathbf{e}_\varphi$ flowing in region I. (For consistency of notation used in the literature we use $\mathbf{H}_A$, rather than $\mathbf{B}_A$, to denote externally applied magnetic induction). We may also caution that

$$\mathbf{B}(\mathbf{r}) \neq \mathbf{H}_A(\mathbf{r}) + \mathbf{M}(\mathbf{r})$$

since $\mu_0\mathbf{H}(\mathbf{r}) \neq \mathbf{H}_A(\mathbf{r})$ for a sample of arbitrary shape, and we have $\mu_0\,\mathbf{H}(\mathbf{r}) = \mathbf{H}_A(\mathbf{r})$ only in the limit $r \to \infty$.

The solution of (6) is obtained by employing Ampere's circuital theorem to the current density $-J_C\mathbf{e}_\varphi$. Using the symbol $\mathbf{M}_{\rm I}(\mathbf{M}_{\rm II})$, for $\mathbf{M}$ in region I(II) we get

$$\mathbf{M}_{\rm I}(u, z) = -\mu_0 J_C[(R^2 - z^2)^{1/2} - u]\mathbf{k} \tag{8a}$$

for $(u, z)$ lying in region I and

$$\mathbf{M}_{\rm II}(u, z) = -\mu_0 J_C[(R^2 - z^2)^{1/2} - u_0(z)]\mathbf{k} \tag{8b}$$

for $(u, z)$ lying in region II, with the point $(u_0, z)$ lying on the surface $r = f(\theta)$. Assuming that the sample is homogeneous to start with (when $\mathbf{B} = 0$), the region II of the sample that has never experienced any $\mathbf{B}$ will remain homogeneous. In this region Div $\mathbf{M}_{\rm II} = 0$. This together with (6) implies $\mathbf{M}_{\rm II}$ is independent of $\mathbf{r}$, so that we get

$$[(R^2 - z^2)^{1/2} - u_0(z)] = \text{constant}. \tag{9}$$

We shall denote this constant by $\xi$ and note that it depends only on the applied field. Expressing $z$ and $u_0(z)$ in polar co-ordinates and noting that $(u_0, z)$ lies on the surface $r = f(\theta)$, we get the expression for $f(\theta)$. We shall denote it by $f(\xi, \theta)$ to show its explicit dependence on the parameter $\xi$. We thus get the equation for the surface separating regions I and II, a surface we shall refer to as the flux-front

$$R^{-1}f(\xi, \theta) = -\xi \sin\theta + [1 - \xi^2 \cos^2\theta]^{1/2}. \tag{10}$$

The parameter $\xi$ is to be determined by satisfying (7). Equation (10) thus provides the solution of the flux-front for a spherical sample provided one assumes that $\mathbf{J}_s = -J_C\mathbf{e}_\varphi$.

We show in Appendix I that it is impossible to satisfy (7) with $f(\xi, \theta)$ given by (10).

We have thus reached the rigorous conclusion that one cannot solve Bean's model for a sphere by assuming $\mathbf{J}_s = -J_C\mathbf{e}_\varphi$ during virgin magnetization curve. A similar conclusion is reached for a circular cylinder with field normal to its axis, as detailed in Appendix I. We thus find that the solution assumed in literature (Wilson 1983; Clem and Kogan 1987), which have $\mathbf{J}_s = -J_C\mathbf{e}_\varphi$ are not correct. In the next section we seek a resolution of this deadlock.

## 3. Consequences of non-zero demagnetization factor

We must now understand why no solution satisfying Bean's model exists, within the framework of §2·3, for the sphere and the transverse circular cylinder. It was assumed, as has been done earlier, that the shielding currents during the virgin magnetization curve will be either 0 (in region II) or $-J_C\mathbf{e}_\varphi$ (in region I). We now argue that for a sample with $N \neq 0$ shielding currents of both senses (i.e. $\mp J_C \mathbf{e}_\varphi$) must flow in region I even during the virgin magnetization curve.

For a sample with $N = 0$ the shielding currents flow in shells such that they do not set up any **H** external to the current shell and

$$\mathbf{H}_{\text{ext}} = \mathbf{H}_{\text{applied}}.$$

As the field is increased during the virgin magnetization, the flux-front moves inwards into the sample and the local field at all **r** increases. The shielding currents set up as the flux-front moves do not affect $\mathbf{B}(\mathbf{r})$ at **r** external to the flux-front.

This situation breaks down for $N \neq 0$. Here $\mathbf{H}_{\text{ext}} \neq \mathbf{H}_{\text{applied}}$, and the shielding currents set up in the interior modify the $\mathbf{B}(\mathbf{r})$ external to the current shell. Eventhough $H_A$ increases monotonically, $\mathbf{B}(\mathbf{r})$ does not. The sense of $J_s$ is to be determined by the *last* local change in the field, and shielding currents $+J_C \mathbf{e}_\varphi$ will flow at those **r** where the setting up of the shielding currents in the inner shells causes a decrease in $\mathbf{B}(\mathbf{r})$. Every change in $J_s$ now modifies $\mathbf{B}(\mathbf{r})$ throughout the sample. At equilibrium, we will now have currents $\mp J_C \mathbf{e}_\varphi$ flowing at various **r** in region I, where the sign at each **r** is to be such as to satisfy (7) and (9) in region II.

In principle it might be possible to incorporate this feature and solve $\text{curl}\, \mathbf{B} = \mu_0 \mathbf{J}_s$ in region I also. We shall, however, carry out a macroscopic averaging in the neighbourhood of each **r**. The spirit of this averaging is the same as that in which we assume a unidirectional shielding current density being fully aware that this results by averaging circular currents around each vortex. The averaged $J_s(\mathbf{r})$ can be smaller than $J_C$ in magnitude, but must satisfy

$$\begin{aligned} \text{curl}\, \mathbf{B}(\mathbf{r}) &= \mathbf{J}_s(\mathbf{r}), \quad \text{in region I,} \\ \mathbf{B}(\mathbf{r}) &= 0, \quad \text{in region II.} \end{aligned} \tag{11}$$

Bean's model will now be solved to obtain $\mathbf{J}_s(\mathbf{r})$, and the flux-front $r = f(\xi, \theta)$.

Equation (11) does not explicity imply $|J_s(\mathbf{r})| = J_C$ for a sample with $N \to 0$. This equality is a consequence of Bean's model as stated in (ii) of §2·1. As noted earlier (Chaddah *et al* 1989) this part of the statement of Bean's model is equivalent to stating that "the direction and magnitude of the shielding current for any change in the external field, is assumed to be such as to minimize the change of the total flux contained in the sample". This statement is general, appears as a logical outcome of Lenz's law, and is seen to be equivalent to the original statement of Bean's model for all the $N = 0$ sample shapes solved so far. We shall now solve the Bean's model as restated above.

## 4. Determination of the shielding current density and the flux-front

It is clear from the above considerations that the solution of Bean's model for a sample with $N \neq 0$ requires the determination of the form of the current density $J_s(\mathbf{r})$ in region I. The extent of region I is governed by flux-front which moves as the applied external field is varied. Since the applied field is constant in direction (along $z$-axis) and only its magnitude is changed, the various flux-fronts should belong to a one parameter family of surfaces, which includes the outer bounding sample surface as a member. We shall first consider the case of a finite sample. The case of an infinite cylindrical sample with applied field transverse to its axis will be dealt with subsequently.

## 4.1 *Finite sample*

4.1.1 *Constraint on the current density and flux-front*: Let us consider a current carrying shell (region I) bounded by two members of a one parameter family of closed surfaces of revolution $r = f(\xi, \theta)$. Let $\xi_1$ and $\xi_2$ correspond respectively to the external and internal boundaries of region I. The current density is assumed to be in the direction $\mathbf{e}_\varphi$ and its magnitude is represented by $J(r, \theta)$. Following (7), we desire that in the volume enclosed by the surface $r = f(\xi_2, \theta)$ (i.e. region II)

$$\mathbf{B}(\mathbf{r}) = \mathbf{H}_A(\mathbf{r}) + \mathbf{B}_J(\mathbf{r}) = 0.$$

Here $\mathbf{B}_J(\mathbf{r})$ is the field created by the shielding currents in region I. A solution of Bean's model thus requires that, for $\mathbf{H}_A(\mathbf{r}) = H_A\mathbf{k}$,

$$\mathbf{B}_J = -H_A\mathbf{k}. \tag{12}$$

We shall first derive necessary and sufficient conditions to be satisfied by $J(r, \theta)$ and $f(\xi, \theta)$ so that the current distribution produces a uniform field (along $z$-axis) in a spherical region (centred at the origin) which lies entirely within region II. For this purpose we examine the vector potential $\mathbf{A}(\mathbf{r})$ generated by the current distribution. The magnetic induction $\mathbf{B}$ is obtained as usual $\mathbf{B} = \operatorname{curl} \mathbf{A}$. Using the standard notation

$$\mathbf{A}(\mathbf{r}) = (\mu_0/4\pi) \int J(r', \theta')\mathbf{e}_{\varphi'}\, \mathrm{d}\mathbf{r}'/|\mathbf{r} - \mathbf{r}'| \tag{13}$$

where the integral extends over the volume of the current carrying region. Substituting for $|\mathbf{r} - \mathbf{r}'|$ in terms of $r, r'$ and $\omega$, (the angle between $\mathbf{r}$ and $\mathbf{r}'$) we write

$$\mathbf{A}(\mathbf{r}) = (\mu_0/4\pi) \int J(r', \theta')\mathbf{e}_{\varphi'}\, \mathrm{d}\mathbf{r}'/(r^2 + r'^2 - 2rr'\cos\omega)^{1/2} \tag{14}$$

where $\cos\omega = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\varphi - \varphi')$ and the unit vector $\mathbf{e}_{\varphi'} = (-\mathbf{i}\sin\varphi' + \mathbf{j}\cos\varphi')$ when expressed in terms of the unit vectors $\mathbf{i}$ and $\mathbf{j}$ along $x$- and $y$- axes respectively. Noting that $\cos\omega$ depends only on the difference $(\varphi - \varphi')$ we introduce a new variable $\eta = \varphi - \varphi'$. Expressing $\mathbf{e}_{\varphi'}$ in terms of $\eta$ and $\varphi$ we can simplify (14) for $\mathbf{A}(\mathbf{r})$ and write

$$\mathbf{A}(\mathbf{r}) = (\mu_0/4\pi)\mathbf{e}_\varphi \int J(r', \theta')\cos\eta\, \mathrm{d}\mathbf{r}'/(r^2 + r'^2 - 2rr'\cos\omega_0)^{1/2} \tag{15}$$

with $\cos\omega_0 = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos\eta$. We now specialize to the case when $\mathbf{r}$ lies in the spherical region mentioned earlier. For this case we have $r < r'$ and we may use a standard expansion involving spherical harmonics and write

$$\mathbf{A}(\mathbf{r}) = (\mu_0/4\pi)\mathbf{e}_\varphi \int J(r', \theta')\cos\eta\, \mathrm{d}\mathbf{r}'$$

$$\times \sum_{l,m} (4\pi/(2l+1)) r^l/r'^{l+1}\, Y_{lm}(\theta, 0)\, Y^*_{lm}(\theta', \eta). \tag{16}$$

Expressing the volume element in spherical polar coordinates we get after some rearrangement

$$\mathbf{A}(\mathbf{r}) = \mu_0 \mathbf{e}_\varphi \sum_{l,m} r^l Y_{lm}(\theta, 0)/(2l+1) \int_0^\pi d\theta' \sin\theta' \times \int dr' J(r', \theta')/r'^{l-1} \int_0^{2\pi} d\eta \cos\eta \, Y^*_{lm}(\theta', \eta). \tag{17}$$

Carrying out the integral over $\eta$ we have

$$\mathbf{A}(\mathbf{r}) = \mu_0 \mathbf{e}_\varphi \sum_l r^l Y_{l1}(\theta, 0)/(2l+1) \times \int_0^\pi d\theta' \int dr' \sin\theta' \, Y_{l1}(\theta', 0) J(r', \theta')/r'^{l-1}. \tag{18}$$

Changing the variable $r'$ to $\xi$ (for a fixed value of $\theta'$) by the substitution $r' = f(\xi, \theta)$, and denoting by $f_\xi$ the partial derivative $\partial f/\partial \xi$, we have $dr' = f_\xi(\xi, \theta')d\xi$. $\xi$ takes values in the range $(\xi_2, \xi_1)$ to cover the region I. We then get

$$\mathbf{A}(\mathbf{r}) = -2\pi\mu_0 \mathbf{e}_\varphi \sum_{l=1}^{\infty} r^l Y_{l1}(\theta, 0)/(2l+1) \times \int_{\xi_2}^{\xi_1} d\xi \int_0^\pi d\theta' \sin\theta' \, Y_{l1}(\theta', 0) J(\xi, \theta') f_\xi(\xi, \theta')/f^{l-1}. \tag{19}$$

The right hand side of (19) resembles the so-called multipole expansion for the vector potential. The magnetic induction $\mathbf{B}(\mathbf{r}) = \text{curl}\, \mathbf{A}(\mathbf{r})$ obtained from the above expression for $\mathbf{A}$ will be uniform *if and only if* the harmonic expansion terminates at $l = 1$ term itself. Thus the condition for obtaining uniform $\mathbf{B}$ inside the spherical region due to the current density in region I is that the double integral appearing in (19) should vanish for all $l \neq 1$. But since $\xi_1$ and $\xi_2$ are arbitrary the integral over $\theta'$ itself should vanish. This leads us to the desired condition on the current density $J(r, \theta)$ and the flux-front $f(\xi, \theta)$

$$\int_0^\pi d\theta' \sin\theta' \, Y_{l1}(\theta', 0) J(\xi, \theta') f_\xi(\xi, \theta')/f^{l-1} = 0 \tag{20}$$

for $l = 2, 3, 4, \ldots$. It is asserted that if $B_J$ is uniform in the spherical region completely lying in region II then it is uniform throughout region II. This follows from the fact that $\mathbf{B}$ satisfies a single differential equation in the whole of region II.

4.1.2 *The form of the current density for a given flux-front*: In view of the preceeding considerations and the results of Appendix-I we must determine the form of the macroscopic current density $J(r, \theta)$ flowing in region I and the equation of the flux-front, which determines the extent of region I, for obtaining the solution of Bean's model for samples with nonzero demagnetization factor. The criteria to be applied in their determination are as follows. The form of the current density flowing in region I should be such that it does not disturb the homogeneity of region II. In particular, as argued earlier, the magnetization $\mathbf{M}_{\text{II}}$ produced by currents in region I must be uniform everywhere in region II. Now let the one parameter family of flux-fronts be given by the equation $r=f(\xi,\theta)$. The region **I** is bounded internally by the surface $r=f(\xi_0,\theta)$ ($\xi_0$ being determined by the applied field) and the sample boundary which we assume to correspond to $\xi=1$. If $u=(x^2+y^2)^{1/2}$ is the usual cylindrical co-ordinate, then applying Ampere's circuit theorem we can write the expression for $\mathbf{M}_{\mathrm{II}}$

$$\mathbf{M}_{\mathrm{II}}=-\mu_0\mathbf{k}\int_{u_{\min}}^{u_{\max}} J(r,\theta)\,\mathrm{d}u \tag{21}$$

where $(u_{\min},z)$ and $(u_{\max},z)$ are points lying respectively on the inner and outer boundary of region I. The integral is to be evaluated for a fixed value of $z$. We can reexpress the integral on the right hand side of (21) as an integral over the parameter $\xi$

$$\mathbf{M}_{\mathrm{II}}=-\mu_0\mathbf{k}\int_{\xi_0}^{1}\mathrm{d}\xi\,J(r,\theta)\,ff_\xi/[f\sin\theta-f_\theta\cos\theta]. \tag{21a}$$

It follows that $\mathbf{M}_{\mathrm{II}}$ would be uniform in region II if the integrand is a function of $\xi$ alone, i.e.,

$$J(r,\theta)\,ff_\xi/[f\sin\theta-f_\theta\cos\theta]=g(\xi) \tag{22}$$

for some function $g(\xi)$ to be determined.

Thus for a given family of flux-fronts the current density in region I that leads to uniform **M** in region II is given by

$$J(r,\theta)=g(\xi)[f\sin\theta-f_\theta\cos\theta]/ff_\xi. \tag{23}$$

The function $g(\xi)$ is to be so chosen as to make the inequality $|J(r,\theta)|\leqslant J_C$ as weak as possible. This would correspond to maximizing the magnetization of the sample for a given applied field, and minimizing the flux change in the sample.

4.1.3 *Determination of the flux-front*: The current density $J(r,\theta)$ given by (23) for the given family of flux-fronts $r=f(\xi,\theta)$ would be adequate if they are consistent with (7). This would be the case if the functions $J$ and $f$ satisfy the necessary and sufficient conditions (20) derived in the last section, namely;

$$\int_0^\pi \mathrm{d}\theta\sin\theta\,Y_{l1}(\theta,0)J(\xi,\theta)f_\xi(\xi,\theta)/f^{l-1}=0$$

for $l=2,3,4,\ldots$.

Using the form (23) for $J$, we can rewrite these conditions in the form

$$\int_0^\pi \mathrm{d}\theta\sin\theta\,Y_{l1}(\theta,0)[f\sin\theta-f_\theta\cos\theta]/f^l=0$$

performing integration by parts and using some well-known identities involving Legendre polynomials we can cast the above conditions in the simpler form

$$\int_0^\pi \mathrm{d}\theta\sin\theta\,P_{l+1}(\cos\theta)/f^{l-1}(\xi,\theta)=0 \tag{24}$$

for $l=2,3,4,\ldots$

We now assume that the sample surface besides being a surface of revolution about the $z$-axis is also symmetric with respect to the $x$-$y$ plane. Since the flux-front is generated by the external magnetic field which is an axial vector remaining invariant under the inversion of the coordinates, the flux-front must also possess the inversion symmetry. In other words we have $f(\xi, \pi - \theta) = f(\xi, \theta)$. Then the integral in (24) vanishes for $l = 2, 4, 6, \ldots$, by the symmetry of the corresponding $P_{l+1}$. For odd values of $l$, i.e. for $l = 2k + 1$, $k = 1, 2, 3, \ldots$, eq. (24) assumes the form

$$\int_0^\pi d\theta \sin\theta P_{2k+2}(\cos\theta)/f^{2k}(\xi, \theta) = 0. \tag{25}$$

It is easy to see that

$$1/f^2 = a_1 + b_1 \cos^2\theta \tag{26}$$

with $a_1$ and $b_1$ being arbitrary functions of the parameter $\xi$ satisfies the conditions (25). It is also the *only* polynomial solution to $1/f^2$ satisfying (25). In order that $r = f$ represent a closed surface $a_1 \neq 0$, and $a_1 > |b_1|$. A nonpolynomial solution for $1/f^2$ becomes relevant for samples of arbitrary shape and will not be attempted here. The special choice of the solution (26), nevertheless, includes samples in the shape of a spheroid (i.e., an ellipsoid of revolution) and in particular also spherical samples.

4.1.4 *Flux-front and current density in a spherical sample*: Using (26), with obvious redefinitions of $a_1$ and $b_1$, we can write the flux-front in the form

$$r = f(\xi, \theta) = a a_1(\xi)/[1 + b_1(\xi)\cos^2\theta]^{1/2} \tag{27}$$

where $a$ represents the radius of the spherical sample. The outer boundary is assumed to correspond to $\xi = 1$, so that $a_1(1) = 1$ and $b_1(1) = 0$. $\xi = 0$ corresponds to full flux penetration so that $a_1(0) = 0$. Further, $a_1(\xi)$ should decrease monotonically to zero as $\xi$ decreases from 1 to 0. Hence, without loss of generality, we may choose $a_1(\xi) = \xi$ and write

$$r = f(\xi, \theta) = a\xi/[1 + b_1(\xi)\cos^2\theta]^{1/2} \tag{28}$$

If $b_1(\xi) = 0$, (28) reduces to

$$r = f(\xi, \theta) = a\xi. \tag{29}$$

This corresponds to a flux-front which is spherical. The expression (23) for the corresponding current density reduces to

$$J = J_C \sin\theta. \tag{30}$$

A solution with $b_1 \neq 0$ is possible. The function $b_1(\xi)$ is determined by demanding that the resulting current density (23) satisfies the inequality $|J(r, \theta)| < J_C$, in its weakest possible form, and the boundary condition that $b_1(1) = 0$. This determines $b_1(\xi) = \xi - 1$ as the solution yielding the largest permissible magnitude of the magnetization, and we get the flux-front in the explicit form

$$= f(\xi, \theta) = a\xi/[1 + (\xi - 1)\cos^2\theta]^{1/2}. \tag{31}$$

The corresponding current density is given by

$$J = \frac{[1 + (\xi - 1)\cos^2\theta]^{1/2}}{[1 + \{(\xi/2) - 1\}\cos^2\theta]} J_C \sin\theta. \tag{32}$$

It is easy to see that for fixed $\theta$ and $\xi$ the current density (32) is larger than that given by (30). Hence the flux-front (31) with the current density (32) is expected to provide more effective shielding of the external field as compared with that provided by the solutions (29) and (30) for the flux-front and the current density respectively.

4.1.5 *Flux-front and current density in a spheroidal sample*: Let the outer boundary of the sample be described by the spheroidal surface, represented in the form

$$r = a/[1 + \{(a^2 - b^2)/b^2\}\cos^2\theta]^{1/2} \tag{33}$$

so that its semi-axis along the axis of $z$ is $b$ and the semi-axis normal to the axis of $z$ is $a$. Comparing this with the flux-front (27), we see as before $a_1(1) = 1$, and $a_1(0) = 0$, and $a_1(\xi)$ should decrease monotonically to zero as $\xi$ decreases from 1 to 0. Therefore we choose $a_1(\xi) = \xi$, leading to (28) for the flux-front. A comparison of (33) and (28) gives us the boundary value $b_1(1) = (a^2 - b^2)/b^2$. We then have two cases, (i) $b_1(\xi) = (a^2 - b^2)/b^2$, i.e., the flux-front retains the ellipticity of the sample surface. The flux-front is given explicitly

$$r = f(\xi, \theta) = a\xi/[1 + \{(a^2 - b^2)/b^2\}\cos^2\theta]^{1/2}. \tag{34}$$

The corresponding current density is given by

$$J = J_C \sin\theta/[1 + \{(a^2 - b^2)/b^2\}\cos^2\theta]^{1/2}; \tag{35}$$

(ii) $b_1(\xi) \neq$ constant, implying that the ellipticity of the flux-front changes continuously from its initial value. Following similar arguments as employed for arriving at (31) and noting the boundary value $b(1)$ in this case, we get $b_1(\xi) = (a^2\xi - b^2)/b^2$. We thus get explicit expressions for the flux-front

$$r = f(\xi, \theta) = a\xi/[1 + \{(a^2\xi - b^2)/b^2\}\cos^2\theta]^{1/2} \tag{36}$$

and the current density ($c/f$ eq. (23))

$$J = J_C \sin\theta \frac{[1 + \{(a^2\xi - b^2)/b^2\}\cos^2\theta]^{1/2}}{[1 + \{((a^2\xi/2) - b^2)/b^2\}\cos^2\theta]}. \tag{37}$$

It is again easy to see that the current density (37) is always larger than that given by (35) for fixed $\xi$ and $\theta$. We choose the flux-front (36) and the current density (37) for subsequent calculation of the magnetic induction **B** and the virgin and hysteretic magnetization of the specimen.

### 4.2 *Infinite cylindrical sample*

We shall now consider infinite cylindrical samples subjected to transverse magnetic field. Let the axis of the cylinder be along the $z$-axis and the applied external field along the $y$-axis (note the change of axis) of a cartesian frame of reference. The transverse cross section of the cylinder shall be assumed to be elliptical in general, with its semi-axes coinciding with the $x$- and $y$-axes. The shielding current $J(x, y)$ set up in the presence of an external field will be along the $z$-direction. The extent of the current carrying region (region I) is governed by the flux-front, cylindrical with its axis coincident with the sample axis. The boundary of the transverse cross section will belong to a one-parameter family of closed curves which will be chosen in the form $u=f(\xi, \theta)$, with $u=(x^2+y^2)^{1/2}$ and $\theta$ being the polar angle measured from the $x$-axis in the counter-clockwise sense. For convenience we shall refer to members of this family of curves as flux-fronts. As noted earlier, a solution of Bean's model requires the determination of the current density $J(u, \theta)$ and the function $f(\xi, \theta)$. This is done in two stages. First we determine the form of $J(u, \theta)$ assuming that $f(\xi, \theta)$ is given and subsequently derive conditions on $f(\xi, \theta)$ so that the derived current density produces a uniform field in the current-free region of the sample (region II). Invoking considerations similar to those leading to (23) for the current density in finite samples we get in the present case

$$J(r, \theta)=g(\xi)[f \cos \theta+f_\theta \sin \theta]/ff_\xi. \tag{38}$$

As remarked earlier the function $g(\xi)$ is to be so chosen as to make the inequality $|J(r, \theta)| \leqslant J_C$ as weak as possible.

To derive the conditions to be satisfied by the flux-front $f(\xi, \theta)$ we begin with the expression for the magnetic induction $\mathbf{B}_J(x, y)$ due to a current density $J(x, y)$ along the fixed direction $\mathbf{k}$ normal to the $x-y$ plane

$$\mathbf{B}_J=\frac{\mu_0}{2\pi}\int\frac{J(x', y')[-(y-y')\mathbf{i}+(x-x')\mathbf{j}]\,dx'\,dy'}{(x-x')^2+(y-y')^2} \tag{39}$$

where $\mathbf{i}$ and $\mathbf{j}$ are unit vectors along the $x$- and $y$-axes of coordinates respectively and the other symbols have their usual meaning. Instead of working with the vector $\mathbf{B}$ it is advantageous to work with a complex number $B(x, y)$ (Brechna 1973)

$$B=\frac{\mu_0}{2\pi}\int\frac{J(x', y')(z-z')\,dx'\,dy'}{|z-z'|^2} \tag{40}$$

with $z=x+iy$ and $z'=x'+iy'$. The components of the induction $\mathbf{B}$ are obtained as $B_x=-\operatorname{Im}(B)$ and $B_y=\operatorname{Re}(B)$. Since the current density is real we may write $B^*$, the complex conjugate of $B$ as

$$B^*=\frac{\mu_0}{2\pi}\int\frac{J(x', y')\,dx'\,dy'}{(z-z')}. \tag{41}$$

If $z$ lies in the circular region (centred at the origin) in the current-free region then $|z|<|z'|$ and we may expand $1/(z-z')$ appearing under the integral as power series in $(z/z')$ and write

$$B^*=-(\mu_0/2\pi)\sum_{n=0}^{\infty} z^n \int J(x', y')\,dx'\,dy'/z'^{n+1}. \tag{42}$$

Writing $z'=u'\exp(i\theta')$ we get

$$B^*=-(\mu_0/2\pi)\sum_{n=0}^{\infty} z^n \int J(u', \theta')\exp[-i(n+1)\theta']\,du'\,d\theta'/u'^n. \tag{43}$$

Changing the variable of integration from $u'$ to $\xi$ by $u' = f(\xi, \theta')$, and noting that $\xi$ will run from $\xi_0$ (defining the inner boundary of region I) to $\xi = 1$ (defining the sample boundary), we may write

$$B^* = -(\mu_0/2\pi) \sum_{n=0}^{\infty} z^n \int_{\xi_0}^{1} \mathrm{d}\xi \int_{-\pi}^{\pi} \mathrm{d}\theta' J(\xi, \theta') \exp[-i(n+1)\theta'] f_\xi / f^n. \quad (44)$$

The magnetic induction resulting from $B^*$ of (44) will be uniform in the circular region under consideration if all the terms on its right hand side vanish except the term with $n = 0$. Since $\xi_0$ is a variable point we must have

$$\int_{-\pi}^{\pi} J(\xi, \theta') \exp[-i(n+1)\theta'] f_\xi \mathrm{d}\theta' / f^n = 0 \quad (45)$$

for $n = 1, 2, 3, \ldots$. The conditions (45) are the analogues of those embodied in (20) for a finite sample. As asserted in§4.1.1 we again note that as **B** satisfies a single differential equation in the whole of region II, a uniform **B** in a finite circular volume in region II ensures that **B** is uniform throughout region II. Using (38) for the current density in (45) we get after some simplification

$$\int_{-\pi}^{\pi} \exp[-i(n+2)\theta'] \mathrm{d}\theta' / f^n = 0 \quad (46)$$

for $n = 1, 2, 3, \ldots$. For the elliptical cylinder under consideration the flux-front must possess the symmetry $f(\xi, \theta) = f(\xi, -\theta)$. The exponential function under the integral in (46) would contribute only the cosine term. The integrand now being an even function of $\theta'$ we may reduce the range of integration to $(0, \pi)$. The integral (46) would be found to vanish for odd values of $n$ due to the fact that cosine of an odd multiple of $\theta$ is an odd function about $\theta = \pi/2$. Hence the conditions to be satisfied by the function $f(\xi, \theta)$ reduce to

$$\int_{0}^{\pi} \cos[2(n+1)\theta'] \mathrm{d}\theta' / f^{2n} = 0 \quad (47)$$

for $n = 1, 2, 3, \ldots$ etc. A solution for $1/f^2$ in a polynomial form can be found in

$$1/f^2 = a_1 + b_1 \sin^2\theta \quad (48)$$

with $a_1$ and $b_1$ being some functions of the parameter $\xi$. Noting the resemblance of (48) and (26), and proceeding as in the case of finite samples we get two kinds of flux-fronts

$$u = f(\xi, \theta) = a\xi / [1 + \{(a^2 - b^2)/b^2\} \sin^2\theta]^{1/2} \quad (49)$$

with constant ellipticity. The corresponding current density obtained by using (38)

$$J = J_C \cos\theta / [1 + \{(a^2 - b^2)/b^2\} \sin^2\theta]^{1/2} \quad (50)$$

and the second solution, with a variable ellipticity, is

$$r = f(\xi, \theta) = a\xi / [1 + \{(a^2\xi - b^2)/b^2\} \sin^2\theta]^{1/2} \quad (51)$$

with the current density given by

$$J = J_C \cos\theta \frac{[1 + \{(a^2\xi - b^2)/b^2\}\sin^2\theta]^{1/2}}{[1 + \{((a^2\xi/2) - b^2)/b^2\}\sin^2\theta]}. \tag{52}$$

We again note that the current density (52) is always larger than that given by (50) for fixed $\xi$ and $\theta$. We will choose the flux-front (51) and the current density (52) in our subsequent calculations involving cylindrical samples.

An explicit calculation of $B^*$ in region II can be done analytically by substituting (51) and (52) in (41), and we do find that $B^*$ is indeed uniform in the entire region II.

## 5. Magnetic field and magnetization due to induced currents

### 5.1 *Cylindrical samples*

5.1.1 *Magnetic field*: We now return to the sample in the shape of an infinite cylinder with elliptical transverse cross-section referred to in §4.2. The region I, bounded by two members of the family given by (51) corresponding to $\xi = \xi_0 < 1$ and the sample boundary corresponding to $\xi = 1$ carries the current density given by (52). We will use (44) to compute the magnetic induction in region II (denoted by $B_{in}$) due to the current distribution in region I. We note that only the term $n = 0$ contributes and $B_{in}$ is along the $y$-direction. We have

$$B_{in} = -(\mu_0/2\pi)\int_{\xi_0}^{1} d\xi \int_{-\pi}^{\pi} d\theta' J(\xi,\theta')\exp[-i\theta'] f_\xi. \tag{53}$$

Substituting the explicit forms for $J$ and $f$ in (53) leads to

$$B_{in} = -\frac{\mu_0 J_C a}{2\pi}\int_{\xi_0}^{1} d\xi \int_{-\pi}^{\pi} d\theta' \frac{\cos\theta' \exp(-i\theta']}{[1 + \{(a^2\xi - b^2)/b^2\}\sin^2\theta]}. \tag{54}$$

The above integral is elementary and we finally have

$$B_{in} = -H^*(2b/a)[1 - \sqrt{\xi_0} - (b/a)\ln\{(1 + a/b)/(1 + a\sqrt{\xi_0}/b)\}] \tag{55}$$

with $H^* = \mu_0 J_C a$. The net magnetic induction in region II is a superposition of $B_{in}$ and $H_A$. To satisfy (7) we must set

$$B_{in} + H_A = 0. \tag{56}$$

Equations (55) and (56) determine $\xi_0$ in terms of $H_A$. We have

$$H_A = H^*(2b/a)[1 - \sqrt{\xi_0} - (b/a)\ln\{(1 + a/b)/(1 + a\sqrt{\xi_0}/b)\}]. \tag{57}$$

It follows that $\xi_0 = 1$ for $H_A = 0$, and $\xi_0 = 0$ corresponds to a field $H_I$ at which full flux penetration occurs.

$$H_I = H^*(2b/a)[1 - (b/a)\ln(1 + a/b)]. \tag{58}$$

A simple calculation shows that for the case where the flux-front retains the ellipticity of the outer surface of the sample would have

$$H_I = H^*[b/(a+b)] = H^*(1-N) \tag{59}$$

where $N = a/(a+b)$ is the demagnetization factor for an elliptical cylinder with field along the major axis ($b$) of the ellipse. As expected $H_I$ given by (58) is always larger than that given by (59).

5.1.2 *Virgin magnetization of a ZFC sample*: To obtain the virgin magnetization we need to evaluate magnetic dipole moment (**M**) due to currents in the sample. We get for the current density $J$ given by (38) the dipole moment **M** as

$$\mathbf{M} = -\mu_0 \mathbf{j} \int g(\xi)\,\mathrm{d}\xi \tag{60}$$

where **j** is the unit vector along positive $y$-axis. Noting that the current density (52) is obtained from (38) and (51) with $g(\xi) = J_C a$ we get

$$M = -\mu_0 J_C a(1-\xi) \quad \xi > \xi_0$$
$$M = -\mu_0 J_C a(1-\xi_0), \quad \xi < \xi_0 \tag{61}$$

The virgin magnetization $m_v$ is obtained as the average dipole moment per unit volume of the sample. Performing the necessary integral of $M$ over the sample volume we get

$$m_v = -(2/5)H^*(1-\xi_0^{5/2}) \tag{62}$$

with $\xi_0$ given in terms of $H_A$ by inverting (57). In the limit of low field limit ($H_A \to 0$) we get

$$m_v = -H_A(a+b)/b \equiv -H_A/(1-N).$$

The virgin magnetization $m_v$ attains its saturation value $m_s = -2H^*/5$ at $H_A = H_I$ given by (58) corresponding to $\xi_0 = 0$. For higher fields the magnetization remains at its saturation value so we have

$$m_v = m_s = -2H^*/5, \quad H_A > H_I. \tag{63}$$

As expected this is larger in magnitude than $m_s = H^*/3$, obtained for the case where the flux-front retains the ellipticity of the sample surface.

5.1.3 *Magnetization during the reverse cycle*: Having reached a sufficiently large field $H_{\max} > 2H_I$ let the field be reduced to $H = H_{\max} - h$. The reduction in the field ($h$) changes the current density in a shell say $(\xi_0', 1)$ from $J$ to $-J$, leaving the current density in the region $0 < \xi < \xi_0'$ unchanged. The net result can be looked upon as a superposition of current density $J$ flowing in the entire sample and a current density $-2J$ in the region $\xi_0' < \xi < 1$ which shields the field $-h$. Thus $\xi_0'$ is determined (57) by the appropriate substitutions.

$$h = H^*(4b/a)[1 - \sqrt{\xi_0'} - (b/a)\ln\{(1+a/b)/(1+a\sqrt{\xi_0'}/b)\}]. \tag{64}$$

Thus the magnetization $m\downarrow$ during the field decreasing case will be obtained as a superposition of two contributions

$$m\downarrow(H_{\max}-h)=m_s+m_v(-h)$$
$$=-2H^*/5+(4H^*/5)(1-\xi'^{5/2}). \qquad (65)$$

This would reach the saturation value $2H^*/5$ at $h=2H_I$ and $m\downarrow$ would remain at the saturation value for further increase in $h$. Thus

$$m\downarrow(H_{\max}-h)=2H^*/5, \quad 2H_I<h<2H_{\max}, \qquad (66)$$

Magnetization during the field increasing cycle form $H_A=-H_{\max}$ is obtained by using the fact

$$m\uparrow(-H)=-m\downarrow(H). \qquad (67)$$

5.2 *Spheroidal sample*

In this case the region I is bounded by two members of the family of surfaces (36) corresponding to $\xi=\xi_0$ representing the flux-front and the sample boundary given by $\xi=1$. This region is assumed to carry induced currents with density given by (37).

5.2.1 *Magnetic induction*: We shall compute the magnetic induction $B_{in}$ in region II due to currents flowing in region I. The vector potential $\mathbf{A}_{in}$ in the region II has nonvanishing component only along $\mathbf{e}_\varphi$ and is given by

$$A_{in}=2\pi\mu_0(r/3)\,Y_{11}(\theta,0)\int_{\xi_0}^{1}\mathrm{d}\xi\int_0^{\pi}\mathrm{d}\theta'\sin\theta' J(\xi,\theta')f_\xi(\xi,\theta')\,Y_{11}(\theta',0). \qquad (68)$$

Using the explicit expressions for the current density $J(\xi,\theta')$ and $f(\xi,\theta')$ we get after some simplifications

$$A_{in}=-\mu_0 J_C ar\sin\theta\int_{\xi}^{1}\mathrm{d}\xi\int_0^{\pi}\mathrm{d}\theta'\frac{\sin^3\theta'}{[1+\{(a^2\xi-b^2)/b^2\}\cos^2\theta']}$$
$$=-(H^*/4)r\sin\theta\,\tilde{A}(a,b,\xi_0) \qquad (69)$$

where $H^*=\mu_0 J_C a$, and $\tilde{A}(a,b,\xi_0)$ stands for the double integral in the first line of (69). The magnetic field $B_{in}$ resulting from $A_{in}$ is along the $z$-direction and is found to be

$$B_{in}=-(H^*/2)\tilde{A}(a,b,\xi_0). \qquad (70)$$

Explicit calculation shows that $\tilde{A}(a,b,\xi_0)$ can be expressed in terms of $k=a/b$. We have for $k<1$,

$$\tilde{A}(a,b,\xi_0)=\frac{2}{k^2}\left[\frac{2(2-k^2\xi_0)}{(1-k^2\xi_0)^{1/2}}\ln\{1+(1-k^2\xi_0)^{1/2}\}\right.$$
$$-\frac{2(2-k^2)}{(1-k^2)^{1/2}}\ln\{1+(1-k^2)^{1/2}\}$$
$$-2\left\{\frac{(2-k^2\xi_0)}{(1-k^2\xi_0)^{1/2}}-\frac{(2-k^2)}{(1-k^2)^{1/2}}\right\}\ln k$$
$$\left.+\left\{2-\frac{(2-k^2\xi_0)}{(1-k^2\xi_0)^{1/2}}\right\}\ln\xi_0\right]. \qquad (71)$$

As argued earlier the total $B$ in region II is a superposition of $H_A$ and $B_{in}$ and setting this equal to zero we get a relation determining $\xi_0$ in terms of $H_A$.

$$H_A = (H^*/2)\tilde{A}(a, b, \xi_0). \tag{72}$$

Full penetration of the flux occurs at an applied field $H_A = H_I$ corresponding to $\xi_0 = 0$ in (72).

$$H_I = \left[ 4\ln 2 - \frac{2(2-k^2)}{(1-k^2)^{1/2}} \ln\{1 + (1-k^2)^{1/2}\} - 2\left\{2 - \frac{(2-k^2)}{(1-k^2)^{1/2}}\right\}\ln k \right] \frac{H^*}{k^2}. \tag{73}$$

As can be worked out, the corresponding value for the case where the flux-front retains the ellipticity of the outer surface is

$$H_I = H^*(1-N) \tag{73a}$$

where $N$, the demagnetization factor for the spheroid for field along the axis of revolution, is given by

$$N = [k^2/(1-k^2)][(1-k^2)^{-1/2}\ln\{(1+(1-k^2)^{1/2})/k\} - 1].$$

As expected the value of $H_I$ given by (73) is larger than that given by (73a).

5.2.2 *The virgin magnetization for a ZFC sample*: The magnetic dipole moment due to the current distribution in region I is given by

$$M = -H^*(1-\xi), \quad \xi > \xi_0$$
$$M = -H^*(1-\xi_0), \quad \xi < \xi_0 \tag{74}$$

The virgin magnetization $m_v$ is obtained by computing the average dipole moment of the sample.

$$m_v = -(2/7)H^*(1-\xi_0^{7/2}). \tag{75}$$

This together with (72) determines $m_v$ as a function of the applied field. In the low field limit ($H_A \to 0$) we get, for all values of $k$,

$$m_v = -H_A/(1-N)$$

where $N$ is the demagnetization factor for the sample. In the case $k > 1$, $N$ is given by the following expression

$$N = [k^2/(k^2-1)][1-(k^2-1)^{-1/2}\sin^{-1}\{k^2-1)^{1/2})/k\}].$$

The virgin magnetization attains a saturation value of $m_s = -2H^*/7$ at $H_A = H_I$ and for $H_A > H_I$ we have

$$m_v = m_s = -2H^*/7, \quad H_A > H_I \tag{76}$$

5.2.3 *Magnetization for the field decreasing case*: Having reached a rather large field $H_A = H_{\max} > 2H_I$, let the field be decreased to $H_A = H_{\max} - h$. This alters the current distribution in a shell corresponding to $\xi'_0 < \xi < 1$, near the sample surface. Following arguments presented earlier $\xi'_0$ is determined in terms of $h$ by the equation

$$h = H^* \tilde{A}(a, b, \xi'_0). \tag{77}$$

The magnetization $m{\downarrow}(H_{\max} - h)$ is obtained by superposition of two contributions

$$\begin{aligned} m{\downarrow}(H_{\max} - h) &= m_s + m_v(-h) \\ &= (2H^*/7)(1 - 2\xi_0'^{7/2}). \end{aligned} \tag{78}$$

This reaches a saturation value of $2H^*/7$ at $h = 2H_I$. Further, $m{\downarrow}$ retains this saturation value for further decrease in the field (i.e., increase in $h$).

$$m{\downarrow}(H_{\max} - h) = 2H^*/7, \quad 2H_I < h < 2H_{\max} \tag{79}$$

When the applied field is increased from $H_A = -H_{\max}$, the magnetization $m{\uparrow}$ during the field increasing cycle is obtained by noting the fact

$$m{\uparrow}(-H) = -m{\downarrow}(H).$$

## 6. Summary and conclusions

In this paper we have considered the isothermal magnetization of a hard superconductor sample with a non-zero demagnetization factor. We have ignored $H_{C1}$ as small and considered only the magnetization caused by the macroscopic shielding currents as the external field is varied isothermally. Assuming only that $\sigma = \infty$ for $J \leqslant J_C$, we have first shown that the only alternative to London's solution for the shielding current distribution is Bean's model. Since London's solutions result in a reversible hysteresis curve, Bean's model provides the viable macroscopic model for a hard type II superconductor.

We have then argued that the shielding currents flowing in the surface layer of a sample with $N \neq 0$ must produce fields both inside and outside the specimen, unlike a sample with $N = 0$, where fields are produced only inside the specimen. The fields outside change sign with **r** and we have used this fact to point out that, unlike the case for $N = 0$, *shielding currents must flow in opposing directions even during the virgin magnetization curve.* As an explicit illustration of this feature, we have rigorously shown that no solution exists, for unidirectional shielding currents of magnitude $J_C$, for a spherical sample and also for a circular cylinder with axis normal to the applied field.

We have then introduced a locally averaged shielding current density. The averaged shielding currents $J_s(r, \theta)$ will flow in region I contained within the flux-front $f(\xi, \theta)$ and the outer surface of the sample. $J_s$ and $f$ are to be determined to satisfy (11) and to minimize the flux in the sample. In §4, we convert (11) into an equation in $1/f$, to which we find polynomial solutions. Amongst these polynomial solutions we are able to find the one that makes the inequality $J_s(r, \theta) \leqslant J_C$ weakest possible, thus maximizing the magnitude of $m_v$ and minimizing the flux. For sample shapes where it may not be possible to find $J_s(r,\theta)$ rigorously, the algorithm would involve a variational method in which $f(\xi,\theta)$ is varied to minimize the flux through the sample.

We have obtained the virgin magnetization curve for a sample in the shape of a spheroid and of a cylinder with elliptical cross-section. We find that at low fields

$$\chi = -1/(1-N)$$

a situation similar to that for soft ($J_C = 0$) type I and type II superconductors. As the field is increased the shape of the region that totally excludes the field changes such that its demagnetization factor decreases. The field at which the magnetization $m$ of the sample reaches its saturation value is thus not $H^*(1-N)$ (as a complete similarity with a soft type II superconductor would require), but is a higher field $H_I$ (eqs (58) and (73)). The saturation magnetization $m_s$ is linearly proportional to $H^*$, where the proportionality factor depends on the sample-shape-family. Just like the peak magnetization of a type I or soft type II superconductor, $m_s$ does not depend on $N$. Some interesting experimental consequences of this case are:

1. For an elliptical cylinder with axis normal to the field, one can measure the saturation magnetization with $H$ along minor axis or along major axis. The latter should be just $b/a$ times the former.
2. The saturation magnetization $m_s$ should be the same for spheroids having the same dimension in the equitorial plane, irrespective of the dimension along the field. Similarly, transverse elliptical cylinders of varying ellipticity but with the same dimension perpendicular to the field will have the same $m_s$.

We have also calculated the hysteresis curves for these sample shapes. This allows us to calculate ac losses in a superconductor wire perpendicular to a field. Earlier results, obtained with approximate solutions for the magnetization (Wilson 1983) would have to be redone. This will be taken up in a subsequent paper.

Our results for the transverse cylinder also provide designs for a dipole magnet that may provide advantages over the currently existing "sin $\theta$" and "intersecting ellipses" design (Brechna 1973; Wilson 1983).

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## Appendix I Inadequacy of shielding current density of constant magnitude

### 1.1 *Spherical sample*

We shall now examine whether the necessary and sufficient conditions represented by (20), viz.,

$$\int_0^{\pi} d\theta \sin\theta Y_{l1}(\theta,0) J(\xi,\theta) f_\xi(\xi,\theta)/f^{l-1} = 0 \tag{A.1}$$

are satisfied when $J(r,\theta) = J_C$ and the flux-front is given by (9)

$$r = f(\xi,\theta) = R[-\xi\sin\theta + (1-\xi^2\cos^2\theta)^{1/2}]. \tag{A.2}$$

Noting that

$$f_\xi = -\sin\theta - \xi\cos\theta/(1-\xi^2\cos^2\theta)^{1/2}$$

we see that both $f$ and $f_\xi$ are even functions of $\theta$, while $Y_{l1}(\theta, 0)$, when $l$ is an even integer, is an odd function of $\theta$ about $\theta = \pi/2$. Hence the conditions (A.1) are indeed satisfied when $l$ is an even integer. To examine the situation when $l$ is an odd integer we consider the special case $l = 3$. In this case we set $l = 3$ in (A.1) and use the appropriate substitutions and denoting by $I$ the relevant integral, we have

$$I = -J_C \int_0^\pi d\theta \sin\theta\, Y_{31}(\theta, 0) \frac{\sin\theta + \xi\cos\theta/(1-\xi^2\cos^2\theta)^{1/2}}{[(1-\xi^2\cos^2\theta)^{1/2} - \xi\sin\theta]^2}. \tag{A.3}$$

It is easy to see that $I = 0$ for $\xi = 0$, however, a straightforward calculation shows that the term linear in $\xi$ in a power series expansion of $I$ survives implying that $I \neq 0$, in general. Hence we conclude that constant current density flowing in region I does not give rise to $B_J$ that is uniform in region II and therefore is not adequate for shielding the region II from an external uniform field.

## 1.2 *Infinite cylindrical sample*

We shall now consider an infinite circular cylindrical sample of radius $a$ with its axis along $z$-axis subjected to an external uniform field along $y$-axis. We assume that the induced current density has a constant absolute magnitude $J_C$ and is given by

$$J(u,\theta) = J_C, \qquad x > 0$$
$$J(u,\theta) = -J_C, \quad x < 0. \tag{A.4}$$

The one parameter family of surfaces representing the flux-fronts would consist of cylindrical surfaces appropriate for the current density (A.4). These are found to be

$$u = (x^2+y^2)^{1/2} = f(\xi,\theta) = a[(1-\xi^2\sin^2\theta)^{1/2} - \xi|\cos\theta|]. \tag{A.5}$$

The pair of functions given by (A.4) and (A.5) would be adequate if they satisfy the condition (45) of §4.2, viz.,

$$\int_{-\pi}^{\pi} J(\xi,\theta)\exp[-i(n+1)\theta] f_\xi\, d\theta/f^n = 0 \tag{A.6}$$

for $n = 1, 2, 3, \ldots$ etc. Using the explicit expressions for $J$ and $f$ given by (A.4) and (A.5) the integral in (A.6) (denoting it by $I_n$) reduces to

$$I_n = J_C \int_{-\pi}^{\pi} d\theta [\exp[i(n+1)\theta] + (-1)^n \exp[-i(n+1)\theta] f_\xi/f^n = 0. \tag{A.7}$$

It follows that $I_n = 0$ for $n = 1, 3, 5, \ldots$ etc. but $I_n$ does not vanish for even values of $n$. In particular, a straightforward calculation shows that $I_2$ is given by the expression

$$I_2 = 2J_C(\partial/\partial\xi)\{[(1-\xi^2)^{1/2} - (\sin^{-1}\xi)/\xi]/\xi\}. \tag{A.8}$$

Since $I_2 \neq 0$, we conclude that the pair of functions (A.4) and (A.5) for the current density and the flux-front are not acceptable.

We must mention that we have also explicitly calculated $B_J(r)$ for the case specified by (A.4) and (A.5) and found that it is not uniform.

The above analysis shows that in contrast with the situation for samples with zero demagnetization factor, the induced shielding current density shall not have a constant absolute magnitude in the case of samples with non-zero demagnetization factor.

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