

Master-Equation Approach to Spontaneous Emission*

G. S. Agarwal

Department of Physics and Astronomy, University of Rochester, Rochester, New York 14627

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Spontaneous emission from a system of N identical two-level atoms is considered using a master equation recently derived by the author. The master equation describing the time evolution of the phase-space distribution function associated with the reduced density operator of the atomic system is obtained. This master equation, which is of the type of a Fokker-Planck equation, is used to derive the equation of motion for the mean values of various atomic operators characterizing the physical properties of the system. This leads to a hierarchy of equations, which is decoupled by making a suitable approximation. The intensity of the spontaneously emitted radiation is then calculated. Next, the spontaneous emission from geometrically small systems is considered. For this case, the master equation is solved exactly, and an exact expression for the radiation rate is obtained. The exact solution of the master equation is also used to calculate the normally ordered correlation functions for the electric field. Section V deals with the spontaneous emission from a system of harmonic oscillators, the size of the system being small compared to a wavelength. The master equation for this problem is also solved exactly, and it is shown that this system also leads to *superradiant emission* in some cases, e.g., if all the oscillators are excited initially to some coherent state $|z_0\rangle$.

I. INTRODUCTION

The problem considered in the present paper is the one studied by Dicke in his classic paper.¹ Dicke calculated the radiation rate and the angular correlation of successive photons emitted from a collection of identical two-level atoms or molecules. He found that under certain conditions the radiation rate is proportional to the square of the number of atoms. This coherent emission of radiation is known as superradiance. This problem has regained interest only recently because of the experimental observation of some interesting related effects such as self-induced transparency² and photon echoes.^{3,4} In these experiments, a system of optically resonant two-level atoms was employed. This particular problem of superradiance has also been recently analyzed by Eberly and Rehler,⁵ who obtained the radiation rate for spontaneous emission and its angular distribution. Their treatment is valid both for small systems (a system whose size is small compared to a wavelength) and large systems.

Another interesting problem which is related to the above is the interaction of N two-level atoms with *one mode* of the radiation field.⁶ Exact solutions to this problem have been obtained.⁷ The work⁸ on this problem has been mainly concentrated on the calculation of the eigenfunctions and eigenvalues of the total Hamiltonian.

In this paper we reexamine Dicke's problem by using the master-equation approach.⁹ Our treatment is valid both for small systems and large systems. For small systems, we have been able to obtain the exact solutions and have calculated the radiation rate and some of the normally ordered

correlation functions. For large systems, an approximate procedure is used to calculate the intensity of spontaneously emitted radiation. The advantage of our method lies in the fact that one can clearly see the nature of approximations made at each stage of the calculation and how to improve upon them.

Our discussion is based on phase-space methods which have been recently developed.^{9,10} In particular, the method of Ref. 9 is the most relevant one, because there we have developed the basic framework for the discussion of problems like the one under consideration. Equations (4.36) and (4.37) of Ref. 9 are the starting equations for our present analysis.

In Sec. II, we consider the spontaneous radiation from a system of N two-level atoms. We obtain the master equation satisfied by the phase-space distribution function associated with the reduced density operator corresponding to the atomic system alone. The master equation is then used to derive the equation of motion for mean values of various atomic operators which characterize the physical properties of our system. It is shown that, in general, one obtains a hierarchy of equations. In Sec. III, this hierarchy is decoupled by making a suitable approximation, and the intensity of the spontaneously emitted radiation is calculated. This result is found to agree with the one obtained by Eberly and Rehler.⁵ In Sec. IV, spontaneous emission from geometrically small systems is considered. The master equation for the reduced density operator is solved exactly. The exact solution is then used to calculate the radiation rate. Finally in Sec. V, spontaneous emission from a system of harmonic oscillators is considered, and it is shown

that this system does lead to superradiant emission if all the oscillators are excited initially to some coherent state $|Z_0\rangle$.

II. MASTER EQUATION FOR THE ATOMIC PHASE-SPACE DISTRIBUTION FUNCTION

In this section, we shall derive the equation of motion obeyed by the phase-space distribution function associated with the reduced density operator corresponding to the atomic system alone. The Hamiltonian, for a collection of N identical two-level atoms interacting with a quantized radiation field, can be shown to be given by

$$H = \omega_0 \sum_{j=1}^N S_j^z + \sum_{ks} \omega_{ks} a_{ks}^\dagger a_{ks} + \sum_{ks} \sum_j (S_j^\dagger a_{ks} g_{jks} + \text{H. c.}), \quad (2.1)$$

In deriving (2.1), the nonresonant terms have been ignored (which is the rotating-wave approximation). Here ω_0 denotes the energy separation of two atomic levels, a_{ks} and a_{ks}^\dagger are the usual boson creation and annihilation operators associated with ks mode of the radiation field. $S_j^{(\pm)}$ are the components of the spin angular momentum operator associated with j th atom.¹¹ g_{jks} is the coupling constant and is given by

$$g_{jks} = - \left(\frac{i\omega_0}{c} \right) \left(\frac{2\pi c}{L^3} \right)^{1/2} \frac{1}{\sqrt{k}} e^{i\vec{k} \cdot \vec{R}_j} (\epsilon_{ks} \cdot \vec{d}), \quad (2.2)$$

where \vec{R}_j is the position vector of the j th atom and \vec{d} is the dipole-moment matrix element. All other symbols have the usual meaning. In order to obtain the equation of motion for the phase-space distribution function, we will express the spin angular momentum operators in terms of boson operators. This can be done in several ways. We will use the Schwinger's coupled boson representation.¹² In this representation, one has

$$S_j^z = \frac{1}{2} (v_j^\dagger v_j - u_j^\dagger u_j), \quad S_j^- = u_j^\dagger v_j, \quad S_j^+ = v_j^\dagger u_j, \quad (j=1, \dots, N) \quad (2.3)$$

where u_j and v_j are two sets of boson operators satisfying the commutation relations

$$[u_j, u_k^\dagger] = [v_j, v_k^\dagger] = \delta_{jk}, \quad (2.4)$$

and all other commutators vanish. From now on we will work in the interaction picture. The interaction Hamiltonian in the interaction picture is given by

$$H_I(t) = \sum_{ks} \sum_j (g_{jks} a_{ks} v_j^\dagger u_j e^{i(\omega_0 - \omega_{ks})t} + \text{H. c.}), \quad (2.5)$$

where use has been made of Eq. (2.3). We will also work with the phase-space distribution function which is obtained from the density operator by using the *normal rule of mapping*.¹⁰ It is evident that the normally ordered equivalent of (2.5) is given by

$$\sum_{ksj} [g_{jks} z_{ks} \alpha_j^* \beta_j e^{i(\omega_0 - \omega_{ks})t} + \text{c. c.}],$$

where the following mapping relations have been used:

$$a_{ks} \rightleftharpoons z_{ks}, \quad u_j \rightleftharpoons \beta_j, \quad v_j \rightleftharpoons \alpha_j. \quad (2.6)$$

Since we are only interested in spontaneous emission, the radiation field is taken to be in the vacuum state at $t=0$. The normally ordered equivalent

$$F_R^{(N)}(\{z_{ks}\}, \{\alpha_j^*\}, 0)$$

of the density operator for the field at the initial time is given by¹³

$$F_R(\{z_{ks}\}, \{\alpha_j^*\}, 0) = \prod_{ks} e^{-|z_{ks}|^2}. \quad (2.7)$$

In this case one can easily show that the matrix Γ defined by Eq. (G4.37b)¹³ reduces to

$$\Gamma_R^{ks, k's'}(t, \tau) = \delta_{kk'} \delta_{ss'} \begin{pmatrix} 0 & e^{-i\omega_{ks}(t-\tau)} \\ 0 & 0 \end{pmatrix}. \quad (2.8)$$

The operators $\mathcal{D}_S^{(1)}$ and $\mathcal{D}_S^{(2)}$, which are defined by Eqs. (G4.35a) and (G4.35b), respectively, are given by

$$\mathcal{D}_S^{(1)} = \prod_{j=1}^N \exp\left(\frac{\bar{\partial}}{\partial \alpha_j} \frac{\bar{\partial}}{\partial \alpha_j^*} + \frac{\bar{\partial}}{\partial \beta_j} \frac{\bar{\partial}}{\partial \beta_j^*} \right), \quad (2.9)$$

$$\mathcal{D}_S^{(2)} = \prod_{j=1}^N \exp\left(\frac{\bar{\partial}}{\partial \alpha_j^*} \frac{\bar{\partial}}{\partial \alpha_j} + \frac{\bar{\partial}}{\partial \beta_j^*} \frac{\bar{\partial}}{\partial \beta_j} \right). \quad (2.10)$$

The subscript S on these operators denotes the fact that these act only on atomic variables. Then, by straightforward but quite lengthy calculations, we may show that Eqs. (G4.36) and (G4.37) lead to the following equation of motion for the normally ordered equivalent $F_S^{(N)}(\{\alpha_j\}, \{\alpha_j^*\}; \{\beta_j\}, \{\beta_j^*\}; t)$ of the reduced density operator:

$$\frac{\partial F_S^{(N)}(t)}{\partial t} + \mathcal{L}_S F_S^{(N)}(t) = 0, \quad (2.11)$$

where

$$\begin{aligned} \mathcal{L}_S F_S^{(N)}(t) = & \sum_i \sum_j \gamma_{ij} \left(\frac{\partial}{\partial \beta_j^*} [(\alpha_i^* \alpha_j \beta_j^* - \delta_{ij} \beta_j^*) F_S^{(N)}] \right. \\ & - \frac{\partial}{\partial \alpha_i} (\beta_i \beta_j^* \alpha_j F_S^{(N)}) - \frac{\partial^2}{\partial \alpha_i \partial \alpha_j^*} (\beta_i \beta_j^* F_S^{(N)}) \\ & \left. + \frac{\partial^2}{\partial \beta_i^* \partial \alpha_j} (\alpha_i^* \beta_j^* F_S^{(N)}) + \text{c. c.} \right). \quad (2.12) \end{aligned}$$

Here the coefficients γ_{ij} are given by

$$\gamma_{ij} = (2\pi^2 \omega_0^2 / c) (1/2\pi)^3 \int d^3\vec{k} \exp[i\vec{k} \cdot (\vec{R}_i - \vec{R}_j)]$$

$$\times [|\vec{d}|^2 - (|\vec{d} \cdot \vec{k}|^2/k^2)] [\delta(kc - \omega_0)/k]. \quad (2.13)$$

In deriving (2.11), we took the limit $L^3 \rightarrow \infty$ and have made use of the Markovian approximation.¹⁴ We have also ignored the frequency shift terms, which cause the renormalization of the energy levels. These terms cause an extra degree of complication and will be discussed elsewhere.

Eq. (2.11) is our Fokker-Planck equation. As is well known, equations of this type are very useful in calculating the correlation functions and in particular the mean values. This equation, as it stands, is too complex to be solved exactly. However, we use it to obtain the equations of motion for the mean values. It is obvious that the mean value of S_i^z is given by

$$\begin{aligned} \langle S_i^z \rangle &= \frac{1}{2} \int (\alpha_i^* \alpha_i - \beta_i^* \beta_i) F_S^{(N)} \\ &\times (\{\alpha_i\}, \{\alpha_i^*\}; \{\beta_i\}, \{\beta_i^*\}; t) \\ &\times d^2 \{\alpha_i/\pi\} d^2 \{\beta_i/\pi\}. \end{aligned} \quad (2.14)$$

We multiply both sides of (2.11) by $\frac{1}{2}(\alpha_i^* \alpha_i - \beta_i^* \beta_i)$ and integrate. The right-hand side of the equation which is obtained is simplified by integration by parts. This procedure leads to the following equation of motion for $\langle S_i^z \rangle$:

$$\frac{\partial}{\partial t} \langle S_i^z \rangle + \sum_j \gamma_{ji} \langle S_i^+ S_j^- \rangle + \sum_j \gamma_{ij} \langle S_i^- S_j^+ \rangle = 0. \quad (2.15)$$

One can similarly obtain the equation of motion for $\langle S_i^+ S_j^- \rangle$ for $i \neq j$. The result of a rather lengthy calculation is

$$\begin{aligned} \frac{\partial}{\partial t} \langle S_i^+ S_j^- \rangle &= -(\gamma_{ii} + \gamma_{jj}) \langle S_i^+ S_j^- \rangle + \gamma_{ij} \langle (1 + 2S_i^z) S_j^z \rangle \\ &+ \gamma_{ij} \langle (1 + 2S_j^z) S_i^z \rangle + \sum_{i \neq j \neq i} 2\gamma_{ji} \langle S_i^+ S_j^z S_i^- \rangle \\ &+ \sum_{i \neq j \neq i} 2\gamma_{ii} \langle S_i^+ S_i^z S_j^- \rangle, \quad (i \neq j) \end{aligned} \quad (2.16)$$

and so on.

One thus sees that the equation of motion for $\langle S_i^z \rangle$ is coupled to that for $\langle S_i^+ S_j^- \rangle$, which in turn is coupled to that for one higher-order correlation function. In this way, one obtains a hierarchy (which is of finite order) of equations. These coupled differential equations are of first order in time and can in principle be solved by using Laplace transform techniques. However, in practice, these can be handled easily if N is very small (e.g. $N=1, 2, 3$). For example, for $N=2$, one has the following:

$$\frac{\partial}{\partial t} \langle S_i^z \rangle + 2\gamma_0 \langle S_i^+ S_i^- \rangle + \gamma_{12} (\langle S_1^+ S_2^- \rangle + \langle S_1^- S_2^+ \rangle) = 0, \quad (2.17)$$

$$\frac{\partial}{\partial t} \langle S_1^+ S_2^- \rangle + 2\gamma_0 \langle S_1^+ S_2^- \rangle = \frac{1}{2} \gamma_{12} (\langle (S_1^z + \frac{1}{2})(S_2^z + \frac{1}{2}) \rangle - \frac{1}{4}), \quad (2.18)$$

$$\frac{\partial}{\partial t} \langle (S_1^z + \frac{1}{2})(S_2^z + \frac{1}{2}) \rangle = -4\gamma_0 \langle (S_1^z + \frac{1}{2})(S_2^z + \frac{1}{2}) \rangle, \quad (2.19)$$

where

$$\gamma_0 = \left(\frac{2\pi^2 \omega_0^2}{c} \right) \left(\frac{1}{2\pi} \right)^3 \int d^3 R \frac{\delta(kc - \omega_0)}{R} \left(|\vec{d}|^2 - \frac{|\vec{d} \cdot \vec{k}|^2}{k^2} \right). \quad (2.20)$$

The last of the Eqs. (2.17)–(2.19) is easily integrable. This solution is to be used in (2.18) to solve for $\langle S_1^+ S_2^- \rangle$, which in turn should be used in (2.17) to solve for $\langle S_i^z \rangle$.

We now consider the expectation value $p_N(t)$ of the operator

$$\begin{aligned} &[\prod_{i=1}^N (S_i^z + \frac{1}{2})]: \\ p_N(t) &= \langle \prod_{i=1}^N (\frac{1}{2} + S_i^z) \rangle. \end{aligned} \quad (2.21)$$

It is obvious that $p_N(t)$ denotes the probability that all the N atoms would be found in an excited state at time t . Using (2.11), we obtain the equation of motion for $p_N(t)$:

$$\frac{\partial p_N(t)}{\partial t} = -2N\gamma_0 p_N(t), \quad (2.22)$$

which leads to

$$p_N(t) = p_N(0) e^{-2N\gamma_0 t}. \quad (2.23)$$

We thus obtain the following very interesting result: The probability that all the atoms would be found in an excited state decays exponentially, regardless of the size of the system, the decay constant being $2N\gamma_0$, which is N times the decay constant for a single atom.

III. APPROXIMATE EXPRESSION FOR THE INTENSITY OF SPONTANEOUSLY EMITTED RADIATION

In order to obtain an approximate expression for the intensity of spontaneously emitted radiation, we consider Eq. (2.15) and make the following *approximation*:

$$\langle S_i^z S_j^z \rangle = \langle S_i^z \rangle \langle S_j^z \rangle \quad (i \neq j). \quad (3.1)$$

It should be noted that this approximation does *not necessarily* imply that the density operator ρ is the direct product of density matrices corresponding to individual atoms.¹⁵ The total energy W of the atomic system, in units of ω_0 , is given by

$$W = \sum_{i=1}^N \langle S_i^z \rangle = \frac{1}{2} \sum_{i=1}^N \langle \sigma_i^z \rangle, \quad (3.2)$$

where σ_l^z are Pauli spin matrices. Equation (2.15), on summing over all values of l , leads to

$$\frac{\partial}{\partial t} \sum_l \langle \sigma_l^z \rangle + \sum_{jl} \gamma_{jl} \langle \sigma_l^+ \sigma_j^- \rangle = 0. \quad (3.3)$$

To simplify (3.3) we use the following relation:

$$\sum_{jl} \langle \sigma_l^+ \sigma_j^- \rangle \gamma_{jl} = \sum_{j \neq l} \gamma_{jl} \langle \vec{\sigma}_l \cdot \vec{\sigma}_j - \sigma_j^z \sigma_l^z \rangle + \sum_j \gamma_{jj} \langle \sigma_j^+ \sigma_j^- \rangle. \quad (3.4)$$

On combining (3.1), (3.3), and (3.4) we obtain

$$\begin{aligned} \frac{\partial}{\partial t} \left(\sum_l \langle \sigma_l^z \rangle \right) + \sum_{j \neq l} \gamma_{lj} \langle 2P_{lj} - 1 \rangle + \sum_j \gamma_{jj} \langle \sigma_j^+ \sigma_j^- \rangle \\ - \left(\frac{\sum_l \langle \sigma_l^z \rangle^2}{N^2} \right) \sum_{j \neq l} \gamma_{lj} = 0, \end{aligned} \quad (3.5)$$

where P_{lj} is the permutation operator defined by

$$P_{lj} = \frac{1}{2} (1 + \vec{\sigma}_l \cdot \vec{\sigma}_j). \quad (3.6)$$

This operator corresponds to the interchange of l th and j th atoms. Since the original Hamiltonian is invariant under the permutation of indices l and j , we conclude that $\langle P_{lj} \rangle = 1$, and then Eq. (3.5) leads to

$$\frac{\partial}{\partial t} \left(\sum_l \langle \sigma_l^z \rangle \right) + \sum_{j \neq l} \gamma_{jl} \left(\frac{1 - \langle \sum_l \langle \sigma_l^z \rangle^2 \rangle}{N^2} \right) + \sum_j \gamma_{jj} \langle \sigma_j^+ \sigma_j^- \rangle = 0. \quad (3.7)$$

The quantity $\sum_{j \neq l} \gamma_{jl}$ is given by

$$\begin{aligned} \sum_{j \neq l} \gamma_{jl} = \left(\frac{2\pi^2 \omega_0^2}{c} \right) \left(\frac{1}{2\pi} \right)^3 \int d^3 \vec{k} \frac{\delta(kc - \omega_0)}{k} \\ \times \left(|\vec{d}|^2 - \frac{|\vec{d} \cdot \vec{k}|^2}{k^2} \right) \sum_{j \neq l} e^{i\vec{k} \cdot (\vec{R}_j - \vec{R}_l)}. \end{aligned} \quad (3.8)$$

The term $\sum_{j \neq l} e^{i\vec{k} \cdot (\vec{R}_j - \vec{R}_l)}$ can be written as

$$\sum_{j \neq l} e^{i\vec{k} \cdot (\vec{R}_j - \vec{R}_l)} = \sum_{jl} e^{i\vec{k} \cdot (\vec{R}_j - \vec{R}_l)} = N^2 [\Gamma(\vec{k}) - (1/N)], \quad (3.9)$$

where $\Gamma(\vec{k})$ is given by

$$\Gamma(\vec{k}) = \left| \langle e^{i\vec{k} \cdot \vec{R}_j} \rangle_{av} \right|^2. \quad (3.10)$$

Here $\langle \rangle_{av}$ denotes the average over the position of all atoms. On combining (3.8) and (3.9) we obtain

$$\begin{aligned} \sum_{j \neq l} \gamma_{jl} = N^2 \gamma_0 \left[\frac{2\pi^2 \omega_0^2}{c \gamma_0} \left(\frac{1}{2\pi} \right)^3 \int \frac{d^3 k}{k} \delta(kc - \omega_0) \Gamma(\vec{k}) \right. \\ \left. \times [|\vec{d}|^2 - (|\vec{d} \cdot \vec{k}|^2 / k^2)] - (1/N) \right] = N^2 \gamma_0 \mu, \end{aligned} \quad (3.11)$$

where

$$\mu = \frac{2\pi^2 \omega_0^2}{c \gamma_0} \left(\frac{1}{2\pi} \right)^3 \int \frac{d^3 \vec{k}}{k} \delta(kc - \omega_0) \Gamma(\vec{k}) \left(|\vec{d}|^2 - \frac{|\vec{d} \cdot \vec{k}|^2}{k^2} \right). \quad (3.12)$$

On using Eqs. (3.2), (3.7), and (3.11), we find that Ψ satisfies the following equation of motion:

$$\frac{\partial W}{\partial t} = (2\mu \gamma_0) \left(W + \frac{1}{2} N \right) \left(W - \frac{1}{2} N - \frac{1}{\mu} \right). \quad (3.13)$$

It should be noted that Eq. (3.13) is exactly the same as the one derived and solved by Eberly and Rehler⁵ using Dicke's expression for the state of the system at time t . The result is

$$W(t) = -\frac{1}{2} N \{ [1 + (\mu N)^{-1}] \tanh \tau^{-1}(t - t_0) - (\mu N)^{-1} \}. \quad (3.14)$$

The constant t_0 is determined from initial condition and τ is given by

$$\tau^{-1} = \gamma_0 (\mu N + 1). \quad (3.15)$$

The expression for the intensity of spontaneous radiation is obtained by using the principle of detailed balance and is given by

$$I(t) = -\omega_0 \dot{W} = (\omega_0 \gamma_0 / 2\mu) (\mu N + 1)^2 \operatorname{sech}^2 \tau^{-1}(t - t_0). \quad (3.16)$$

We thus see that we have been able to obtain the radiation rate by just making the approximation (3.1). At present it is not clear how good this approximation^{15a} is. This approximation is also inherent in Dicke's work, where he has used an explicit expression for states of the system at time t . This explicit expression is essentially of the form^{1,5}

$$\rho(t) = \Pi_j \left[\frac{1}{2} - S_j^z \cos \theta + \frac{1}{2} i (S_j^+ \alpha_j^- - S_j^- \alpha_j^+) \sin \theta \right], \quad (3.17)$$

where

$$\cos \theta = -2 \langle S_j^z \rangle, \quad (3.18a)$$

$$i \alpha_j = \exp(-\omega_0 t + i\vec{k} \cdot \vec{R}_j + i\varphi_0), \quad (3.18b)$$

Again there does not appear to be any *a priori* justification for using this expression except some analogies drawn from the semiclassical treatment.¹¹ We believe that an improvement over the approximation (3.1) can be obtained by using Eq. (2.16).

IV. SPONTANEOUS EMISSION FROM SMALL SYSTEMS: SOME EXACT RESULTS

We now consider the spontaneous emission from small systems, i. e., systems whose size is small compared to the wavelength. In this case it is not necessary to make the approximation (3.1), and it is possible to obtain exact results. For small sys-

tems, we may introduce the following operators:

$$S^z = \sum_i S_i^z, \quad S^\pm = \sum_i S_i^\pm. \tag{4.1}$$

The operators S^z, S^\pm satisfy the angular momentum commutation relations. Furthermore, it can be shown that S^2 is a constant of motion.¹ We may show that the phase-space equation of motion (2.11) is equivalent to the following equation for the reduced density operator ρ_S :

$$\frac{\partial \rho_S}{\partial t} = - \sum_{ij} \gamma_{ij} [S_i^+ S_j^- \rho - S_j^- \rho S_i^+] + \text{H. c.}] \tag{4.2}$$

The proof of this result is given in the Appendix. For small systems, we can make dipole approximation and find that $\gamma_{ij} \rightarrow \gamma_0$ independent of i and j . Using (4.1) and (4.2), we obtain the equation¹⁶

$$\frac{\partial \rho}{\partial \tau} = - \frac{1}{2} (S^+ S^- \rho - 2S^- \rho S^+ + \rho S^+ S^-), \tag{4.3}$$

where

$$\tau = 2\gamma_0 t. \tag{4.4}$$

Let $|m\rangle$ denote the angular momentum eigenstate corresponding to a given eigenvalue $S(S+1)$ of S^2 and m of S^z . Then on taking the matrix element of (4.3), we find that

$$\frac{\partial \rho_{m,n}}{\partial \tau} = (\nu_{m+1} \nu_{n+1})^{1/2} \rho_{m+1,n+1} - \frac{1}{2} (\nu_m + \nu_n) \rho_{m,n}, \tag{4.5}$$

where we have set

$$\nu_m = (s - m + 1)(s + m). \tag{4.6}$$

We can solve the set of different differential equations (4.5) by iteration. It turns out that we have¹⁷

$$\rho_{m,m}(\beta) = \sum_{i \geq 0} \left(\prod_{k=m+1}^{m+i} \nu_k / \prod_{k=m}^{m+i} (\beta + \nu_k) \right) \rho_{m+i,m+i}(0), \tag{4.7}$$

where $\rho_{m,m}(\beta)$ is the Laplace transform of $\rho_{m,m}(t)$, i. e.,

$$\rho_{m,m}(\beta) = \int_0^\infty e^{-\tau\beta} \rho_{m,m}(\tau) d\tau \quad (\text{Re}\beta \geq 0). \tag{4.8}$$

We assume that all the atoms were in the excited state at time $t=0$, so that we have $s = \frac{1}{2}N$ and $\rho_{m,m}(0) = \delta_{m, N/2}$. Then the exact expression for the energy of the system is

$$\begin{aligned} \langle S^z(\beta) \rangle &= \int_0^\infty e^{-\tau\beta} \langle S^z(t) \rangle dt \\ &= \sum_{m=-N/2}^{+N/2} \binom{m}{N/2} \frac{N! (\frac{1}{2}N - m)!}{(\frac{1}{2}N + m)!} \prod_{k=m}^{N/2} \frac{1}{(\beta + \nu_k)}. \end{aligned} \tag{4.9}$$

$\langle S^z(t) \rangle$ is obtained by inverting (4.9). It should be noted that the expression (4.9) involves poles of order 1 and 2, and that the poles of order 2 complicate the final expression for $\langle S^z(t) \rangle$. As an example, when $N = \text{odd}$, $\langle S^z(\beta) \rangle$ can be written as

$$\begin{aligned} \langle S^z(\beta) \rangle &= \sum_{m=1/2}^{N/2} \frac{m(N!) (\frac{1}{2}N - m)!}{(\frac{1}{2}N + m)!} \prod_{k=m}^{N/2} \frac{1}{(\beta + \nu_k)} - \frac{1}{2} N(N!)^2 \left(\beta(\beta + \nu_{1/2}) \prod_{k=-(N/2)-1}^{-1/2} (\beta + \nu_k)^2 \right)^{-1} \\ &+ \sum_{m=-(N/2)-1}^{-1/2} \frac{m(N!) (\frac{1}{2}N - m)!}{(\frac{1}{2}N + m)!} (\beta + \nu_{1/2})^{-1} \prod_{k=(2-m)}^{N/2} (\beta + \nu_k)^{-1} \prod_{l=m}^{-1/2} (\beta + \nu_l)^{-2}. \end{aligned} \tag{4.10}$$

In this expression we have shown explicitly the simple and double poles of $\langle S^z(\beta) \rangle$. Now the standard complex variable techniques lead to the following expression for $\langle S^z(\tau) \rangle$:

$$\begin{aligned} \langle S^z(\tau) \rangle &= \sum_{m=1/2}^{N/2} \frac{m(N!) (\frac{1}{2}N - m)!}{(\frac{1}{2}N + m)!} \sum_{k=m}^{N/2} e^{-\nu_k \tau} \prod_{l=m \neq k}^{N/2} (\nu_l - \nu_k)^{-1} \\ &- (\frac{1}{2}N)(N!)^2 (\nu_{1/2})^{-1} \prod_{k=-(N/2)-1}^{-1/2} (\nu_k)^{-2} + (\frac{1}{2}N)(N!)^2 (\nu_{1/2})^{-1} \prod_{k=-(N/2)-1}^{-1/2} (\nu_k - \nu_{1/2})^{-2} e^{-\nu_k \tau} \\ &- (\frac{1}{2}N)(N!)^2 \sum_{k=-(N/2)-1}^{-1/2} e^{-\nu_k \tau} \left([\tau \nu_k (\nu_k - \frac{1}{2}) + (2\nu_k - \nu_{1/2})] \nu_k^{-2} (\nu_k - \nu_{1/2})^{-2} \prod_{l=-(N/2)-1 \neq k}^{-1/2} (\nu_l - \nu_k)^{-2} \right) \\ &+ 2 \sum_{l=-(N/2)-1 \neq k}^{-1/2} \nu_k^{-1} (\nu_k - \nu_{1/2})^{-1} (\nu_k - \nu_l)^{-2} \prod_{r=-(N/2)-1 \neq l \neq k}^{-1/2} (\nu_r - \nu_k)^{-2} \\ &+ \sum_{m=-(N/2)-1}^{-1/2} \frac{m(N!) (\frac{1}{2}N - m)!}{(\frac{1}{2}N + m)!} \prod_{k=2-m}^{N/2} (\nu_k - \nu_{1/2})^{-1} \prod_{l=m}^{-1/2} (\nu_l - \nu_{1/2})^{-2} e^{-\nu_{1/2} \tau} \end{aligned}$$

$$\begin{aligned}
& + \sum_{m=-[(N/2)-1]}^{-1/2} \frac{m(N!) (\frac{1}{2}N - m)!}{(\frac{1}{2}N + m)!} \sum_{k=2-m}^{N/2} (\nu_{1/2} - \nu_k)^{-1} \prod_{l=2-m \neq k}^{N/2} (\nu_l - \nu_k)^{-1} \prod_{r=m}^{-1/2} (\nu_r - \nu_k)^{-2} e^{-\nu_k \tau} \\
& + \sum_{m=-[(N/2)-1]}^{-1/2} \frac{m(N!) (\frac{1}{2}N - m)!}{(\frac{1}{2}N + m)!} \sum_{k=m}^{-1/2} [\tau(\nu_{1/2} - \nu_k) - 1] (\nu_{1/2} - \nu_k)^{-2} \prod_{l=2-m}^{N/2} (\nu_l - \nu_k)^{-1} \prod_{r=m \neq k}^{-1/2} (\nu_r - \nu_k)^{-2} e^{-\nu_k \tau} \\
& + \sum_{m=-[(N/2)-1]}^{-1/2} \frac{m(N!) (\frac{1}{2}N - m)!}{(\frac{1}{2}N + m)!} \sum_{k=m}^{-1/2} \left[\sum_{l=2-m}^{N/2} (\nu_{1/2} - \nu_k)^{-1} \prod_{p=2-m \neq l}^{N/2} (\nu_p - \nu_l)^{-1} (\nu_l - \nu_k)^{-2} \right. \\
& \left. \times \prod_{r=m \neq k}^{-1/2} (\nu_r - \nu_k)^{-2} - \sum_{r=m \neq k}^{-1/2} 2(\nu_r - \nu_k)^{-3} (\nu_{1/2} - \nu_k) \prod_{l=2-m}^{N/2} (\nu_l - \nu_k)^{-1} \prod_{p=m \neq k \neq r}^{-1/2} (\nu_p - \nu_k)^{-2} \right] e^{-\nu_k \tau}. \quad (4.11)
\end{aligned}$$

As an example, for $N=3$, one easily finds that

$$\langle S^z(t) \rangle = [6e^{-4\tau} - 3e^{-3\tau} + 12\tau e^{-3\tau} - \frac{3}{2}], \quad (4.12)$$

where τ is given by (4.4). A similar expression holds when N is an even number. These expressions can then be used to compute the radiation rate. In particular, when $N=1-4$ and 8 , these expressions lead to the results of Dillard and Robl.¹⁸

It can further be shown that the Fourier component of the spontaneously radiated intensity is given by

$$I(\omega) = \int_0^\infty I(\tau) e^{-i\omega\tau} d\tau = (i\omega \langle S^z(i\omega) \rangle - \frac{1}{2}N), \quad (4.13)$$

where $\langle S^z(i\omega) \rangle$ is given by (4.9) with β replaced by $i\omega$.

For numerical analysis, it appears to be advantageous to solve a partial differential equation which we will now derive. Consider the following generating function:

$$Q(x, \tau) = \langle \exp[-x(\frac{1}{2}N + S^z(\tau))] \rangle. \quad (4.14)$$

Since $[\frac{1}{2}N + S^z]$ is a positive semidefinite bounded operator, it is clear that the expectation value (4.14) is bounded. From (4.14) and (4.3) we find that

$$\begin{aligned}
\frac{\partial Q}{\partial \tau} &= \frac{1}{2} \langle [S^+, e^{-xS^z}] S^- \rangle \exp(-x\frac{1}{2}N) \\
&+ \frac{1}{2} \langle S^+ [e^{-xS^z}, S^-] \rangle \exp(-x\frac{1}{2}N). \quad (4.15)
\end{aligned}$$

The commutator $[S^+, e^{-xS^z}]$ can be evaluated by using an identity¹⁹ due to Kubo:

$$\begin{aligned}
[S^+, \exp(-xS^z)] &= - \int_0^x dy e^{-(x-y)S^z} [S^+, S^z] e^{-yS^z} \\
&= \int_0^x dy e^{-(x-y)S^z} S^+ e^{-yS^z}; \quad (4.16)
\end{aligned}$$

but we have

$$\begin{aligned}
e^{yS^z} S^+ e^{-yS^z} &= S^+ + (y/1!) [S^z, S^+] \\
&+ (y^2/2!) [S^z, [S^z, S^+]] + \dots \\
&= S^+ + (y/1!) S^+ + (y^2/2!) S^+ + \dots \\
&= e^y S^+, \quad (4.17)
\end{aligned}$$

which on substitution in (4.16) leads to

$$[S^+, e^{-xS^z}] = e^{-xS^z} \int_0^x dy e^y S^+ = (e^x - 1) e^{-xS^z} S^+. \quad (4.18)$$

Substituting (4.18) in (4.15) leads to

$$\frac{\partial Q}{\partial \tau} = 2e^{-x(N/2)} (e^x - 1) \cdot \frac{1}{2} \langle S^+ S^- e^{-xS^z} \rangle, \quad (4.19)$$

which on using the relation

$$S^+ S^- = \frac{1}{2}N(\frac{1}{2}N + 1) - (S^z)^2 + S^z$$

reduces to

$$\frac{\partial Q}{\partial \tau} = - (e^x - 1) \left(\frac{\partial^2}{\partial x^2} + (N+1) \frac{\partial}{\partial x} \right) Q. \quad (4.20)$$

This differential equation should be solved subject to the initial condition $Q(x, 0) = \exp(-Nx)$ and the boundary condition $Q(0, t) = 1$. Once the solution is known, expectation value of energy can be obtained from the formula

$$\langle S^z(t) \rangle = \left[- \frac{\partial Q(x, \tau)}{\partial x} \Big|_{x=0} - \frac{1}{2}N \right]. \quad (4.21)$$

Having obtained the exact expression for the intensity of the spontaneously emitted radiation, we now proceed to calculate the correlation functions for the emitted radiation. It is seen from (4.8) that

$$\begin{aligned}
\langle S^+(\beta) \rangle &= \sum_m \rho_{m, m+1}(\beta) (\nu_{m+1})^{1/2} \\
&= \sum_m (\nu_{m+1})^{1/2} \sum_{n \geq 0} \left[\left(\prod_{k=1}^n \nu_{m+k} \nu_{m+1+k} \right)^{1/2} \right] / \\
&\prod_{k=0}^n [\beta + \frac{1}{2}(\nu_{m+k} + \nu_{m+1+k})] \rho_{m+n, m+n+1}(0). \quad (4.22)
\end{aligned}$$

This equation can also be written as

$$\begin{aligned}
\langle S^+(\beta) \rangle &= \sum_m (\nu_{m+1})^{1/2} \sum_{n \geq 0} \left[\left(\prod_{k=1}^n \nu_{m+k} \nu_{m+1+k} \right)^{1/2} \right] / \\
&\prod_{k=0}^n [\beta + (\nu_{m+k} + \nu_{m+1+k})] (\nu_{m+n+1})^{-1/2}
\end{aligned}$$

$$\times \langle S^+ | n+m \rangle \langle n+m | \rangle. \quad (4.23)$$

In order to calculate $\langle S^+(\beta)S^-(0) \rangle$ we now make use of the quantum regression theorem,²⁰ which allows us to calculate the two time correlation functions in terms of one time correlation functions. The result of this calculation can be shown to be

$$\begin{aligned} \langle S^+(\beta)S^-(0) \rangle &= \sum_m (\nu_{m+1})^{1/2} \sum_{n \geq 0} \left[\left(\prod_{k=1}^n \nu_{m+k} \nu_{m+1+k} \right)^{1/2} / \right. \\ &\quad \left. \prod_{k=0}^n \left[\beta + \frac{1}{2}(\nu_{m+k} + \nu_{m+k+1}) \right] \right] (\nu_{m+n+1})^{-1/2} \\ &\times \langle S^+ | n+m \rangle \langle n+m | S^- \rangle, \quad (4.24) \end{aligned}$$

which on simplification leads to

$$\begin{aligned} \langle S^+(\beta)S^-(0) \rangle &= \sum_m (\nu_{m+1})^{1/2} \sum_{n \geq 0} \left[\left(\prod_{k=1}^n (\nu_{m+k} \nu_{m+1+k}) \right)^{1/2} / \right. \\ &\quad \left. \prod_{k=0}^n \left[\beta + \frac{1}{2}(\nu_{m+k} + \nu_{m+k+1}) \right] \right] (\nu_{m+n+1})^{1/2} \\ &\times \rho_{m+n+1, m+n+1}(0). \quad (4.25) \end{aligned}$$

The initial condition is

$$\rho_{m+n+1, m+n+1}(0) = \delta_{m+n+1, (1/2)N}. \quad (4.26)$$

On substituting (4.26) in (4.25), we find that

$$\begin{aligned} \langle S^+(\beta)S^-(0) \rangle &= \sum_{m=-N/2}^{+(N/2-1)} \left(\nu_{m+1} \cdots \nu_{N/2} / \right. \\ &\quad \left. \prod_{k=m}^{(N/2-1)} \left[\beta + \frac{1}{2}(\nu_k + \nu_{k+1}) \right] \right) \quad (4.27) \end{aligned}$$

Thus the correlation function is given by

$$\begin{aligned} \langle S^+(\beta)S^-(0) \rangle &= \int_0^\infty e^{-\tau\beta} \langle S^+(\tau)S^-(0) \rangle d\tau \\ &= \sum_{m=-N/2}^{+(N/2-1)} \frac{N!(N/2-m)!}{(N/2+m)!} \left(\prod_{k=m}^{(N/2-1)} \left[\beta + \frac{1}{2}(\nu_k + \nu_{k+1}) \right] \right)^{-1}. \quad (4.28) \end{aligned}$$

As an example for $N=2$, we find that

$$\langle S^+(t)S^-(0) \rangle = [4 \exp(-\gamma_0 t) - 2 \exp(-2\gamma_0 t)]. \quad (4.29)$$

It can further be shown, by using the Heisenberg equations of motion, that the normally ordered correlations function of order 2 for the electric field in the radiation zone is directly proportional to $\langle S^+(t)S^-(0) \rangle$. Hence, (4.28) also gives the power spectrum of spontaneously emitted radiation. We can use similar methods to obtain intensity correlations.²¹

V. SPONTANEOUS EMISSION FROM A SYSTEM OF HARMONIC OSCILLATIONS

In this section we shall consider the spontaneous

emission from a system of harmonic oscillators. Then the procedure outlined in Sec. II leads to the following equation of motion for the antinormally ordered equivalent of the density operator:

$$\frac{\partial F_S^{(A)}}{\partial t} = \gamma_0 \left(\sum_{ij} \frac{\partial}{\partial z_i} (z_j F_S^{(A)}) + \text{c. c.} \right). \quad (5.1)$$

This equation is the generalization of Eq. (G5.19) for the case of N harmonic oscillators emitting spontaneously. The solution to Eq. (5.1) is

$$\begin{aligned} F_S^{(A)}(\{z_i\}, \{z_i^*\}; t) &= \exp\left(\gamma_0 t \sum_{ij} \frac{\partial}{\partial z_i} (z_j)\right) \\ &\times \exp\left(\gamma_0 t \sum_{ij} \frac{\partial}{\partial z_i^*} (z_j^*)\right) F_S^{(A)}(\{z_i\}, \{z_i^*\}; 0) \\ &= \exp\left(\gamma_0 t \frac{\partial}{\partial A} (A)\right) \exp\left(\gamma_0 t \frac{\partial}{\partial A^*} (A^*)\right) \\ &\times F_S^{(A)}(\{z_i\}, \{z_i^*\}; 0), \quad (5.2) \end{aligned}$$

where

$$A = \sum_{i=1}^N z_i. \quad (5.3)$$

We recall the following identity, which is easily proved¹²:

$$\exp\left(\gamma_0 t \frac{\partial}{\partial A} (A)\right) = \exp(N\gamma_0 t) \left[\exp\left((e^{N\gamma_0 t} - 1) \frac{1}{N} A \frac{\partial}{\partial A}\right) \right]_d, \quad (5.4)$$

where $[]_d$ means that in the expansion of the term in parentheses all the derivatives $\partial/\partial A$ stand to the right of A . On using (5.4) we may show that (5.2) reduces to

$$F_S^{(A)}(\{z_i\}, \{z_i^*\}, t) = F_S^{(A)}(\{\bar{z}_i\}, \{\bar{z}_i^*\}, 0) e^{2\gamma_0 t N}, \quad (5.5)$$

where

$$\bar{z}_i = z_i + \frac{1}{N} (e^{\gamma_0 t N} - 1) \sum_{i=1}^N z_i. \quad (5.6)$$

Relations (5.6) are easily inverted to give

$$z_j = \bar{z}_j - \frac{1}{N} (1 - e^{-\gamma_0 t N}) \sum_{j=1}^N \bar{z}_j. \quad (5.7)$$

We now compute the average energy (in units of ω_0) of the system which is given by

$$\sum_i \langle a_i^\dagger(t) a_i(t) \rangle = \sum_i \int d^2\{z_i/\pi\} |z_i|^2 F_S^{(A)}(\{z_i\}, \{z_i^*\}, t). \quad (5.8)$$

It is obvious from (5.8) and (5.5) that the energy of the system depends on the initial state of the system. We shall discuss two cases below and show that in one case the system leads to superradiant emission and that in the other case it does

not.

We first assume that at $t = 0$ that phase-space distribution function $F_S^{(A)}$ satisfies the following conditions:

$$F_S^{(A)}(\{z_i\}, \{z_i^*\}, 0) = \prod_{j=1}^N F_j^{(A)}(z_j, z_j^*, 0), \quad \langle a_i(0) \rangle = 0. \quad (5.9)$$

On using Eqs. (5.5), (5.6), and (5.9) we can easily show that (5.8) reduces to

$$\sum_i \langle a_i^\dagger(t) a_i(t) \rangle = \sum_i \langle a_i^\dagger(0) a_i(0) \rangle \frac{1}{N} [(N-1) + e^{-2\gamma_0 t N}]. \quad (5.10)$$

We thus find that the radiation rate is given by the following:

$$\begin{aligned} I(t) &= -\omega_0 \frac{d}{dt} \sum_i \langle a_i^\dagger(t) a_i(t) \rangle \\ &= 2\gamma_0 \omega_0 e^{-2\gamma_0 t N} \sum_i \langle a_i^\dagger(0) a_i(0) \rangle \\ &= 2\gamma_0 \omega_0 e^{-2\gamma_0 t N} W(0), \end{aligned} \quad (5.11)$$

where $W(0)$ is the energy of the system at time $t = 0$. We therefore conclude that a system of harmonic oscillators, which are excited in a state described by Eq. (5.9) does not lead to superradiant emission.

We now consider a system of harmonic oscillators which are excited at $t = 0$ so that each of them is in a coherent state $|z_0\rangle$. Then $F_S^{(A)}(\{z_i\}, \{z_i^*\}, 0)$ is given by

$$F_S^{(A)}(\{z_i\}, \{z_i^*\}, 0) = \prod_{i=1}^N \pi \delta^{(2)}(z_i - z_0). \quad (5.12)$$

On substituting (5.12) and (5.5) in (5.8), we obtain the following expression for the energy of the system:

$$\sum_i \langle a_i^\dagger(t) a_i(t) \rangle = N |z_0|^2 e^{-2\gamma_0 t N}. \quad (5.13)$$

The radiation rate is then given by

$$\begin{aligned} I(t) &\equiv -\omega_0 \frac{d}{dt} \sum_i \langle a_i^\dagger(t) a_i(t) \rangle \\ &= 2\gamma_0 \omega_0 |z_0|^2 N^2 e^{-2\gamma_0 t N}, \end{aligned} \quad (5.14)$$

which leads us to conclude that the radiation rate is proportional to the square of the number of oscillators, when each of the oscillators is initially excited to a coherent state $|z_0\rangle$. Therefore, state $|z_0\rangle$ for the case of harmonic oscillators is the analog of the state $|N/2, 0\rangle$ (in which half of the atoms are in excited state and half of them in ground

state) for two-level atoms. It is also worth noting that the decay constant is N times larger than that for a single oscillator.

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APPENDIX: PROOF OF THE RELATION (4.2)

We apply the mapping operator $\Omega^{(N)}$ corresponding to the normal rule of mapping to both sides of Eq. (2.11), to find the following:

$$\frac{\partial}{\partial t} \Omega^{(N)} F_S^{(N)} + \Omega^{(N)} \mathcal{L}_S F_S^{(N)} = 0, \quad (A1)$$

or

$$\frac{\partial \rho_S}{\partial t} + \Omega^{(N)} \mathcal{L}_S F_S^{(N)} = 0.$$

We will now use the following identities²²:

$$[a, G(a, a^\dagger)] = \Omega^{(N)} \frac{\partial}{\partial z^*} [F^{(N)}(z, z^*)], \quad (A2)$$

$$[G(a, a^\dagger), a^\dagger] = \Omega^{(N)} \frac{\partial}{\partial z} [F^{(N)}(z, z^*)]. \quad (A3)$$

Here $G(a, a^\dagger)$ is an arbitrary function of boson operators a and a^\dagger , and $F^{(N)}(z, z^*)$ is the normally ordered equivalent of $G(a, a^\dagger)$. Using these identities we can show, by straightforward but long calculations, that

$$\begin{aligned} \Omega^{(N)} \mathcal{L}_S F_S^{(N)} &= \sum_{jj'} \{ [u_j, (v_j^\dagger u_j^\dagger, \rho_S v_j - \delta_{jj'} u_j^\dagger \rho_S)] \\ &\quad - [u_j^\dagger, \rho_S v_j, u_j, v_j^\dagger] - [[v_j, u_j, \rho_S u_j], v_j^\dagger] \\ &\quad + [u_j, [v_j, u_j^\dagger, v_j^\dagger \rho_S]] \} \gamma_{jj'} + \text{H. c.} \end{aligned} \quad (A4)$$

Equation (A4), on simplification, leads to

$$\begin{aligned} \Omega^{(N)} \mathcal{L}_S F_S^{(N)} &= \sum_{jj'} \gamma_{jj'} (v_j^\dagger u_j v_j, u_j^\dagger, \rho_S - u_j^\dagger, v_j, \rho_S v_j^\dagger u_j) \\ &\quad + \text{H. c.}, \end{aligned} \quad (A5)$$

which on using Schwinger's coupled boson representation (2.3) reduces to

$$\Omega^{(N)} \mathcal{L}_S F_S^{(N)} = \sum_{jj'} \gamma_{jj'} (S_j^\dagger S_j^-, \rho_S - S_j^-, \rho_S S_j^\dagger) + \text{H. c.} \quad (A6)$$

On substituting (A6) in (A1) we obtain the desired result (4.2), viz.,

$$\frac{\partial \rho_S}{\partial t} = - \left[\sum_{jj'} \gamma_{jj'} (S_j^\dagger S_j^-, \rho_S - S_j^-, \rho_S S_j^\dagger) + \text{H. c.} \right]. \quad (A7)$$

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$$\rho = \frac{1}{4} + \sum_{i=1}^3 \sigma_1^{(i)} a_1^{(i)} + \sum_{j=1}^3 \sigma_2^{(j)} \sigma_2^{(j)} + \sum_{ij=1}^3 \sigma_1^{(i)} \sigma_2^{(j)} a^{ij},$$

where $\vec{\sigma}_1$ and $\vec{\sigma}_2$ are the Pauli spin matrices corresponding to particles 1 and 2, respectively, and $a_1^{(i)}$, $a_2^{(j)}$, and a^{ij} are parameters. It is easily seen that condition (3.1) is satisfied when $a^{33} = a_1^{(3)} a_2^{(3)}$, but this does not necessarily imply that $\rho = \rho_1 \rho_2$.

^{15a}Footnote added in proof: Since this paper was submitted for publication, we have been able to obtain the corrections to the result (3.16). These arise because a finite correlation is induced between different atoms due to spontaneous emission and may be obtained by decoupling the higher-order equations of motion. As an example we find that the corrections to (3.1) are of the order $1/N$.

¹⁶For notational convenience we omit the subscript S from ρ in this section.

¹⁷The fact that the solution is of the form (4.7) is also seen to be apparent if transformation is made to a new representation [see, P. Résibois, *Physica* **27**, 541 (1961)] in which $\rho = m - n$, $q = \frac{1}{2}(m + n)$. In this representation (4.5) reads as

$$\frac{\partial \rho_{p,q}}{\partial \tau} = \rho_{p,q+1} \left\{ [(S-q)^2 - \frac{1}{4}p^2] [(S+q+1)^2 - \frac{1}{4}p^2] \right\} q^{1/2} - [(S-q+1)(S+q) - \frac{1}{4}p^2] \rho_{p,q},$$

and therefore it is evident that $\rho_{b,g}(\tau)$ is related to $\rho_{b,l}(0)$ (with p fixed), where l takes all possible values.

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Quantum-Mechanical Aperiodicity in the Resonant Interaction between a Field Mode and a Spin System

I. R. Senitzky

Institute for Exploratory Research, U.S. Army Electronics Command, Fort Monmouth, New Jersey 07703

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The absence of exact periodicity in the oscillation of the energy expectation value between a coupled linear oscillator and an angular momentum oscillator, even when the oscillators are macroscopic, is shown to be a consequence of the uncertainty principle.

The resonant interaction between a field mode and a spin system (or a number of two-level systems) has received considerable theoretical attention,¹⁻⁴ because it represents an idealized form of the interaction involved in quantum-electronic systems such as lasers and optical parametric ampli-

fiers. The analysis has been carried out both classically and quantum mechanically, in a perturbative and nonperturbative manner. The quantum-mechanical nonperturbative solutions of greatest accuracy have been obtained by numerical methods (for special cases)²⁻⁴ since exact solutions with easily dis-