Comparing Classical Generating Methods with an Evolutionary Multi-Objective Optimization Method

Pradyumn Kumar Shukla, Kalyanmoy Deb, and Santosh Tiwari

Kanpur Genetic Algorithms Laboratory (KanGAL)
Indian Institute of Technology Kanpur
Kanpur, PIN 208016, India
{shukla,deb,tiwaris}@iitk.ac.in

Abstract. For the past decade, many evolutionary multi-objective optimization (EMO) methodologies have been developed and applied to find multiple Pareto-optimal solutions in a single simulation run. In this paper, we discuss three different classical generating methods, some of which were suggested even before the inception of EMO methodologies. These methods specialize in finding multiple Pareto-optimal solutions in a single simulation run. On visual comparisons of the efficient frontiers obtained for a number of two and three-objective test problems, these algorithms are evaluated with an EMO methodology. The results bring out interesting insights about the strengths and weaknesses of these approaches. Further investigations of such classical generating methodologies and their evaluation should enable researchers to design a hybrid multi-objective optimization algorithm which may be better than each individual method.

1 Introduction

Multi-objective optimization has been a rapidly growing area in modern optimization. There exist a plethora of methods and algorithms for solving multi-objective optimization problems. The methods can be divided in two categories: (i) classical methods which use direct or gradient-based methods following some mathematical principles and (ii) non-traditional methods which follow some natural or physical principles. Of them, the evolutionary multi-objective optimization (EMO) has been getting growing attention over the past decade. The classification is also appropriate from two other perspectives. The classical approaches usually use deterministic transition rules, whereas non-traditional approaches use stochastic rules. They are also different from each other from another vital consideration. Classical methods mostly attempt to scalarize multiple objectives and perform repeated applications to find a set of Pareto-optimal solutions. On the other hand, EMO methods attempt to find multiple Pareto-optimal solutions in a single simulation run.

However, there exist a few classical generating methods (stochastic and deterministic) which attempt to find multiple Pareto-optimal solutions in a single simulation run, very much similar to the way EMO methods work. In this paper, we present three such algorithms and provide simulation results on a number of two and three-objective optimization problems. We also compare their performance with an EMO methodology and unveil the problem classes where the classical generating methods are better and the problem classes where the EMO methods have their niche. The study reveals important insights about the working of the algorithms, which can be combined together in a hybrid manner to develop an algorithm even better than individual algorithms.

2 Classical Generating Methods

Although most classical generating multi-objective optimization methods use an iterative scalarization scheme of standard procedures such as weighted-sum or epsilon-constraint methods [8], we have found at least three generating methods which attempt to find multiple Pareto-optimal solutions in a single simulation run. In the following subsections, we describe these methods.

2.1 Schäffler's Stochastic Method (SSM)

A stochastic method for the solution of unconstrained multi-objective optimization problems was proposed by Schäffler et. al. [9] in 2002. The method is based on the solution of a set of stochastic differential equations. This method requires the objective functions to be twice continuously-differentiable. It may be used for the computation of all or a large number of Pareto-optimal solutions. In each iteration, a trace of non-dominated points is constructed by calculating at each point $\mathbf{x}$, a direction $(-q(\mathbf{x}))$ in the decision space which is a direction of descent for *all* objective functions. The direction of descent is obtained by solving a quadratic subproblem. The following initial value problem (IVP) for a multi-objective optimization problem is then set up:

$$\dot{\mathbf{x}} = -q(\mathbf{x}(t)), \quad \mathbf{x}(0) = \mathbf{x}_0,$$

where x_0 is a starting point. The numerical solution of the above IVP gives a single point where the first-order weak Pareto-optimality conditions are fulfilled. After such a solution is obtained, a set of non-dominated solutions is obtained by perturbing it using a Brownian motion concept. The following stochastic differential equation is employed for this purpose:

$$d\mathbf{X}_t = -q(\mathbf{X}_t)d(t) + \varepsilon dB_t, \quad \mathbf{X}_0 = \mathbf{x}_0, \tag{1}$$

where $\varepsilon > 0$ and B_t is a n-dimensional Brownian motion having the following properties:

1. The expected value is zero,
2. The increments B_0, $(B_{t_1} - B_{t_0})$, $(B_{t_2} - B_{t_1})$ for every $t_0(= 0) < t_1 < t_2 < \ldots$ are stochastically independent, and

3. For every $s < t$, the increment $(B_s - B_t)$ is normally distributed with mean equal to zero and a variance equal to $(s-t)I_n$, where I_n is a n-dimensional identity matrix.

Thus, starting from an initial solution, a number of solutions converging to the efficient frontier are expected to be generated by this procedure. The $-q(\mathbf{X}_t)d(t)$ term in Equation 1 is the deterministic descent part, while the Brownian motion is the local random search term. In all simulations here, to solve the above equation numerically, we employ the Euler's method. The approach needs two parameters to be set properly: (i) the parameter ε which controls the amount of local search and (ii) the step size σ used in the Euler's approach which controls the accuracy of the integration procedure. At the end of a pre-specified number of iterations, a non-domination check of the obtained solutions is performed and the resulting solutions are declared as the obtained Pareto-optimal solutions. For more information on this algorithm, interested readers may refer to the original study [9].

2.2 Timmel's Population Based Method (TPM)

As early as in 1980, Timmel [10] proposed a population-based stochastic approach for finding multiple Pareto-optimal solutions of a differentiable multi-objective optimization problem. In this method, first a feasible solution set (we call it a population) is randomly created. The non-dominated solutions $(\mathbf{X}_0 = \{\mathbf{x}_1^0, \mathbf{x}_2^0, \ldots, \mathbf{x}_s^0\})$ are identified and they serve as the first approximation to the Pareto-optimal set. Thereafter, from each solution $\mathbf{x}_k^0$, a child solution is created in the following manner:

$$\mathbf{x}_k^1 = -\sum_{i=1}^{M} t_i u_i \nabla f_i(\mathbf{x}_k^0),$$

where u_i is a uniformly distributed random number (between 0 and 1) and t_i is step-length in the i-th objective. It is a simple exercise to show that the above formulation ensures that not all functions can be worsened simultaneously. Thus, the child solution is either non-dominated to the parent solution $\mathbf{x}_k^0$, or it dominates the parent. However, the variation of the step-length over iterations must be made carefully to ensure convergence to the efficient frontier. The original study suggested the following sequence for updating t_i:

$$\lim_{i\to+\infty} t_i = 0, \quad \sum_{i=1}^{\infty} t_i = \infty, \quad \sum_{i=1}^{\infty} t_i^2 < \infty.$$

After the child population is created, it is combined with the parent population and only the non-dominated solutions are retained. This set then becomes the second approximation to the Pareto-optimal set. This procedure is continued for a pre-specified number of iterations. Note that the population size can vary with iterations. In fact, in most problems, an increase in the population size is expected.

The step-length variation mentioned above ensures the following aspects:

1. The step size should slowly decrease to zero as solutions closer to the Pareto-optimal set are found and
2. The decrease of the step size must not be slow enough so that the algorithm gets caught in sub-optimal points.

Thus, it is clear that the update of the step length is a crucial part of the working of the algorithm and a tuning of the update strategy may have to be done for every problem. Here, we use the following strategy: $t_i = c/i$ (where c is a positive constant), which satisfies all the above-mentioned conditions. For the interested readers, we refer to the original study [10, 11] for further details. It is interesting to note that this algorithm uses an elitist strategy, in which best of parent and offspring populations is retained.

2.3 Normal Boundary Intersection method (NBI)

The NBI method was developed by Das et. al. [1] for finding a uniform spread of Pareto-optimal solutions for a general nonlinear multi-objective optimization problem. The weighted-sum scalarization approach has a fundamental drawback of not being able to find a uniform spread of Pareto-optimal solutions given a uniform spread of weights. The NBI approach uses a scalarization scheme with a property that a uniform spread in parameters will give rise to a uniform spread in points on the efficient frontier. Also, the method is independent of the relative scales of different objective functions. The scalarization scheme is briefly described below.

Let us consider a multi-objective problem as $\min_{\mathbf{x} \in S} F(\mathbf{x})$, where $S = \{\mathbf{x} \mid h(\mathbf{x}) = 0; g(\mathbf{x}) \leq 0, a \leq \mathbf{x} \leq b\}$ be the constraint set. Let $F^* = (f_1^*, f_2^*, \ldots, f_M^*)^T$ be the utopia point of the multi-objective optimization problem with M objective functions and n variables. Let the individual minima of the functions be attained at $\mathbf{x}_i^*$ for each $i = 1, 2, \ldots, M$. The convex hull of the individual minima is then obtained. The simplex obtained by the convex hull of the individual minima can be expressed as $\Phi\beta$, where $\Phi = (F(\mathbf{x}_1^*), F(\mathbf{x}_2^*), \ldots F(\mathbf{x}_M^*))$ is a $M \times M$ matrix and $\beta = \{(b_1, b_2, \ldots, b_M)^T \mid \sum_{i=1}^{M} b_i = 1\}$. The original study suggested a systematic method of setting β vectors in order to find a uniformly distributed set of efficient points. The NBI scalarization scheme takes a point on the simplex and then searches for the maximum distance along the normal pointing towards the origin. This chosen point may or may not be a Pareto-optimal point. In non-convex situations, even the Pareto-optimal points which cannot be obtained by the usual weighted-sum schemes, are possible to be obtained by this method. The NBI subproblem (NBI_β) for a given vector β is as follows:

$$\begin{aligned} &\max_{(\mathbf{x},t)} \quad t, \\ &\text{subject to } \Phi\beta + t\hat{n} = F(\mathbf{x}), \\ &\qquad\qquad\quad \mathbf{x} \in S, \end{aligned} \tag{2}$$

where $\hat{n}$ is the normal direction at the point $\Phi\beta$ pointing towards the origin. The solution of the above problem gives the maximum t and also the corresponding

Pareto-optimal solution, $\mathbf{x}$. The method works even when the normal direction is not an exact one, but a quasi-normal direction. The following quasi-normal direction vector is suggested in Das et al. [1]: $\hat{n} = -\Phi e$, where $e = (1, 1, \ldots, 1)^T$ is a $M \times 1$ vector. The above quasi-normal direction has the property that NBI_β is independent of the relative scales of the objective functions. A modified version of the NBI approach (called the recursive knee approach) was developed elsewhere [2] for convex problems. Another study extended the approach by using a suitable inequality constraint to define a subproblem [7].

3 Comparison with NSGA-II

In this section, we compare the above three classical generating methods with NSGA-II on a number of two and three-objective test problems. The test problems are chosen in such a way so as to systematically investigate various aspects of an algorithm. In the test problems, the exact knowledge of the Pareto-optimal front is available. For classical methods, a limited parametric study is performed for each test problem and results from the best parameter setting are presented. For NSGA-II, we use a standard real-parameter SBX and polynomial mutation operator with $\eta_c = 10$ and $\eta_m = 10$, respectively [3]. For all problems solved using NSGA-II, we use a population of size 100.

3.1 Two-Objective Test Problems

First, we consider two-objective ZDT test problems [3, 4]. The test problems are slightly modified so that they become unconstrained multi-objective optimization problems, as the SSM method is only able to tackle unconstrained problems. A constrained version of SSM algorithm is currently being investigated by the authors.

Modified ZDT1 Test Problem: The modified ZDT1 test problem can be stated as follows:

$$\begin{aligned} &\text{Minimize } f_1(\mathbf{x}) = x_1, \\ &\text{Minimize } f_2(\mathbf{x}) = g(\mathbf{x}) \left(1 - \sqrt{\frac{x_1}{g(\mathbf{x})}}\right), \\ &\quad \text{where } g(x) = 1 + \frac{9}{n-1} \sum_{i=2}^{n} x_i^2, \end{aligned} \tag{3}$$

where the box constraints are $x_1 \in [0, 1]$, and $x_i \in [-1, 1]$ for $i = 2, 3, \ldots, n$. Here, we choose $n = 30$. This modified ZDT1 problem has a convex Pareto-optimal front. The Pareto-optimal solutions correspond to $0 \leq x_1^* \leq 1$ and $x_i^* = 0$ for $i = 2, 3, \ldots, n$. This problem offers a difficulty in handling a large number of variables.

The Euler's method with a step size of $\sigma = 0.8$ along with $\epsilon = 0.05$ is used in SSM. An initial starting point is randomly created using the box constraints. It is to be noted that the SSM method requires gradient information. To make a fair comparison with an EMO methodology, gradients are calculated numerically here and the overall function evaluations is recorded. Figure 1 shows the obtained distribution of efficient solutions after 20,000 (inset plot) and 100,000 function evaluations. Due to the use of a descent direction, the SSM method quickly converges near to the efficient frontier in this problem. However, the spread of

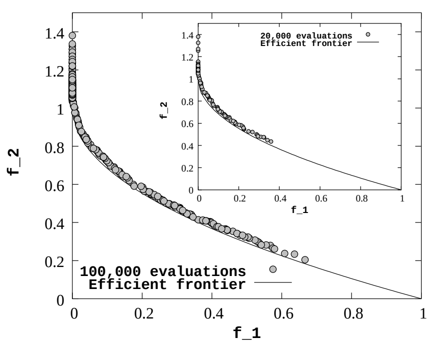

Fig. 1. Performance of SSM method on ZDT1.

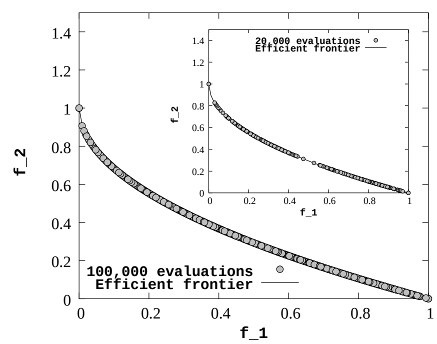

Fig. 2. Performance of TPM method on ZDT1.

solutions along the efficient frontier is very slow. Notice that from 20,000 function evaluations till 100,000 function evaluations, the procedure finds a spread from $x_1 = 0.4$ to $x_1 = 0.7$. The use of a Brownian motion for spread seems to be too generic to get a faster spread along the efficient frontier. After even 100,000 function evaluations, the solutions are not quite on to the efficient frontier. The numerical gradient evaluation is costly, requiring $2n$ function evaluations for each gradient. With a large number of variables, such methods may become computationally expensive. However, the simulation results show that this test problem does not offer too much of a difficulty to the SSM method in quickly converging near to the efficient frontier.

Next, we apply the TPM method. We begin the search with a single solution ($s = 1$), randomly created satisfying the box constraints. Figure 2 shows the obtained front after 20,000 (inset plot) and 100,000 function evaluations. It is clear that the TPM method performs extremely well on ZDT1 both in terms of convergence and maintenance of diversity.

The NBI method needs the computation of the utopia point. This requirement causes an added difficulty for the NBI method. Here, the subproblems are solved using the sequential quadratic programming (SQP) method. Figure 3 (inset) shows that the NBI method is capable of finding a good spread of Pareto-optimal solutions even with 20,000 function evaluations. Since, a systematic initial points are considered in this approach, a good spread is obtained. If more β-vectors are used, a more dense set of solutions can be found.

Finally, we apply NSGA-II for a total of 20,000 (inset plot) and 100,000 function evaluations. Figure 4 shows that even with 20,000 evaluations, a good

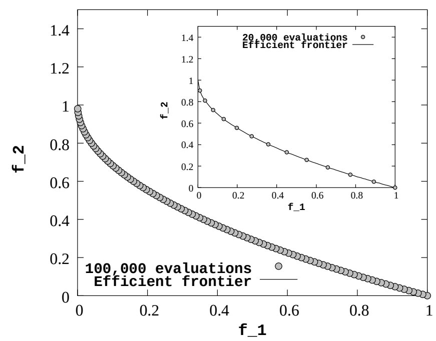

Fig. 3. Performance of NBI method on ZDT1.

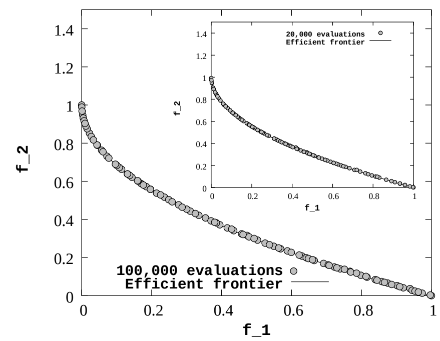

Fig. 4. Performance of NSGA-II method on ZDT1.

distribution is achieved. Based on all these simulations, it can be concluded that the ZDT1 problem is best solved by using a systematic procedure such as the NBI method, whereas the TPM or NSGA-II also performs well on this problem.

Modified ZDT2 Test Problem: The modified ZDT2 test problem can be stated as follows:

$$\begin{aligned} &\text{Minimize } f_1(\mathbf{x}) = x_1, \\ &\text{Minimize } f_2(\mathbf{x}) = g(\mathbf{x}) \left(1 - \left(\tfrac{x_1}{g(\mathbf{x})}\right)^2\right), \\ &\quad \text{where } g(x) = 1 + \tfrac{9}{n-1} \textstyle\sum_{i=2}^{n} x_i^2, \end{aligned} \tag{4}$$

where the box constraints are $x_1 \in [0, 1]$ and $x_i \in [-1, 1]$ for $i = 2, 3, \ldots, n$. Here again we use $n = 30$. This problem has a non-convex efficient frontier. The Pareto-optimal solutions correspond to $0 \le x_1^* \le 1$ and $x_i^* = 0$ for $i = 2, 3, \ldots, n$. This problem provides two difficulties to an optimization algorithm: (i) large number of variables and (ii) a non-convex efficient frontier.

The Euler's method with a step size of $\sigma = 0.1$ along with $\epsilon = 0.01$ is used in the SSM algorithm. Figure 5 shows the obtained distribution of solutions after 20,000 and 100,000 function evaluations. Although the convergence near the efficient front is quick similar to that in ZDT1, the distribution is poor. In the TPM method, we use a population of size 100 randomly created satisfying the box constraints. Solutions after 20,000 and 100,000 evaluations are shown in Figure 6. A good convergence and diversity of solutions is observed. Figure 7 shows the solutions obtained using the NBI method. A good set of solutions even with 20,000 function evaluations is apparent from the figure. Figure 8 shows the NSGA-II solutions for 20,000 function evaluations. It is clear that the non-convexity of the efficient frontier did not provide any problem to NBI and TPM approaches, while the performance of the SSM approach is poor.

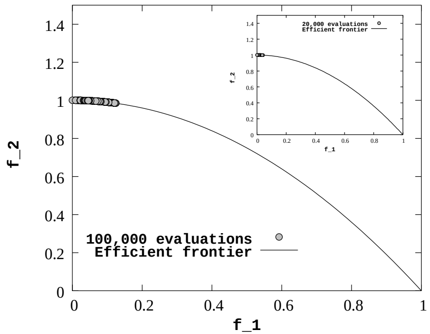

Fig. 5. Performance of SSM method on ZDT2.

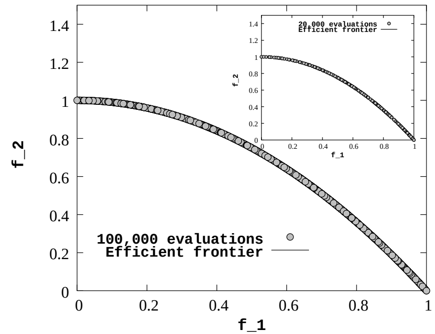

Fig. 6. Performance of TPM method on ZDT2.

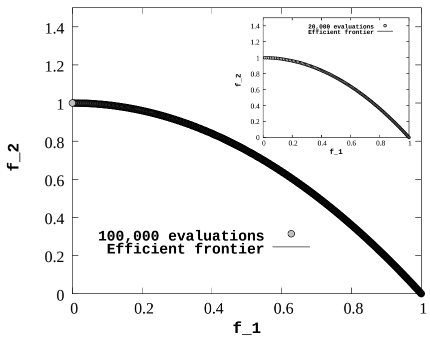

Fig. 7. Performance of NBI method on ZDT2.

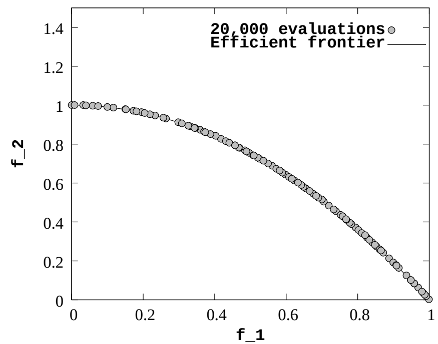

Fig. 8. Performance of NSGA-II method on ZDT2.

Modified ZDT3 Test Problem: The modified ZDT3 test problem can be stated as follows:

$$
\begin{aligned}
&\text{Minimize } f_1(\mathbf{x}) = x_1, \\
&\text{Minimize } f_2(\mathbf{x}) = g(x)\left(1 - \sqrt{\frac{x_1}{g(\mathbf{X})}} - \frac{x_1}{g(\mathbf{X})}\sin(10\pi x_1)\right), \\
&\quad \text{where } g(x) = 1 + \frac{9}{n-1}\sum_{i=2}^{n} x_i^2,
\end{aligned} \tag{5}
$$

where the box constraints are $x_1 \in [0, 1]$, and $x_i \in [-1, 1]$ for $i = 2, 3, \ldots, n$. We use $n = 30$. This problem has a convex discontinuous efficient frontier. The Pareto optimal solutions correspond to $0 \leq x_1^* \leq 1$ and $x_i^* = 0$ for $i = 2, 3, \ldots, n$. The Euler's method with a step size $\sigma = 0.5$ along with an $\epsilon = 0.01$ is used in SSM. Figure 9 shows the obtained distribution after 100,000 functions evaluations. Only a portion of the efficient frontier is discovered by this method. The TPM method is applied with an initial population of size 5,000, randomly created satisfying the box constraints. Figure 10 shows the obtained solutions after 100,000 evaluations. The figure shows that all disconnected efficient fronts are discovered by this method. It is noteworthy that with 20,000 evaluations the complete front was not fully discovered by the TPM method.

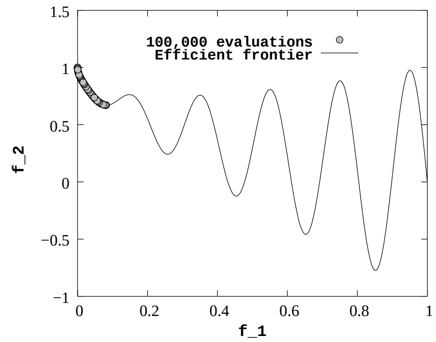

Fig. 9. Performance of SSM method on ZDT3.

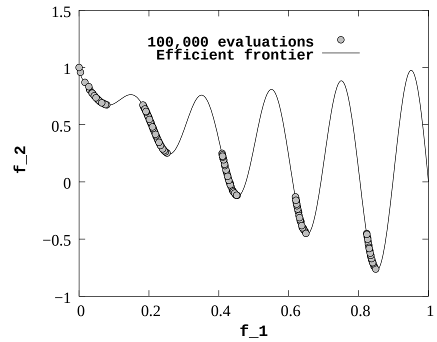

Fig. 10. Performance of TPM method on ZDT3.

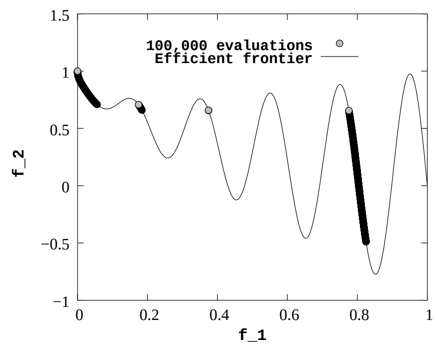

Fig. 11. Performance of NBI method on ZDT3.

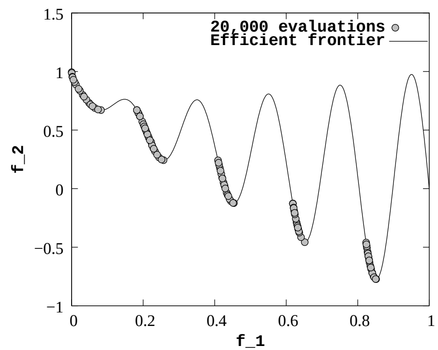

Fig. 12. Performance of NSGA-II method on ZDT3.

Figure 11 shows the distribution of the NBI method after 100,000 evaluations. It is apparent that not all disconnected efficient fronts are discovered by this approach. Although the NBI method performed very well on ZDT1 and ZDT2 problems having a continuous efficient front, the disconnectedness of the efficient frontier seems to have provided difficulty to this approach. Since not all fronts are discovered, this method ends up finding a dominated portion of the true efficient frontier. Since the idea of non-domination is not built in the NBI approach, it ends up finding some non-efficient points. A comparison with the NSGA-II results (Figure 12) indicates the NSGA-II with only 20,000 evaluations is able to find all disconnected efficient fronts.

ZDT4 Test Problem: Next, we use the 10-variable ZDT4 test problem [3]. This problem has a total of 100 distinct local efficient fronts in the objective space. The global Pareto-optimal solutions correspond to $0 \leq x_1^* \leq 1$ and $x_i^* = 0$ for $i = 2, 3, \ldots, n$. The algorithms face a difficulty in overcoming a large number of local fronts and converging to the global front.

The Euler's method with a step size of $\sigma = 0.1$ along with $\epsilon = 0.001$ is used in SSM. Only a few weak Pareto-optimal solutions ($f_1 = 0$ and $f_2 = 70$ to 70.4) are found after 20,000 evaluations. Since the SSM method requires functions to be twice continuously-differentiable and since ZDT4 is not twice differentiable precisely at $x_1 = 0$, the gradient computation is erroneous at $x_1 = 0$, resulting in a failure of the method.

The TPM method is applied with 2,000 initial solutions randomly created satisfying the box constraints. Figure 13 shows that a set of dominated local-Pareto-optimal solutions is discovered after 100,000 evaluations. The optimization algorithm used in the TPM method can get stuck to a local-optimal solution and the ZDT4 problem with many local efficient frontier provides enough difficulty to this approach for finding the true global efficient frontier. The multi-

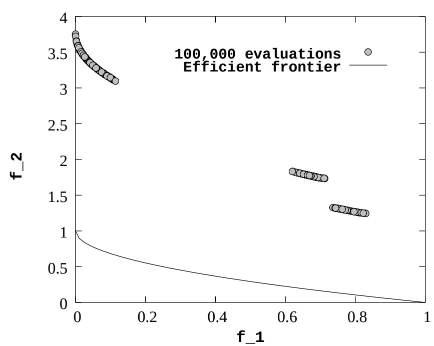

Fig. 13. Performance of TPM method on ZDT4.

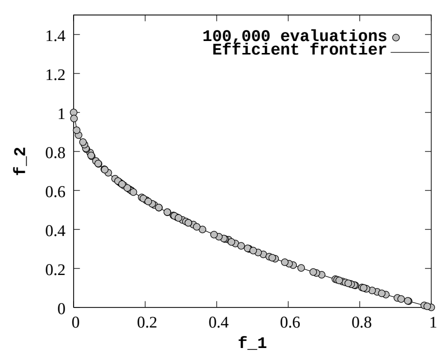

Fig. 14. Performance of NSGA-II method on ZDT4.

modality of the search space also causes the NBI method to not find the global efficient frontier. The SQP method is inadequate to find the global optimal solutions. Figure 14 shows that NSGA-II with 100,000 evaluations is able to converge to the global efficient frontier.

The problem ZDT4 provides difficulty in terms of multi-modality of the search space. It is evident from the simulation results that the classical generating methods face enormous problems in overcoming the multi-modalities, whereas in this type of problems evolutionary multi-objective (EMO) methods are found to be useful.

Modified ZDT6 Test Problem: The $n = 10$ variable modified ZDT6 test problem is as follows:

$$\begin{aligned} \text{Minimize } & f_1(\mathbf{x}) = 1 - \exp(-4x_1)\sin^6(4\pi x_1), \\ \text{Minimize } & f_2(\mathbf{x}) = g(\mathbf{x})\left(1 - \left(\frac{f_1(\mathbf{x})}{g(\mathbf{x})}\right)^2\right), \\ \text{where } & g(x) = 1 + 9\left(\sum_{i=2}^{n} x_i^2/(n-1)\right)^{0.25}, \end{aligned} \quad (6)$$

where the box constraints are $x_1 \in [0, 1]$ and $x_i \in [-1, 1]$ for $i = 2, 3, \ldots, n$. This problem has a non-convex and non-uniformly spaced Pareto-optimal solutions. The Pareto-optimal solutions correspond to $0 \le x_1^* \le 1$ and $x_i^* = 0$ for $i = 2, 3, \ldots, n$. The Euler's method with a step size of $\sigma = 0.15$ along with $\epsilon = 0.001$ is used in SSM. Figure 15 shows the distribution of obtained solution after 100,000 function evaluations. The algorithm is not able to find a well-converged set of solutions. Although there is no local efficient frontier at the location where the algorithm gets stuck, parameters play an important role in the success of SSM and in this problem it is seen that for small values of parameters there is an ascent in functions, instead of a descent in them. A theoretical analysis suggests that at each point $\mathbf{x}$ the direction $(-q(x))$ is a descent direction for all functions, however with a finite step size this result does not hold.

The TPM method with 1,000 initial random solutions produces a set of solutions closer to the efficient frontier, but there are only a few solutions found even after 100,000 function evaluations (Figure 16). For this problem, there is a slow improvement in each iteration and by the time the solution reaches near the efficient frontier, the step size t_i becomes very small and it would take a long time before the solutions fall on the efficient frontier.

The NBI method (Figure 17) performs poorly on this problem. Since the density of solutions along the frontier is non-uniform, the SQP method along with the NBI strategy is unable to find a good distribution. On the other hand, NSGA-II is able to find a good convergence and distribution with 100,000 function evaluations (Figure 18).

Based on these simulations, we infer that a non-uniform density of solutions in the objective space (which occurs in many real-world problems [3]) provides enough difficulty to the classical generating methods. These are another class of optimization problems in which EMO methodology performs comparatively better than the classical methods.

3.2 Three-Objective Test problems

Now, we consider a couple of three-objective test problems developed elsewhere [6] to study the behavior of all four algorithms.

DTLZ2 Test Problem: First, we consider the 12-variable DTLZ2 test problem having a spherical efficient front satisfying $f_1^2 + f_2^2 + f_3^2 = 1$ in the range $f_1, f_2 \in [0, 1]$. The Euler's method with a step size of $\sigma = 0.1$ and $\epsilon = 0.01$ is used

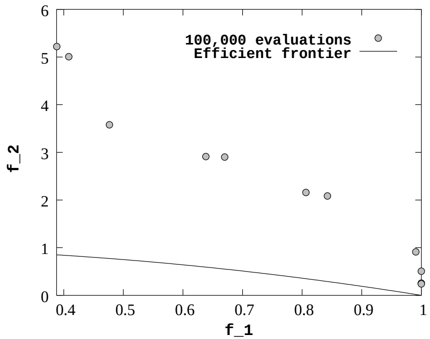

Fig. 15. Performance of SSM method on ZDT6.

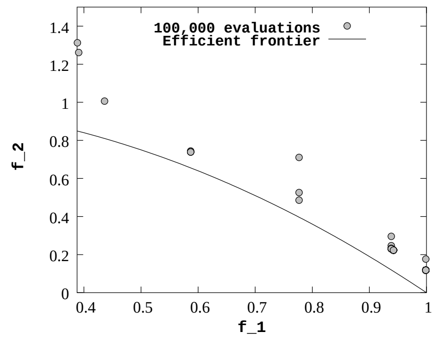

Fig. 16. Performance of TPM method on ZDT6.

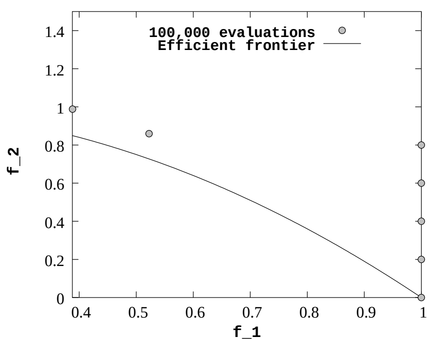

Fig. 17. Performance of NBI method on ZDT6.

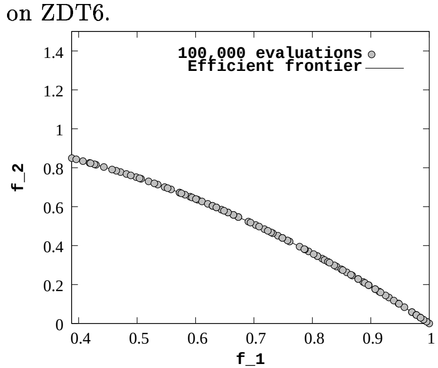

Fig. 18. Performance of NSGA-II method on ZDT6.

in SSM. Figure 19 shows all obtained solutions after 100,000 evaluations. It is clear that the SSM approach is able to get the solutions on the frontier, but the distribution of solutions (obtained mainly by the Brownian approach) is not adequate. It will take enormous number of evaluations for the algorithm to find a distribution across the complete efficient frontier. However as apparent from the figure, due to the descent direction it needs only few iteration to reach the efficient frontier.

The Timmel's method is applied next with 1,000 initial random solutions. After 100,000 evaluations, the approach is able to find a good coverage of the entire efficient frontier (Figure 20). It is interesting that the boundary solutions are adequately discovered by this approach. The NBI approach, after 100,000 evaluations, finds a few well-distributed solutions (Figure 21). If more evaluations are allowed, the remaining portion of the efficient frontier may also be discovered by this method, however the requirement of a large number of evaluations for high-dimensional objective space is a drawback of this algorithm.

The spread of solutions using NSGA-II (with 20,000 evaluations) is shown in Figure 22. Although the distribution is not as regular as in the NBI approach,

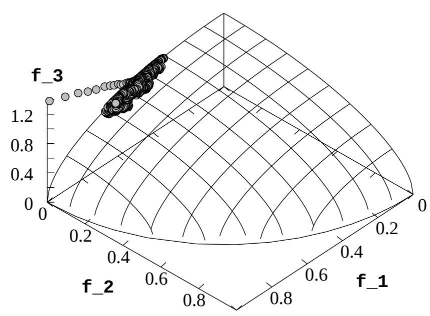

Fig. 19. Performance of SSM method on DTLZ2.

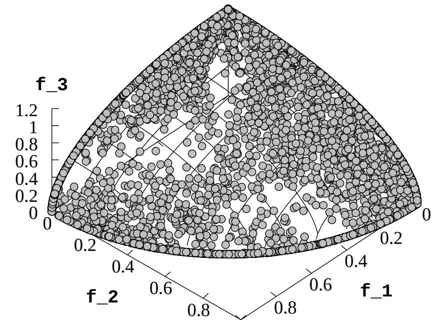

Fig. 20. Performance of TPM method on DTLZ2.

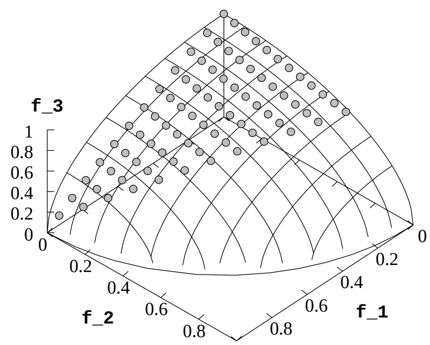

Fig. 21. Performance of NBI method on DTLZ2.

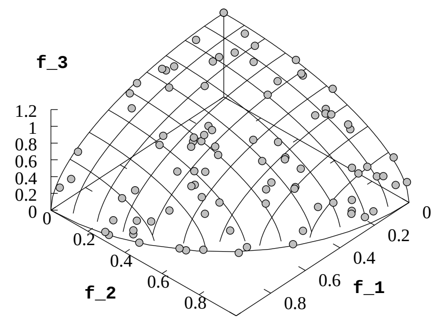

Fig. 22. Performance of NSGA-II method on DTLZ2.

the obtained solutions spread across the entire front. As pointed elsewhere, a better niching operator than the crowding-distance operator, such as a clustered NSGA-II [5] or another EMO such as SPEA2 can employ a better distribution of solutions in problems having more than two objectives.

If these algorithms are applied on DTLZ3 which has a number of local efficient frontiers as in ZDT4, the classical algorithms will have similar difficulties in converging to the true efficient frontier. Thus, we do not show the results on DTLZ3.

DTLZ5 Test Problem: The DTLZ5 is a 12-variable problem having a Pareto-optimal curve: $f_3^2 = 1 - f_1^2 - f_2^2$ with $f_1 = f_2 \in [0, 1]$. This problem, although a three-objective one, has a one-dimensional efficient frontier. The SSM, using Euler's method with a step size of $\sigma = 0.5$ and $\epsilon = 0.01$, finds the partial front after 100,000 evaluations, as shown in Figure 23. The TPM approach (with 500 initial random solutions) finds the complete front, as shown in Figure 24. On the other hand, the NBI approach finds a different one-dimensional curve as the efficient frontier (Figure 25).

On the other hand, like TPM, NSGA-II (with 20,000 function evaluations) does not have any problem in finding a good distribution on the true frontier, as shown in Figure 26.

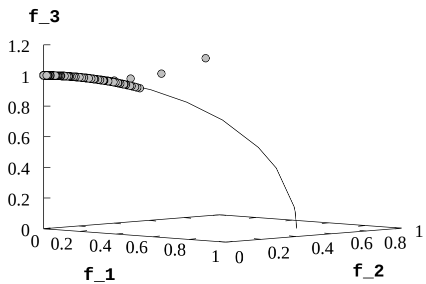

Fig. 23. Performance of SSM method on DTLZ5.

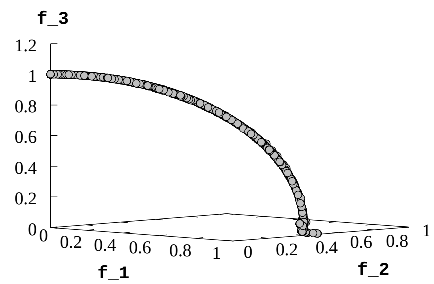

Fig. 24. Performance of TPM method on DTLZ5.

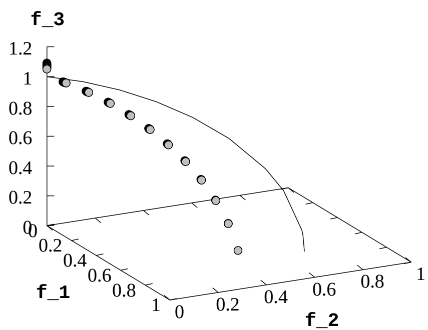

Fig. 25. Performance of NBI method on DTLZ5.

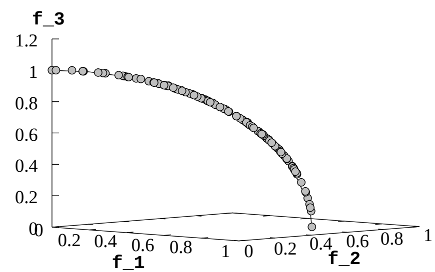

Fig. 26. Performance of NSGA-II method on DTLZ5.

4 Conclusions

This study brings into light three different classical generating methods which can be used to find a set of Pareto-optimal solutions in a single simulation run. The comparison of these methods with NSGA-II on a number of test problems have adequately demonstrated that these methods perform very well when the problem size and search space complexity is small. Among the three methods, the SSM approach seems to find only a part of the entire efficient front. However, due to the use of a direction of descent on all objective functions simultaneously, it usually reaches a local efficient front quickly. The TPM approach is similar to an elite-preserving population-based EMO approach with an exception that with iterations the population size can increase indefinitely, thereby making the latter iterations slow. The approach also requires fixing a step-size update scheme, which requires fine-tuning for every problem. The NBI approach is a systematic mathematical programming approach in which a number of searches are performed from a uniformly-distributed set of points in the objective space.

On a number of two and three-objective test problems, it has been observed that the TPM and NBI are better than the SSM approach. However, for problems having multi-modal efficient fronts or non-uniform density of points in the objective space, all three methods do not perform well. They either get stuck to a local efficient frontier or to suboptimal solutions. On the other hand, on all problems considered here, NSGA-II with an identical parameter setting, has performed well. One way to extend the study would be to replace the SQP or classical optimization approach embedded to these classical algorithms with an evolutionary algorithm. Another approach would be to use some of the classical principles as an additional operator in an EMO methodology. Some such extensions would be an immediate focus for useful research and application in the area of multi-objective optimization.

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