

A Fermion-like description of condensed Bosons in a trap

R.K.Bhaduri and M. V. N. Murthy*

Department of Physics and Astronomy, McMaster University,

Hamilton, Ontario, Canada L8S 4M1

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Abstract

A Bose-Einstein condensate of atoms, trapped in an axially symmetric harmonic potential, is considered. By averaging the spatial density along the symmetry direction over a length that preserves the aspect ratio, the system may be mapped on to a zero temperature noninteracting Fermi-like gas. The “mock fermions” have a state occupancy factor ($\gg 1$) proportional to the ratio of the coherence length to the “hard-core” radius of the atom. The mapping reproduces the ground state properties of the condensate, and is used to estimate the vortex excitation energy analytically. The “mock-fermion” description predicts some novel collective excitation in the condensed phase.

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*Permanent Address: The Institute of Mathematical Sciences, Madras 600 113, India

Recently there has been a renewed interest in the Bose-Einstein condensation(BEC) of a gas after its experimental demonstration [1] with rubidium vapour in a trap at a temperature of 170 nanokelvin and at a number density $\rho = 2.5 \times 10^{12}$ atoms per cc. This experiment has been followed by others using alkali atoms [2]. At these temperatures the atoms form a weakly interacting metastable gas of Bosons. For a non-technical account see the review by Burnett [3]. In a typical device, atoms are trapped in a potential which is well described by an axially symmetric parabolic confinement. The oscillator frequency in the symmetry direction is larger than the frequency in the plane perpendicular to it. The experimental situation of interest to us is the one with rubidium vapour, where the s-wave scattering length between two atoms is known to be positive. The effect of the interatomic interaction may be mocked up by a repulsive pseudo-potential [4]. The interaction energy is proportional to $a\rho^2$, where a is the s-wave scattering length and ρ is the number density of the atoms. The properties of the condensate have been studied by constructing the density functional involving this repulsive interaction energy, and the potential energy of the atoms in the trap [5–7]. In this paper, we first note that by averaging the spatial density of the condensed bosons along one direction, it may be reduced to the same form as the density of a non-interacting Fermi gas. We chose the averaging direction to be the symmetry axis (the z-direction), along which the harmonic confinement is steeper. *This enables us to use the Fermi gas model to compute the low-lying planar excitations of the condensate in the shallow well.* Moreover, the averaging distance is chosen to preserve the aspect ratio (the ratio of the length scales in the planar to perpendicular directions) of the original trap. The Bose-condensate is now described by a three-dimensional noninteracting “Fermi” gas, trapped in the same planar parabolic potential as the original system, but free to move along the z-direction within the averaging distance. One peculiarity of these mock-fermions, as we call them, is their occupancy factor per state. Instead of being one (or zero) at $T = 0$, it is multiplied by a large factor proportional to $\lambda_F/2a$, where $\lambda_F = h/p_F$, and $2a$ the apparent size of the atom. In fact, the Fermi momentum $\hbar k_F$ is such that $k_F^{-1} = \xi$, where $\xi = (8\pi\rho a)^{-1/2}$ is just the coherence length in the bose condensate. It is remarkable that the kinetic energy of these

mock fermions exactly reproduces the condensate energy in the large-N limit. The latter is in fact calculated by neglecting the kinetic energy of the bosons. After having shown this equivalence, we go on to use this model to calculate some other properties. These include the velocity of sound, and the vortex excitation energy. The sound velocity in the mock-fermion ideal gas is the same as in the Bose-condensate. A simple estimate of the vortex excitation energy is made by “digging” a hole in the central density, i.e., promoting all the s-state mock-fermions out of the Fermi sea to states of non-zero angular momentum. This reproduces the numerical results of Dalfovo and Stringari [6] satisfactorily. The latter calculation involved solving a nonlinear Schrödinger equation that was obtained from the density-functional formalism. Finally, our description also predicts some novel collective excited states with zero angular momentum involving a large number of mock-fermions.

We begin with the ground state energy for condensed bosons given by the Ginzberg-Gross-Pitaevskii [8] energy functional,

$$E[\psi] = \int d^3r \left[\frac{\hbar^2}{2m} |\nabla\psi|^2 + \frac{m}{2} (\omega_{\perp}^2 r_{\perp}^2 + \omega_3^2 z^2) |\psi(r)|^2 + \frac{2\pi\hbar^2 a}{m} |\psi(r)|^4 \right], \quad (1)$$

where m is the mass of the atom, ω_{\perp}, ω_3 denote the oscillator frequencies in the transverse direction and in the direction of the symmetry axis (z-axis), and a is the s-wave scattering length which defines the strength of the interaction in the pseudo-potential method. The condensate wave function is usually denoted as

$$\psi(r) = \sqrt{\rho(r)} \exp is(r). \quad (2)$$

We work in the limit of strong repulsive interaction where the kinetic energy term can be neglected in the condensate phase. In the large N-limit we can then obtain the density $\rho(r)$ by minimising $(E - \mu N)$, where μ is the chemical potential. This is the Thomas-Fermi expression for the density within the classical turning points [6] :

$$\rho(r) = \frac{m}{4\pi\hbar^2 a} \left[\mu - \frac{m}{2} (\omega_{\perp}^2 r_{\perp}^2 + \omega_3^2 z^2) \right]. \quad (3)$$

The density is zero outside the turning points. Note that when the scattering length a is positive, the chemical potential μ is necessarily positive since $\rho(r) \geq 0$. This is the situation

with the BEC of rubidium atoms. It is now straight-forward to calculate the particle number and the energy using the density given above. The particle number is given by,

$$N = \int d^3r \rho(r) = \frac{a_3^2}{15aa_\perp} \left(\frac{2\mu}{\hbar\omega_\perp}\right)^{5/2}, \quad (4)$$

where

$$a_\perp = \sqrt{\frac{\hbar}{m\omega_\perp}}; \quad a_3 = \sqrt{\frac{\hbar}{m\omega_3}}, \quad (5)$$

The integration limits are set by the turning points in the z - and the r_\perp -directions. These may be obtained by first performing the z - integration for a fixed r_\perp , and then allowing the latter to vary within the prescribed limits. The condensate energy is obtained by substituting the expression (3) for the density in Eq.(1), and is given by

$$E = \frac{a_3^2}{42aa_\perp} \left(\frac{2\mu}{\hbar\omega_\perp}\right)^{7/2} \hbar\omega_\perp. \quad (6)$$

As noted before, the kinetic energy term in Eq. (1) is neglected. These are known results. Before we go further, we note that the ratio $E/N = 5\mu/7$. This is indeed the energy per particle in a non-interacting fermionic system whose single-particle density of states $g(E) \propto E^{3/2}$, obtained by convolving the state density in a planar parabolic potential, with the density in the transverse direction.

We now take the crucial step by averaging the density in the direction of the symmetry axis over the a scale L_3 :

$$\rho_a(r_\perp) = \frac{1}{L_3} \int dz \rho(r) . \quad (7)$$

We refer to ρ_a as the mock-fermion density for reasons that will be clear soon. The length scale L_3 ($-L_3/2 \leq z \leq L_3/2$) will be fixed later by fitting the aspect ratio. A little algebra shows that the result may be written as

$$\rho_a(r_\perp) = \frac{1}{\alpha} \rho_{TF}(r_\perp), \quad (8)$$

where

$$\rho_{TF}(r_{\perp}) = \frac{1}{6\pi^2} \left[\frac{2m}{\hbar^2} \left(\mu - \frac{m\omega_{\perp}^2}{2} r_{\perp}^2 \right) \right]^{3/2}, \quad (9)$$

and the dimensionless parameter α is given by

$$\alpha = \frac{L_3 a}{\pi a_3^2}. \quad (10)$$

The Thomas-Fermi expression for the density, ρ_{TF} , given by Eq. (9), is in fact the density of spinless fermions confined in a harmonic potential in the plane and free to move in the z-direction. Note from Eqs. (8, 10) that for $a_3 \rightarrow 0$, the system becomes two-dimensional, and there is no condensation since $\rho_a \rightarrow 0$. Similarly, the condensation density is depleted as the scattering length a is increased. Note that $1/\alpha$ plays the role of the occupancy factor which is not necessarily unity. It is in this sense we call these particles mock-fermions.

To proceed further we first compute the number of particles N and the condensate energy E , using $\rho_a(r_{\perp})$ as given by Eq. (8). Once again the number of particles is defined by

$$N = L_3 \int d^2 r_{\perp} \rho_a(r_{\perp}) = \frac{L_3}{15\pi\alpha a_{\perp}} \left(\frac{2\mu}{\hbar\omega_{\perp}} \right)^{5/2}. \quad (11)$$

Substituting for α from Eq.(10) reproduces the particle number given by Eq.(4). The total energy in fermion-like picture is given by,

$$E[\rho_a] = \frac{1}{\alpha} \int d^3 r \left[\frac{\hbar^2}{2m} \tau(\vec{r}) + U(\vec{r}) \rho_{TF}(\vec{r}) \right], \quad (12)$$

where $\tau(\vec{r})$ is the kinetic energy density of the fermions in the Thomas-Fermi approximation,

$$\tau(\vec{r}) = \frac{1}{10\pi^2} (6\pi^2 \rho_{TF})^{5/3} \quad (13)$$

and $U(\vec{r})$ is the confining oscillator potential in the plane. The particles are of course free to move in the z-direction, but confined within the length L_3 . Substituting for the density and the kinetic energy density in Eq.(12), we immediately obtain the total energy

$$E = \frac{L_3}{42\pi\alpha a_{\perp}} \left(\frac{2\mu}{\hbar\omega_{\perp}} \right)^{7/2} \hbar\omega_{\perp}. \quad (14)$$

Again, substituting for α yields the energy of the Bose condensate as given by Eq.(6). This is remarkable since the energy of the interacting bosons in the BEC is identical to that of the non-interacting mock-fermions.

Note, from Eqs. (11, 14), that both the particle number N and energy E are independent of the choice of L_3 . This is because it comes in the combination L_3/α . This does not mean that the choice of L_3 may be arbitrary, since we shall presently see that that the excited state properties depend crucially on it. A physically meaningful way of fixing the length is to set it through the aspect ratio $\sqrt{\langle x^2 \rangle / \langle z^2 \rangle} = \omega_3/\omega_\perp$. Consequently the size of the cloud is approximately the same in BEC and in the mock-fermion picture. We have cylindrical symmetry in the latter instead of ellipsoidal symmetry of the BEC. In the fermion picture $\langle z^2 \rangle = L_3^2/12$. The average distance in the plane is ($\langle r_\perp^2 \rangle = \frac{2a_\perp^2 \mu}{7\hbar\omega_\perp}$) is the same as in the BEC. Equating the aspect ratios in both pictures we get,

$$L_3 = \sqrt{\frac{24\mu}{7m\omega_3^2}} . \quad (15)$$

This then determines the occupancy factor α uniquely,

$$\alpha = \frac{a}{\pi} \sqrt{\frac{12}{7}} \sqrt{\frac{2m\mu}{\hbar^2}} , \quad (16)$$

where a is the scattering length. Note that α now depends on the chemical potential. To obtain the sound velocity in the interior of the cloud, we set the density ρ_a from Eq. (8) to its central value:

$$\rho_a = \frac{1}{6\pi^2\alpha} \left[\frac{2m}{\hbar^2} \mu \right]^{3/2} . \quad (17)$$

But from Eq. (16), α is proportional to $\sqrt{\mu}$, yielding the correct linear dependence between the density ρ and μ as in BEC. We then obtain the same sound velocity u_s as in BEC :

$$u_s^2 = \frac{\rho_a}{m} \frac{\partial \mu}{\partial \rho_a} = \frac{\mu}{m} . \quad (18)$$

By contrast, $u_s^2 = 2\mu/3m$ in a free Fermi gas with constant α . By setting $k_F^2 = \frac{2m\mu}{\hbar^2}$, we see from Eq. (16) that α is proportional to the dimensionless quantity ak_F . This results in the occupancy factor $1/\alpha$ being

$$\frac{1}{\alpha} = \sqrt{\frac{7}{12}} \frac{\lambda_F}{2a}. \quad (19)$$

Since the de Broglie wave length $\lambda_F = 2\pi/k_F$ may be regarded as the resolving power, we may interpret the degeneracy described by the parameter $1/\alpha$ as the collective number of the atoms which can be accommodated within a wave length. The quantum mechanical wave functions of all these atoms overlap substantially over a wave length. Thus a single “fermion” in our picture is as if made of $1/\alpha$ number of mock-fermions. It is also easy to check that

$$k_F^{-1} = (8\pi a\rho)^{-1/2} = \xi, \quad (20)$$

where ξ is the coherence length of the bosons in the condensate [6]. Using the parameters of the experiment [1], it is straight-forward to get a numerical estimate of this collectivity. For example, for $N = 5000$, we find $1/\alpha = 140$. Note that in our picture the mock-fermions are essentially free apart from the confinement in the plane and over a length in the z-direction. The single fermion excitations in this picture may now be described as the excitation of $1/\alpha$ mock-fermions. This has interesting consequences in computing the excitation energies of the system.

As an immediate application we may consider the vortex states discussed by Dalfalvo and Stringari [6]. In our Fermi gas picture, we may make an estimate for a vortex excitation in the static “shell model” picture. For making this estimate, consider the ground state in which mock-fermions multiply occupy the states of the two-dimensional harmonic oscillator upto the Fermi energy. Since the density profile of a vortex has a hole in the centre, we may simulate it by promoting all the s-state mock-fermions out of the Fermi sea to states with $l \neq 0$. Each of these s-states contain $1/\alpha$ mock-fermions in our picture. Keeping N fixed, the number of $l = 0$ states below the chemical potential μ is given by $\mu/(2\hbar\omega)$ for large μ . Since the degeneracy of the harmonic oscillator state just above the the Fermi energy is $(\mu + 1)/\hbar\omega_{\perp}$, it is possible to promote all the s-state particles to this state (upto a maximum angular momentum). The excitation energy per particle in the presence of a vortex is then

given by,

$$\Delta E/N = \frac{1}{N\alpha} \left(\frac{\mu}{2\hbar\omega_{\perp}} \right)^2 \hbar\omega_{\perp}. \quad (21)$$

This should be compared with the energy $\hbar\Omega_c$ calculated by Dalfovo and Stringari [6] for $\kappa = 1$, where Ω_c is the critical angular velocity and $\hbar\kappa$ the angular momentum of the vortex. In Fig.1, we display the result of our estimate for Ω_c , obtained from Eq. (21), with the graph given in [6] for the same. The agreement for large N is good between our calculations and those obtained by solving the non-linear Schrödinger equation. For comparison we have also displayed the calculated values of Ω_c obtained in the large N limit by Baym and Pethick [5]. While the power law dependence on the chemical potential is the same in our calculation and in [5], we cannot get the dependence on the logarithm of the chemical potential in the static shell model type of analysis. In our naive model, the vortex energy is insensitive to the vortex angular momentum $\hbar\kappa$ upto a maximum.

As pointed out in the introduction, the mock-fermion picture also predicts collective particle-hole excitations from the filled Fermi level to an excited state, where all $\frac{1}{\alpha}$ mock fermions of a given angular momentum state are excited together. Since the adjacent shells of the harmonic oscillator have states of opposite parity, excitations to the next higher state of the $\frac{1}{\alpha}$ mock fermions would entail a huge change in angular momentum. Such high-spin collective excitations are unlikely. There could be, however, collective excitations of energy $\frac{2}{\alpha}\hbar\omega_{\perp}$ and zero angular momentum. Both the above types of excitations are of a novel collective type, and involve a hundred or more mock-fermions. As such, they are much larger energy excited states than the ones predicted and analysed by Stringari [9] recently.

To conclude, the Bose-Einstein condensate of interacting bosons is described in this paper by a noninteracting Fermi gas. We call them mock fermions since they obey a generalised Pauli principle with the occupancy of a state i given by

$$\begin{aligned} n_i &= \frac{1}{\alpha}, \quad \epsilon_i < \mu; \\ &= 0, \quad \epsilon_i > \mu. \end{aligned} \quad (22)$$

This occupancy factor is directly proportional to the coherence length $\xi = (8\pi a\rho)^{-1/2}$ divided by the “size” $2a$ of the atom. The collectivity is generated by the combined motion of $1/\alpha$ mock fermions. In passing, it is worth mentioning that an occupancy factor of the type (22) at $T = 0$ arises naturally in systems that obey the so-called Haldane statistics [10]. It has also been noted that in such systems that the single particle density is scaled by the occupancy factor [11]. Irrespective of this aspect, the present analysis of the bosonic condensate in terms of mock-fermions is interesting in its own right, although one should only expect it to hold for global properties, and not for the analysis of correlations.

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FIGURES

FIG. 1. The critical frequency Ω_c (in units of ω_{\perp}) is shown as a function of the particle number in the fermion-like picture (solid line). The dotted line is based on the graph given by Dalfovo and Stringari[6]. The dashed line is the large N prediction[5].