Hybridization of SBX Based NSGA-II and Sequential Quadratic Programming for Solving Multi-objective Optimization Problems

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***Abstract*— Most real-world search and optimization problems involve multiple conflicting objectives and results in a Pareto-optimal set. Various multi-objective optimization algorithms have been proposed for solving such problems with the goals of finding as many trade-off solutions as possible and maintaining diversity among them. Since last decade, Evolutionary Multi-objective Optimization (EMO) algorithms have been applied successfully to various test and real-world optimization problems. These population based algorithms provide a diverse set of non-dominated solutions. The obtained non-dominated set is close to the true Pareto-optimal front but it's convergence to the true Pareto-optimal front is not guaranteed. Hence to ensure the same, a local search method using classical algorithm can be applied.**

In the present work, SBX based NSGA-II is used as a population based approach and the sequential quadratic programming (SQP) method is used as a local search procedure. This hybridization of evolutionary and classical algorithms approach provides a confidence of converging near to the true Pareto-optimal set with a good diversity. The proposed procedure is successfully applied to 13 test problems consisting two, three and five objectives. The obtained results validate our motivation of hybridizing evolutionary and classical methods.

I. Introduction

Real-world optimization and test problems deal with simultaneous optimization of objectives. The outcome of such an optimization problem is a set of compromised solutions of different objectives. These are known as 'Pareto-optimal' solutions [1]. Among the different multi-objective algorithms, it is observed that an elitist non-dominated sorting algorithm (popularly known as NSGA-II) can converge near to the true Pareto-optimal front as well as maintain the diversity of population on the Pareto-optimal front [2], [4]. The non-dominated solutions obtained from NSGA-II can be improved by local search technique using classical procedure.

This paper is prepared for the special session devoted to performance assessment and competition of different multi-objective algorithms on a set of 13 test problems [6]. In this paper, we employ a population-based optimization algorithm (NSGA-II) as a global optimizer and the sequential quadratic programming (SQP) for locally improving the non-dominated set of solutions. Most of the classical optimization methods are designed to solve a derived single-objective optimization problem in order to handle a multi-objective optimization problem. The procedure needs to be scalarized to a single objective problem. In this work, we have used an ϵ-constraint procedure [1] to convert multiple objectives to a single-objective optimization problem.

The test problems consist of various properties in terms of number of objectives (separability and deception), unimodality and multi-modality, convexity and concavity, and with complex geometry, and others. It is our intuition that to solve such wide variety of test problems with a reasonable level of satisfaction, a hybrid approach consisting of evolutionary and classical algorithms, has to be used. In the following section, we describe the description of algorithm and proposed procedure. In Section III and IV, we present the parameters setting and simulation results in tabular and graphical forms respectively and the paper is concluded in section V.

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II. Description of the Proposed Procedure

A. Elitist Non-dominated Sorting Genetic Algorithm

Elitist Non-dominated Sorting Genetic Algorithm (NSGA-II) uses an elite-preserving strategy as well as an explicit diversity preserving mechanism [4]. The offspring population Q_t is first created by using the parent population P_t. Then, the two population are combined together and a non-dominated sorting is performed. To preserve diversity, a density metric called *Crowding Distance* is used. The different steps of the algorithm are described below:

Step 1: A random population is initialized.
Step 2: Objective functions for all objectives and constraint are evaluated.
Step 3: Front ranking of the population is done based on the dominance criteria.
Step 4: Crowding distance is calculated.
Step 5: Selection is performed using crowded binary tournament selection operator.
Step 6: Crossover and mutation operators are applied to generate an offspring population.
Step 7: Parent and offspring populations are combined and a non-dominated sorting is done.
Step 8: The parent population is replaced by the best members of the combined population.

In Step 3, Each solution is assigned a non-domination rank (a smaller rank to a better non-dominated front). In Step 4, for each i-th solution of a particular front, density of solutions in its surrounding is estimated by taking average distance of two solutions on its either side along each of the objective[4]. This average distance is called the crowding distance.

Selection is done based on the front rank of an individual and for solutions having same front rank, selection is done on the basis of their crowding distances (larger, the better). To create new offspring, *simulated binary crossover (SBX)* operator [3] and *polynomial mutation* operator [5] are used. In Step 8, initially solutions of better fronts replace the parent population. When it is not possible to accommodate all solutions of a particular front, that front is sorted on the basis of crowding distance and as many individuals are selected on the basis of higher crowding distance, which makes the population size of the new population same as the previous population.

Inspiration of using the SBX operator arrives from the property of creating offsprings in proportion to the distance between two parent solutions. The SBX operator biases solutions near each parent more favorably than solutions away from the parents [3]. In the present work, NSGA-II is used as a population based algorithm which helps in finding a non-dominated set of solutions with a good diversity.

B. Sequential Quadratic Programming

Sequential Quadratic programming (SQP) method uses a quadratic model for the objective function and a linear model for constraint(s) [7]. A nonlinear program in which the objective function is quadratic and constraints are linear is called a *quadratic program* (QP). SQP method solves an approximated QP at each iteration. The gradients are numerically calculated using forward finite difference technique. In this study, the ϵ-constraint method is used to convert multiple objectives to a single-objective optimization problem, following which SQP is applied to the single-objective problem. Here the SQP is employed as a local search procedure. Steps of the proposed algorithm are described below:

1) Random population is initialized.
2) NSGA-II is applied to the initial population.
3) After finding a diverse set of non-dominated solutions of NSGA-II, the SQP is employed on these solutions.
4) Solutions obtained from SQP, are again given to NSGA-II to ensure non-domination and for achieving any further improvement in terms of diversity among them in the final iteration.

Test problems with $M = 2$ and $M = 3$ objectives are solved with the above procedure. For M = 5 objectives, the obtained SQP solutions are supplied to NSGA-II and then, an archive of non-dominated solutions is created during the final three iterations of NSGA-II to make sure that an adequate number of non-dominated solutions are obtained, as demanded in [6].

III. Parameters Setting

A. Test Suite

The performance of the hybrid algorithm is tested on a set of 19 benchmark problems [6], which include seven two-objective test problems, six three-objective and six five-objective test problems.

B. PC Configuration

- System: Mandrake Linux 10.1
- CPU: P-IV 2.8 GHz
- RAM: 1 GB
- Language: ANSI-C
- Compiler Used: GCC version-3.2.2

C. Parameters Setting for NSGA-II

1) For $M = 2$ objective problems: Population size $(N) = 100$; Probability of crossover $(P_c) = 0.9$; Probability of mutation $(P_m) = 0.033$; Distribution index for crossover $(\eta_c) = 5$; Distribution index for mutation $(\eta_m) = 15$.
2) For $M = 3$ objective problems: Population size $(N) = 150$; Probability of crossover $(P_c) = 0.9$; Probability of mutation $(P_m) = 0.033$; Distribution index for crossover $(\eta_c) = 15$; Distribution index for mutation $(\eta_m) = 10$.
3) For $M = 5$ objective problems: Population size $(N) = 300$; Probability of crossover $(P_c) = 0.9$; Probability of mutation $(P_m) = 0.033$; Distribution index for crossover $(\eta_c) = 15$; Distribution index for mutation $\eta_m = 15$.

D. Parameters Setting and Termination Criteria of SQP

1) Norm of descent direction: $\| d \| \leq \epsilon$; where $\epsilon = 10^{-9}$ or
2) Maximum number of iterations allowed (τ): 50, 50 and 20 for two, three and five objective problems respectively.

As soon as, any of the above criteria of SQP is satisfied, it terminates and the solutions are supplied back to NSGA-II. Terminating criteria of hybrid algorithm is based on the total number of function evaluations taken by NSGA-II including its final iteration(s) and by SQP method. It should be less or equal to the maximum FES allowed $(5(10^5))$ [6]. We calculate the number of FES required by SQP approximately as follows. For n number of variables, SQP requires on an average $(n + 2)$ number of function evaluations in each iteration to calculate gradient and objective function values. Since a maximum of τ iterations are allowed, it will take $\tau \times (n + 2)$ number of FES for each solution and for N non-dominated points, total SQP function evaluations are $\tau \times N(n+2)$. Thus, the function evaluations left to NSGA-II is $(5(10^5) - \tau \times N(n + 2))$. On the basis of above calculations, the number of generations allowed by NSGA-II is $(5(10^5) - \tau \times N(n + 2))/N$. For example, for S_ZDT1 problem with $n = 30$, $N = 100$, and $\tau = 50$, the number of generations allowed for NSGA-II procedure is $3,400$. It is clear that the results shown at $5(10^5)$ FES used SQP procedure.

As an initial parameter study, we have chosen 4 different values of η_c (range: 2–20) and η_m (range: 5–50) and three different values of τ (range: 20–50). A parametric study on nine of 19 test problems was performed to find good combinations of η_c, η_m and τ parameters. We have chosen three test problems with 2, 3 and 5-objective test problems. A total of $(4 \times 4 \times 3 \times 9)$ or 432 runs were executed with $5(10^5)$ function evaluations (FES) for each run. Hence, a maximum of $2.16(10^8)$ FES were performed for tuning the parameters.

IV. Simulation Results

A. R, Hypervolume and Covered Sets Indicators

First, we show the R-indicator values, which computes difference between the maximum values of the augmented Tchebycheff utility function of the supplied reference set and our obtained solutions. A negative R-indicator means a better obtained utility function value than that of the reference set. A value close to zero means almost similar utility function value between reference and obtained solutions. The results obtained are presented in Tables I, II, III, IV, V, VI and VII. 25 runs are performed for each test problem and the best, median and worst results of R and $I_{\overline{H}}$ indicators are presented along with these mean and standard deviation values.

First three tables show the values of R indicator at $5(10^3)$, $5(10^4)$ and $5(10^5)$ FES. Tables I, II and III indicate that the algorithm converges at $5(10^4)$ FES and SQP helps in improving the solution locally. In case of S_ZDT1, S_ZDT2 and S_ZDT4, good improvement in R-indicator values can be observed after the local search. It is also observed that 11 out of 19 problems, a negative R-indicator value is obtained after $5(10^5)$ FES, meaning that a better utility function value than the supplied reference set is found by our procedure. In the other eight problems, the R-indicator is very close to zero, meaning that a similar utility function value to that of the reference set is found. On the basis of R-indicator values, the proposed algorithm has performed well for the given test suite.

Tables V to VII show hypervolume indicator $I_{\overline{H}}$ at $5(10^3)$, $5(10^4)$ and $5(10^5)$ FES. A lower value of $I_{\overline{H}}$ indicator corresponds to a better approximated set. $I_{\overline{H}}$ indicators show better hypervolume values of our hybrid algorithm than the supplied reference set in nine out of 19 problems after $5(10^5)$ FES. Except both three and five-objective WFG1 problems, in eight other problems, the $I_{\overline{H}}$ indicator value is close to zero.

Table IV shows the covered set indicator values for the SYMPART test problem only. Best, median, worst, mean and standard deviation values at $5(10^3)$, $5(10^4)$ and $5(10^5)$ FES are presented.

B. Attainment Surface Plots

Attainment surface signifies a combination of both convergence and diversity of obtained solutions. Figures 1, 2, 3 and 4 show 0%, 50% and 100% attainment surfaces after $5(10^5)$ FES along with Pareto front. Figures 5, 6 and 7 show the 50% attainment surface after $5(10^5)$ FES. It can be observed from the plots that for all two objectives test problems except R_ZDT4 and S_ZDT6, the proposed procedure shows excellent convergence and spread. Good attainment surfaces for the shifted DTLZ problems are also obtained, whereas performances on rotated DTLZ and WFG1 problems are not quite satisfactory. In the case of S_DTLZ3, WFG8 and WFG9 problems, the attainment surface plots depict a partial convergence and diversity. When the range for each objective is fixed between $[1, 2.5]$ for S_DTLZ3 problem, the 50% attainment surface plot shows a clear Pareto-optimal front, indicating that the proposed procedure finds an adequate set of solutions.

Figure 8 shows pair-wise interaction among five-objective problems for WFG8 (above diagonal) and WFG9 (lower diagonal) problems. The function values are normalized in the range $[1, 2]$ using lower and upper bounds given in the reference data set [6]. Median approximation set with respect to R-indicator at $5(10^5)$ is used for plotting the same. Definite structures between objective pairs are visible from the plots.

C. Algorithm Complexity

Table VIII shows the complexity of the algorithm. $T1 = (\sum_{i=1}^{N} t1_i)/N$, where $t1_i$ is the computing time for 10,000 function evaluations for problem i and N is the total number of test problems. Here $N = 19$.

$T2 = (\sum_{i=1}^{N} t2_i)/N$, where $t2_i$ is the computing time for the algorithm with 10,000 function evaluations for problem i. Time complexity of the hybrid algorithm is 2.9295 which depicts fast convergence capability.

V. Conclusions

In this paper, we have presented a hybrid approach consisting of evolutionary and classical algorithms. A state-of-the-art evolutionary multi-objective algorithm (NSGA-II) and a local search procedure (SQP) have been put together. The performance of the algorithm has been tested on 19 test problems and assessment of performance has been done on the basis of R-indicator, $I_{\overline{H}}$-indicator, covered sets indicators, and the attainment surface plots. The proposed hybrid procedure has shown a good performance for problems OKA2, SYMPART, S_ZDT1, S_ZDT2, S_ZDT4, S_DTLZ2 and S_DTLZ3. For test problems S_ZDT6, WFG8 and WFG9, the procedure has exhibited a fair performance, whereas for problems WFG1, R_ZDT4 and R_DTLZ2 the performance has been reasonably well. Even in case of higher number of objectives, the proposed procedure has shown a good convergence and diversity. The time complexity estimate has revealed a fast convergence property of the proposed procedure. In this paper, we have introduced a fast and efficient hybrid multi-objective optimization procedure which has solved a wide variety of problems with a reasonable satisfaction. We are pursuing further investigation on problems the proposed procedure did not perform to our satisfaction.

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TABLE I

THE RESULTS FOR R-INDICATOR ON TEST PROBLEMS 1-7 FOR TWO-OBJECTIVE PROBLEMS.

FES		1.OKA2	2. SYMPART	3. S_ZDT1	4. S_ZDT2	5. S_ZDT4	6. R_ZDT4	7. S_ZDT6
5×10^3	Best	-0.1059e-2	0.8908e-3	0.1077e-1	0.3019e-1	0.3419e-1	0.5902e-2	0.1038e-1
	Median	-0.3735e-3	0.1538e-2	0.1343e-1	0.4382e-1	0.4147e-1	0.9405e-2	0.1118e-1
	Worst	0.1854e-1	0.2244e-2	0.1870e-1	0.7153e-1	0.5387e-1	0.1724e-1	0.1180e-1
	Mean	0.8095e-3	0.1543e-2	0.1349e-1	0.4809e-1	0.4164e-1	0.1034e-1	0.1111e-1
	Std	0.3947e-2	0.3519e-3	0.2097e-2	0.1090e-1	0.4301e-2	0.3229e-2	0.4161e-2
5×10^4	Best	-0.1065e-2	0.8419e-4	0.7988e-4	-0.3617e-4	0.3230e-3	0.7503e-3	0.2002e-1
	Median	-0.1064e-2	0.1380e-3	0.1096e-3	0.3548e-4	0.8402e-3	0.3062e-2	0.2561e-1
	Worst	-0.1063e-2	0.2035e-3	0.1727e-3	0.1752e-3	0.1650e-2	0.5635e-2	0.2942e-1
	Mean	-0.1064e-2	0.1449e-3	0.1143e-3	0.4377e-4	0.8775e-3	0.2947e-2	0.2522e-1
	Std	0.3121e-6	0.3010e-4	0.2319e-4	0.4200e-4	0.3112e-3	0.1453e-2	0.1923e-2
$5(10^5)$	Best	-0.1065e-2	0.1408e-4	-0.4727e-7	-0.2062e-3	-0.1410e-8	0.3253e-3	0.8514e-2
	Median	-0.1065e-2	0.1902e-4	0.1967e-6	-0.2061e-3	0.2866e-7	0.1908e-2	0.1191e-1
	Worst	-0.1065e-2	0.2923e-4	0.7923e-6	-0.2057e-3	0.1134e-6	0.5039e-2	0.1417e-1
	Mean	-0.1065e-2	0.2024e-4	0.2092e-6	-0.2061e-3	0.3036e-7	0.2126e-2	0.1191e-1
	Std	0.0	0.4737e-5	0.2172e-6	0.1227e-6	0.2400e-7	0.1304e-2	0.1670e-2

TABLE II

THE RESULTS FOR R-INDICATOR ON TEST PROBLEMS 8-13 FOR THREE-OBJECTIVE PROBLEMS.

FES		8. S_DTLZ2	9. R_DTLZ2	10. S_DTLZ3	11. WFG1	12. WFG8	13. WFG9
5×10^3	Best	0.4213e-4	0.3726e-3	0.1931e-3	0.7293e-1	0.1323e-2	-0.8088e-2
	Median	0.6445e-4	0.4535e-3	0.2320e-3	0.7428e-1	0.3666e-2	-0.6207e-2
	Worst	0.8864e-4	0.5499e-3	0.2641e-3	0.7659e-1	0.6047e-2	0.1104e-2
	Mean	0.6528e-4	0.4522e-3	0.2320e-3	0.7457e-1	0.3667e-2	-0.5182e-2
	Std	0.1152e-4	0.4624e-4	0.1457e-4	0.1003e-2	0.1225e-2	0.2689e-2
5×10^4	Best	0.6797e-4	0.1148e-3	0.1062e-4	0.5026e-1	-0.8896e-2	-0.1291e-1
	Median	0.9662e-4	0.1466e-3	0.1744e-4	0.5434e-1	-0.7800e-2	-0.9202e-2
	Worst	0.1246e-3	0.1710e-3	0.2644e-4	0.5735e-1	-0.5940e-2	-0.9090e-2
	Mean	0.9901e-4	0.1422e-3	0.1775e-4	0.5364e-1	-0.7643e-2	-0.9790e-2
	Std	0.1563e-4	0.1639e-4	0.4404e-5	0.2865e-2	0.7587e-3	0.1260e-2
$5(10^5)$	Best	0.7081e-4	0.1348e-4	0.1354e-7	0.1543e-1	-0.1280e-1	-0.1420e-1
	Median	0.1014e-3	0.2127e-4	0.5996e-7	0.2326e-1	-0.1249e-1	-0.9279e-2
	Worst	0.1194e-3	0.3077e-4	0.1483e-6	0.2957e-1	-0.1176e-1	-0.9242e-2
	Mean	0.9541e-4	0.2137e-4	0.6402e-7	0.2254e-1	-0.1238e-1	-0.1021e-1
	Std	0.1481e-4	0.4241e-5	0.3266e-7	0.3607e-2	0.2853e-3	0.1899e-2

TABLE III

THE RESULTS FOR R-INDICATOR ON TEST PROBLEMS 8-13 FOR FIVE-OBJECTIVE PROBLEMS.

FES		8. S_DTLZ2	9. R_DTLZ2	10. S_DTLZ3	11. WFG1	12. WFG8	13. WFG9
5×10^3	Best	-0.9778e-5	0.1183e-4	-0.1391e-7	0.6185e-1	-0.3134e-3	-0.2922e-4
	Median	-0.7284e-5	0.1985e-4	0.2956e-5	0.6423e-1	0.2893e-2	0.1677e-2
	Worst	0.5061e-4	0.3300e-4	0.1777e-4	0.6591e-1	0.4856e-2	0.5751e-2
	Mean	0.2352e-5	0.2134e-4	0.5304e-5	0.6398e-1	0.2796e-2	0.1964e-2
	Std	0.1779e-4	0.6508e-5	0.5494e-5	0.1096e-2	0.1178e-2	0.1213e-2
5×10^4	Best	-0.9820e-5	-0.1162e-4	-0.1473e-7	0.4820e-1	-0.9025e-2	-0.2232e-2
	Median	-0.9820e-5	-0.9776e-5	-0.1470e-7	0.4971e-1	-0.7763e-2	-0.2084e-2
	Worst	-0.3798e-5	0.4056e-4	-0.1182e-7	0.5066e-1	-0.7142e-2	-0.1870e-2
	Mean	-0.9508e-5	-0.5551e-5	-0.1432e-7	0.4955e-1	-0.7838e-2	-0.2089e-2
	Std	0.1200e-6	0.1165e-4	0.7007e-9	0.6795e-3	0.4335e-3	0.7862e-4
$5(10^5)$	Best	-0.9820e-5	-0.1186e-4	-0.1473e-7	0.3408e-1	-0.1239e-1	-0.2337e-2
	Median	0.3086e-5	-0.1186e-4	-0.1473e-7	0.3727e-1	-0.1168e-1	-0.2320e-2
	Worst	0.2175e-4	-0.1135e-4	-0.1473e-7	0.3980e-1	-0.1117e-1	-0.2253e-2
	Mean	0.2811e-5	-0.1183e-4	-0.1473e-7	0.3694e-1	-0.1181e-1	-0.2253e-2
	Std	0.1273e-4	0.1097e-6	0.0	0.1727e-2	0.3587e-3	0.1895e-4

TABLE IV

THE RESULTS FOR COVERED SETS FOR TEST PROBLEM SYMPART.

FES	5×10^3	5×10^4	$5(10^5)$
Best	1.0000e-0	1.0000e-0	1.0000e-0
Median	1.0000e-0	1.0000e-0	1.0000e-0
Worst	1.0000e-0	1.0000e-0	1.0000e-0
Mean	1.0000e-0	1.0000e-0	1.0000e-0
Std	0.0	0.0	0.0

TABLE V

THE RESULTS FOR HYPERVOLUME INDICATOR $I_{\overline{H}}$ ON TEST PROBLEMS 1-7 FOR TWO-OBJECTIVE PROBLEMS.

FES		1.OKA2	2. SYMPART	3. S_ZDT1	4. S_ZDT2	5. S_ZDT4	6. R_ZDT4	7. S_ZDT6
5×10^3	Best	-0.9124e-3	0.2593e-2	0.4323e-1	0.6483e-1	0.1011e-0	0.1771e-1	0.2533e-0
	Median	0.1217e-2	0.4457e-2	0.4790e-1	0.9467e-1	0.1241e-0	0.2824e-1	0.2763e-0
	Worst	0.2625e-1	0.6491e-2	0.6285e-1	0.1453e-0	0.1605e-0	0.5292e-1	0.2916e-0
	Mean	0.1968e-2	0.4473e-2	0.4983e-1	0.9915e-1	0.1244e-0	0.3192e-1	0.2741e-0
	Std	0.5235e-2	0.1015e-2	0.5287e-2	0.1957e-1	0.1278e-1	0.9994e-2	0.1119e-1
5×10^4	Best	-0.1085e-2	0.2503e-3	0.5984e-3	-0.1169e-1	0.8615e-3	0.2377e-2	0.4467e-1
	Median	-0.1048e-2	0.4089e-3	0.7706e-3	-0.1156e-1	0.2281e-2	0.9371e-2	0.5772e-1
	Worst	-0.9996e-3	0.6012e-3	0.9585e-3	-0.1130e-1	0.4755e-2	0.1698e-1	0.6655e-1
	Mean	-0.1047e-2	0.4291e-3	0.7811e-3	-0.1156e-1	0.2391e-2	0.9266e-2	0.5677e-1
	Std	0.2104e-4	0.8844e-4	0.8364e-4	0.8540e-4	0.9101e-3	0.4095e-2	0.4493e-2
$5(10^5)$	Best	-0.1090e-2	0.4457e-4	0.2743e-3	-0.1214e-1	0.1233e-5	0.1471e-2	0.1839e-1
	Median	-0.1052e-2	0.5944e-4	0.3063e-3	-0.1211e-1	0.1520e-5	0.5727e-2	0.2625e-1
	Worst	-0.9906e-3	0.9000e-4	0.3748e-3	-0.1207e-1	0.2027e-5	0.1502e-1	0.3121e-1
	Mean	-0.1047e-2	0.6302e-4	0.3104e-3	-0.1211e-1	0.1507e-5	0.6521e-2	0.2618e-1
	Std	0.2217e-4	0.1422e-4	0.2567e-4	0.1430e-4	0.1602e-6	0.3829e-2	0.3618e-2

TABLE VI

THE RESULTS FOR HYPERVOLUME INDICATOR $I_{\overline{H}}$ ON TEST PROBLEMS 8-13 FOR THREE-OBJECTIVE PROBLEMS.

FES		8. S_DTLZ2	9. R_DTLZ2	10. S_DTLZ3	11. WFG1	12. WFG8	13. WFG9
5×10^3	Best	0.7192e-3	0.6519e-2	0.2351e-2	0.6476e-0	-0.2597e-0	-0.4135e-1
	Median	0.8781e-3	0.1019e-1	0.3618e-2	0.6595e-0	-0.2235e-0	-0.2463e-1
	Worst	0.1213e-2	0.1457e-1	0.4960e-2	0.6773e-0	-0.1951e-0	0.1555e-1
	Mean	0.8817e-3	0.1058e-1	0.3565e-2	0.6603e-0	-0.2238e-0	-0.2242e-1
	Std	0.1194e-3	0.2526e-2	0.6222e-3	0.8864e-2	0.1556e-1	0.1687e-1
5×10^4	Best	0.4969e-3	0.3230e-3	0.5142e-6	0.5273e-0	-0.3792e-0	-0.5860e-1
	Median	0.1134e-2	0.5792e-3	0.1985e-5	0.5371e-0	-0.3370e-0	-0.4609e-1
	Worst	0.2286e-2	0.9148e-3	0.8502e-5	0.5464e-0	-0.3075e-0	-0.3337e-1
	Mean	0.1116e-2	0.6018e-3	0.2800e-5	0.5360e-0	-0.3405e-0	-0.4615e-1
	Std	0.3593e-3	0.1597e-3	0.2169e-5	0.6005e-2	0.1847e-1	0.7021e-2
$5(10^5)$	Best	0.4857e-3	0.2223e-5	0.5884e-11	0.3213e-0	-0.4246e-0	-0.7746e-1
	Median	0.1335e-2	0.6243e-5	0.1207e-8	0.4249e-0	-0.3911e-0	-0.4877e-1
	Worst	0.1976e-2	0.2564e-4	0.5189e-8	0.4622e-0	-0.3742e-0	-0.3631e-1
	Mean	0.1294e-2	0.7885e-5	0.1475e-8	0.4231e-0	-0.3991e-0	-0.5158e-1
	Std	0.3796e-3	0.5197e-5	0.1183e-8	0.2736e-1	0.1756e-1	0.1134e-1

TABLE VII

THE RESULTS FOR HYPERVOLUME INDICATOR $I_{\overline{H}}$ ON TEST PROBLEMS 8-13 FOR FIVE-OBJECTIVE PROBLEMS.

FES		8. S_DTLZ2	9. R_DTLZ2	10. S_DTLZ3	11. WFG1	12. WFG8	13. WFG9
5×10^3	Best	-0.6328e-5	0.1104e-3	0.1024e-7	0.6818e-0	-0.2454e-0	-0.1845e-0
	Median	0.7930e-5	0.1703e-3	0.1459e-4	0.7050e-0	-0.2107e-0	-0.1611e-0
	Worst	0.1737e-3	0.2991e-3	0.8320e-4	0.7207e-0	-0.1750e-0	-0.1095e-0
	Mean	0.3752e-4	0.1878e-3	0.2507e-4	0.7025e-0	-0.2120e-0	-0.1573e-0
	Std	0.5447e-4	0.6238e-4	0.2618e-5	0.1054e-1	0.1524e-1	0.1573e-1
5×10^4	Best	-0.6815e-5	-0.1805e-3	-0.1776e-14	0.5498e-0	-0.3780e-0	-0.2144e-0
	Median	-0.6815e-5	-0.1722e-3	0.1998e-14	0.5644e-0	-0.3563e-0	-0.2124e-0
	Worst	-0.5044e-6	-0.1682e-4	0.1464e-8	0.5753e-0	-0.3011e-0	-0.2079e-0
	Mean	-0.6518e-5	-0.1557e-3	0.5859e-10	0.5629e-0	-0.3498e-0	-0.2124e-0
	Std	0.1260e-5	0.3828e-4	0.2928e-9	0.7024e-2	0.1988e-1	0.1375e-2
$5(10^5)$	Best	-0.6815e-5	-0.1830e-3	-0.1110e-14	0.4115e-0	-0.4302e-0	-0.2160e-0
	Median	-0.3104e-6	-0.1830e-3	0.0	0.4415e-0	-0.4209e-0	-0.2153e-0
	Worst	0.3992e-4	-0.1829e-3	0.1998e-14	0.4621e-0	-0.3870e-0	-0.2139e-0
	Mean	0.5535e-5	-0.1830e-3	0.2042e-15	0.4381e-0	-0.4165e-0	-0.2154e-0
	Std	0.1428e-4	0.1363e-7	0.9108e-15	0.1510e-1	0.1375e-1	0.4374e-3

TABLE VIII

COMPUTATIONAL COMPLEXITY.

$T1$	$T2$	$(T2-T1)/T1$
0.240s	0.9431s	2.9295

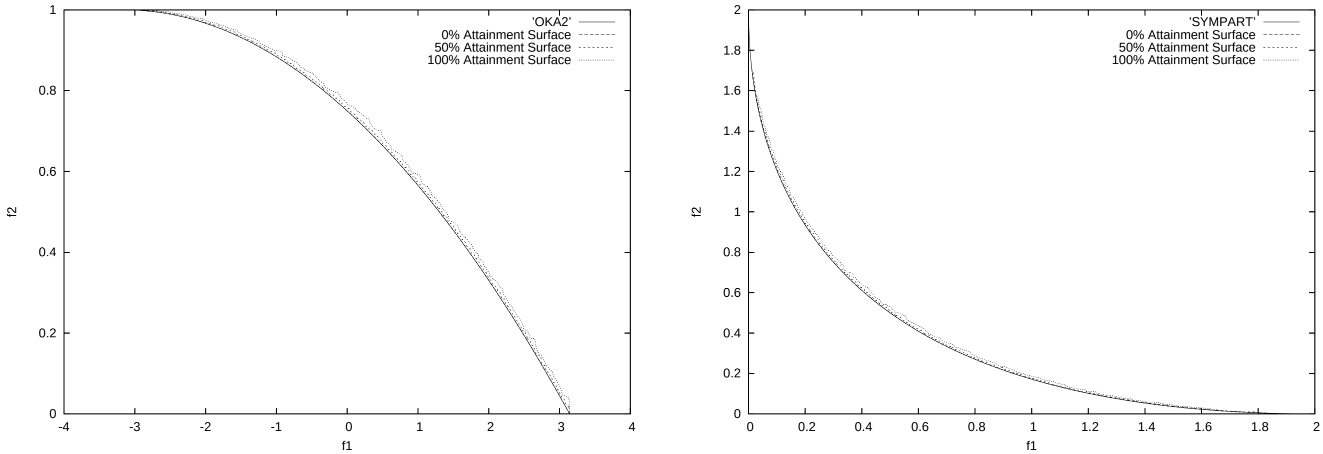

Fig. 1. Attainment plots of OKA2 and SYMPART.

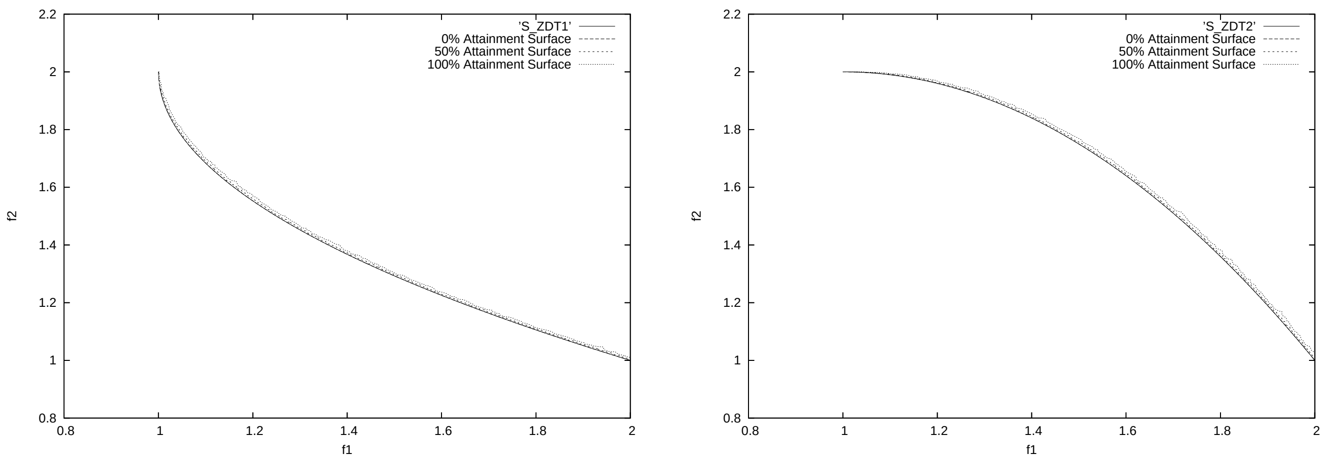

Fig. 2. Attainment plots of S_ZDT1 and S_ZDT2.

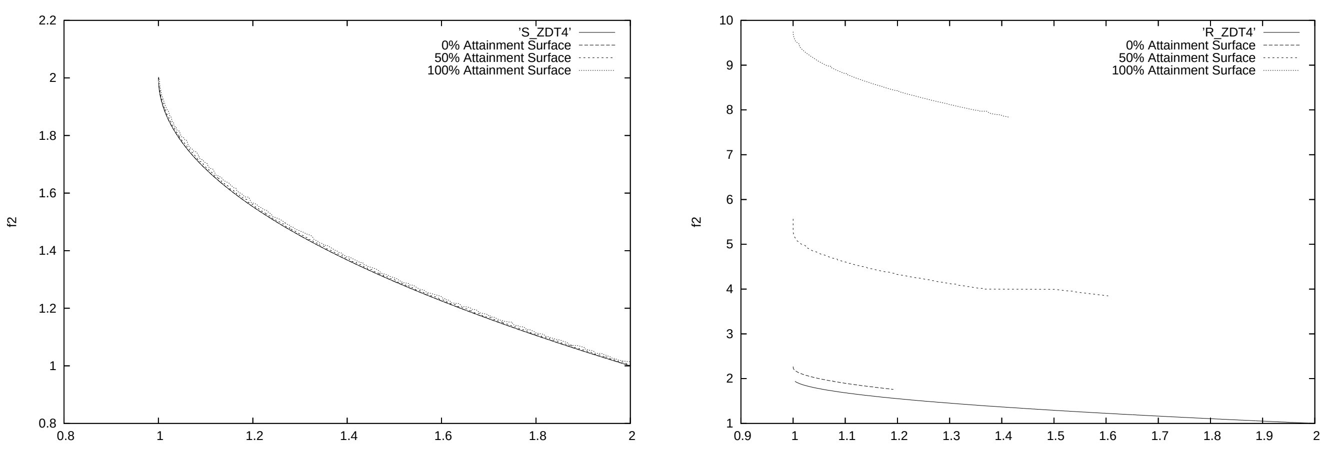

Fig. 3. Attainment plots of S_ZDT4 and R_ZDT4.

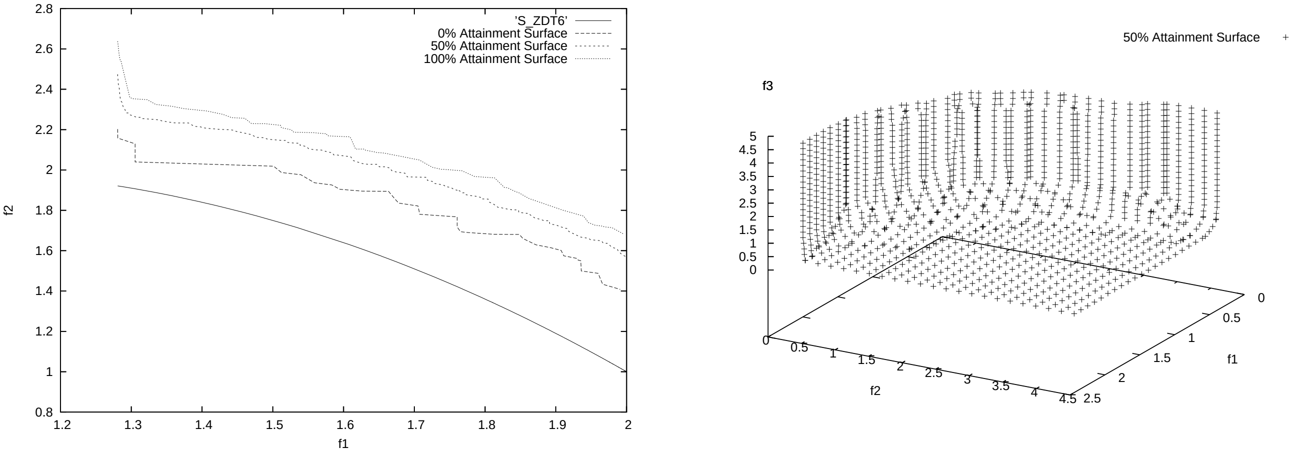

Fig. 4. Attainment plots of S_ZDT6 and WFG1_M3.

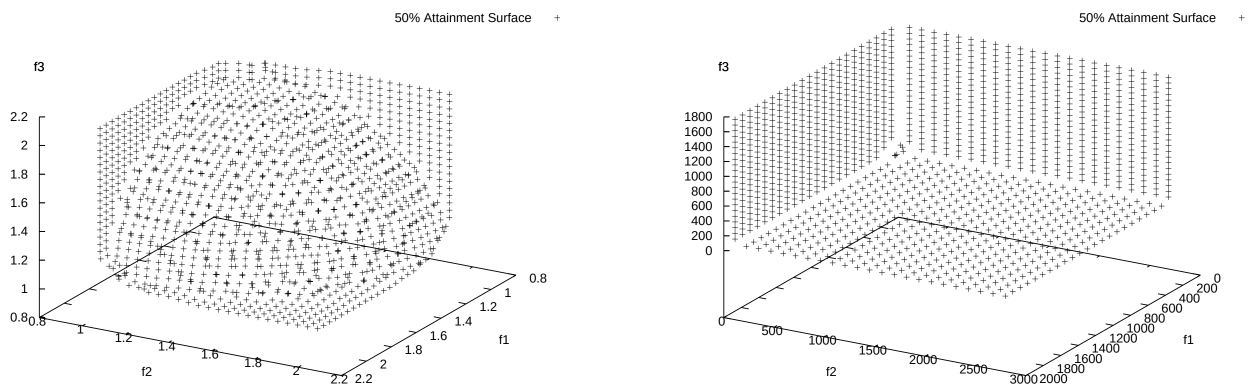

Fig. 5. Attainment surfaces of S_DTLZ2_M3 and R_DTLZ2_M3.

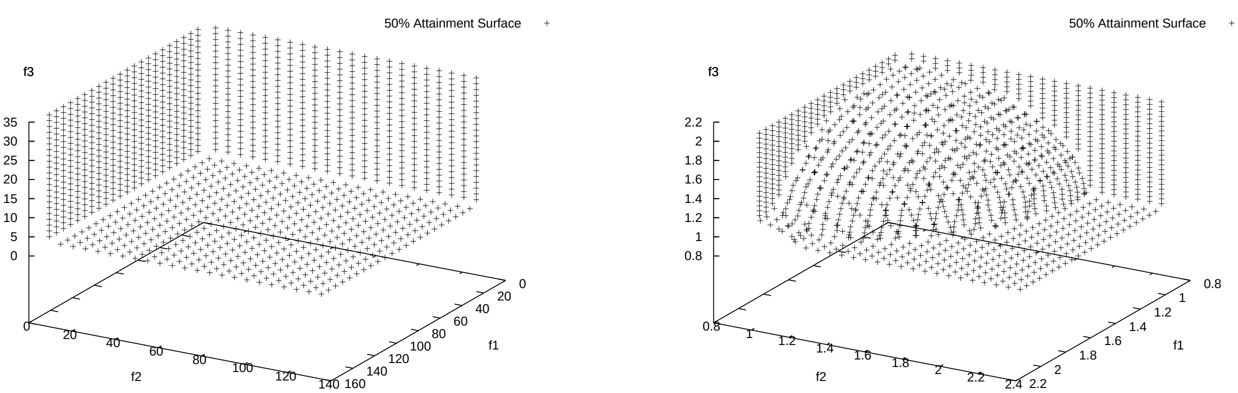

Fig. 6. Attainment surfaces of S_DTLZ3_M3 with supplied bound and with $f_i \in [1.0, 2.5]$.

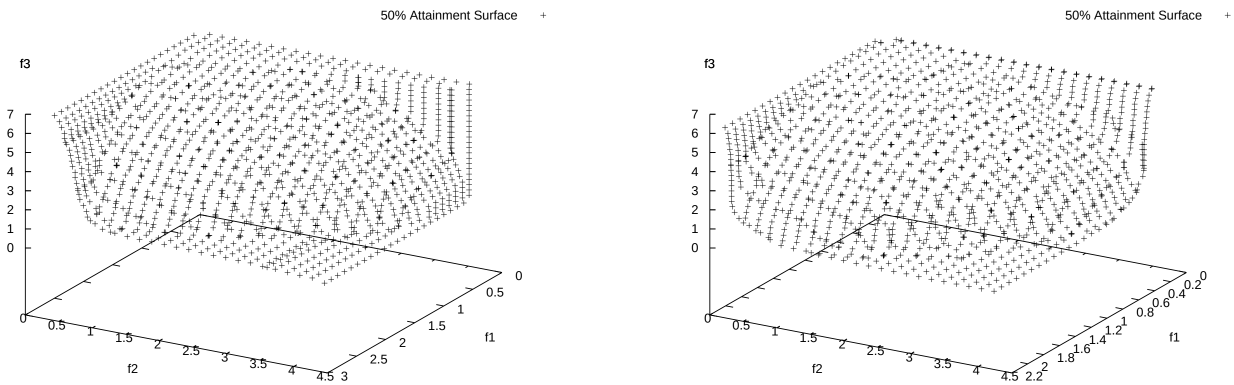

Fig. 7. Attainment surfaces of WFG8_M3 and WFG9_M3.

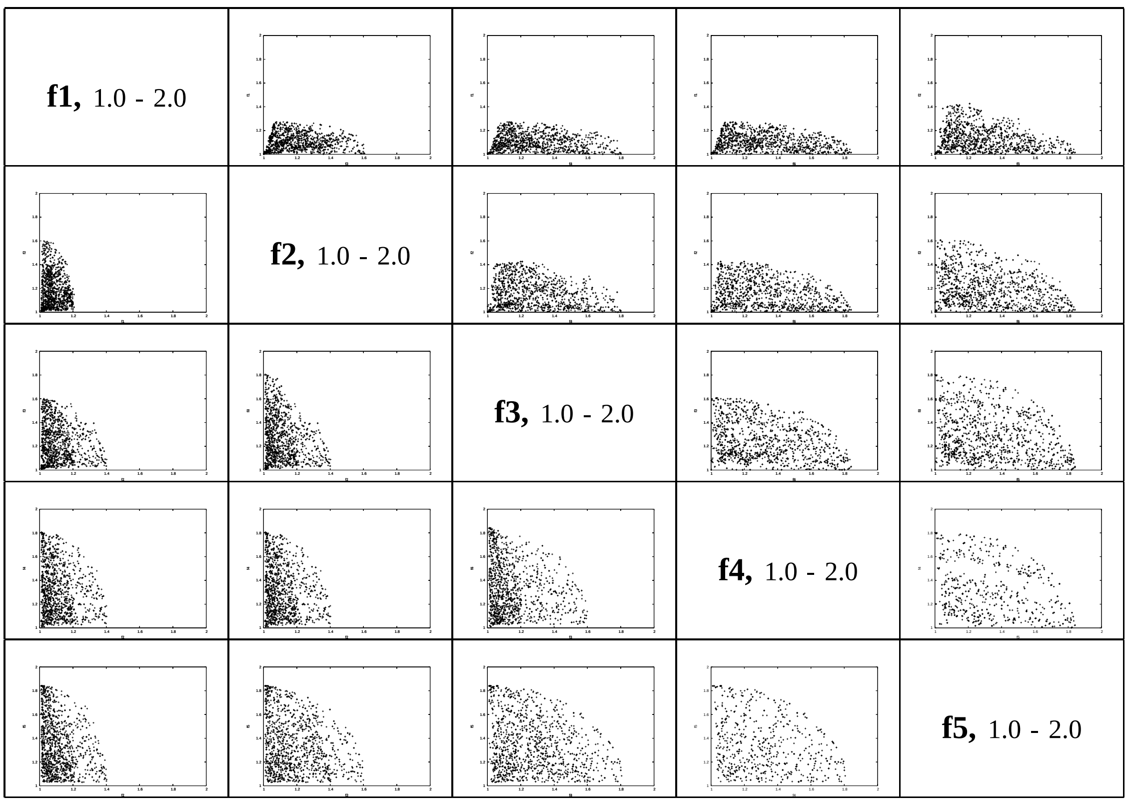

Fig. 8. Upper diagonal plots for WFG8 ($M = 5$) and lower diagonal plots are for WFG9 ($M = 5$) with respect to the median approximation set of R-indicator at $5(10^5)$ FES.

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