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Classical and Quantum Mechanics of Anyons

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Abstract

We review aspects of classical and quantum mechanics of many anyons confined in an oscillator potential. The quantum mechanics of many anyons is complicated due to the occurrence of multivalued wavefunctions. Nevertheless there exists, for arbitrary number of anyons, a subset of exact solutions which may be interpreted as the breathing modes or equivalently collective modes of the full system. Choosing the three-anyon system as an example, we also discuss the anatomy of the so called "missing" states which are in fact known numerically and are set apart from the known exact states by their nonlinear dependence on the statistical parameter in the spectrum.

Though classically the equations of motion remains unchanged in the presence of the statistical interaction, the system is non-integrable because the configuration space is now multiply connected. In fact we show that even though the number of constants of motion is the same as the number of degrees of freedom the system is in general not integrable via action-angle variables. This is probably the first known example of a many body pseudo-integrable system. We discuss the classification of the orbits and the symmetry reduction due to the interaction. We also sketch the application of periodic orbit theory (POT) to many anyon systems and show the presence of eigenvalues that are potentially non-linear as a function of the statistical parameter. Finally we perform the semiclassical analysis of the ground state by minimizing the Hamiltonian with fixed angular momentum and further minimization over the quantized values of the angular momentum.

Contents

1 Introduction

In 1977, Leinaas and Myrheim[1] showed that in two space dimensions it is possible to have particles obeying intermediate statistics different from the well known Bose and Fermi statistics. Wilczek[2] later coined the name *anyons* for particles obeying these peculiar statistics. In the last two decades, a lot of work has been done in this field. Many of these developments have been brought up to date and nicely summarized in the books by Lerda[3] and Khare[4] as also in the collection of articles on anyon superconductivity[5]. One aspect that has not been covered in detail in these books and other reviews[6] concerns the question of integrability properties of a system of many anyons and its classification. This review is intended to focus on these aspects which have not been discussed in detail earlier or have been largely left out of earlier attempts.

Here we focus on the non-relativistic classical and quantum mechanics of particles in two spatial dimensions which obey fractional statistics [1, 2]. Generically such particles are referred to as anyons. Anyons as a research area is now over twenty years old. These systems have emerged as being interesting in their own right from the point of view of mathematical physics at both the classical and the quantum level. These systems also constitute an example of inequivalent quantizations due to non trivial fundamental group of the configuration space [7]. It is well known that the existence of fractional statistics is intimately connected to having multivalued wavefunctions which naturally occur in quantum mechanics on multiply connected spaces [8]. The classification of multivalued wave functions is provided by one dimensional representations of the fundamental group of the multiply connected configuration space. The configuration space Q^N is the d-dimensional Euclidean space given by R_2^N with all the diagonal points, Δ, removed. The fundamental group of Q^N turns out to be

$$\pi_1(Q^N) = S_N \quad if \quad d \geq 3$$

and

$$\pi_1(Q^N) = B_N \quad if \quad d = 2$$

where B_N is the braid group of N objects which contains the permutation group S_N as a subgroup. This immediately brings out the difference between two and three dimensions. Although there are several known examples of a kinematic classification [7] (in terms of the representations of the fundamental group), very little is known about the dynamics on such spaces. The motivation for studying such systems is two fold: they are a good and physically relevant example of quantum mechanics on multiply connected spaces. It is physically relevant because anyons were proposed as candidates for explaining fractional quantum Hall effect [9] and their possible connection to high T_c superconductivity [10]. It is a good example because at least some exact solutions to the energy eigenvalue problem for N anyons in some external confining potential are known—an exception in many-body quantum mechanics with non-separable Hamiltonians.

The quantum spectrum of anyons has been analyzed by many authors. Broadly speaking, several authors have pointed out the existence of a subset of exact solutions[11] whereas in the numerical analysis of the low lying spectrum of three and four anyons, many non-trivial features of the spectrum have been shown to exist[12]. The quantum mechanical spectrum (with harmonic potential added) shows two distinct qualitative features: a) eigenvalues which depend *linearly* on the statistical parameter, α, *all* of which are exactly known and b) eigenvalues which depend *non-linearly* on α and *none* of which is exactly known[12]. This is also borne out by a perturbative analysis of the three-anyon ground state[13, 14]. Another interesting feature of the spectrum is the additional degeneracy around the semion point which is related to the reflection symmetry[15]. These features may be traced to the properties of partial separability and pseudo-integrability manifested at the classical level [16].

Partial separability of the Hamiltonian may be exhibited explicitly in terms of two collective degrees of freedom and the remaining "relative" degrees of freedom. The total Hamiltonian, after removing the center of mass, may be written as a sum of $H_1(collective) + H_2(collective, relative)$. The commutator of H_1 and H_2 turns out to be proportional to H_2 [16]. This implies that the subspace of eigenstates of the full H on which H_2 vanishes are exact eigenstates of H_1 and these give *all* the exactly known eigenvalues and eigenstates.

Pseudo-integrability, a concept first introduced by Richens and Berry [17], may be described by exhibiting 2N constants of motion in involution, which fail to lead to integrability via usual action-angle variables. This was conjectured to be the reason for the level repulsion seen numerically in the non-linearly interpolating spectrum [16]. The precise meaning of pseudo-integrability in this context, however, was not elaborated earlier. In particular, how this classical feature translates into there being only two good quantum numbers, was not analyzed. A further fallout of the pseudo-integrability property of the anyon system is its effect on the so called nearest neighbour spacing distribution in the energy spectrum. It is now known that the spectrum exhibits features of both regular and irregular spectra as a function of the statistical parameter[18].

Yet another method of approaching the quantum spectrum of many anyons is through the stationary phase approximation (SPA). In this, an approximation to the propagator $G(E + i\epsilon)$ is developed using a suitable path integral representation [19]. In this approach the propagator is typically obtained as a sum over (families of) periodic trajectories in the classical phase space. This is entirely given in terms of classical quantities. One therefore expects to see directly the effects of non-trivial fundamental group of the phase space, pseudo-integrability and partial separability in a semiclassical framework. Since at the classical level, anyons with oscillator confinement, are *locally* identical to the oscillator system (but differ in the global topology of the phase space), the periodic orbits are known. Thus application of the periodic orbit theory (POT) to anyons, may be expected to be tractable.

There is one clarification to be noted. At the quantum level, the statistical parameter, α_q, enters via the stipulation of multivaluedness of the wave functions and is *dimensionless*. At the classical level its counterpart, α_c, enters as the coefficient of a total derivative term in the Lagrangian and has the dimensions of $\hbar$. These two must be related as $\alpha_c = \hbar\alpha_q$. In the SPA computation, α_c is held fixed with $\hbar$ going to zero. In the leading approximation, the spectrum will depend *linearly* on α_c and thus also on α_q. Alternatively, for comparison with the quantum spectrum to leading order, one will want to keep α_q fixed implying that α_c to go to zero with $\hbar$. Viewed either way, one may *not* expect to see *non-linear* dependence on the statistical parameter at the level of the leading approximation. Nonetheless, if the non-linearly interpolating eigenvalues also have a linear piece, one should be able to see it in the leading SPA.

This article is concerned with the discussion of the many points raised above. While many of these points have emerged from previously published papers, we attempt a coherent presentation of all these aspects with appropriate connections noted along with new material. We begin in section 2 with the basic formalism where we build the Hamiltonian of the anyons in two dimensions by requiring that their exchange statistics depend on a continuous parameter. In section 3, we review the quantum spectrum of many anyons. We first prove the existence of a subset of exact solutions. We then discuss, albeit in the case of three anyons, the so called missing states. Noting that eigenvalues themselves are independent of where the differential equation is solved, we expect the asymptotic form of the differential equation to determine the spectrum. The three anyon problem in the asymptotic regions reduces to solving a Confluent Hypergeometric Equation (CHE) in the region $R \leq x < \infty$. This equation has a solution regular at $x = 0$ and a solution which is irregular at $x = 0$. The regular solution leads to linearly interpolating energies while the irregular one may (R is strictly greater than zero) lead to nonlinearly interpolating solutions.

Though we are unable to make definite predictions about the spectrum, we point out the inherent difficulty which involves the question "what is the quantization condition".

In section 4, we analyze the many anyon system classically and show that the subset of exactly solvable solutions arise from the quantization of the collective coordinates (after the trivial centre of mass degrees are separated). This is made possible by the property of partial separability of the system for certain initial conditions and consistent with the equations of motion. In fact these solutions do not carry any information on the internal dynamics of anyons which may be frozen as far these solutions are concerned. These eigenvalues also goes as N^2 for N anyons. Thus the solutions are trivial dynamically. The real anyon dynamics therefore resides in the solutions which depend non-linearly on the statistical parameter. In the analysis of the quantum virial expansion for the equation of state of the anyon gas, it was shown that these trivial states conspire to cancel the divergent parts while the nonlinear solutions give the finite part which then defines the equation of state[20]. Thus it is of utmost importance to obtain these non-linear states to study the thermodynamic properties of the system. The low lying states have been obtained numerically[12] and perturbatively [13, 14] in the case of three and four anyons. In the case of large N anyonic systems, Thomas-Fermi approach has been used to obtain the properties of the ground and low lying excited states[21]. However to establish a virial expansion for the equation of state, by going beyond the third and the fourth virial coefficients, much more work on the *nonlinear* states is needed.

In section 5, we discuss semiclassical approach via the periodic orbit theory. we consider the question of classical integrability- the equations of motion of course do not depend on the statistical parameter, however the configuration space of N- particles is $R_2^N - \Delta$, where Δ denotes the set of all diagonal points which are removed. All the orbits which pass through these diagonal points are now modified. This leads us to conjecture that the system may be non-integrable a-la Richens and Berry [17]. We demonstrate that the classical analysis is adequate for exhibiting quantum symmetries. In this process we sharpen and clarify the meaning of pseudo-integrability and role of the fundamental group of the phase space. The system is shown to be identical to the isotropic oscillator *locally* but not at the global topological level. We then classify the classical trajectories using symmetry transformations and explain how the non-trivial global topology reduces the symmetry group. Furthermore, we sketch the application of the periodic orbit theory to many anyons. In the process we show the presence of eigenvalues which are potentially non-linearly interpolating. As mentioned earlier such eigenvalues have been seen in the numerical spectrum for three and four anyons [12] and have been conjectured to be present for general N [22]. We obtain these systematically from POT. Further, both the classical modeling and the exact propagator for two anyons, indicate an ambiguity regarding possibility of closed orbits with half the basic period and its inclusion in the trace formula. We discuss this ambiguity in some detail. We show that it is possible to regularise the point-like statistical flux and deduce the existence of half period trajectories. We look for their evidence in the exact propagator for two anyons and point out their ambiguous status. In Section 6, we present a detailed semiclassical analysis of the ground state of many anyons using a two step minimization. While this may seem unusual procedure to follow, we show that the method (even at the classical level) retains some of the features of the quantum spectrum.

Finally the results are summarized in Section 7. The appendices contain many results to keep this review self contained. Appendix A contains a discussion of regularisation method for the so-called reflecting orbits which arise in the zero angular momentum sector. Appendix B contains details of $OSp(4N, R)$ symmetry. Appendix C contains details of the Symplectic diagonalisation procedure relevant for Section 6.

2 Basic Formalism

In this section, we first develop the concept of anyons in two dimensions through multivalued wavefunctions. The exchange statistics of anyons then involves a continuous parameter, α—the so called statistical parameter[1]. It is the peculiarity of two dimensions that the quantization does not depend of the particular value of α, unlike in higher dimensions where we require α to be an integer.

In the following, by "anyons" we mean a quantum mechanical system of N particles in two dimensions with wave functions which have a stipulated multi-valuedness to be specified below. To make this explicit let us denote a generic wave function as $\psi(\vec{r}_1, ..., \vec{r}_N)$, where $\vec{r}_i$ denotes the position vector of a particle. Let $[P_{ij}]_\gamma$ denote the operation of taking the ith particle coordinate around the jth coordinate along a closed path γ. The path γ does not enclose any other particle coordinate and is taken in an anti-clockwise sense, say. Then let us stipulate that under such an operation ψ acquires a phase namely,

$$[P_{ij}]_\gamma \psi(\vec{r}_1, ...\vec{r}_N) = exp(i2\pi\alpha)\psi(\vec{r}_1, ...\vec{r}_N). \tag{1}$$

If a path γ encloses other particle coordinates as well then such a path can be broken into a set of closed paths each of which encloses exactly one particle. Applying the stipulation above, one can compute the total phase, for such a path. If the sense of the path is reversed then $\alpha \to -\alpha$. Clearly the phase acquired depends only on the homotopy class of the path (i.e., it is the same for two paths γ and γ' if γ and γ' can be continuously deformed into each other).

Let us introduce the complex notation for particle coordinates: $z_j = x_j + iy_j$, $\bar{z}_j = x_j - iy_j$. Clearly z_{ij}, where $z_{ij} = z_i - z_j$, has the property that if z_j is taken around z_i, z_{ij}^α changes by $\exp(i2\pi\alpha)$. This allows us to write any generic wave function satisfying eq.(1) as,

$$\psi(z_i, \bar{z}_i) = \left[\prod_{i<j} \left(\frac{z_{ij}}{\bar{z}_{ij}}\right)^{\alpha/2}\right] \tilde{\psi}(z_i, \bar{z}_i), \tag{2}$$

with the bracketed expression being a phase and now $\tilde{\psi}(z_i, \bar{z}_i)$ is a single valued function.

Clearly,

$$\nabla_k \psi(z_i, \bar{z}_i) = \prod_{i<j} \left(\frac{z_{ij}}{\bar{z}_{ij}}\right)^{\alpha/2} \left[\nabla_k \tilde{\psi}(z_i, \bar{z}_i) + \nabla_k ln \left(\prod_{i<j} \left(\frac{z_{ij}}{\bar{z}_{ij}}\right)^{\alpha/2}\right) \tilde{\psi}(z_i, \bar{z}_i)\right]. \tag{3}$$

which can be rewritten as,

$$\nabla_k \psi(z_i, \bar{z}_i) = \prod_{i<j} \left(\frac{z_{ij}}{\bar{z}_{ij}}\right)^{\alpha/2} \left[\nabla_k \tilde{\psi}(z_i, \bar{z}_i) + i\alpha \sum_{j\neq k} \frac{\hat{z} \times \vec{r}_{kj}}{\mid \vec{r}_{kj} \mid^2} \tilde{\psi}(z_i, \bar{z}_i)\right]. \tag{4}$$

Since $\tilde{\psi}$ is single valued, the right hand side of the above equation has exactly the same multivaluedness as the left hand side. In other words we have,

$$\nabla_k \left[\prod_{i<j} \left(\frac{z_{ij}}{\bar{z}_{ij}}\right)^{\alpha/2} \tilde{\psi}(z_i, \bar{z}_i)\right] = \left[\prod_{i<j} \left(\frac{z_{ij}}{\bar{z}_{ij}}\right)^{\frac{\alpha}{2}}\right] D_k \tilde{\psi}, \tag{5}$$

[1]Later we replace α with α_q to denote the fact that this is the statistical parameter appearing in the quantum mechanics of anyons. While this is a dimensionless parameter here, it is not so in the classical case. This distinction is important in the discussion of semiclassics.

where

$$D_k\tilde{\psi} = [\nabla_k + \vec{A}_k]\tilde{\psi}$$

and

$$A_k(\vec{r}_k) = i\alpha \sum_{j\neq k} \frac{\hat{z}\times\vec{r}_{kj}}{\mid \vec{r}_{kj} \mid^2}.$$

This allows us to write any higher order differential operators on ψ in terms of corresponding covariant differential operators on the single-valued wave function $\tilde{\psi}$. In particular a Hamiltonian operator, typically $-\sum_i \nabla_i^2 + V$ can be written similarly. An eigenvalue equation written in terms of ψ can then be recast as a corresponding equation in terms of $\tilde{\psi}$ involving the covariant derivative.

Although both formulations are equivalent, dealing with operators on multivalued wave functions is much less transparent than dealing with operators on single-valued wave functions. Naive commutation rules, symmetries that one would expect by looking at an operator on single valued functions are not at all true in general for the "same" differential operators acting on multivalued wave function.

Considering eigenvalue problem in terms of $\tilde{\psi}$ has other advantages too. Since all the subtleties of multivaluedness are equivalently transcribed in terms of additional "interaction" terms (the so called statistical interactions), the eigenvalue problem is amenable to approximations. One is also on firmer ground in doing usual algebraic manipulations with operators. With these in mind we will work with single-valued wave functions with "statistical interactions".

As a first step one would like to understand the system of "free anyons". However, the statistical interaction falls off as $\mid r_{ij} \mid^{-2}$ as $\mid r_{ij} \mid \to \infty$. So one is not sure whether the Hamiltonian with only statistical interactions has only discrete eigenvalues. One can put the system in a box to ensure discrete eigenvalues but then one needs suitable boundary condition. An oscillator potential ensures discrete spectrum without introducing a finite size. One could take some other confining potential but in the limit $\alpha \to 0$ one should know the spectrum. One then has hope of doing at least the perturbative analysis [23]. Since the statistical interaction depends only on relative separations, the Centre of Mass (CM) dynamics should play a trivial role and oscillator potential also allows a separation of CM and relative coordinate dynamics. The oscillator potential problem can also be mapped on to a problem of anyons in a real, constant external magnetic field along the z-axis [24]. Bearing these facts in mind, we choose the oscillator potential without further justification. In order to derive the thermodynamic properties of a system of anyons there exist well defined methods of eliminating the dependence on the oscillator frequency [13, 20].

The Hamiltonian we consider is – after carrying out the usual scaling of variables – in terms of dimensionless quantities

$$H = \hbar\omega[\frac{1}{2}\sum_{i=1}^{N} p_i^2 + \frac{1}{2}\sum_{i=1}^{N} r_i^2 - \alpha \sum_{j>i=1}^{N} \frac{\ell_{ij}}{r_{ij}^2} + \frac{\alpha^2}{2}\sum_{i\neq j,k}^{N} \frac{\vec{r}_{ij}.\vec{r}_{ik}}{r_{ij}^2 r_{ik}^2}], \tag{6}$$

where

$$\ell_{ij} = (\vec{r}_i - \vec{r}_j)\times(\vec{p}_i - \vec{p}_j).$$

and all distances have been expressed in units of $\sqrt{\frac{\hbar}{m\omega}}$. Notice that the statistical interaction is independent of the centre of mass coordinates. This is the Hamiltonian we analyze in the following sections both quantum mechanically and classically. In the quantum analysis, the Hamiltonian is considered to act on wave functions which vanish suitably at the coincident points.

3 Quantum Spectrum of Many Anyons

In this section we summarize the known results on the quantum mechanics of N-anyons confined in an oscillator potential. A detailed discussion of the results may be found in the books by Lerda[3] and Khare[4]. The latter contains the results for several other types of confinement potentials as well.

We first briefly summarize the results in the case of two anyons which is exactly solvable. In the case of N anyons ($N \geq 3$) the problem is not exactly solvable though there exists a subset of exact solutions. The results of this section provide the necessary framework to discuss further results in later sections.

3.1 Spectrum of two-anyons

The quantum Hamiltonian in the case of two anyons may be written as

$$H = H_{cm} + H_{rel}, \tag{7}$$

where H_{cm} is the Hamiltonian that describes the dynamics of the centre-of-mass. Since the statistical interaction is translation invariant, the centre-of-mass part is independent of this. As a result the spectrum of this Hamiltonian is the same as the spectrum of a two dimensional oscillator. That is,

$$E_{cm} = \hbar\omega[2n_{cm} + |l_{cm}| + 1]. \tag{8}$$

Here n_{cm} and l_{cm} denote the radial and angular momentum quantum numbers of the centre-of-mass excitations.

The relative Hamiltonian is given by,

$$H_{rel} = \hbar\omega[p^2 + r^2 - \alpha\frac{l}{r^2} + \frac{\alpha^2}{2}\frac{1}{r^2}], \tag{9}$$

where p is the momentum operator. The eigenvalue equation is easily solved by noting that the additional α dependent terms, may be combined with the centrifugal barrier with l shifted by $l - \alpha$. The spectrum of the Hamiltonian is then given by,

$$E_{rel} = \hbar\omega[2n + |l - \alpha| + 1]. \tag{10}$$

Together with the spectrum of H_{cm}, this provides the complete spectrum of two anyons in an oscillator potential. Indeed this is the only case that is solved fully.

We next prove the existence of a subset of exact solutions when the number of anyons is more than two. Later we show that these solutions indeed correspond to the quantization of collective coordinates of the N-anyon system. We close the discussion with an analysis of the so called "missing states" by specifically taking three anyons as an example.

3.2 Exact Solutions

For discussing the known class of exact solutions it is convenient to use the complex coordinates $z_i, \bar{z}_i$ in terms of which the N-anyon Hamiltonian eq.(6) takes the form (here after we set $\hbar = c = 1$),

$$H = -2\sum_i \partial_i\bar{\partial}_i + \frac{1}{2}\sum_i z_i\bar{z}_i - \alpha\sum_{i<j}(\frac{\partial_{ij}}{\bar{z}_{ij}} - \frac{\bar{\partial}_{ij}}{z_{ij}}) + \frac{\alpha^2}{2}\sum_{i\neq j,k}\frac{1}{\bar{z}_{ij}z_{ik}}, \tag{11}$$

where $\partial_i = \frac{\partial}{\partial z_i}$; $\partial_{ij} = \partial_i - \partial_j$, etc., and the eigenvalue equation is,

$$H\psi(z_i, \bar{z}_i) = E\psi(z_i, \bar{z}_i). \tag{12}$$

The conserved angular momentum J with eigenvalues denoted by j, is given by,

$$J = z_i\partial_i - \bar{z}_i\bar{\partial}_i. \tag{13}$$

At large distances, the oscillator confinement dominates which immediately signals the presence of the Gaussian factor in the wave function. Since the potential is singular as the relative distances go to zero, one needs to regulate the wave function in this limit to get the valid spectrum. We do this first by defining the variables:

$$X = \prod_{i<j}^{N} z_{ij}; \qquad t = \sum_{i=1}^{N} z_i\bar{z}_i, \tag{14}$$

We may classify the known exact solutions as follows:

(a) $j < 0$ solutions:

$$\psi_j = |X|^\alpha \ \phi_j(\bar{z}_i) \ e^{-t/2} \ ; \quad j < 0 \tag{15}$$

with the energy eigenvalues given by,

$$E = N - j + \alpha\frac{N(N-1)}{2}. \tag{16}$$

Here and in what follows ϕ_j generically denotes an eigenfunction of the angular momentum operator J with eigenvalue j. For elucidating the solutions the specific form of ϕ is irrelevant.

(b) $j = 0$ solutions:

$$\psi_0 = |X|^\alpha \ \phi_0(t) \ e^{-t/2} \ ; \quad j = 0 \tag{17}$$

with $\phi_0(t)(= \sum_{k=1}^{m} C_k t^k)$ a polynomial of degree m in t. The corresponding energy eigenvalues are given by

$$E = N + 2m + \alpha\frac{N(N-1)}{2}. \tag{18}$$

The second solution is necessarily bosonic since t is symmetric, whereas the first solution needs explicit symmetrization and antisymmetrization of the wavefunction in terms of $\bar{z}_i$ to obtain the bosonic and fermionic wavefunctions. Since this is always possible, the degeneracy of the first type of solution is exactly the same for both bosonic and fermionic type solutions for any given angular momentum j (< 0). We may also take a combination (product) of the solutions of the types discussed above, to get further $j < 0$ solutions. This then generates an infinite tower of radial excitations for each value of j.

(c) $j > 0$ solutions:

$$\psi_j = |X|^{-\alpha} \ \phi_j(z_i) \ e^{-t/2} \ ; \quad j > 0 \tag{19}$$

with the energy eigenvalues given by,

$$E = N + j - \alpha\frac{N(N-1)}{2}. \tag{20}$$

Caution must be exercised in choosing the value of j for these solutions since the wave function is not square-integrable for all values of j. In fact the lower bound is obtained by requiring that the wave function be square-integrable over the whole domain of $\alpha(0 \leq \alpha \leq 1)$. This means that

$j > (N-1)(N-2)/2$. If however this condition is not satisfied then the wave function remains regular only for some values of α ($0 \leq \alpha \leq 2j/N(N-1)$) but not for all $0 \leq \alpha \leq 1$ which gives rise to the so called non-interpolating solutions which have also been discussed in the literature[25].

All these solutions for the energy eigenvalues have a linear dependence on α with a coefficient $\pm N(N-1)/2$ while the corresponding eigenfunctions are finite order polynomials apart from the overall $\mid X \mid^{\pm\alpha}$ and the Gaussian factors. These solutions (a)–(c) cover all the known exact solutions. However it is by now known that these exact solutions span only a subspace of the full Hilbert space and existence of nonlinear solutions has been shown numerically as well as through mean-field calculations. It is our aim here to understand the reason for the existence of this dichotomy.

3.3 Missing states: Asymptotic analysis of the three-anyon problem

We now discuss the question of "missing" states, a terminology used by Yong-Shi Wu[11]. The existence of these states has been shown numerically [12] for three and four anyons, and through perturbative analysis [13, 14] for three anyon problem. Thomas-Fermi method has been used to obtain approximate ground state and excitations for large number of anyons [21]. We focus on these states now. We cannot easily analyze these in general for a general N-anyon system. Therefore we restrict ourselves to the three anyon system, where it is well known that solutions which have nonlinear dependence on α exist. Even here, we can only comment on the nature of such solutions, but we do not see a simple method of obtaining even low lying excitations analytically.

Recall that the two asymptotic regions, namely $z_{ij} \to \infty$ and $z_{ij} \to 0$ are always admissible. Hence we may still write the full solution as,

$$\psi(z_i, \bar{z}_i) = \exp\left(-\frac{1}{2}\sum_i z_i \bar{z}_i\right) \prod_{i<j} |z_{ij}|^{\alpha}\ \psi_2(z_i, \bar{z}_i), \tag{21}$$

with ψ_2 satisfying the equation,

$$\left[-2\sum_i \partial_i \bar{\partial}_i + \sum_i (z_i \partial_i + \bar{z}_i \bar{\partial}_i) - 2\alpha \sum_{i<j} \frac{\partial_{ij}}{\bar{z}_{ij}}\right] \psi_2 = \left[E - N - \frac{\alpha^2}{2} N(N-1)\right] \psi_2 \tag{22}$$

The known exact solutions for the energy eigenvalues, discussed previously, have a linear dependence on the statistical parameter α while the corresponding eigenfunctions are finite order polynomials, ψ_2, apart from the $\mid X \mid^{\alpha}$ and the Gaussian factor. In fact a simple scaling argument shows that if ψ_2 is a polynomial (i.e. has a finite degree), then the corresponding eigenvalue must be linear in α. For, if ψ_2 is a polynomial in $z_i, \bar{z}_i$, with highest total degree d then for $z_i, \bar{z}_i \to \infty$ the polynomial becomes a monomial and only the scaling operator term, $z_i \partial_i$, dominates. This gives,

$$E = N + d + \alpha \frac{N(N-1)}{2}. \tag{23}$$

Thus to have non-linear dependence on α, ψ_2 can not be a polynomial.

Suppose ψ_2 admits a power series representation with infinite radius of convergence but the series does not truncate then the scaling argument fails. However the analysis of such a series solution always seems to lead to exponentially divergent behaviour making the solution non-normalizable, ie., the power series has to truncate. But then only linear solutions are possible

because of the scaling argument above. The other possibility then is that the power series has a finite radius of convergence, i.e., a "scale" R has to enter if nonlinearity is to be possible.

Let us pursue this scaling argument further, consider $\psi_2(\lambda z_i, \lambda \bar{z}_i)$ where λ is the scaling parameter. Having a scale R, radius of convergence, means that ψ_2 has two different representations as the scale parameter $\lambda \to 0$ and $\lambda \to \infty$, each being valid for $\lambda < \lambda_{max}$ and $\lambda > \lambda_{min}$ respectively. For $\lambda \geq \lambda_{min}$ the scaling argument can still work but now the series need not have only integer powers, i.e., d can be a nonlinear function of α. We conclude then that if a nonlinearly interpolating eigenvalue is possible at all, the corresponding eigenfunction must have two different series representations as $\lambda \to 0$ and $\lambda \to \infty$. One then has to try to match the two series suitably.

Exploring this possibility is extremely complicated for general N. The first non-trivial case of $N = 3$ is analyzed in [26]. While this analysis shows that the possibility of finite radius of convergence is viable, one is not able to obtain any specific non-linearly interpolating solution. Observe that to get a discrete spectrum, the solutions of the eigenvalue equation must be regular and normalizable. For a single power series, this forces us to truncate the series and one obtains linearly interpolating spectrum. For multiple power series for the eigenfunctions, while non-linear dependence is conceivable, it is hard to see regularity/normalizability to get discrete spectrum. One can get a glimpse of the intricacies through the following analysis.

Consider the method of analyzing solutions in the three anyon problem consists of using the Fourier expansion of the wavefunction. Since the center of mass degree of freedom is irrelevant for further arguments, we may separate that out and write the wavefunction as

$$\psi = \exp\left(-\frac{1}{2}\sum_i r_i^2\right)\ |X|^\alpha\ \phi(r_1, r_2, \theta_1, \theta_2)\ \Phi_{CM}(R, \Theta),$$

where $\vec{r}_1, \vec{r}_2$ are relative coordinates chosen appropriately. In general ϕ may be written as,

$$\phi_j(r_1, r_2, \theta_1, \theta_2) = \sum_{n_1, n_2} e^{in_1\theta_1} e^{in_2\theta_2}\ \chi_{n_1,n_2}(r_1, r_2) = \sum_n e^{ij\frac{(\theta_1+\theta_2)}{2}} e^{in\frac{(\theta_1-\theta_2)}{2}}\ \chi_{j,n}(r_1, r_2) \qquad (24)$$

for a given angular momentum j, where $j = n_1 + n_2$, $n = n_1 - n_2$. We can now use this representation to look at the solutions for a given j. The eigenvalue equation now becomes $H_{rel}\phi_j = \mathcal{E}\phi_j$, where H_{rel} is the three-anyon Hamiltonian in relative coordinates and $\mathcal{E} = E - 2 - 3\alpha$ as before. Substituting for ϕ_j the eigenvalue equation takes the form,

$$\begin{aligned}
& 3x_2\left[4x_1\frac{\partial^2}{\partial x_1^2} + 4(1+\alpha-x_1)\frac{\partial}{\partial x_1} - \frac{(j+n)(j+n-4\alpha)}{4x_1}\right. \\
+\quad & \left. 4x_2\frac{\partial^2}{\partial x_2^2} + 4(1+2\alpha-x_2)\frac{\partial}{\partial x_2} - \frac{(j-n)(j-n-8\alpha)}{4x_2} + 2\mathcal{E}\right]\chi_{j,n} \\
=\quad & x_1\left[4x_1\frac{\partial^2}{\partial x_1^2} + 4(1+3\alpha-x_1)\frac{\partial}{\partial x_1} - \frac{(j+n+4)(j+n+4-12\alpha)}{4x_1}\right. \\
+\quad & \left. 4x_2\frac{\partial^2}{\partial x_2^2} + 4(1-x_2)\frac{\partial}{\partial x_2} - \frac{(j-n-4)^2}{4x_2} + 2\mathcal{E}\right]\chi_{j,n+4},
\end{aligned}$$

where $x_i = r_i^2$. This expression is exact, representing an infinite set of coupled equations. They relate Fourier modes differing by 4, i.e., if $\chi_0, \chi_1, \chi_2, \chi_3$ are known then $\chi_{4k}, \chi_{4k+1}, \chi_{4k+2}, \chi_{4k+3}$ get determined in terms of $\chi_0, ..., \chi_3$ etc. However there is no relation among $\chi_0, \chi_1, \chi_2, \chi_3$ themselves. In a sense these four functions will give four independent solutions of the eigenvalue equations. We can then deal with a given "tower" separately and independently and this is true for every given j. Let us concentrate on one tower. Now three distinct cases arise naturally:

(a) Only one member of the tower is nonzero, ie., $\chi_{j,n} = \chi_{j,m}\delta_{n,m}$.
(b) Only a finite number of $\chi_{j,n}$'s are nonzero, ie., $\chi_{j,n} = 0 \ \ \forall \ \ n \geq n_1$ and $\forall \ \ n \leq n_2$ with $n_2 < n_1$.
(c) Infinitely many $\chi_{j,n}$'s are nonzero. This gives raise to the following cases:

$$\begin{aligned} \chi_{j,n} &= 0 \ \ \forall \ \ n \geq n_1 \\ \chi_{j,n} &= 0 \ \ \forall \ \ n \leq n_2 \\ \chi_{j,n} &\neq 0 \ \ \forall \ \ n. \end{aligned}$$

Case (a) is simple to analyze and it reproduces the known exact solutions. In the exact eigenvalue equation relating $\chi_{j,n}$ and $\chi_{j,n+4}$ given above, we set the left hand side to zero which gives an equation for the $\chi_{j,n}$. But the right hand side also gives rise to another equation for the same $\chi_{j,n}$ when the coupling to $\chi_{j,n-4}$ is taken into account. However the form of the equations suggests that the two equations are separable but must be consistently solved. The consistency conditions immediately yield the linear exact solutions which are already outlined above. Cases (b) and (c) seem quite complicated and nonlinearly interpolating states must be in one of these cases. Consider the normalization condition on the full wavefunction,

$$\| \ \psi_j \ \|^2 = \sum_n \int_0^\infty dx_1 dx_2 |X|^{2\alpha} |\chi_{j,n}|^2 e^{-x_1+x_2} \equiv \sum_n C_n. \tag{25}$$

If ψ_j is given by case (b) then there are only a finite number of terms in the norm and if C_n's are finite then ψ_j is normalizable. Thus the quantization condition for the nonlinear states must arise by the demand that C_n must be finite. For the case (c) even if all C_n's are finite we may still get the norm of the wavefunction to be infinite because the sum above may not converge and then the quantization condition would be the convergence of $\sum_n C_n$. It is therefore apparent that non-linear dependence on α must come about in a very complicated manner.

4 Classical Analysis of Many Anyons and Integrability

Having noted the difficulties inherent in describing the non-linearly interpolating solutions, it is striking that the subspace spanned by the exact solutions are obtained so easily. Obviously these two sets must have very different origins. Indeed as we will see below, the exact solutions have their origin in the quantization of collective degrees of freedom. To explore this aspect we analyze the system classically.

We may now, therefore, ask the question, what is the classical Lagrangian which gives raise to the Hamiltonian of Eq.(6)? In fact it is quite straight forward to see that the Lagrangian is given by,

$$L = \frac{1}{2} \sum_{i=1}^{N} \ (\dot{\vec{r}}_i)^2 + \alpha \sum_{i<j}^{N} \dot{\theta}_{ij} - V(\vec{r}_i); \qquad \theta_{ij} = tan^{-1} \left(\frac{y_i - y_j}{x_i - x_j} \right), \tag{26}$$

where $\vec{r}_i$ denote particle coordinates and $V(\vec{r}_i)$ is some confining potential which we choose to be the harmonic oscillator potential $V(\vec{r}_i) = \sum_{i=1}^{N} \vec{r}_i{}^2$. The α-dependent term is the statistical interaction. The harmonic oscillator potential is convenient since the dynamics of the system is well understood in the limit $\alpha = 0$. As mentioned in the previous section, the corresponding quantum system displays the existence of a subset of exact solutions, though the full spectrum is not analytically solvable for many anyons. It is possible that the system is at least partially

separable to allow for the known exact solutions. We will return to this question to the next subsection and consider the question of integrability first. If one considers the classical Lagrangian, the statistical interaction is a total time derivative. The Euler-Lagrange equations of motion are therefore the same as those of a 2N-dimensional oscillator and this of course is an integrable system. However the quantum spectrum displays not only the exact solutions but it has been demonstrated numerically in the case of three and four anyons that the spectrum also shows evidence of many level crossings and level repulsions [12, 18] which is often taken to be an indication of non-integrability. Here we argue that even though we have $2N$ constants of motion in involution the invariant surfaces do not have the topology of a $2N$ dimensional torus.

We begin with the analysis of the classical Lagrangian given by Eq.(26). It is convenient to write the Lagrangian in the form,

$$L = \frac{1}{2}\sum_{i=1}^{N}\left[(\dot{\vec{r}}_i)^2 - \vec{r}_i^{\,2}\right] + \alpha\sum_{i<j}^{N}\frac{\vec{r}_{ij}\times\dot{\vec{r}}_{ij}}{\vec{r}_{ij}\,.\,\vec{r}_{ij}}, \tag{27}$$

where the dots indicate the time derivatives. The first step is to introduce relative coordinates, also called Jacobi coordinates and separate the trivial centre of mass degree of freedom. To this end we write,

$$\vec{\rho}_a = \frac{1}{\sqrt{a(a+1)}}\sum_{k=1}^{a}\vec{r}_k - \sqrt{\frac{a}{a+1}}\vec{r}_{a+1}; \quad a = 1, ..., N-1; \tag{28}$$

with the inverse relation given by,

$$\vec{r}_i = \left[-\sqrt{\frac{i-1}{i}}\vec{\rho}_{i-1} + \sum_{k=i}^{N-1}\frac{\vec{\rho}_k}{\sqrt{k(k+1)}}\right] + \vec{R}_{cm} \equiv A_i^a\vec{\rho}_a + \vec{R}_{cm}, \tag{29}$$

where $\vec{\rho}_a$, $a = 1, ..., N-1$ are dimensionless relative coordinates and $\vec{R}_{cm}$ is the centre of mass coordinate. It follows that,

$$\sum_i A_i^a = 0; \qquad \sum_i A_i^a A_i^b = \delta^{ab}.$$

It is straight forward to see that,

$$L = L_{CM} + L_{rel},$$

where

$$L_{CM} = \frac{1}{2}[\dot{\vec{R}}_{CM}^2 - \vec{R}_{CM}^2],$$

$$L_{rel} = \frac{1}{2}\left[\dot{\vec{\rho}}_a^{\,2} - \vec{\rho}_a^2\right] + \alpha\sum_{i<j}\frac{A_{ij}^a\,A_{ij}^b\;\vec{\rho}_a\times\dot{\vec{\rho}}_b}{A_{ij}^c\,A_{ij}^d\;\vec{\rho}_c\cdot\vec{\rho}_d},$$

where $A_{ij}^a = A_i^a - A_j^b$. From now on we concentrate only on the L_{rel} and drop the subscript. It is easy to see that the Euler-Lagrange equations of motion are,

$$\ddot{\vec{\rho}}_a = -\vec{\rho}_a. \tag{30}$$

There is no α dependence in these equations since the corresponding term is a total derivative. From these equations it follows that

$$\mathcal{E}_a \equiv \frac{1}{2}[\dot{\vec{\rho}}_a^2 + \vec{\rho}_a^2]; \quad l_a \equiv \vec{\rho}_a\times\dot{\vec{\rho}}_a; \quad a = 1, ..., N-1 \tag{31}$$

are constants of motion. In the Hamiltonian formulation the $\dot{\vec{\rho}}_a$ are expressed in terms of conjugate momenta and coordinates which *do contain* the α dependence,

$$P_{ax} = \frac{\partial L}{\partial \dot{\rho}_{ax}} = \dot{\rho}_{ax} - \alpha \mathcal{A}_{ax}; \quad \mathcal{A}_{ax} = \sum_{i<j} \frac{A^a_{ij}\, A^b_{ij}\, \rho_{by}}{A^c_{ij}\, A^d_{ij}\, \vec{\rho}_c.\vec{\rho}_d}, \tag{32}$$

$$P_{ay} = \frac{\partial L}{\partial \dot{\rho}_{ay}} = \dot{\rho}_{ay} + \alpha \mathcal{A}_{ay}; \quad \mathcal{A}_{ay} = \sum_{i<j} \frac{A^a_{ij}\, A^b_{ij}\, \rho_{bx}}{A^c_{ij}\, A^d_{ij}\, \vec{\rho}_c.\vec{\rho}_d}, \tag{33}$$

and the constants of motion in relative coordinates are,

$$\begin{aligned} \mathcal{E}_a &= \frac{1}{2}\left[(P_{ax} + \alpha \mathcal{A}_{ax})^2 + (P_{ay} - \alpha \mathcal{A}_{ay})^2 + \vec{\rho}_a{}^2\right] \\ l_a &= \vec{\rho}_a \times \vec{P}_a - \alpha(\rho_{ax}\mathcal{A}_{ay} + \rho_{ay}\mathcal{A}_{ax}). \end{aligned} \tag{34}$$

Clearly the Hamiltonian is given by

$$H = \sum_{a=1}^{N-1} \mathcal{E}_a.$$

Note that the α-dependent term in L is well defined only if $\vec{r}_{ij} \neq 0$ for all i, j, that is, only if

$$A^a_{ij}\, A^b_{ij}\, \vec{\rho}_a.\vec{\rho}_b \;\neq\; 0 \;\; \forall \;\; i, j.$$

Consequently the expressions for $\mathcal{E}_a, l_a$ and H are also valid only if $\vec{r}_{ij}$ is not zero. In effect the classical configuration space on which L is well defined is the space

$$Q^{N-1} = R_2^{N-1} - \Delta; \quad \Delta = \{\ \vec{\rho}_a \ / \ A^c_{ij} A^d_{ij}\ \vec{\rho}_c.\vec{\rho}_d \ = \ 0 \ \text{ for some pair(s) } \ i, j\}. \tag{35}$$

The space Q^{N-1} is not simply connected. Its fundamental group π_1 is the same as the fundamental group of $\{R_2 - (N-1)\}$ points which is known to be nontrivial. For $N = 2$, $\pi_1 = Z$ while for $N \geq 3$, π_1 is non-Abelian.

The corresponding phase space is the cotangent bundle of Q on which $\mathcal{E}_a, l_a$ and H are well defined. This phase space is topologically nontrivial and for our purposes we do not need the full machinery for handling this topologically nontrivial phase space. It is sufficient to pretend that the full space is still R_2^{N-1} but simply avoid coincident points.

It is straightforward then to prove the following Poisson bracket relations,

$$\{\mathcal{E}_a, \mathcal{E}_b\} = \{l_a, l_b\} = \{\mathcal{E}_a, l_a\} = 0 \quad \forall \quad a, b = 1, ..., N-1, \tag{36}$$

$$\{H, \mathcal{E}_a\} = \{H, l_a\} = 0. \tag{37}$$

Thus for $4(N-1)$ degrees of freedom P_a, ρ_a we have $2(N-1)$ constants of motion in involution. So a necessary condition for a system to be integrable is satisfied. We will tentatively refer to this system as being integrable (or potentially integrable) in the "Liouville sense". However, for the system to be integrable via action angle variables, further conditions have to be satisfied (see page 271 of [27]): If $M(\mathcal{E}_a, l_a)$ denote the set of points in phase space at which the constants of motion have values $\mathcal{E}_a, l_a$ then (a) M is a $2(N-1)$ dimensional submanifold iff $\mathcal{E}_a, l_a$ are all independent, (b) if M is a submanifold which is compact and connected then M is a $2(N-1)$ dimensional torus. If these conditions are satisfied then one can introduce action-angle variables in the neighbourhood of M. The orbits on M will all be conditionally periodic in general and for the periodic orbits one may use Bohr-Sommerfeld quantization to get a subset of eigenvalues.

We will show in the next section that there exist $M(\mathcal{E}_a, l_a)$ for which action-angle variables can be introduced. These correspond to the collective motion of N-anyons and their semiclassical quantization reproduces the exact eigenvalues of the quantum system that are already known. However, if we go away from these initial conditions some of the orbits of the oscillator will cease to be periodic when α dependence is introduced. For instance the Euler-Lagrange equations

$$\ddot{\vec{r}}_i = -\vec{r}_i \Rightarrow \ddot{\vec{r}}_{ij} = -\vec{r}_{ij} \quad \text{(independent of } \alpha\text{)}.$$

For any pair $ij,\ i \neq j$, $\vec{r}_{ij}$ in general describes an ellipse in the configuration space. However for zero orbital angular momentum, the ellipse degenerates to a straight line, hence the point $\vec{r}_{ij} = 0$ is on such an orbit. Since these points are not admissible when $\alpha \neq 0$ these orbits cannot lead to periodic orbits in the phase space. For all choices of $\mathcal{E}_a, l_a$ such that for some initial conditions the phase space orbits will have r_{ij} approaching arbitrarily close to zero, $M(\mathcal{E}_a, l_a)$ cannot be a torus. (Even if M is a manifold it will fail to be compact and/or connected.) In appendix A, we consider the explicit example of an angular momentum zero orbit in the case of two anyons, which necessarily passes through the diagonal point. Even though in configuration space this may appear to be periodic, in the phase-space the orbits stay on the surface of a cylinder which can be broken into two disconnected sheets. Therefore for $\alpha \neq 0$, although we have a potentially integrable system it is not generically integrable via action-angle variables. The existence of such an M indicates the possibility of "extreme sensitivity" to the initial conditions. This is a possible reason for the "level repulsions" seen numerically. This bears a resemblance to the billiard system considered by Richens and Berry [17] though the reasons for the failure of integrability via action-angle coordinates are different. Following Richens and Berry we conclude that our system is *Pseudo-integrable*. We will sharpen and formalize this notion later in Section 5.

4.1 Collective modes and relation to exact states

We may now ask the question, what causes these states, though a subset of the full Hilbert space, to be exact solutions of the N-anyon Hamiltonian which is in general non-separable. Indeed as we will show now, these solutions may be obtained by quantizing the collective coordinates of the N-anyon system in relative coordinates consistent with the equations of motions for a set of initial conditions. In some sense these collective coordinates do not give us any information on the internal dynamics of the system. These are therefore trivial solutions dynamically. The anyonic interaction term in L is invariant under two sets of time independent transformations:

$$\vec{\rho}_a \ \rightarrow \ \vec{\eta}_a = U(\theta)\vec{\rho}_a \quad \forall \quad a, \tag{38}$$

$$\vec{\rho}_a \ \rightarrow \ \vec{\eta}_a = \lambda\vec{\rho}_a \quad \forall \quad a, \tag{39}$$

where $U(\theta)$ denotes a rotation of the vector by an angle θ in two dimensions. Therefore for any given configuration $\{\vec{\rho}_a\}$ at any given time t we can always rotate the axes so that $\rho_{1y} = 0$ (say). Thus by making a time dependent rotation we can ensure that $\eta_{1y} = 0$ for all t. Defining,

$$\vec{\rho}_a = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \vec{\eta}_a, \tag{40}$$

The following identities are easy to prove:

$$\vec{\rho}_a.\vec{\rho}_b = \vec{\eta}_a.\vec{\eta}_b, \tag{41}$$

$$\vec{\rho}_a \times \vec{\rho}_b = \vec{\eta}_a \times \vec{\eta}_b, \tag{42}$$

$$\dot{\vec{\rho}}_a{}^2 = \dot{\vec{\eta}}_a{}^2 + 2\dot{\theta}\,\vec{\eta}_a \times \dot{\vec{\eta}}_a + \dot{\theta}^2\,\vec{\eta}_a{}^2, \tag{43}$$

$$\vec{\rho}_a \times \dot{\vec{\rho}}_b = \dot{\theta}\,\vec{\eta}_a.\vec{\eta}_b + \vec{\eta}_a \times \dot{\vec{\eta}}_b. \tag{44}$$

Now the set $\{\eta_a; a = 1, ..., N-1\}$ is effectively a $2(N-1)-1$ dimensional vector since $\eta_{1,y} = 0$ for all time t. We can therefore introduce the standard spherical coordinates by the usual procedure,

$$\vec{\eta}_a \equiv R\vec{\xi}_a; \quad \xi_{1y} = 0; \quad \sum_a \xi_a^2 = 1, \tag{45}$$

where

$$\begin{aligned}
\xi_{N-a,x} &= s_1 s_2 ... s_{2a-2} c_{2a-1}, & (46)\\
\xi_{N-a,y} &= s_1 s_2 ... s_{2a-1} c_{2a}, & (47)\\
\xi_{1,x} &= s_1 s_2 ... s_{2N-4}, & (48)\\
\xi_{1,y} &= 0, & (49)
\end{aligned}$$

where $a = 2, ..., N-1$ and $s_\mu = \sin\theta_\mu,\ c_\mu = \cos\theta_\mu$. In terms of these variables the Lagrangian may be rewritten as

$$\begin{aligned}
L &= \frac{1}{2}\left[\dot{R}^2 - R^2 + R^2\dot{\theta}^2 + 2R^2\dot{\theta}\sum_a \vec{\xi}_a \times \dot{\vec{\xi}}_a + R^2 \sum_a \dot{\vec{\xi}}_a{}^2\right] \\
&+ \alpha\frac{N(N-1)}{2}\dot{\theta} + \alpha\sum_{i<j}\frac{A_{ij}^a\, A_{ij}^b \vec{\xi}_a \times \dot{\vec{\xi}}_b}{A_{ij}^c\, A_{ij}^d \vec{\xi}_c . \vec{\xi}_d}.
\end{aligned} \tag{50}$$

Since $\vec{\eta}_a$ are obtained from $\vec{\rho}_a$ by the same rotation matrix for all a, the angle $\theta(t)$ clearly describes a collective rotation of all the N anyons about some reference axis. The anyonic interaction term (a total time derivative) is manifestly independent of $R(t)$. This may also be regarded as a collective mode as discussed below. It is the semiclassical quantization of these two modes that yields the exactly known energy eigenvalues. To elaborate these points let us consider the Euler-Lagrange equations of motion. Since the α dependent part of the Lagrangian is a total time derivative we can ignore it for analyzing the classical equations of motion. Clearly these equations are identical to that of the oscillator equations of motions. Translating this into the spherical coordinates we obtain,

$$[\ddot{R} + R - R\dot{\theta}^2]\vec{\xi}_a - [R\ddot{\theta} + 2\dot{R}\dot{\theta}]V\vec{\xi}_a + 2[\dot{R} - R\dot{\theta}V]\dot{\vec{\xi}}_a + R\ddot{\vec{\xi}}_a = 0, \tag{51}$$

where

$$V = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

and in the matrix equation above the $\vec{\xi}$ is a column vector with two elements. The matrix V between two vectors essentially generates their cross product. Taking the dot product with $\vec{\xi}_a$ and summing over a we find,

$$[\ddot{R} + R - R\dot{\theta}^2] = 2R\dot{\theta}\sum_a \vec{\xi}_a \times \dot{\vec{\xi}}_a + R\sum_a \dot{\vec{\xi}}_a^2, \tag{52}$$

where we have made use of the identities,

$$\sum_a \vec{\xi}_a . \dot{\vec{\xi}}_a = 0, \quad \sum_a \vec{\xi}_a . \ddot{\vec{\xi}}_a = -\sum_a \dot{\vec{\xi}}_a^2 .$$

Taking the cross product with $\vec{\xi}_a$ and summing over a we find,

$$[R\ddot{\theta} + 2\dot{R}\dot{\theta}] = -2\dot{R}\sum_a \vec{\xi}_a \times \dot{\vec{\xi}}_a - R\sum_a \vec{\xi}_a \times \ddot{\vec{\xi}}_a. \tag{53}$$

Using the above two relations Eq.(51) can be rewritten as,

$$R\ddot{\vec{\xi}}_a + 2[\dot{R} - R\dot{\theta}V]\dot{\vec{\xi}}_a + [2R\dot{\theta}\vec{\xi}_b \times \dot{\vec{\xi}}_b + R\dot{\vec{\xi}}_b^2]\vec{\xi}_a + [2\dot{R}\vec{\xi}_b \times \dot{\vec{\xi}}_b + R\vec{\xi}_b \times \ddot{\vec{\xi}}_b]V\vec{\xi}_a = 0, \tag{54}$$

where sum over b is assumed. Here the equations of motion for R and θ are still coupled to all the other internal coordinates, $\vec{\xi}_a$ and as yet they are not collective coordinates. We therefore need to impose a set of initial conditions which may lead to the separation of these two modes as collective modes from the internal variables θ_μ. To realize this let at time $t = 0$, all velocities $\dot{\vec{\xi}}_a(t = 0) = 0$ for all a. Then the above equations reduce to,

$$R[\ddot{\vec{\xi}}_a + (\sum_b \vec{\xi}_b \times \ddot{\vec{\xi}}_b)V\vec{\xi}_a] = 0, \tag{55}$$

$$\ddot{R} + R - R\dot{\theta}^2 = 0, \tag{56}$$

$$R\ddot{\theta} + 2\dot{R}\dot{\theta} = -R\sum_b \vec{\xi}_b \times \ddot{\vec{\xi}}_b. \tag{57}$$

Now consider the first equation for $R \neq 0$ and $a = 1$. From the initial conditions it is obvious that,

$$\ddot{\xi}_{1x} = 0; \qquad \xi_{1x}\sum_b (\vec{\xi}_b \times \ddot{\vec{\xi}}_b) = 0$$

because $\xi_{1y} = 0 \ \ \forall \ \ t$.

Now *if* ξ_{1x} is nonzero, then $\sum_b \vec{\xi}_b \times \ddot{\vec{\xi}}_b = 0$, and hence

$$\ddot{\vec{\xi}}_b = 0 \ \ \forall \ \ b.$$

Since the equations of motion are second order in time t, we have proved that: *if* $\dot{\vec{\xi}}_a(0) = 0 \ \ \forall \ \ a$, *and* $R(0)$, $\xi_{1x}(0)$ are non zero *then* $\forall \ \ t$,

$$\vec{\xi}_a(t) = \vec{\xi}_a(0),$$

$$\ddot{R} + R - R\dot{\theta}^2 = 0,$$

$$R\ddot{\theta} + 2\dot{R}\dot{\theta} = 0.$$

For oscillator ($\alpha = 0$) R(0) and $\xi_{1x}(0)$ being non zero is a special class of initial condition. Indeed in general for a second order equation system if both first and second order derivatives vanish at any t then this can hold only for some restricted class of coordinate values. For the equations of motions considered in the configuration space R_2^{N-1} the restricted class is precisely characterized by $R(0) \neq 0$ and $\xi_{1x}(0) \neq 0$. However, when $\alpha \neq 0$ the configurations space is Q^{N-1} which does not contain points which have R = 0 and $\xi_{1x} = 0$. Thus there are no restrictions on the initial conditions in Q^{N-1}.

The above initial conditions amount to freezing the "internal motion" of the anyons. The remaining motion is a collective motion described by R(t) and $\theta(t)$ which is just a $2(N-1)$ dimensional oscillator. R(t) being nonzero implies that the angular momentum must be nonzero.

This explains in what sense R(t) can be interpreted as a collective mode. For describing the motion of collective mode the effective Lagrangian is

$$L_{eff} = \frac{1}{2}[\dot{R}^2 + R^2\dot{\theta}^2 - R^2] + \alpha\frac{N(N-1)}{2}\dot{\theta}. \tag{58}$$

This is identical to the Lagrangian in the relative coordinate for a two anyon system with $\alpha \to \alpha\frac{N(N-1)}{2}$ and in $2(N-1)$ dimensions. Semiclassical quantization will then reproduce all exactly known energy eigenvalues summarized in Sect. 3. In this effective Lagrangian picture the only known memory of N resides in the coefficient of α. This can be understood by noting that when all the N-anyons are rotated by 2π about the centre of mass they also circle each other to pick up the extra phase.

If on the other hand we consider $\dot{R} = \dot{\theta} = 0$ at $t = 0$ class of initial conditions then we see that $\ddot{R}, \ddot{\theta}$ are nonzero and hence $R(t), \theta(t)$ do depend on θ_μ. Thus the collective motion is not fully decoupled from the "internal motion". We therefore refer to this system as partially separable.

We also note that the same conclusions may be drawn from the Hamiltonian formulation. For completeness we give the relevant expressions and arguments. The conjugate momenta for the hyper-spherical variables are given by,

$$P_R = \dot{R}, \tag{59}$$

$$P_\theta = R^2\dot{\theta} + R^2\sum_\mu \dot{\theta}_\mu F_\mu + \alpha\frac{N(N-1)}{2}, \tag{60}$$

$$P_\mu = R^2 r_\mu^2\dot{\theta}_\mu + R^2\dot{\theta}F_\mu + \alpha G_\mu, \tag{61}$$

where,

$$r_\mu = s_1 s_2 ... s_{\mu-1}, \quad \mu = 1, ..., 2N-3, \quad r_1 = 1$$

and F_μ and G_μ are defined through,

$$\sum_{\mu=1}^{2N-4} \dot{\theta}_\mu F_\mu \equiv \sum_{a=1}^{N-1} \vec{\xi}_a \times \dot{\vec{\xi}}_a$$

$$\sum_{\mu=1}^{2N-4} \dot{\theta}_\mu G_\mu \equiv \sum_{i<j} \frac{A_{ij}^a\, A_{ij}^b\, \vec{\xi}_a \times \dot{\vec{\xi}}_b}{A_{ij}^c\, A_{ij}^d\, \vec{\xi}_c.\vec{\xi}_d}.$$

The Hamiltonian is then given by

$$H = H_1 + H_2, \tag{62}$$

where

$$H_1 = \frac{1}{2}[P_R^2 + R^2] + \frac{(P_\theta - \alpha\frac{N(N-1)}{2})^2}{2R^2}, \tag{63}$$

$$H_2 = \frac{1}{2R^2}\left[\sum_\mu \frac{(P_\mu - \alpha G_\mu)^2}{r_\mu^2} + \frac{(P_\theta - \alpha\frac{N(N-1)}{2} - \sum_\mu \frac{F_\mu(P_\mu - \alpha G_\mu)}{r_\mu^2})^2}{1 - \sum_\mu \frac{F_\mu^2}{r_\mu^2}} - (P_\theta - \alpha\frac{N(N-1)}{2})^2\right]. \tag{64}$$

It is easy to see that for the special initial conditions $\dot{\theta}_\mu = 0$, $H_2 = 0$. Also the Poisson bracket $\{H_1, H_2\} \propto H_2$ vanishes for these initial conditions. The corresponding quantum statement would

$[H_1, H_2] \propto H_2$, and therefore if we consider the subspace of states $\{\psi\}$ on which $H_2\psi = 0$ then H_1 will act invariantly on this subspace. Further $H = H_1$ on this subspace and thus the eigenstates of H_1 will be exact eigenstates of the full system and conversely. But the problem of solving H_1 is analogous to that of two anyon problem with $\alpha \to \alpha\frac{N(N-1)}{2}$. Therefore these solutions are given by,

$$\begin{aligned} E &= 2m + |j - \alpha\frac{N(N-1)}{2}| + (N-1), && (65)\\ \psi &= C\, R^{|j-\alpha|}\, e^{-R^2/2}\, f(R), && (66)\\ H_1\psi &= E\psi && (67) \end{aligned}$$

where $f(R)$ is some polynomial of degree $2m$ in R. In general the normalization constant C in ψ will have dependence on θ_μ's, restricted by $H_2\psi = 0$. These are precisely all the known solutions and the above argument shows that there are no more solutions satisfying $H_1\psi = E\psi$ and $H_2\psi = 0$. Thus $H_2\psi = 0$ characterizes the subspace spanned by *all* the known exact solutions. So the quantum counterpart of classical partial separability implies the existence of these solutions.

5 Semiclassical Analysis — Periodic Orbit Theory

We have seen the difficulty in obtaining the non-linearly interpolating eigenvalues. We have also discussed their possible relation to, *at classical level*, pseudo-integrability on the one hand and the possibility of two different asymptotic series at the quantum level on the other hand. There is yet another possible approach to gain some understanding of these eigenvalues, namely semi-classical analysis. This is what we discuss now. We begin by classifying the classical orbits for N-anyon system and discuss the trace formula in the subsequent section. Most of the discussion of this section is based on reference [28].

5.1 Classification of orbits and symmetry reduction

We will first carry out a classification of orbits for N anyons and see how the pseudo-integrability amounts to a reduction of the dynamical symmetry. We will also see the role played by the fundamental group of the phase space in this regard. Noting that a trajectory is specified by giving an initial point, we will use the dynamical symmetry of the oscillator system to group various orbits into continuous families. For anyons, certain points and hence certain orbits are disallowed (removed) with the result that one gets several continuous families.

5.1.1 The case of two anyons

As usual we can separate the center-of-mass (CM) dynamics from the relative coordinate dynamics. The CM dynamics is trivial – identical to an oscillator locally and globally. The anyonic feature is contained entirely in the relative dynamics described as:

$$\begin{array}{lcllcl} Q &=& R^2 - \vec{0} & , & \Gamma &= Q \times R^2 \\ L &=& \frac{1}{2}(\dot{\vec{r}}^2 - \vec{r}^2) + \alpha_c\dot{\theta} & , & \theta &\equiv \tan^{-1} y/x \\ H &=& \frac{1}{2}(\vec{\pi}^2 + \vec{r}^2) & , & \vec{\pi} &\equiv \vec{p} - \alpha_c\frac{\hat{k}\times\vec{r}}{r^2} \end{array} \qquad (68)$$

The classical trajectories are trivially known in these cases. These are ellipses with two constants of motion, the energy (E) and the angular momentum ($J = (\vec{r}\times\dot{\vec{r}})_z$). The sign of J gives the sense of traversal and a trajectory reaches the point $\vec{r} = \vec{0}$ if and only if $J = 0$.

These degenerate trajectories ($J = 0$) are not allowed for the anyons though are allowed for the oscillator.

Consider the set of all possible trajectories with a given energy. Fix two such trajectories with angular momenta J, J'. Suppose there exists a continuous family of interpolating trajectories connecting these two. Clearly if the angular momenta have opposite signs, then at least one of the interpolating trajectories must be degenerate. But this is disallowed for anyons. Hence, for anyons there must be *at least* **two** 'orientation' classes of trajectories distinguished only by the sign of angular momenta.

While we have shown that there must be at least two families we do not know yet if there *must* be *precisely* two families. We will use the symmetries to show that there are precisely two families for anyons and precisely one family for the oscillator.

As far as the PB algebra is concerned, the oscillator and anyons are identical and hence we do have *infinitesimal* symmetries forming the Lie algebra of $U(2)$ (or $OSp(4,R)$) for both cases. The constant H surface (S^3 in the phase space R^4 for oscillator) is of course invariant under the infinitesimal symmetry transformations. For anyons, the phase space has $\vec{r} = \vec{0}$ removed and the constant H surface has a great circle (one dimensional) removed from the S^3, say a "longitude".

In the appendix B we have given the details of the $OSp(4N,R)$ symmetry of the oscillator. In the present case of two anyons with center of mass coordinate separated, we have effectively, $N = 1$ and the symmetry generators form $OSp(4,R)$. We have only the 4 generators of the $\mathbf{T}_i$ type. Of these four, $\mathbf{u}_1$ generates rotations of position and momentum while $\mathbf{u}_2$ generates time evolution. The generic integral curves are given in the appendix B.

From appendix B we also recall that $OSp(4,R)$ acts transitively on $H = E$ sphere. This shows that for the oscillator there is precisely ONE family of trajectories. For anyons we need analogous result with the extra condition that degenerate trajectories are avoided to conclude that there are precisely **two** families.

Claim: Given any two trajectories with the same sign of their angular momenta, there exists a one parameter subgroup connecting the two without changing the sign of angular momenta.

Proof: Observe that $\mathbf{u}_1, \mathbf{u}_2$ transformations leave the angular momentum invariant. Choose $\mathbf{u} = \cos(\beta)\mathbf{u}_3 + \sin(\beta)\mathbf{u}_4$. The one parameter group generated by this generator transforms the angular momentum as, (appendix B)

$$2J(\sigma) = 2J - \sin^2(\sigma)\{2J - \bar{\omega}\bar{\mathbf{u}}\mathbf{L}\mathbf{u}\omega\} + \sin(2\sigma)\{\bar{\omega}\mathbf{L}\mathbf{u}\omega\} \tag{69}$$

For the choice of $\mathbf{u}$ made, $\bar{\mathbf{u}}\mathbf{L}\mathbf{u} = -\mathbf{L}$ and putting $J_u \equiv \frac{\bar{\omega}\mathbf{L}\mathbf{u}\omega}{2}$,

$$\begin{aligned} J(\sigma) &= J\ \cos(2\sigma) + \sin(2\sigma)\ J_u \\ &= \sqrt{J^2 + J_u^2}\ \cos(2\sigma - \delta)\ , \qquad \delta \equiv \tan^{-1}(J_u/J) \in (-\pi/2, \pi/2). \end{aligned} \tag{70}$$

This is valid for all choices of β. Observe that J_u depends on β. Furthermore it satisfies,

$$\begin{aligned} \frac{\partial^2 J_u(\beta)}{\partial\beta^2} &= -J_u(\beta) && \Rightarrow \\ J_u(\beta) &= J_u(0)\cos(\beta) + \frac{\partial J_u}{\partial\beta}(0)\sin(\beta) && \Rightarrow \\ (J_u)_{max} &= \left\{ (J_u(0))^2 + (\frac{\partial J_u}{\partial\beta}(0))^2 \right\}^{1/2} = \sqrt{E^2 - J^2} \end{aligned} \tag{71}$$

The last equality follows by explicit evaluation. Choosing β to maximize J_u then implies that the square root in the equation for $J(\sigma)$ is just the energy E. Since for any given energy, we must have $J^2 \leq E^2$, it follows that $J(\sigma)$ covers the entire range of possible J values.

From this it follows that we can always choose a $\hat{\sigma}$ such that $J(\hat{\sigma}) = J'$ without passing through zero angular momentum. This proves the claim. Hence there are precisely *two* families of trajectories each spanning the whole orientation class. This gives the classification of trajectories.

One may view the S^3 as compactification of R^3 with the south pole as the origin of R^3 and the north pole as the point at infinity. The removal of Δ now corresponds to removal of an infinite line, say, the z-axis of the R^3. It is immediate that the fundamental group is the group of integers, Z, and therefore the basic trajectories winding around the z-axis belong to two orientation classes. In this case the fundamental group directly shows how the basic trajectories are naturally classified into the "orientation classes".

Consider now the Hamiltonian vector fields generating the three one-parameter subgroups of $SU(2), \mathbf{u}_1$ generates rotations about the z-axis. Its orbit through any generic point avoids the z-axis and the vector field remains complete (its integral curves range over the full real line). The other two subgroups on the other hand have orbits *necessarily* cutting through the z-axis. These vector fields therefore are necessarily incomplete (for the anyon case). The transformations generated by these vector fields do *not* form a group and the dynamical symmetry group for anyons is reduced from $U(2)$ to $U(1) \times U(1)$.

Thus the removal of the set Δ has altered the fundamental group, has classified trajectories into orientation classes and has also reduced the dynamical symmetry group form $U(2)$ to $U(1) \times U(1)$ due to the incompleteness of vector fields.

Note that while the symmetry group is reduced from $U(2)$ to $U(1) \times U(1)$ for anyons, the rank has remained the same and therefore the two-anyon system continues to be integrable. The integral curves being circles, we have integrability via action-angle variables and we get the exact spectrum from the semiclassical approximation. In fact one can follows the EBK quantization procedure to reproduce the exact spectrum.

5.1.2 The case of many anyons

The $N > 2$ case differs from the $N = 2$ case in many ways. The potential symmetry group, $OSp(4N, R)$, (see appendix B) generating families of trajectories is lot more complicated and so is the fundamental group for the phase space of anyons. However the analysis of the previous section already suggests a strategy for obtaining a classification of the families of trajectories. A generic trajectory may be viewed as a collection of $\frac{N(N-1)}{2}$ ellipses traced by the $\vec{r}_{ij}, i < j$. To get a convenient handle on the disallowed trajectories, define $J_{ij} \equiv (\vec{r}_{ij} \times \dot{\vec{r}}_{ij})_z$. Clearly a trajectory will cut through the set Δ if and only if J_{ij} vanishes for at least one pair of indices. We will term such a trajectory as degenerate. For anyons only non-degenerate trajectories are allowed. Note that J_{ij} are constants of motion just as J_i and $E_i \equiv \frac{1}{2}(r_i^2 + p_i^2)$ [16] are. Each of these elliptical trajectories will have a sense of traversal ($\epsilon_{ij} \equiv$ sign of J_{ij}). Thus a basic trajectory of N particles has an associated set: $\{\epsilon_{ij}\}$. Exactly as in the previous sub-section, the set of non-degenerate trajectories can be classified by these 'orientations'. Provided we can exhibit trajectories which realize all possible choices of ϵ_{ij}, we will have *at least* $2^{N(N-1)/2}$ families for anyons and precisely as many if every member of a class is connected to every other member of the same class by a symmetry transformation without crossing Δ. It is easy to show that every choice of ϵ_{ij} is realized by considering concentric circular trajectories for each of the particles with various possible signs for the angular momenta J_i and various possible ordering of the radii of the circular trajectories. For oscillator we will have precisely one family if $OSp(4N, R)$ acts

transitively on the constant energy sphere, S^{4N-1}.

The proof of transitivity in the case of oscillator is most directly obtained by noting that $OSp(4N,R)$ is isomorphic to $U(2N,C)$ and it is well known (and easy to see) that the coset space $U(2N,C)/U(2N-1,C)$ is diffeomorphic to S^{4N-1}. This immediately shows that for the oscillator, there is precisely *one* family of periodic trajectories. The counting of families is more tricky in the case of the anyons. Let us see the symmetry reduction in a slightly different manner.

Let Δ' denote the set of all degenerate trajectories i.e. at least one of the J_{ij}'s is zero. Note that Δ and Δ' are different, the former being a proper subset of the latter. Suppose we find the subgroup of symmetries which leave Δ' invariant, then the same subgroup will also leave the set of all non-degenerate trajectories invariant. Since we are looking for a group, we can consider the infinitesimal action. Let $J \equiv \prod_{i<j} J_{ij}$. Clearly, $J = 0$ characterizes the set Δ'. If we have a degenerate trajectory with two or more J_{ij}'s zero, then every infinitesimal action will keep within the set of degenerate trajectories. If only one J_{mn} is zero then infinitesimal action will act invariantly if and only if J_{mn} is invariant. Of course we could have degenerate trajectories with different angular momenta being zero and hence for the invariance of Δ' it is necessary that the generators of the subgroup must leave each of the J_{ij} 's invariant *whenever* J_{ij} *itself is zero*. This subgroup is identified in the appendix B. There are two forms of generators. Any generator with all the block matrices being the same $\mathbf{u}$ matrix with $\mathbf{u} \in OSp(4,R)$ will leave all the J's invariant. These transformations act on the center-of-mass variable alone and constitute the expected $OSp(4,R)$ symmetry. In addition, generators with only diagonal blocks being the same $\mathbf{u}$ and all the off-diagonal blocks being $\mathbf{0}$ also leave the J's invariant provided $\mathbf{u} = \mathbf{u_1}$ or $\mathbf{u}_2$ (or a linear combination thereof). These are just the total Hamiltonian and the total angular momentum ($\sum_i J_i$) which generate time evolutions and common rotations of all the positions and momenta. These are the only subgroups leaving the set of degenerate (and non-degenerate) trajectories invariant. The vector fields corresponding only to these will remain complete. Thus we see that the symmetry is reduced to $OSp(4,R)\times O(2)\times O(2)$. This is identical to the symmetry in the case of two anyons.

It is apparent from the above discussion that pseudo-integrability of anyons really means that not all the Lie algebra level symmetries exponentiate to Lie group symmetries. One should distinguish between "infinitesimal integrability" and integrability. A general discussion is given in the sub-section 5.3 .

This purely classical analysis has already explained the qualitative result that for anyons only the total energy and the total angular momentum are the conserved quantities (good quantum numbers). Since we also have a classification of periodic trajectories we will take the next step of attempting semiclassical approximation to see what further information can be obtained.

5.2 Semiclassical Spectrum of anyons

To develop a semiclassical approximation the central quantity of interest is the propagator defined as,

$$\begin{aligned} G(E+i\epsilon) &\equiv Tr(\frac{1}{E-H+i\epsilon}) \\ &= \sum_n (E-E_n+i\epsilon)^{-1} \\ &= \sum_n \mathcal{P}((E-E_n)^{-1}) - i\pi \sum_n \delta(E-E_n). \end{aligned} \tag{72}$$

In the last equality, $\mathcal{P}$ denotes the principle value while the second term is of course the density

of states. Defining $\tilde{G} \equiv iG$ allows us to write the density of states, $g(E)$ as ,

$$\begin{aligned} g(E) &= \frac{1}{2\pi} \{\tilde{G}(E+i\epsilon) + \tilde{G}^*(E-i\epsilon)\} \\ &= \frac{1}{\pi} Re\{\tilde{G}(E+i\epsilon)\} \end{aligned} \tag{73}$$

The derivation of the semiclassical trace formula begins with the definition of the propagator, $G(E)$, with trace operation expressed via a path integral representation. The trace operation combined with the stationary phase approximation (SPA) gives the propagator as a sum over periodic orbits (in the phase space), of terms with an amplitude given by a suitable Van-Vleck determinant times a phase whose exponent is the value of the 'action', $\int \mathbf{p} \cdot \mathbf{dq}$ around the periodic orbit together with a contribution from the Maslov indices. This works very well for systems having only isolated periodic orbits. For systems with symmetries, as for the oscillator or anyons, the periodic orbits come in continuous families and a modification is needed.

In cases with symmetries, the sum over orbits gets replaced by a sum over families of orbits together with a measure factor. If the family arises due to a symmetry, as is usually the case, the amplitude and the phase is same for all members of the family and as such can be computed from any one member. This is discussed by Littlejohn et al [29] in detail. The origin of families of periodic orbits in the case of oscillator is of course the $OSp(4,R)$ dynamical symmetry and for anyons essentially the same transformations generate families of trajectories. This is already discussed in the previous sub-section.

A system could also admit the so called "diffractive" orbits. Generically these are orbits which are closed in the configuration space but not in the phase space (or that the Hamiltonian flow is discontinuous)[30]. Within the framework of so called "uniformly continuous approximation" [31], these lead to further contributions to the semiclassical propagator. The anyon system presents us with both these features .

In the context of semiclassical approximation, one point needs to be noted. Observe that in terms of the $\vec{\pi}_i, \vec{r}_i$ coordinates there is no explicit reference to α and Δ is also precisely the set on which one gets δ-function singular Poisson brackets with α appearing as the coefficient. If one did inverse Legendre transformation to get a Lagrangian one would not get the α dependent term, but of course the configuration space will have the Δ removed. Using $\vec{p}_i, \vec{r}_i$ variables and then doing inverse Legendre transformation will produce the Lagrangian with the α dependent total derivative term. Now the removal of Δ is made explicit by the α dependent term. It is this explicit form which is convenient for computation of action for periodic orbits.

The action for the oscillator is equal to $(2\pi E)/\omega$ and the action for anyons is the same as that for the oscillator except for the contribution from the total derivative term in the Lagrangian. Its value is $\pm 2\pi\alpha$ with sign depending on the orientation class(es). We have already seen that for $N = 2$ we have precisely 2 orientation classes while for general N we have *at least* $2^{N(N-1)/2}$ classes.

Are there "diffractive contributions"? To appreciate this, let us consider the two anyon case in more detail. The propagator can be computed exactly from the known exact spectrum [32]. The exact spectrum is given by,

$$E_{n,j} = \hbar\omega(2n + |j - \alpha_q| + 1), \quad n \geq 0, \quad j \in Z. \tag{74}$$

From this the partition function is obtained as,

$$Z(\beta) = \frac{\cosh(\beta\hbar\omega\ (\alpha_q - 1)\ ) + \cosh(\beta\hbar\omega\alpha_q\ )}{2\sinh^2(\beta\hbar\omega)}, \tag{75}$$

and the exact density of states is obtained as:

$$\begin{aligned} g(E) &\equiv \frac{1}{2\pi i}\int_{\epsilon-i\infty}^{\epsilon+i\infty} e^{\beta E} Z(\beta) \\ &= \frac{E}{(\hbar\omega)^2}\{1+\sum_{k\geq 1}\{\cos(\frac{2\pi kE}{\hbar\omega}+2\pi k\alpha_q)+\cos(\frac{2\pi kE}{\hbar\omega}-2\pi k\alpha_q)\} \\ &\quad -\frac{1}{(\hbar\omega)}\sum_{k\geq 1}\{2\alpha_q\ \sin(2\pi k\alpha_q)\ \sin(\frac{2\pi kE}{\hbar\omega})\} \\ &\quad +\frac{1}{(\hbar\omega)}\sum_{k\geq 1}\{(-1)^k\ \sin(\pi k\alpha_q)\ \sin(\frac{\pi kE}{\hbar\omega})\}. \end{aligned} \tag{76}$$

Rewriting the products of sines as differences of cosines, we get

$$\begin{aligned} g(E) &= \frac{E}{(\hbar\omega)^2} \\ &\quad +\frac{1}{\hbar\omega}\ (\ \frac{E}{\hbar\omega}+\alpha_q\ )\sum_{k\geq 1}\ \cos(\frac{2\pi kE}{\hbar\omega}+2\pi k\alpha_q) \\ &\quad +\frac{1}{\hbar\omega}\ (\ \frac{E}{\hbar\omega}-\alpha_q\ )\sum_{k\geq 1}\ cos(\frac{2\pi kE}{\hbar\omega}-2\pi k\alpha_q) \\ &\quad -\frac{1}{2\hbar\omega}\ \sum_{k\geq 1}\ (-1)^k\ \cos(\frac{\pi kE}{\hbar\omega}+\pi k\alpha_q) \\ &\quad +\frac{1}{2\hbar\omega}\ \sum_{k\geq 1}\ (-1)^k\ \cos(\frac{\pi kE}{\hbar\omega}-\pi k\alpha_q) \end{aligned} \tag{77}$$

Remarks:

1. The partition function is manifestly invariant under $\alpha_q \to 1-\alpha_q$. The convergence of integrals in computing the density of states for $\alpha_q \in (0,1)$ requires $\frac{E}{\hbar\omega}$ to be greater than or equal to 1.

2. The first term above is the usual Thomas-Fermi term [32]. This will be suppressed in expressions below.

3. In the limit $\hbar \to 0$ keeping E, α_q fixed, the exact propagator effectively looses all dependence on α_q. This is also the semiclassical trace formula for the oscillator showing that the semiclassical approximation is exact for the oscillator. All the other terms are sub-leading.

4. For a comparison with semiclassical approximation, a different limit is implied. One should use $\alpha_c = \hbar\alpha_q$ and the SPA will be done keeping α_c fixed. Then only the last two terms will be sub-leading and the propagator will continue to have α_c dependence. This is the limit considered in the following.

5. In the last two sub-leading terms the argument of cosine is half of that in second and the third term. As is well known, the argument of cosines is the classical action, $\int \vec{p}\cdot d\vec{r}$, around a classical orbit including possible contributions from "Maslov indices". The last two terms are therefore suggestive of contribution from "half orbits". Indeed if one takes a Fourier transform of $g(E)$ [32] , then one sees a peak at 1/2 period from the last two terms. *One may therefore suspect a contribution from classical trajectories with half the period.* Do we have such classical trajectories?

Recall that anyons are fundamentally defined quantum mechanically and all the energy eigenfunctions of the system vanish on the set Δ of coincident points. We modeled the classical system by removing Δ and explored the consequences in the previous sections. In particular we just *omitted* trajectories that could cut through Δ.

From a purely classical point of view, though, this is a little unsatisfactory. One could legitimately ask just what happens to trajectories (e.g. $J = 0$ for $N = 2$) that attempt reaching the disallowed region? To decide this one has to *extend* the classical modeling by supplementing the equations of motions with a specified "boundary" condition. The choice is helped by noting that the curl of the vector potential is a δ-function which may be "regulated" to stipulate the "boundary condition". This analysis is given in appendix A.

The result is that *all the elliptical trajectories are identical to those of the oscillator and only the degenerate trajectories get modified. A degenerate trajectory must reflect back from the coincident point.* Clearly such a trajectory will have half the period of the generic trajectories and hence we do have such "half trajectories". While this can been seen directly in the oscillator confinement from the form of the density of states, it may also be seen in the numerical computation of the Fourier transform of the density of states of a particle moving on a two-dimensional disc in the presence of the flux line[32].

While classically there *are* half-period orbits, the inference that they are manifested in the two sub-leading terms of eq.(78), is ambiguous. To see this let us express the propagator in a sum-over-orbits form. To do this rewrite the the last two terms to resemble the second and the third terms.

The last two terms also have a $(-1)^k$ in the summation over k and this can be handled in two ways.

(A) Separate these sums into even and odd integer sums. All the sums over k being geometric series can be done explicitly ($i\epsilon$ needs to be added). Defining $\mathcal{E} = E/(\hbar\omega)$, we get,

$$\begin{aligned}\hbar\omega g(E) &= \mathcal{E} + \\ &\frac{\frac{1}{2}\,(\mathcal{E}+\alpha_q-\frac{1}{2}\,)e^{2\pi i(\mathcal{E}+\alpha_q)}+\frac{1}{4}e^{i\pi(\mathcal{E}+\alpha_q)}}{1-e^{2\pi i\alpha_q(\mathcal{E}+\alpha_q)}} + \text{ C. C.} \\ + &\frac{\frac{1}{2}\,(\mathcal{E}-\alpha_q+\frac{1}{2}\,)e^{2\pi i(\mathcal{E}-\alpha_q)}-\frac{1}{4}e^{i\pi(\mathcal{E}-\alpha_q)}}{1-e^{2\pi i\alpha_q(\mathcal{E}-\alpha_q)}} + \text{ C. C.}\end{aligned} \tag{78}$$

Here C. C. means complex conjugation. Now we can "read-off" $G(E+i\epsilon)$ and get (suppressing the Thomas-Fermi term),

$$\begin{aligned}(-2\pi i)^{-1}\hbar\omega G(E+i\epsilon) &= \frac{1}{2}\,\{\mathcal{E}+\alpha_q-\frac{1}{2}(1-e^{-i\pi(\mathcal{E}+\alpha_q+i\epsilon)})\,\}\;\{\frac{e^{2\pi i(\mathcal{E}+\alpha_q+i\epsilon)}}{1-e^{2\pi i(\mathcal{E}+\alpha_q+i\epsilon)}}\}+ \\ &\frac{1}{2}\,\{\mathcal{E}-\alpha_q+\frac{1}{2}(1-e^{-i\pi(\mathcal{E}-\alpha_q+i\epsilon)})\,\}\;\{\frac{e^{2\pi i(\mathcal{E}-\alpha_q+i\epsilon)}}{1-e^{2\pi i(\mathcal{E}-\alpha_q+i\epsilon)}}\}.\end{aligned} \tag{79}$$

Noting that the second group of braces is sum of a geometric series:

$$\frac{e^{2\pi i(\mathcal{E}\pm\alpha_q+i\epsilon)}}{1-e^{2\pi i(\mathcal{E}\pm\alpha_q+i\epsilon)}}=\sum_{k\geq 1}\,e^{2\pi ik(\frac{E}{\hbar\omega}\pm\alpha_q+i\epsilon)}, \tag{80}$$

we see that the propagator *is* expressed in a form suggestive of a sum over periodic orbits. The first group within the braces being the "amplitude" while the second group being contributions from multiple traversals of a basic periodic orbit. The two terms can be seen to come from families

of elliptical orbits going clockwise and anti-clockwise and that there are *no* term corresponding to degenerate ellipses or half orbits. Note that the terms in the "amplitudes" other than $\mathcal{E} \pm \alpha_q$ are sub-leading relative to $\mathcal{E} \pm \alpha_q$. This leading part of the "amplitudes" can also be seen to come from POT in presence of continuous families of periodic orbits [29]. The "amplitudes" however are complex. For $\alpha_q = 0$ there is cancellation of these sub-leading terms and one recovers the exact result for the oscillator. It appears that though the system is integrable, the trace formula does *not* give exact propagator.

The poles in $G(E)$ come from the two terms and are given by:

$$E_+(n_+) = \hbar\omega(n_+ - \alpha_q) \qquad n_+ \geq 2 \tag{81}$$

$$E_-(n_-) = \hbar\omega(n_- + \alpha_q) \qquad n_- \geq 1. \tag{82}$$

The residues at these poles ($\mathcal{E} \pm \alpha_q = n_\pm$) are $n_\pm$ if $n_\pm$ is *even* and $n_\pm - 1$ if $n_\pm$ is *odd*. These residues of course give the degeneracy. *Note that the sub-leading terms are important for this.*

It is easy to see that the locations of all the poles can be re-expressed in the form given by the exact spectrum. The degeneracies at the poles also match exactly as expected.

(B) We can also write $(-1)^k cos(k\theta) = cos(k(\theta \pm \pi))$. Proceeding exactly as in the case (A), we can "read-off" $G(E + i\epsilon)$ and get,

$$\begin{aligned}\left(\frac{-\hbar\omega}{2\pi i}\right) G(E+i\epsilon) &= \frac{1}{2}\{\mathcal{E}+\alpha_q\}\{\frac{e^{2\pi i(\mathcal{E}+\alpha_q+i\epsilon)}}{1-e^{2\pi i(\mathcal{E}+\alpha_q+i\epsilon)}}\} + \frac{1}{2}\{\mathcal{E}-\alpha_q\}\{\frac{e^{2\pi i(\mathcal{E}-\alpha_q+i\epsilon)}}{1-e^{2\pi i(\mathcal{E}-\alpha_q+i\epsilon)}}\} \\ &\quad -\frac{1}{4}\{\frac{e^{\pi i(\mathcal{E}-1+\alpha_q+i\epsilon)}}{1-e^{\pi i(\mathcal{E}-1+\alpha_q+i\epsilon)}}\} + \frac{1}{4}\{\frac{e^{\pi i(\mathcal{E}+1-\alpha_q+i\epsilon)}}{1-e^{\pi i(\mathcal{E}+1-\alpha_q+i\epsilon)}}\}. \end{aligned} \tag{83}$$

These have poles which are subsumed by the poles from the first two terms. The overall pole structure and residues of course are exactly same as before as they should be.

The exponents in the last two terms however do *not* look like the action integral. A reflecting trajectory will not receive a contribution from the α_q dependent total derivative term. The reflecting trajectories will also form a single family of trajectories and thus should give only one term and not two terms. The last two terms can be combined and the propagator may be re-expressed as

$$\begin{aligned}\left(\frac{-\hbar\omega}{2\pi i}\right) G(E+i\epsilon) &= \sum_{k\geq 1}\Big[\,(\frac{\mathcal{E}+\alpha_q}{2})\; e^{2\pi i k(\mathcal{E}+\alpha_q+i\epsilon)} + (\frac{\mathcal{E}-\alpha_q}{2})\; e^{2\pi i k(\mathcal{E}-\alpha_q+i\epsilon)} \\ &\quad + (\frac{-i\;\sin(k\pi\alpha_q)}{2})\; e^{i\pi k(\mathcal{E}-1)}\,\Big]. \end{aligned} \tag{84}$$

The propagator now does have form indicating a contribution from the half period orbits including a contribution from a Maslov index due to reflection. However it has a complex amplitude with a dependence on α_c and the multiple traversals index k.

Thus the exact propagator can be expressed in two different forms mimicking the trace formula with and without the half period trajectories. Although the phases are as expected, the amplitudes are not. The "amplitudes" are complex in general and also have terms with different orders of $\hbar$.

The different orders of $\hbar$ seen can be understood in the context of trace formula as due to *families* of periodic trajectories and not due to any higher order corrections to the SPA. When periodic trajectories come in continuous families, the number of Gaussian integrations is reduced since the integration over the family must be done exactly [29] and this increases the powers

of $\hbar$ in the denominator. The elliptic trajectories form a three parameter family while the half trajectories form a one parameter family which explains the difference in the powers of $\hbar$.

While the powers of $\hbar$ can be understood, the precise matching of the amplitudes from the POT does not follow. One may conclude from this that *the exact propagator is not of the form implied by POT and that additional contributions are necessary.* The "half period" peaks in the Fourier transform is a property of the exact propagator and need not imply presence of "half orbits" in a POT, a "diffractive" contribution may also generate such a peak.

This analysis of the exact propagator shows that even though the system is integrable, the propagator is not expressible as a POT sum, with or without "half orbits". In the light of the work on "diffractive" contributions cited above, the exact propagator seems to need such contributions. However we have not done a precise demonstration that the exact propagator *can* be obtained by inclusion of "diffractive" contributions. This is beyond the scope of present review.

Needless to say, the structure of "half trajectories" will be more complicated for $N > 2$. One needs to develop/adapt the machinery of diffractive contributions to this case to strengthen a semiclassical demonstration of the non-linearly interpolating eigenvalues.

Observe that *even without* the knowledge of precise number of families, we have precisely $2^{N(N-1)/2}$ sums in the sum over periodic trajectories. This is because, the contribution of the action integral depends only on the orientation class and not on further possible subclasses of trajectories. The measure factors therefore add up within each orientation class and this is sufficient to get the semiclassical eigenvalues.

The sum over families in the trace formula contains as many terms. To get the eigenvalues we need to look at only the action integral which differs from that of the oscillator only by the additional $\oint \frac{\alpha_c}{2} \sum_{i\neq j} \dot{\theta}_{ij}$ contribution. The oscillator's contribution to the action integral is of course just $\frac{2\pi}{\omega}E$, where E is the classical energy and ω is the oscillator frequency.

The net α_c dependent contribution to the action integral is given by,

$$\frac{\alpha_c}{2} \oint \sum_{i\neq j} \dot{\theta}_{ij} \;=\; 2\pi\alpha_c \sum_{i<j} \epsilon_{ij}. \tag{85}$$

Here ϵ_{ij} is the sign of traversal of $\vec{r}_{ij}$ and is ± 1. The $\sum_{i<j}$ is trivial to evaluate.

Since the classical trajectories are identical to those of the oscillator, we will have the same Maslov indices which incorporate the usual zero point energy namely, $N(N-1)$ for $2N$ dimensional oscillator. Including this contribution, the semiclassical energies are given by :

$$E_n\{\epsilon_i\} \;=\; \hbar\omega \left[n \;+\; \alpha_q \sum_{i<j} \epsilon_{ij} \;+\; N(N-1) \right] \qquad n \;\geq 0. \tag{86}$$

The coefficient of α_q takes all possible integer values from $-N(N-1)/2$ to $N(N-1)/2$ in steps of 2. The extreme values correspond to all the particles traversing the same way and represent the contribution of "collective" motion shown in [16]. The remaining values represent contributions of "relative" motion. These are new eigenvalues albeit at the leading $\hbar$ level. Numerical spectra available for $N = 3, 4$ show eigenvalues matching with the above at the $\alpha_q = 0$ and $\alpha_q = 1$ [22]. These numerical eigenvalues however are non-linearly interpolating while our semiclassical eigenvalues are linearly interpolating.

In reference [22] also it was conjectured that there will be linearly interpolating eigenvalues with various slopes based on a different semiclassical argument. We have obtained these eigenvalues by direct application of the periodic orbit theory and proved their existence for all N.

Observe that the α_q dependence in the eigenvalues (locations of poles of $G(E)$) arises only from the explicit $\alpha_c \equiv \alpha_q \hbar$ in the action. This therefore must be at the most linear in α_q. Since α_q is fixed, the semiclassical limit of $\hbar \to 0$ implies $\alpha_c \to 0$. Thus one may at the most see linear α_c dependence in the leading approximation which could however be indicative of exact non-linearly interpolating eigenvalues.

Thus the leading level semiclassical propagator (with POT only) already indicates the presence of eigenvalues such that $E(\alpha_q = 1) - E(\alpha_q = 0)$ takes all possible integer values between $\pm N(N-1)/2$ in steps of 2.

As there does not appear to be any scope for going beyond the leading SPA (all intermediate exponents of phases are quadratic), it may be conjectured that diffractive contributions are needed to improve these semiclassical eigenvalues.

5.3 Pseudo-integrability

In the previous sections we encountered two important features, one related to the fundamental group and one related to symmetry reduction due to incomplete vector field. Both were caused by the same source, namely removal of coincident points implying topologically non-trivial phase space. Both are relevant for periodic orbit theory in a general way and a few general remarks are in order.

The stationary phase approximation to the propagator naturally leads to periodic orbits in the phase space. These orbits may be isolated or come in continuous families or both. The continuous families may be generated by a full group of symmetries or by only a subset of symmetry transformations. Since each periodic orbit is also a map of S^1 to the phase space Γ, clearly every orbit must belong to one and only one homotopy class of the fundamental group, $\pi_1(\Gamma)$, of the phase space. If an orbit is a member of a continuous family then the entire family must belong to a single homotopy class. Note that a given homotopy class may contain no orbit (solution of equation of motion) or several isolated orbits and/or several families of orbits. But a family can not spill over two distinct homotopy classes. In sub-section 5.1 above, we saw precisely the splitting of a single basic family for oscillator into two basic families for anyons because of the non-trivial fundamental group. Multiple traversals of course belong to different homotopy classes and are explicitly summed over. Thus a non-trivial π_1 may (but not necessarily) provide an obstruction to a symmetry. How exactly may such an obstruction manifest itself? For this we have to consider vector fields generating symmetries.

Recall [33] that every function on the phase space generates *infinitesimal* symplectic diffeomorphisms (canonical transformations) via its corresponding (globally) Hamiltonian vector field. The Lie algebra of such vector fields is isomorphic to the Poisson Bracket (PB) algebra of functions on Γ. However such infinitesimal transformations exponentiate to give a one parameter group of transformations *only if* the vector field is *complete* i.e. the integral curves of the vector field can be extended so as to have these as a map from the full R. Only complete vector fields—and this is a global statement—give rise to groups of symmetries. (A corresponding quantum mechanical statement for continuous symmetries is contained in the Stone's theorem: every one parameter group of unitary transformations is generated by a self-adjoint operator and conversely.) Since we usually want classical symmetries to be reflected at the quantum level with observable generators we have to have *groups* of symmetries and hence complete vector fields.

The criterion of integrability in terms of vanishing PB's guarantees only the existence of *infinitesimal symmetries* which is of course a prerequisite. An "infinitesimally integrable" system may thus be: (a) integrable via action-angle coordinates if the vector fields are complete and the integral curves are closed; (b) integrable, but not by action-angle variables if the vector fields are

complete but only a subset of these have closed integral curves and (c) partially integrable if only a subset of vector fields are complete. The "pseudo integrability" property of anyons pointed out in ref. [16] falls in the category (c). As explained in the appendix B, apart from the Hamiltonian only the total angular momentum has a complete vector field and hence is the only quantum number that survives.

A non-trivial fundamental group by itself however does not imply possibility of incomplete vector fields. For, on a compact manifold all non-singular vector fields are complete[33] and it can of course have a non-trivial π_1. In many cases constant energy surfaces are compact and one does not have to worry about incomplete vector fields. For the anyons however removal of the set Δ makes the constant energy surface non-compact which admits possibility of incomplete vector fields and corresponding loss of symmetry. In the above, we have seen manifestation of all these features.

6 Semi-classical analysis of the ground state

The semiclassical analysis of the last section is a general one for all the states of the N-anyon system. However, the ground state of this system deserves special attention, being somewhat unusual. As pointed out by Chitra and Sen [21], due to the level-crossing between pairs of energy levels, the ground state consists of many pieces at different α values. This is because states of different J have lowest energy at different values of alpha. In this section we present briefly a semiclassical analysis of the many-anyon ground state via another approach, which is motivated by the above observation.

We start from the classical solutions of lowest energy, which are the stable configurations ("classical ground states") of the system. These states are determined by minimizing the total energy of the system. The assumption is then made that in the semiclassical regime, the quantum ground state energy is given by the classical energy to the zeroth approximation with corrections to first order given by quadratic, quantum fluctuations about this value.

Now the quantum ground state doesn't interpolate smoothly between the extreme values of the statistical parameter. This behaviour can never be captured if we try to find the ground state by simply minimizing the classical Hamiltonian. In fact, such a step gives a "ground state" which is completely independent of the angular momentum, and therefore of α.

The rotational symmetry of the Hamiltonian also means that the angular momentum of the system is conserved. To find the minima of the energy, one can proceed in two steps: first minimize the Hamiltonian for a fixed angular momentum and then minimize the resulting energy with respect to the angular momentum. In the second step, one could carry out the minimization over *quantized* values of the angular momentum. This method, as we shall shortly see, retains the dependence on angular momentum of the lowest energy, even at the classical level and also has some features observed in the Thomas-Fermi approach to the ground state [21].

6.1 Classical ground states

The energy of N free anyons in a harmonic confinement is represented by the classical hamiltonian

$$H = \frac{1}{2}\sum_{i}^{N}\left((\mathbf{p}_i - \mathbf{a}_i)^2 + r_i^2\right),$$
$$\text{where} \quad \mathbf{a}_i = \alpha\hat{\mathbf{z}} \times \sum_{j\neq i}\frac{\mathbf{r}_{ij}}{r_{ij}^2}. \tag{87}$$

Constancy of angular momentum is applied as a constraint,

$$C \equiv \sum_i^N j_i - J \qquad \text{where} \qquad j_i = x_i p_{y_i} - y_i p_{x_i}. \tag{88}$$

Introducing the Lagrange multiplier λ, the constrained function

$$F = H - \lambda C, \tag{89}$$

must be minimised. That is, $\delta F = 0$ gives the extrema which will be (local) minima if $\delta^2 F > 0$. The equations for equilibrium are thence derived to be the following:

$$\begin{aligned} p_{x_i} &= (a_{x_i} - \lambda) y_i, \\ p_{y_i} &= (a_{y_i} + \lambda) x_i, \end{aligned} \tag{90}$$

are conditions arising from the variation of F with respect to the momenta, while variation with respect to the positions gives the equation

$$(1 - \lambda^2)\mathbf{r}_i = 0. \tag{91}$$

Since this is true for all i, this fixes the Lagrange multiplier: $\lambda = \pm 1$. The constant angular momentum becomes

$$J = -\lambda \sum_i^N r_i^2 - \frac{\alpha}{2} N(N-1). \tag{92}$$

The equilibrium configurations are thus dynamic paths with non-zero momentum and angular momentum, the positions being given by those satisfying the above relation for a given angular momentum sector. The energy in such a sector is given by

$$\begin{aligned} E &= \sum_i^N r_i^2 \\ &= \pm (J + \frac{\alpha}{2} N(N-1)). \end{aligned} \tag{93}$$

To find the ground state energy E_0, one must now minimize this energy with respect to J. Since the main aim of the present exercise is to find a semiclassical route to anyon ground state energy, the above expression can be minimised with respect to the *quantized*, integer values of J. Now the term $\frac{\alpha}{2} N(N-1)$ can be expressed as a fraction plus an integer: $\mu + n$ where $n \in \mathbb{Z}$. So the energy is quantized as $E_0 = m + \mu$ where m is an integer and μ is a fraction. E_0 is evaluated and plotted as a function of α for different N in figure 1.

A remarkable feature of this picture is that the the ground state for different values of α has a different J. Note that the ground state energy for $N = 2$ is exact, even quantum-mechanically. For higher N, the "sawtooth" envelope of the minimum energy over the whole range of α reflects the behaviour observed in the earlier numerical studies on the quantum ground state energy [12, 21]. Level crossing effects are seen at the classical level itself, as different J values contribute to the ground state at different α. Quantum corrections could be expected to smooth out this graph.

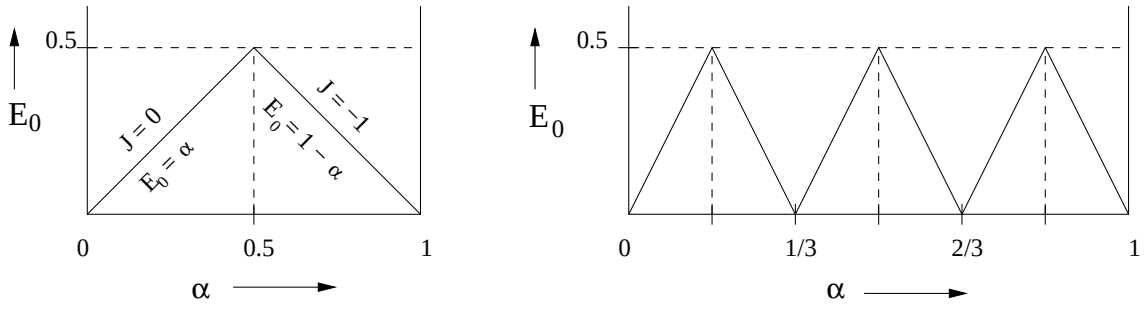

Figure 1: *Classical ground state energy of two and three anyons as a function of* α

6.2 Towards Semiclassical corrections

Quantum mechanically, the ground state energy will be corrected by the zero-point energy due to fluctuations about the classical minimum energy configuration. Now we can make the assumption that the potential is approximately harmonic near the minimum positions. The quantum wave function is localized around these points. To calculate the corrections to the energy, one can expand the potential as a Taylor series about the minimum. The zeroth order term is of course the classical ground state energy. The first order term is zero and the second order term is quadratic in the fluctuations. The matrix of second derivatives of the potential, or the Hessian, can be diagonalised in the basis of the normal modes of the system. Then the second order term for the expanded potential is just a set of oscillators in the normal modes, whose squared frequencies are the eigenvalues ω_i^2 of the Hessian. These oscillators can be quantized to give corrections $\Delta E = 1/2 \sum_{i=1}^{N} \hbar\omega_i$, to first order in $\hbar$. This of course is valid only if one can neglect the higher order terms in the expansion of V (see, for example, Rajaraman[34], chapter 5). One hitch in this program is the occurrence of *zero modes*: when any one of the ω_i's is zero, then this approximation is no longer valid. This happens when the configuration is one of neutral stability, i.e. there exist directions of symmetry in the model. Perturbations along the symmetry direction do not change the energy of the system. The quantum energy eigenfunctions cannot be said to be localized near the minimum: they will spread along the flat "valley". Zero modes can be dealt with in various ways. One of the most straightforward methods is discussed in chapter 8 of [34]. The basic idea is to work in a system of coordinates where the symmetry directions get separated when calculating the quantum corrections. The method we will adopt is to perform a transformation in the coordinates such that the direction of symmetry as the collective coordinate becomes apparent and can be isolated. The Hamiltonian is manifestly independent of this coordinate and the corresponding conjugate momentum is conserved. Hence the fluctuations along this direction can be ignored and the Hessian, being only in terms of the remaining coordinates, picks up the contribution from fluctuations along directions that are not unchanged by the symmetry transformation. This is equivalent to transforming the Hessian to a basis in which it looks block-diagonal, such that one block contributes the zero eigenvalues alone and the others do not. This block can be separated out. One then diagonalises the remaining part of the matrix.

In the N-anyon case, we work with the Lagrangian

$$L = \frac{1}{2} \sum_{1}^{N} \dot{\mathbf{r}}_i^2 + \sum_{1}^{N} \dot{\mathbf{r}}_i . \mathbf{a}_i - V(r_i), \tag{94}$$

where $\mathbf{a}_i$ is the usual statistical gauge potential and $V(r_i)$ is some confining potential.

Overall rotations is a symmetry and motion along the rotation coordinate can be treated as a collective coordinate and separated out. This was already seen in section 4: we perform the time-dependent transformation

$$\mathbf{r}_i = R(\theta)\boldsymbol{\eta}_i \quad \text{with} \quad \eta_{y_N} = 0, \tag{95}$$

$$\text{where} \quad R(\theta) = \begin{pmatrix} \cos\theta(t) & -\sin\theta(t) \\ \sin\theta(t) & \cos\theta(t) \end{pmatrix}. \tag{96}$$

The Lagrangian then becomes

$$L = \frac{1}{2}\sum_1^N \left(\dot{\boldsymbol{\eta}}_i{}^2 + 2\dot{\theta}\boldsymbol{\eta}_i \times \dot{\boldsymbol{\eta}}_i + \dot{\theta}^2\eta_i^2 \right) + \dot{\theta}\sum_1^N \boldsymbol{\eta}_i \times \mathbf{A}_i + \sum_1^N \dot{\boldsymbol{\eta}}_i.\mathbf{A}_i - V(\eta_i), \tag{97}$$

The conjugate momenta are defined by the equations

$$P_{x_i} = \dot{\eta}_{x_i} - \dot{\theta}\eta_{y_i} + A_{x_i}, \tag{98}$$

$$P_{y_i} = \dot{\eta}_{y_i} + \dot{\theta}\eta_{x_i} + A_{y_i}, \tag{99}$$

$$P_{x_N} = \dot{\eta}_{x_N} + A_{x_N}, \tag{100}$$

$$P_\theta = \dot{\theta}\sum_{i=1}^N \boldsymbol{\eta}_i^2 + \sum_1^{N-1} \boldsymbol{\eta}_i \times \dot{\boldsymbol{\eta}}_i - \frac{\alpha}{2}N(N-1). \tag{101}$$

It is convenient to perform a canonical transformation to new canonical momenta

$$\boldsymbol{\pi}_i = \mathbf{P}_i - \mathbf{A}_i. \tag{102}$$

Taking $\eta_{x_N} \equiv X$, the Hamiltonian is then

$$H = \frac{1}{2}\sum_1^N \pi_i^2 + \frac{1}{2X^2}\left(P_\theta + \frac{\alpha}{2}N(N-1) - \sum_1^{N-1} \boldsymbol{\eta}_i \times \boldsymbol{\pi}_i \right)^2 + V(\eta). \tag{103}$$

and constrained minimization gives

$$\begin{aligned} \pi_{x_i} &= -\lambda\eta_{y_i}, \\ \pi_{y_i} &= \lambda\eta_{x_i}, \\ \pi_X &= 0, \\ \boldsymbol{\nabla}_i V &= \lambda^2\boldsymbol{\eta}_i. \end{aligned} \tag{104}$$

For harmonic confinement, $V = \frac{1}{2}\sum \eta_i^2$ and the Lagrange multiplier gets fixed to ± 1, the choice depending on which minimizes the energy. The equilibrium conditions can then be written down as

$$\begin{aligned} \boldsymbol{\pi}_i &= \lambda\hat{\mathbf{k}} \times \boldsymbol{\eta}_i, \\ J &= \lambda\sum_i^N \eta_i^2. \end{aligned} \tag{105}$$

This means that the classical extrema lie on a $(2N-1)$-dimensional hyper-sphere in coordinate space whose radius is the square root of the magnitude of the total angular momentum. The

energy is

$$\begin{aligned} E_0(J) =& \sum_1^N \eta_i^2 \\ =& |J| \\ =& |P_\theta + \frac{\alpha}{2} N(N-1)| \end{aligned} \tag{106}$$

which is the same as obtained before. The advantage is that in these coordinates the Hessian will not contain the zero mode corresponding to rotations. The Hessian matrix, evaluated at the equilibrium positions, in terms of the coordinates $\eta_{x_i}, \eta_{y_i}, i = 1 \ldots N-1$, momenta $\pi_{x_i}, \pi_{y_i}, i = 1 \ldots N-1$ and finally the coordinate X and momentum π_X of the Nth particle, looks like

$$M = \begin{pmatrix} \Delta + \zeta\zeta^T & 2\zeta & 0 \\ 2\zeta^T & 4 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{107}$$

where

$$\begin{aligned} \zeta^T &= \frac{1}{X} \begin{pmatrix} \eta_{x_i} & \eta_{y_i} & -\lambda\eta_{y_i} & \lambda\eta_{x_i} \end{pmatrix}, \\ \Delta &= \begin{pmatrix} \mathbb{I}|_{2N-2} & \Omega|_{2N-2} \\ -\Omega|_{2N-2} & \mathbb{I}|_{2N-2} \end{pmatrix}, \\ \Omega &= \begin{pmatrix} 0 & \mathbb{I}|_{N-1} \\ -\mathbb{I}|_{N-1} & 0 \end{pmatrix}, \quad \text{the symplectic form in } 2N-2 \text{ dimensions.} \end{aligned}$$

This matrix must now be diagonalised to get a quadratic form in the normal modes. In this case, since the ground states are not static this matrix is in terms of all the phase-space variables and not merely the coordinates. The diagonalization must therefore preserve the symplectic structure of the phase space. The details of this symplectic diagonalization are given in appendix C.

We find that the matrix has one eigenvalue equal to 1, $2N-2$ eigenvalues equal to 2, one eigenvalue equal to $2\left(1 + \frac{|J|}{X^2}\right)$ and finally, extra $2N-2$ *zero* modes. The eigenvectors corresponding to these extra zero modes are tangential to the equilibrium phase space coordinates. Displacement along these directions preserves the $2N-1$ dimensional sphere of equilibrium, so they correspond to displacements along this sphere. These zero modes cannot however, be removed using similar techniques as used above. The main reason for this is that they correspond to infinitesimal displacements along the $2N-1$ dimensional sphere mentioned above, which is however punctured at the positions of coincidence of particles. The displacements cannot be made large, and therefore these zero modes do not actually represent a global symmetry direction of the system.

This path to finding the exact ground states semiclassically seems to face difficulties here, and the origin seems to be precisely the analytic intractability of the non-trivial energy levels that will descend through level crossings.

7 Summary and Discussion

We have presented in this review several issues related to the classical and quantum mechanics of anyons. Since several books and reviews dealing with various aspects of anyon physics already

exist, we have focussed on aspects that have received little or no attention until now. With this in view, we have specifically focussed on the integrability properties of the anyon system and semiclassical analysis of the spectrum. We have therefore discussed two properties of the many anyon system: (1) The partial separability of collective and internal degrees of freedom and (2) its identification as a pseudo-integrable system. We have shown that the first property explains the existence of the exactly known eigenvalues which is some what uncommon for a generic many body system. *This also shows that the exactly known spectrum incorporates only a some what trivial aspect of anyon dynamics.* The nontrivial aspects although partially uncovered by numerical results are still remain elusive.

The second property enables one to understand qualitatively the origin of regular and irregular features in the spectrum of many anyons. We have clarified and sharpened the meaning of pseudo-integrability. The system is *locally* identical to the isotropic oscillator in two dimensions but not at the *global* topological level. It is the non-trivial global topology that reduces the symmetry group from that of the oscillator.

We also considered application of the periodic orbit theory to this locally trivial (integrable) but globally non-trivial system. As a by-product, we saw how the exact spectrum for two anyons is reproduced by semiclassical methods. We saw that while the dynamical symmetry is reduced from $SU(2)$ to $U(1)$, the rank remained the same and hence integrability property is preserved.

For $N \geq 3$, we reproduced the previously known exact eigenvalues [16]. In addition, we obtained further *new* linearly interpolating eigenvalues. In the language of reference [16], these correspond to the signature of 'relative' dynamics. In this case the symmetry reduction was drastic, from $OSp(4N, R)$ to $OSp(4, R) \times O(2, R) \times O(2, R)$. The rank was reduced from $2N$ to 6, destroying the integrability property.

We have also discussed in detail the issue of "half" trajectories and pointed out their ambiguous role. Taking the view-point that "half" trajectories be excluded, and noting that there does not seem to be any scope for computing higher order corrections to the semiclassical spectrum with a possible non-linear dependence on α_q, it seems that the non-linearly interpolating eigenvalues are genuine quantum consequences beyond what semiclassical analysis could give. Semi-classical analysis is nevertheless sufficient to indicate the presence of these eigenvalues. Thus, the impact of pseudo-integrability of the anyon system on the classical periodic orbit theory is quite non-trivial.

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Appendix A : Regularised classical dynamics and reflecting orbits

Consider without loss of generality the case of two anyon in the relative coordinates. Generic orbits in the configure space, $R^2 - \{\vec{0}\}$, are of course an ellipse and only a degenerate elliptical orbit attempts to pass through the origin. Noting that this system can be thought of as a charged particle in presence of a singular ("statistical") magnetic field at the origin, one may regularise the magnetic field or the flux to study the orbits and obtain the limiting behaviour to deduce "boundary condition" at the origin.

To do this we imagine the relative Hamiltonian to be that of a particle in a magnetic field along the z axis. The field is of course to be effectively confined to a small disc around the origin. Now observe that for an axially symmetric magnetic field along the z-axis, $B(\vec{r},\theta) = B(r)\hat{k}$ where r, θ are the usual spherical coordinates in two dimensions, we may write the vector potential as,

$$\vec{A} = A_r\hat{r} + A_\theta\hat{\theta} \tag{108}$$

This implies,

$$B(r) = \partial_r A_\theta + \frac{A_\theta}{r} - \frac{1}{r}\partial_\theta A_r, \tag{109}$$

In a symmetric gauge, the vector potential is independent of the polar angle and by choosing it to be divergence free one can set the radial component to zero.

The flux $\Phi(R)$, through a disc of radius R is given by,

$$\Phi(R) \;=\; 2\pi\int_0^R drrB(r) \;\; = R\oint A_\theta d\theta \;=\; 2\pi R A_\theta(R). \tag{110}$$

This implies,

$$A_\theta(r) = \frac{\Phi(r)}{2\pi r} = \frac{1}{r}\int_0^r dr' r' B(r') \tag{111}$$

The choice, $\Phi(r) \;=\; 2\pi\alpha \;\forall\; r > 0$, gives the vector potential used in the quantum mechanical calculation. It also implies the $rB(r) \;=\; \alpha\delta(r)$ and this of course is the singular nature of the magnetic field. Notice that smearing the $\delta(r)$ will not make the magnetic field non-singular at the origin because of the explicit $1/r$.

For a regulated system one wants the magnetic field to be non-singular every where and effectively confined to a disk of radius ϵ around the origin. This requires the vector potential A_θ also to be non-singular and therefore vanishing at the origin. If $A_\theta(r) \;\rightarrow\; cr^\beta$ as r approaches zero then, $B(r) \;\rightarrow\; c(\beta+1)r^{\beta-1}$. For a finite, nonzero $B(0)$ we must have $\beta \;=\; 1$ and therefore $\Phi(r) \;\rightarrow\; 2\pi c r^2$ near the origin. For $r \geq \epsilon$ we still retain the flux to be $2\pi\alpha$. Continuity at $r = \epsilon$ then gives $c = \alpha/\epsilon^2$. Notice that this limiting behaviour is fixed by the demand of non-singularity of the fields and as such must be reflected in any explicit choice for the magnetic field.

Thus our regulation involves choosing,

$$\begin{aligned} \Phi(r) \;&=\; 2\pi\alpha \qquad\quad \forall \;\; r \geq \epsilon \\ &\rightarrow \;\; 2\pi\alpha\frac{r^2}{\epsilon^2} \quad \text{as} \;\; r \;\rightarrow 0 \end{aligned} \tag{112}$$

One could choose a uniform nonzero magnetic field inside the disk as an explicit choice but it will not be necessary. Since we are interested in the limiting behaviour of trajectories as ϵ is taken to zero, the limiting behaviour of the flux is all that we need.

Consider now the orbit equation for ϵ nonzero. Integrating the equations of motion once using the two constants, energy E and angular momentum ℓ, the orbit equations in r, θ coordinates become :

$$\begin{aligned} \dot{r} &= \pm\sqrt{2E - r^2\dot{\theta}^2 - r^2} \\ r^2\dot{\theta} &= \ell - rA_\theta(r) \\ &= \begin{cases} \ell - \alpha\frac{r^2}{\epsilon^2} & r < \epsilon \\ \ell - \alpha & r \geq \epsilon \end{cases} \end{aligned} \tag{113}$$

There are three types of orbits possible: those which are fully inside the disc, those which are fully outside the disc and those which go both inside and outside the disc. In the limit of ϵ going to zero, the first type of orbits are clearly irrelevant. It is easy to see that for the second type of orbits one must have $\ell \neq \alpha$. These are insensitive to the flux in the limit and are thus identical to the orbits of the oscillator.

For the last type, an interior turning point is possible only for $0 \leq \ell \leq \alpha$. The condition that such an orbit must also have an exterior turning point limits ℓ to α. We are interested in computing the change in the angular coordinate from the entry into the disk till exit from it. Explicit computation shows that the change in the angular coordinate goes to zero as ϵ goes to zero. Thus such an orbit *reflects* at the origin. Note that these are precisely the radial orbits.

To summarize, a regulated classical modeling for anyons is generically stipulated by giving the behaviour of the flux near the origin. It amounts to cutting out a disk of radius ϵ and filling it up with non-singular fields. The classical non-radial orbits then are exactly same as those of the oscillator except for the replacement $l \to l - \alpha$. The radial orbits reflect at the the origin and are termed *half orbits* since their period is half of that for the other orbits.

Appendix B : OSp(4N, R) dynamical symmetry and classification of trajectories

In this appendix we collect together some of the well known facts about the dynamical symmetry of the n dimensional isotropic oscillator.

The isotropic oscillator in n dimensions has $OSp(2n, R)$ as the group of dynamical symmetries. This is a group of $2n \times 2n$ order matrices, $\mathbf{g}$, which are both orthogonal and symplectic. Denoting by $\bar{\mathbf{g}}$ the transpose of $\mathbf{g}$, we have the defining equations:

$$\begin{array}{lllll} & \bar{\mathbf{g}}\ \mathbf{g} & = & \mathbf{I}_{2n} & : \text{ (orthogonal)} \\ & \bar{\mathbf{g}}\ \mathbf{\Omega}\ \mathbf{g} & = & \mathbf{\Omega} & : \text{ (symplectic)} \\ \text{and with} & \mathbf{g} & \approx & \mathbf{I}_{2n} + \epsilon \mathbf{T} & \text{generators } T \text{ satisfy} \\ & \bar{\mathbf{T}} & = & -\mathbf{T} & \\ & \bar{\mathbf{T}}\ \mathbf{\Omega} & = & -\mathbf{\Omega}\ \mathbf{T} & \end{array}$$

Here $\mathbf{I}_{2n}$ is the identity matrix of order $2n$ while $\mathbf{\Omega}$ is a suitable matrix defining the symplectic condition. This will be chosen below.

The symplectic condition ensures that we have a (linear) canonical transformation while orthogonality ensures that the Hamiltonian, $\sum_i (p_i^2 + q_i^2)/2$, is invariant. It is easy to see that dimension of this group is n^2 while its rank is n. In fact one can show that the group $OSp(2n, R)$ is isomorphic to the group $U(n, C)$. Making an explicit choice of Ω in a block form, one can obtain the block form for $\mathbf{g}$ using the symplectic condition. The real block matrices can be combined into complex block matrices. The orthogonality condition in terms of real matrices then translates into the unitarity condition for the complex matrices thereby proving the group isomorphism. This isomorphism immediately implies that the ortho-symplectic group acts transitively on the constant energy sphere. This is used in the section 5.1 .

We are interested in finding the action of this group on the phase space via canonical transformations (symplectic diffeomorphisms). For notational convenience let us group the standard canonical variables together and denote them by ω_μ , $\mu = 1, 2, ..., 2n$, the first n being coordinates and the last n being the momenta. The dual of the symplectic form has components, $\Omega^{\mu\nu}$ given by the $\mu\nu$ th element of the matrix $\mathbf{\Omega}$. With this notation, the PB of functions on the phase space and the infinitesimal canonical transformations are given by,

$$\begin{aligned} \{F(\omega)\ ,\ G(\omega)\} &= \Omega^{\mu\nu}\ \partial_\mu F\ \partial_\nu G \\ \delta_\epsilon \omega^\mu &= \epsilon\ \Omega^{\mu\nu}\ \partial_\nu F(\omega) \end{aligned} \tag{114}$$

Functions purely quadratic in ω generate linear canonical transformations and are also closed under the PB's and those which leave the Hamiltonian invariant give the symmetry transformations. Explicitly,

$$\begin{array}{lllllllll} F & \equiv & \frac{1}{2}\bar{\omega}\ \mathbf{A}\ \omega & , & G & \equiv & \frac{1}{2}\bar{\omega}\ \mathbf{B}\ \omega & \Rightarrow \\ \{F, G\} & = & \frac{1}{2}\bar{\omega}\ (\mathbf{A\Omega B} - \mathbf{B\Omega A})\ \omega & & & & & \end{array}$$

The Hamiltonian corresponds to $\mathbf{A} = \mathbf{I}_{2n}$. The Poisson bracket of any G with H vanishes provided the matrix $\mathbf{B}$ commutes with $\mathbf{\Omega}$. Thus generic matrices defining quadratic functions are real, symmetric matrices commuting with $\mathbf{\Omega}$. The matrices $\mathbf{\Omega B}$ are then antisymmetric and provide an isomorphism of quadratic functions to the generators $\mathbf{T}$ of the group $OSp(2n, R)$.

Action of the one parameter group generated by a function G is found from the integral curves of the corresponding Hamiltonian vector field. These curves are defined by the matrix equations,

($\mathbf{T} \equiv \mathbf{\Omega B}$)

$$\frac{d\omega(\sigma)}{d\sigma} \;=\; \mathbf{\Omega B}\omega \tag{115}$$

$$\omega(\sigma) \;=\; (e^{\sigma \mathbf{\Omega B}})\omega(0) \tag{116}$$

It is convenient to choose a particular grouping of the phase space coordinates and corresponding choice of $\mathbf{\Omega}$. Firstly let us put $n = 2N$, relevant for the present context, so that the phase space is $4N$ dimensional. Arrange the coordinates and momenta as $x_1, y_1, p_{1x}, p_{1y}, ..., x_N, y_N, p_{Nx}, p_{Ny}$ and denote by ω_i the coordinates and momenta of the i^{th} particle. The index i now runs from 1, ..., N and each ω_i is a 4×1 matrix. Correspondingly we choose $\mathbf{\Omega}$ as a block diagonal matrix with N blocks and each block being 4×4 matrix $\mathbf{\Lambda}$. We choose,

$$\mathbf{\Lambda} \;=\; \begin{pmatrix} \mathbf{0}_2 & \mathbf{I}_2 \\ -\mathbf{I}_2 & \mathbf{0}_2 \end{pmatrix} \tag{117}$$

One can choose a basis for $OSp(4N, R)$ as follows. Let us denote the generators as $\mathbf{T}_i$, $i = 1, 2, ..., N$ and $\mathbf{T}_{ij}$ with $i < j$. Each of these are expressed in the block form. The $\mathbf{T}_i$ are block diagonal with a nonzero 4×4 matrix $\mathbf{u}_i$ as the i^{th} block element. The $\mathbf{T}_{ij}$ have a matrix $\mathbf{v}_{ij}$ at the i^{th} row and j^{th} column ($(ij)^{th}$ block) and $-\bar{\mathbf{v}}_{ij}$ at the $(ji)^{th}$ block. In equations,

$$\begin{aligned} (\mathbf{T}_i)_{mn} \;&=\; \mathbf{u}_i \;\delta_{im}\; \delta_{mn} \\ (\mathbf{T}_{ij})_{mn} \;&=\; \mathbf{v}_{ij}\; \delta_{im}\delta_{jn} - \;\bar{\mathbf{v}}_{ij}\; \delta_{in}\delta_{jm} \end{aligned} \tag{118}$$

With these definitions it is easy to translate the conditions on the generators in terms of the 4 matrices **u , v** as:

$$\bar{\mathbf{u}}_i \;=\; -\,\mathbf{u}_i \qquad\qquad \mathbf{u}_{ij}\mathbf{\Lambda} \;=\; \mathbf{\Lambda}\mathbf{u}_{ij} \;\; ; \qquad\qquad \mathbf{v}_{ij}\mathbf{\Lambda} \;=\; \mathbf{\Lambda}\mathbf{v}_{ij} \tag{119}$$

Thus the **u**'s generate $OSp(4, R)$ while the **v**'s are required to commute with $\mathbf{\Lambda}$. The number of independent **u**'s is 4 while number of independent **v**'s is 8 of which 4 are 'diagonal' and 4 are 'off-diagonal'. The dimension of $OSp(4N, R)$ is thus $4N + 8N(N-1)/2 \;=\; 4N^2$. These independent matrices can be explicitly chosen in terms of 2×2 Pauli matrices and the identity matrix as:

$$\begin{aligned} \mathbf{u}_{(1,2,3,4)} \quad &\sim \quad \begin{pmatrix} i\sigma_2 & \mathbf{0}_2 \\ \mathbf{0}_2 & i\sigma_2 \end{pmatrix}, \begin{pmatrix} \mathbf{0}_2 & \mathbf{I}_2 \\ -\mathbf{I}_2 & \mathbf{0}_2 \end{pmatrix}, \begin{pmatrix} \mathbf{0}_2 & \sigma_1 \\ -\sigma_1 & \mathbf{0}_2 \end{pmatrix}, \begin{pmatrix} \mathbf{0}_2 & \sigma_3 \\ -\sigma_3 & \mathbf{0}_2 \end{pmatrix}; \\ \mathbf{v}_{(1,2,3,4)} \quad &\sim \quad \begin{pmatrix} \mathbf{I}_2 & \mathbf{0}_2 \\ \mathbf{0}_2 & \mathbf{I}_2 \end{pmatrix}, \begin{pmatrix} \sigma_1 & \mathbf{0}_2 \\ \mathbf{0}_2 & \sigma_1 \end{pmatrix}, \begin{pmatrix} i\sigma_2 & \mathbf{0}_2 \\ \mathbf{0}_2 & i\sigma_2 \end{pmatrix}, \begin{pmatrix} \sigma_3 & \mathbf{0}_2 \\ \mathbf{0}_2 & \sigma_3 \end{pmatrix}, \\ \mathbf{v}_{(5,6,7,8)} \quad &\sim \quad \begin{pmatrix} \mathbf{0}_2 & \mathbf{I}_2 \\ -\mathbf{I}_2 & \mathbf{0}_2 \end{pmatrix}, \begin{pmatrix} \mathbf{0}_2 & \sigma_1 \\ -\sigma_1 & \mathbf{0}_2 \end{pmatrix}, \begin{pmatrix} \mathbf{0}_2 & i\sigma_2 \\ -i\sigma_2 & \mathbf{0}_2 \end{pmatrix}, \begin{pmatrix} \mathbf{0}_2 & \sigma_3 \\ -\sigma_3 & \mathbf{0}_2 \end{pmatrix}, \end{aligned} \tag{120}$$

We are interested in one parameter groups generated by some element $\mathbf{T}$ of the Lie algebra. If $\mathbf{T}$ satisfies: $\mathbf{T}^2 = -\mathbf{P}$ with $\mathbf{P}$ satisfying $\mathbf{P}^2 = \mathbf{P}$, $[\mathbf{T}, \mathbf{P}] = 0$, then it follows that,

$$e^{\sigma \mathbf{T}} \;=\; \mathbf{I}_{4N} - \mathbf{P} + \mathbf{P}\,[cos(\sigma) + sin(\sigma)\mathbf{T}]\,\mathbf{P} \tag{121}$$

For the basis generators $\mathbf{T}_i$ we have $\mathbf{P}$ to be block diagonal with $\mathbf{I}_4$ as the i^{th} block while for $\mathbf{T}_{ij}$ we have $\mathbf{P}$ to be the block diagonal matrix with $\mathbf{I}_4$ as the i^{th} and the j^{th} blocks. Using these,

the exponentials of basis generators can be evaluated to get the integral curves as, (with obvious notation)

$$\begin{aligned} \omega(\sigma) &= \left[\mathbf{I}'_i + \mathbf{I}_i \{ cos(\sigma) + sin(\sigma) \mathbf{T}_i \} \mathbf{I}_i \right] \omega(0) \quad \text{and,} \\ \omega(\sigma) &= \left[\mathbf{I}'_{ij} + \mathbf{I}_{ij} \{ cos(\sigma) + sin(\sigma) \mathbf{T}_{ij} \} \mathbf{I}_{ij} \right] \omega(0) \end{aligned} \tag{122}$$

Therefore integral curves of the basis generators are periodic curves. Further, the $\mathbf{T}_i$'s affect only the i^{th} particle position and momenta while the $\mathbf{T}_{ij}$ mix the i^{th} and j^{th} particles only.

We need to study how the angular momenta J_i and J_{ij} vary under the action of one parameter subgroups. Define the (block) matrices,

$$\begin{aligned} (\mathbf{L}_i)_{mn} &\equiv \mathbf{L}\delta_{im}\delta_{mn} \qquad \mathbf{L} \equiv \mathbf{v}_7 \\ (\mathbf{L}_{ij})_{mn} &\equiv \mathbf{L}\{ \delta_{mn}(\delta_{im}+\delta_{jm}) - \delta_{im}\delta_{jn} - \delta_{in}\delta_{jm} \} \end{aligned} \tag{123}$$

In terms of these matrices the angular momenta are given by:

$$\begin{aligned} J_i &= \frac{1}{2} \bar{\omega} \mathbf{L}_i \omega \qquad = \frac{1}{2} \bar{\omega}_i \mathbf{L} \omega_i \\ J_{ij} &= \frac{1}{2} \bar{\omega} \mathbf{L}_{ij} \omega \qquad = \frac{1}{2} \bar{\omega}_{ij} \mathbf{L} \omega_{ij} , \qquad \omega_{ij} \equiv \omega_i - \omega_j. \end{aligned} \tag{124}$$

Under the group generated by the basis generator, $\mathbf{T}_i$, the J_m for instance varies as:

$$\begin{aligned} 2J_m(\sigma) = 2J_m + \delta_{im} \big[& -sin^2(\sigma) \{2J_i - \bar{\omega}_{\mathbf{i}} (\bar{\mathbf{u}} \mathbf{L} \mathbf{u}) \omega_{\mathbf{i}}\} \\ & + sin(\sigma)cos(\sigma) \{ \bar{\omega}_{\mathbf{i}} (\bar{\mathbf{u}}\mathbf{L} + \mathbf{L}\mathbf{u}) \omega_{\mathbf{i}} \} \big] \end{aligned} \tag{125}$$

while under the group generated by the basis generator, $\mathbf{T}_{ij}$, the J_m varies as:

$$\begin{aligned} 2J_m(\sigma) = 2J_m + \delta_{im} \big[& -sin^2(\sigma) \{2J_i - \bar{\omega}_{\mathbf{j}} (\bar{\mathbf{v}} \mathbf{L} \mathbf{v}) \omega_{\mathbf{j}}\} \\ & + 2sin(\sigma)cos(\sigma) \{ \bar{\omega}_{\mathbf{i}} (\mathbf{L}\mathbf{v}) \omega_{\mathbf{j}} \} \big] \\ \delta_{jm} \big[& -sin^2(\sigma) \{2J_j - \bar{\omega}_{\mathbf{i}} (\mathbf{v} \mathbf{L} \bar{\mathbf{v}}) \omega_{\mathbf{i}}\} \\ & - 2sin(\sigma)cos(\sigma) \{ \bar{\omega}_{\mathbf{i}} (\mathbf{v}\mathbf{L}) \omega_{\mathbf{j}} \} \big] \end{aligned} \tag{126}$$

Similar but more complicated expressions follow for the $J_{mn}(\sigma)$ also.

Remark: For the case of $N = 1$ we have only one block. This could be either a single two dimensional oscillator *or* the relative coordinate dynamics of two anyons. The generators are of course only $\mathbf{u}$'s. Of these $\mathbf{u}_2$ corresponds to the Hamiltonian itself while $\mathbf{u}_1$ generates same rotations of both $\vec{r}, \vec{p}$. These two matrices commute with $\mathbf{L}$ while the remaining two anti-commute with $\mathbf{L}$. This leads to the result quoted in the section 5.1 .

To deduce the surviving symmetry group for many anyons we need infinitesimal variations of J_{mn} for arbitrary generator $\mathbf{T}$. This is easily derived and is given by,

$$\delta J_{mn} = \sum_j \bar{\omega}_{\mathbf{mn}} \mathbf{L} (\mathbf{T}_{mj} - \mathbf{T}_{nj}) \omega_j \tag{127}$$

In section 5.1 we needed to determine $\mathbf{T}$ such that δJ_{mn} is zero *whenever* J_{mn} is zero. That is, at all points in the phase space where any single J_{mn} is zero, we want its infinitesimal variation induced by $\mathbf{T}$ to be zero.

Fix a particular m and n. Fix ω_{mn}. We can consider points with ω_j, $j \neq m, n$ such that no other J_{ij} is zero. $\delta J_{mn} = 0$ then implies $\mathbf{T}_{mj} = \mathbf{T}_{nj}$ $\forall$ $j \neq m, n$. We can also consider

points with different ω_m, ω_n keeping ω_{mn} fixed and maintaining all other conditions. This implies $\mathbf{T}_{mm} - \mathbf{T}_{nm} + \mathbf{T}_{mn} - \mathbf{T}_{nn} = 0$. Repeating this for all $m \neq n$ fixes the form of $\mathbf{T}$ in terms of arbitrary generators of $OSp(4,R)$, $\mathbf{u}$, $\mathbf{v}$ as,

$$(\mathbf{T})_{ij} \;=\; \mathbf{u}\delta_{ij} \;+\; \mathbf{v}(1-\delta_{ij}) \;\;,\;\; [\,\mathbf{L}\;,\;\mathbf{u}-\mathbf{v}\,] \;=\; 0 \tag{128}$$

The choice $\mathbf{u} = \mathbf{v}$ corresponds to $\mathbf{T}_{ij} \;=\; \mathbf{u}\;\forall\; i,j$. It follows that *all* J_{mn} 's are invariant independent of their values. It is easy to see that these *four* $\mathbf{T}$'s effect transformations of the center-of-mass variables which are insensitive to the anyonic features. This $OSp(4,R)$ symmetry is thus always present for all $N \geq 2$.

The choice $\mathbf{v} = 0$ implies $\mathbf{T}$ is block diagonal with the same $\mathbf{u}$ on all the diagonal blocks. Further, $[\mathbf{L}, \mathbf{u}] = 0$ implies that $\mathbf{u}$ must be a linear combination of $\mathbf{u}_1$ and $\mathbf{u}_2$. These *two* $\mathbf{T}$'s can be seen to correspond to the total Hamiltonian and the total angular momentum. This result is used in the section 5.1 to deduce that the surviving symmetry for N anyons is $OSp(4,R) \times O(2,R) \times O(2,R)$.

Appendix C : Symplectic Diagonalization

In our method of calculating semiclassical corrections to a classical ground state, we need to diagonalise the Hessian matrix, the matrix of second derivatives of the potential at the ground state.. In the case of anyons, since the ground state is not static this matrix is in terms of all the phase-space variables and not merely the coordinates. The diagonalization must hence preserve the symplectic structure of the phase space. The Hamiltonian expanded about the ground state is

$$H = H_0 + \xi^T M \xi + \quad \text{higher order terms}, \tag{129}$$

where ξ is the column vector made up of the phase space variables.

The phase space structure is preserved if the matrix M is diagonalised by a symplectic matrix S such that $S^T \mathbb{J} S = \mathbb{J}$, $\mathbb{J}$ being the $2N-2$-dimensional symplectic form. Now there exists a theorem[2] known as Williamson's theorem [36] (a compact summary is presented in Appendix 6 of Arnold [27]) part of which implies that if M is a real symmetric positive definite matrix then it can be symplectically diagonalised by such an S. Moreover, if $S^T M S = D^2$ where $D = \mathrm{diag}(D_1, D_2)$, D_1, D_2 being $N-1$ dimensional diagonal matrices, and $\tilde{M} = -\iota M^{-1/2} \mathbb{J} M^{-1/2}$ is a Hermitian matrix with real eigenvalues $\pm\omega_i$ then it can be shown that $D_1^{-1} D_2^{-1} = \mathrm{diag}(\omega_i)$. When this happens, then if the normal modes are $\xi' \equiv \binom{Q_i}{P_i} = S^{-1}\xi$, the quadratic form looks like

$$\begin{aligned} \xi'^T D^2 \xi' =& \sum_i (d_i^2 Q_i^2 + \omega_i^{-2} d_i^{-2} P_i^2) \\ =& \omega_i^{-2} d_i^{-2} (P_i^2 + \omega_i^2 d_i^4 Q_i^2). \end{aligned} \tag{130}$$

This is a harmonic oscillator in the space of normal modes, on quantizing which we get the correction to the ground state energy as

$$\begin{aligned} \delta E_0 =& \quad \tfrac{1}{2}\hbar \textstyle\sum_i \omega_i^{-2} d_i^{-2} \omega_i d_i^2 \\ =& \qquad \tfrac{1}{2}\hbar \textstyle\sum_i \omega_i^{-1}. \end{aligned} \tag{131}$$

Thus one needs just the positive eigenvalues of the matrix $\tilde{M}$, which should further be non-zero. This will require M itself to have non-zero eigenvalues.

Now to find the eigenvalues of the Hessian (107): The last row and column, coming from the derivatives with respect to P_X, contribute an eigenvalue 1. Consider the reduced $4N-3$ dimensional matrix M' formed by removing the last row and the last column. It turns out that this matrix does have zero eigenvalues, inherited from the eigenvalues of Δ with eigenvectors of the form $k\big(\zeta, -\frac{1}{2}\zeta^T\zeta\big)^T$, k being an arbitrary proportionality factor. There are exactly $2N-2$ of these. The remaining eigenvalues consist of an eigenvalue equal to 2 which is $2N-2$ fold degenerate and a non-degenerate eigenvalue equal to $2\left(1 + \frac{|J|}{X^2}\right)$.

[2] We are indebted to Prof. R. Simon for bringing this theorem to our notice, as well as for the simple proof in the paper [35]

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