



Visualizing Multi-Dimensional Pareto-Optimal Fronts with a 3D Virtual Reality System

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Abstract—In multiobjective optimization, there are several targets that are in conflict, and thus they all cannot reach their optimum simultaneously. Hence, the solutions of the problem form a set of compromised trade-off solutions (a Pareto-optimal front or Pareto-optimal solutions) from which the best solution for the particular problem can be chosen. However, finding that best compromise solution is not an easy task for the human mind. Pareto-optimal fronts are often visualized for this purpose because in this way a comparison between solutions according to their location on the Pareto-optimal front becomes somewhat easier. Visualizing a Pareto-optimal front is straightforward when there are only two targets (or objective functions), but visualizing a front for more than two objective functions becomes a difficult task. In this paper, we introduce a new and innovative method of using three-dimensional virtual reality (VR) facilities to present multi-dimensional Pareto-optimal fronts. Rotation, zooming and other navigation possibilities of VR facilities make easy to compare different trade-off solutions, and fewer solutions need to be explored in order to understand the interrelationships among conflicting objective functions. In addition, it can be used to highlight and characterize interesting features of specific Pareto-optimal solutions, such as whether a particular solution is close to a constraint boundary or whether a solution lies on a relatively steep trade-off region. Based on these additional visual aids for analyzing trade-off solutions, a preferred compromise solution may be easier to choose than by other means.

I. INTRODUCTION

IN MANY real-world problems, decision making with multiple conflicting objectives in every day operation can be demanding. Moreover, an unfavorable decision can be financially expensive or even hazardous in some situations. There exist different ways to support a decision making process [1], and some of them are based on multiobjective optimization. Multiobjective optimization methods are capable of handling multiple conflicting objectives at the same time. Solutions of the multiobjective optimization problem form a Pareto-optimal front, i.e. a set of compromised trade-off solutions. However, even when different Pareto-optimal solutions are found,

choosing a particular optimal compromise solution is not a trivial task. Furthermore, the objectives in a multiobjective optimization task do not need to be commensurable. In such a case, the multiobjective decision making task gets more difficult, especially when the number of objectives is larger than two. This is why there is a need for developing new methodologies for supporting the decision making process.

A Pareto-optimal front, from where an individual final solution can be chosen (e.g. by a decision maker [2]) is often studied with different visualization tools. In this way, the decision maker can extract useful information from the results and thus, a comparison between solutions gets easier. A Pareto-optimal front is quite simple to visualize when there are only two objective functions. However, visualizing with more than two objectives has so far been problematic, and few attempts have been made to visualize a higher dimensional Pareto-optimal front [3], [4], [5]. Virtual reality (VR) is a visualization environment that offers facilities to present high-dimensional spaces and it has also been applied for Pareto-optimal fronts, see [6], [7], [8]. In this paper, we suggest the use of the VR environment not only to visualize a higher dimensional Pareto-optimal front, but also to analyze and understand the nature and relative location of solutions in order to help choosing the best solution for the particular problem.

Basically, the VR is a computer created environment which can be used for visualizing three-dimensional (3D) objects (see, e.g. [9], [10]). Hence it makes possible to visualize and compare solutions which are on a 3D Pareto-optimal front. Thus, a visualized 3D Pareto-optimal front can be examined in many ways: it can be zoomed and rotated, and it also allows the decision maker to dive into the front to get a feel of the nature of the solutions. Moreover, VR enables the user to interact with the visualized Pareto-optimal solutions. This makes easier to compare neighboring solutions and allows the decision maker to learn about the problem and

the interrelationships among objectives. Based on these VR facilities the decision maker can identify a particular solution which is an adequate compromise.

In this paper, there are two case studies utilizing 3D VR facilities presented. First, we use an evolutionary computation-based multiobjective optimization scheme for generating a large number of Pareto-optimal solutions. In addition, an interactive visualization scheme in the VR is used to decipher some interesting features of the solutions obtained by using existing methodologies such as the concept of *innovization* [11]. Second, there is an industrial example having four conflicting objectives presented. Although we show a few capabilities of a VR system here for decision making purpose, certainly many other innovative methodologies are possible, and this paper should encourage execution of further studies in the coming years.

II. MULTIOBJECTIVE OPTIMIZATION AND VIRTUAL REALITY

A multiobjective optimization problem is often defined as follows:

$$\begin{aligned} & \text{minimize } \{f_1(\mathbf{x}), \dots, f_k(\mathbf{x})\} \\ & \text{subject to } \mathbf{x} \in S, \end{aligned} \quad (1)$$

where \mathbf{x} is a vector of decision variables from the feasible set $S \subset \mathbb{R}^n$ defined by linear, nonlinear and box constraints. An objective vector can be denoted by $\mathbf{f}(\mathbf{x}) = (f_1(\mathbf{x}), f_2(\mathbf{x}), \dots, f_k(\mathbf{x}))^T$. Here we minimize, but if an objective function f_i is to be maximized, it is equivalent to consider minimization of $-f_i$.

Optimality in multiobjective optimization is understood in the sense of Pareto-optimality or non-dominated solutions [12]. The Pareto-optimality is defined as follows: a decision vector $\mathbf{x}' \in S$ is Pareto-optimal if there does not exist another decision vector $\mathbf{x} \in S$ such that $f_i(\mathbf{x}) \leq f_i(\mathbf{x}')$ for all $i = 1, \dots, k$ and $f_j(\mathbf{x}) < f_j(\mathbf{x}')$ for at least one index j . These Pareto-optimal solutions form a Pareto-optimal set or a Pareto-optimal front. There are two concepts often used in multiobjective optimization: an ideal objective vector $z^* \in \mathbb{R}^k$ and a nadir objective vector $z^{nad} \in \mathbb{R}^k$ that give lower and upper bounds, respectively, for the objective functions in the Pareto-optimal front (see [2] for details). All the Pareto-optimal solutions are equally good compromises from a mathematical point of view, and there exists no trivial mathematical tool to find the best solution in the Pareto-optimal front. Typically a decision maker, who is an expert in the field from where the problem has arisen, is needed in order to find the best or the most satisfying solution. The decision maker can participate in the process of finding the solution in the different ways and also the different phases of solving process by determining which of the Pareto-optimal solutions is the most satisfying to be the final solution. However, decision making is sometimes tricky, because comparing the numerical values of the solutions is difficult. Thus, some additional information and aids are needed to support decision making process.

A. Virtual Reality Environment

Virtual reality is a medium which makes it possible to visualize and experience objects from an animated world having visual, sound and haptic experiences realized through immersion, interaction, and collaboration of the VR elements. According to [10] four key elements create the VR environment: a virtual world, immersion, sensory feedback, and interactivity. *The virtual world* is a content of given medium including a collection of objects, and their relationships and rules in the space. *Immersion* into an alternative reality means possibility to perceive something besides the world in which one is living currently. Immersion is sometimes divided into physical and mental immersions, but often they both exist in a VR system. *The sensory feedback* is the third key element. It is based on user's physical position, and the aim is that the objects and the whole space alter depending on the user's position. The last element is *interactivity* which means real-time response to the user's actions. There are many applications that can utilize virtual reality technology: visualizing scientific results, interior design in architecture, and prototype testing in industry [13], [14], for example.

In the VR laboratory objectives can be examined in their real size or small objectives can be enlarged, which makes VR usable in several applications. VR can be also build in PC environment with feasible equipment, software, active stereo glasses etc. Then navigation is not comprehensive as in laboratory, but the user is still able to interact with VR similarly to laboratory environment.

B. 3D Virtual Reality Utilized in Multiobjective Optimization

The VR system can be used in visualizing Pareto-optimal solutions, and thereby supporting the decision making in a multiobjective optimization process. The flexibility associated with a 3D VR system makes it an alternative way of visualizing the Pareto-optimal front and suing visual information in the decision making process. In addition, analyzing the solutions in order to understand the interactions of objective functions and decision variables comes easier. In Fig. 1 the users utilize a VR environment in order to examine an approximated Pareto-optimal front. The front is controlled (zoomed, rotated and scaled) by the user using a wand (also called a 3D-mouse). In the VR, the users can study the relationships between objective functions and then get ideas what kind of compromises between the multiple objectives can be made. This process will then aid in selecting the final compromised solution. When complexity of the data increases, valuable information of the Pareto-fronts and problem's behaviour can be extracted from graphical presentation efficiently. One should note that the real immersion and 3D objects can be experienced only in a virtual reality laboratory, not in the figures presented in this paper.

Integration of multiobjective optimization and VR requires a computerized algorithm for calculating Pareto-optimal solutions and a hardware system for a VR environment coupled with a software for visualization in stereo [10]. The VR environment used in this research has been built at the University of Kuopio in Finland, and it is based on OpenDX visualization

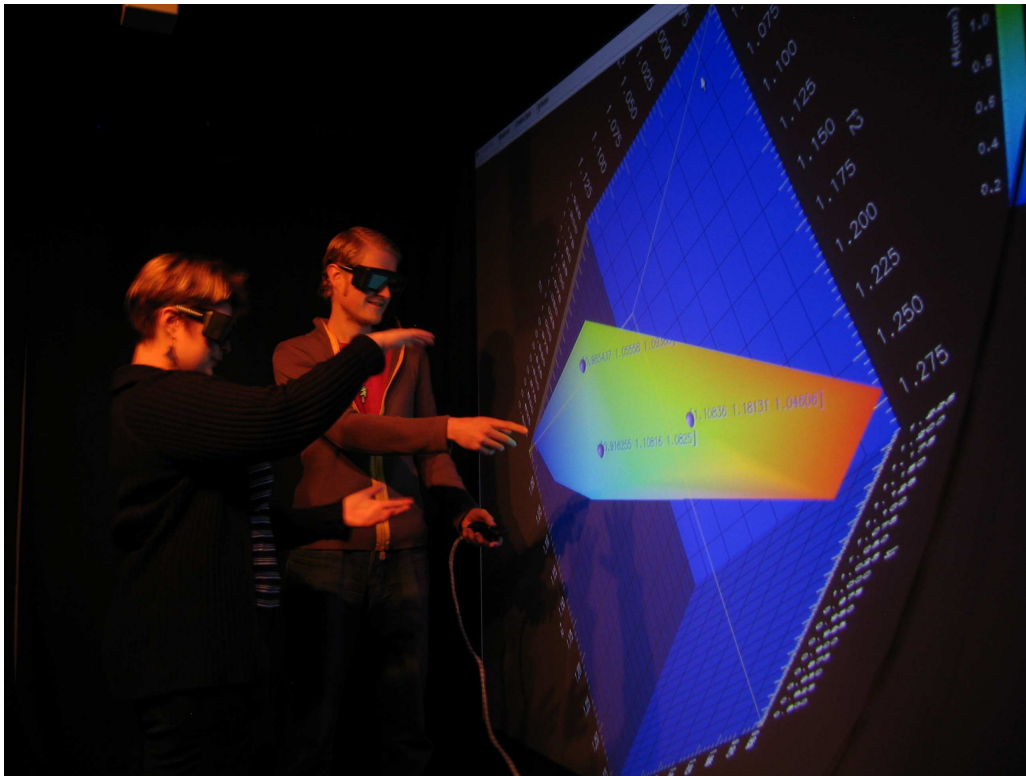


Fig. 1. Analyzing a Pareto-optimal front with the 3D VR system. A 3D Pareto-optimal front visualized in the VR system is presented in a two-dimensional figure here, because the real immersion on a front can be experienced only in a laboratory.

software. Graphics Computer SGI Prism with 8 CPUs (a 64-bit, 1.5 GHz, Intel Itanium 2, 24 GB memory, 48 Gflops) is used with SuSe Linux Enterprise Server 9.3 as an operating system. 3D-effects are generated through wireless set of liquid crystal shutter eye wear (active stereo glasses). Stereo glasses shut alternately left and right eye view with frequency about 45 pictures per eye per second. Visualized 3D-objects (in this paper Pareto-optimal fronts) are controlled through the wand. Polhemus equipment is used to follow the wand's movements to control the objects in the VR environment.

III. VISUALIZATION EXAMPLES

In this section, we present two examples illustrating the new visualizing aspects which a VR facility can offer. In the first example, a standard three-objective test problem having a disconnected set of non-linear Pareto-optimal fronts was solved by evolutionary multiobjective optimization (EMO). EMO procedures are generic population-based meta-heuristic optimization algorithms [12]. They use natural evolutionary principles, such as reproduction, mutation and recombination, iteratively to attempt to find a set of Pareto-optimal solutions. EMO methodologies are capable of finding a large set of trade-off solutions as presented in the first example.

In the second example, we consider a real-world paper-making optimization problem with four objective functions in which the advantages of decision making aspects with the VR system are presented. This example was solved by a classical multiple criteria decision making method. The solution process

contained two steps, and in the first step a genetic algorithm with a scalarizing function [2] was used. Then neighboring solutions were connected with a hyper-plane in visualization, and any point in the hyper-plane could be chosen as a reference point. Thus, in the second step the corresponding Pareto-optimal solution could be obtained by solving an achievement scalarizing function [2], [5], [15].

A. A Test Problem with Highlighted Solutions

First, we considered a multiobjective optimization test problem (DTLZ6) [16]. In the general form of this problem, there are k objective functions with a complete decision variable vector partitioned in k non-overlapping groups $\mathbf{x} \equiv (\mathbf{x}_1, \dots, \mathbf{x}_k)^T$. We solved a three-objective version of the problem that is written as follows [16]:

$$\begin{aligned} & \text{minimize} && \{f_1(\mathbf{x}), f_2(\mathbf{x}), f_3(\mathbf{x})\} \\ & \text{subject to} && 0 \leq x_i \leq 1 \text{ for } i = 1, \dots, 22, \end{aligned} \quad (2)$$

where the objective functions were defined as $f_1(\mathbf{x}_1) = x_1$, $f_2(\mathbf{x}_2) = x_2$ and $f_3(\mathbf{x}) = (1 + g(\mathbf{x}_3))h(f_1, f_2, g)$. The functionals were $g(\mathbf{x}_3) = 1 + \frac{9}{|\mathbf{x}_3|} \sum_{x_i \in \mathbf{x}_3} x_i$ and $h(f_1, f_2, g) = 3 - \sum_{i=1}^2 \left[\frac{f_i}{1+g} (1 + \sin(3\pi f_i)) \right]$. The functional $g(\mathbf{x}_3)$ required $|\mathbf{x}_3| = 20$ variables and n was the total number of variables, here $n = 22$. In this test problem, there were $2^2 = 4$ disconnected Pareto-optimal regions.

The NSGA-II procedure [12] was used as an EMO algorithm in this study and it was run with 1,000 population

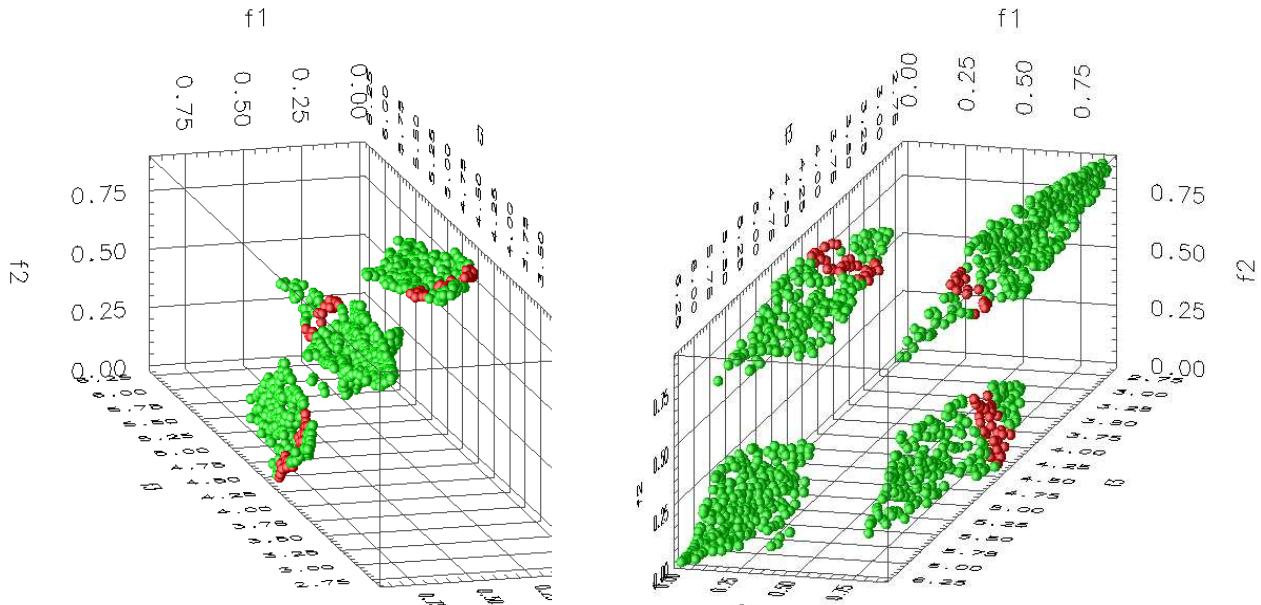


Fig. 2. Disconnected set of nonlinear Pareto-optimal regions (forming the Pareto-optimal front) in evolutionary computation example (DTLZ6) in a VR environment. Solutions having $4.2 \leq f_3 \leq 4.4$ are colored red.

members. The final solutions were visualized with the proposed VR system and they are shown also in Fig. 2. With the 3D visualization capabilities, the nonlinear feature of the disconnected Pareto-optimal regions was much easier to see compared to the earlier studies [16]. In this example, the decision maker was first interested in seeing all the solutions in which $4.2 \leq f_3 \leq 4.4$ as presented in Fig. 2. As one can see in the figure, this constraint made possible quite different compromises between the other two objectives: highlighted solution were located into three separate Pareto-optimal regions. Thus, understanding the trade-offs between different targets was more clear. Moreover, because of zooming, rotating, and immersion possibilities, the Pareto-optimal front was easy to comprehend. Also, the trade-offs between targets were easy to understand, and hopping from one Pareto-optimal region to another got simplified.

In the concept of *innovization* [11], the task of evolutionary multiobjective optimization is followed by a search of hidden interactions among decision variables and objective functions within obtained solutions. This concept has revealed interesting and important insights about design and optimization problems. Here, we argue that the proposed VR based visualization tool can be used as an aid to assist in the *innovization* task. Combining these two concepts allows the decision maker to test the validity of different interrelationships among the decision variables and objective functions. For example, the existence of a given relationship, such as $\Phi(\mathbf{f}, \mathbf{x}) = 0$, can be tested by marking all solutions (among the obtained EMO solutions) which restrict the absolute value of Φ within a threshold, say $\epsilon = 10^{-6}$, in red. The location and trace of these solutions on the Pareto-optimal front will provide a plethora of information to the decision maker about the importance of the above relationship before choosing a particular solution.

To illustrate, we return to the DTLZ6 test problem and investigate the existence of Pareto-optimal solutions satisfying the following relationships:

$$\begin{aligned} \Phi_1(\mathbf{f}, \mathbf{x}) &: x_1 = 0 \quad (\text{Red}), \\ \Phi_2(\mathbf{f}, \mathbf{x}) &: x_2 = 0 \quad (\text{Blue}), \\ \Phi_3(\mathbf{f}, \mathbf{x}) &: x_1 = 1 \quad (\text{Brown}), \\ \Phi_4(\mathbf{f}, \mathbf{x}) &: x_2 = 1 \quad (\text{Purple}). \end{aligned}$$

The above conditions check if any Pareto-optimal solution made box constraints on variables x_1 and x_2 active. In Fig. 3, there are marked all such solutions with $\epsilon = 10^{-6}$. It is interesting to note that there were no solutions on the Pareto-optimal front close to the upper bound of these two variables and there were a number of solutions which were close to their lower bounds. Only a few solutions made x_1 close to zero, but there exist a number of solutions which made x_2 close to zero. Furthermore, all these solutions seemed to lie on only one of the four Pareto-optimal regions. It could be useful to identify solutions close to constraint boundaries and a further investigation and relaxation of active constraints could lead to better solutions. Such information was not only interesting but could be useful if problem-specific relationships were tested.

As seen from this example, the VR environment can be used as a 3D visualization tool for Pareto-optimal solutions obtained with an EMO procedure. These solutions can be studied with a VR tool not only to make a better visualization of the front, but also to gather more useful information and properties of Pareto-optimal solutions. Next, we present a more complex real-world industrial decision making problem.

B. Industrial Example: Papermaking Optimization

In papermaking, the aim is to produce paper as much as possible with as low costs as possible [17], [18], [19]. In addition, there are several quality properties which should

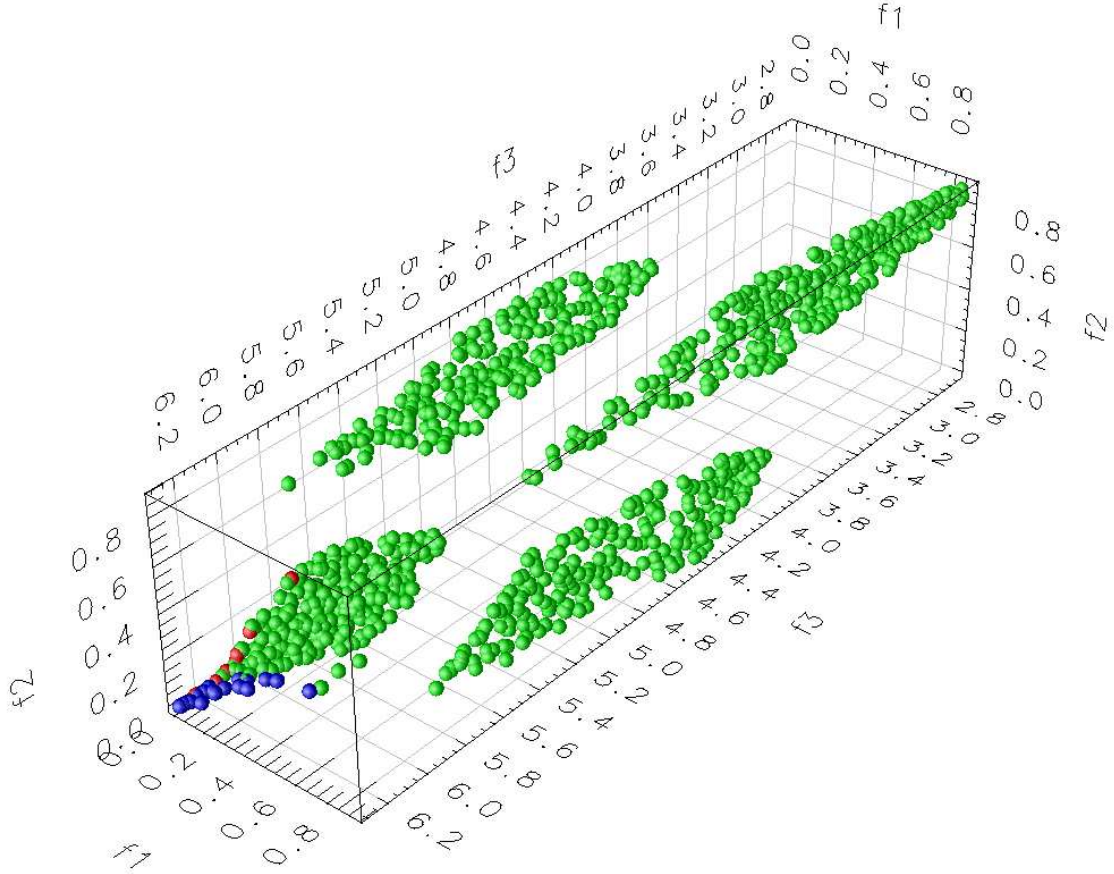


Fig. 3. Pareto-optimal solutions close to the constraint boundaries are highlighted with red and blue colors using the VR environment for the DTLZ6 test problem.

obtain acceptable values at the same time. These targets are often conflicting, and thus the optimization problems become naturally multiobjective. In this example a papermaking optimization problem is studied. Because of the long computational time, the decision maker wanted to compute as few solutions as possible. The Pareto-optimal solutions computed were visualized as 3D points in the VR, and an approximation of a Pareto-optimal front was formed with these few solutions.

In this example, there were four papermaking objectives and eight decision variables. The problem was formed as a model-based optimization problem [18], where the objective function values could be evaluated based on the solution of equations describing the system, i.e. a simulation model of a paper machine. Thus, the optimization problem was written as follows:

$$\begin{aligned} & \text{optimize } \{f_1(\mathbf{x}, \mathbf{q}_1, \dots, \mathbf{q}_{27}), \dots, f_4(\mathbf{x}, \mathbf{q}_1, \dots, \mathbf{q}_{27})\} \\ & \text{subject to } \begin{cases} A_1(\mathbf{x}, \mathbf{q}_1) = 0 \\ A_2(\mathbf{x}, \mathbf{q}_1, \mathbf{q}_2) = 0 \\ \vdots \\ A_{27}(\mathbf{x}, \mathbf{q}_1, \dots, \mathbf{q}_{27}) = 0 \end{cases} \\ & \mathbf{x} \in S, \end{aligned} \quad (3)$$

where f_1 presented paper tensile strength ratio and it was given the desired value 3.4. The objective functions f_2 and

f_3 were paper formation and basis weight, which were given the desired values 0.36 g/m^2 and 50.5 g/m^2 , respectively. The fourth objective function f_4 was evaporated water which was to be maximized. A vector $\mathbf{x} \in S$ contained all the decision variables that were typical controls of paper machine and the feasible set S was formed of their box constraints. Mappings A_i for all $i = 1, \dots, 27$ denoted unit-process models constituting a simulation model for the entire papermaking process, and \mathbf{q}_i , $i = 1, \dots, 27$ were the simulation model outputs [18].

The optimization process contained two separate steps. *In the first step*, a set of the trade-off solutions were calculated with a genetic algorithm with scalarization by achievement scalarizing function. Then, an approximation of the Pareto-optimal front was generated in the VR environment using these solutions. The left plot in Fig. 4 shows the solutions and the approximated front obtained after the first step. The values of the objective functions f_1 , f_2 and f_3 are presented on the axes and f_4 is presented by colour in the figure. Here, the proposed VR environment was found to be quite effective tool to explore the multidimensional Pareto-optimal solutions and the approximated front between them. The decision maker observed that there was a conflict between the first two objective functions, i.e. a good tensile strength ratio caused a large formation value which was not desired and vice versa. Thus, there exists a trade-off. Another observation was that a

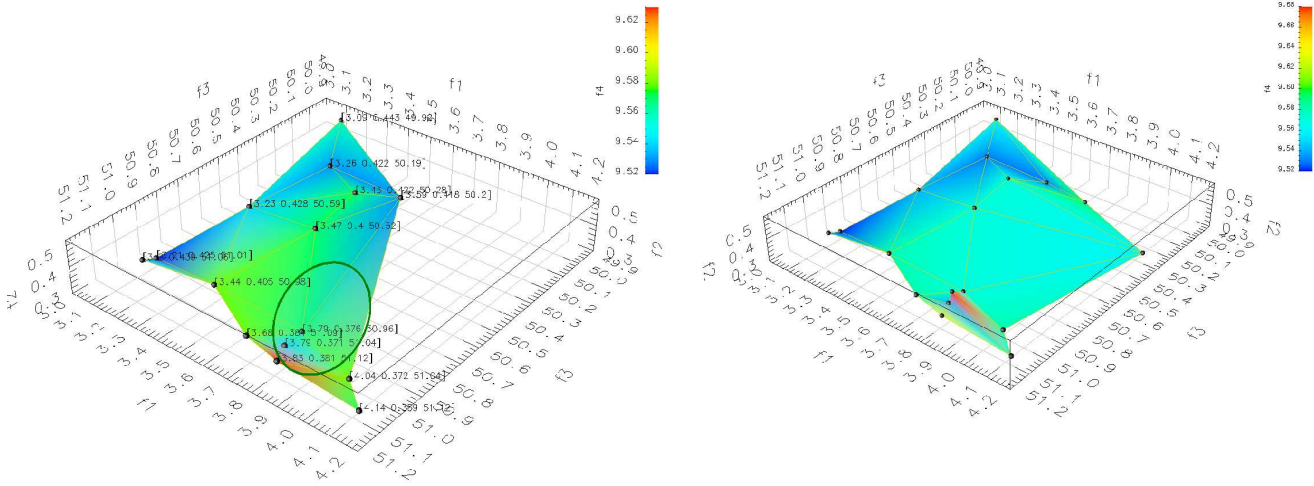


Fig. 4. On the left, solutions after the first step, and on the right, all solutions after the first and the second steps. The objective functions from f_1 to f_3 are presented as a 3D surface and f_4 is presented as a variation in color.

large value of the fourth objective function came with a large value of the third objective function, thereby producing also a conflict between these two objective functions: the desired value of f_3 could not be achieved at the same time with a good value of f_4 . However, there were good compromise solutions between the objective functions on middle and front part of the approximated set. Based on these observations, the region highlighted by an ellipse (shown in the figure) was chosen and the optimization process was re-directed towards this region in the second step. This preference information was obtained with the help of visualization through the VR tool, where the decision maker could examine the existing solutions in many ways by rotating and zooming the Pareto-optimal set.

In the second step, three new solutions were calculated with help of the reference point method and the gradient-based optimizer. The decision maker's preferences, the circled region in Fig. 4, was utilized in defining the reference points. Unfortunately, only one of the three new solutions generated was located into the preferred region and other two were located in such a part of the solution space, where there were no solutions after the first step. The plot on the right side in Fig. 4 shows all the solutions, that is the solutions produced in the first step complemented by three solutions produced in the second step. Two of the new solutions were interesting from the papermaking point of view: one located inside the preferred region (circled in Fig. 4) and another one located on the right side having values $(\mathbf{f} = (3.78, 0.39, 50.19, 9.59)^T)$, which presented also a good compromise between the objective functions. However, the first-mentioned solution (inside the circled part) had objective function values: $\mathbf{f} = (3.79, 0.41, 51.02, 9.68)^T$ and it was the most satisfying compromise solution to be the final one according to the decision maker's knowledge.

The ability to visualize trade-off information among objective functions through the 3D VR system makes it possible to focus on interesting part of the solution space. This will certainly enhance the decision making ability in computation-

ally demanding real-world optimization problems and reduce the number of uninteresting solutions needed be calculated. In addition, better visualization technique allows one to get more information about the relationships between the solutions and objective functions than a simple plot of the numerical data. We believe that the VR tool will help the decision maker to understand and analyze the Pareto-optimal front, and thus make it easier to choose a single preferred solution.

IV. DISCUSSIONS AND CONCLUSIONS

In multiobjective decision making, Pareto-optimal fronts are often visualized because in this way a comparison between solutions becomes easier. A Pareto-optimal front is easy to visualize when there are only two objective functions, but visualizing more than two objective functions is problematic. In this paper, we have integrated multiobjective optimization with the 3D VR tool to study the Pareto-optimal solutions and approximated Pareto-optimal fronts to help to make a better decision when choosing the final solution. The 3D VR tool makes easier to compare solutions, navigate from one solution to the other by zooming and rotating the front. Thus, it allows a better comprehension of solutions with desired properties through highlighting. In addition, using sophisticated visualization tools means that fewer solutions need to be computed in order to learn and understand the interrelationships of the conflicting objectives. This is important especially if a problem is computationally expensive (e.g. in real-world industrial cases).

In this paper, different kinds of 3D visualizations with the VR environment have been discussed and demonstrated. First, a large number of solutions forming a dense set of Pareto-optimal solutions obtained by EMO was visualized. In this set, some interesting features of the solutions were highlighted and studied. In the second problem, a few Pareto-optimal solutions were calculated with different optimization techniques and they were visualized using the proposed VR environment. The information gathered from this exercise helped to find

an interesting Pareto-optimal region for the decision maker to concentrate. Such a technique will be valuable for handling large number of objective functions.

This paper, so far, has shown a number of advantages of using a VR environment in making a better realization of the Pareto-optimal front in a multiobjective optimization task. In addition, we have emphasized capabilities of VR that helps decision making in real-world applications, which we see as one of the potential application of the VR systems. These initial results are promising and open up a number of challenging research issues, such as handling a large number of objective functions, simultaneous visualization of objective and solution spaces, faster an optimization software and a VR hardware interactions, etc. The purpose of this paper is to discuss the power and usefulness of the VR environment in multiobjective optimization, and to bring out the technique as a new and promising mean of visualizing and understanding complex interactions among objectives and solutions.

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