CLASSICAL PARTICLES WITH INTERNAL STRUCTURE: A SYSTEMATIC ANALYSIS

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ABSTRACT

A comprehensive analysis of theories of classical relativistic particles with internal structure, based on group theoretic and differential geometric methods, is presented. The Lagrangian formulation of dynamics is used.

INTRODUCTION

Classical relativistic particles with internal structure have been the subject of a considerable amount of study recently. In addition to space-time position x^μ, they possess a set of translationally invariant internal variables q^r which describe an internal space Q admitting an action of the homogeneous Lorentz group. Such classical indecomposable objects can be viewed as a useful starting point for the description of, and an approximation to, the concept of Regge trajectories.

The Lagrangian approach to classical particles with spin was pioneered very early by Frenkel[1]. A great deal of work has been done since then by several authors. A recent development of considerable significance is the work of Hanson and Regge[2] on the relativistic spherical top, where systematic use was made of the powerful methods of Dirac's constrained Hamiltonian dynamics. Somewhat simpler models with interesting features were subsequently constructed by Mukunda *et al*[3].

For several reasons the Lagrangian approach to these problems is particularly attractive, in contrast to an equations-of-motion method or a direct Hamiltonian one. Both manifest covariance and the existence of the conservation laws are easily ensured, while possible couplings to external fields are also easily analysed. Moreover, as exemplified by the work of Hanson and Regge, the physical requirement of reparametrization invariance of the action leads in a natural way to a mass-spin trajectory condition arising as a constraint in Dirac's algorithm.

In this paper, we use group theoretic and differential geometric methods to develop a systematic classification and to analyse all possible internal structures for classical relativistic particles. It is shown that this problem is related to that of classifying all the essentially distinct coset spaces of the group $G = SL(2, C)$, with respect to its continuous subgroups H. Methods to determine the cases that allow non-trivial coupling of the internal and space-time variables, and so permit construction of interesting Lagrangians, are also outlined.

PHYSICAL REQUIREMENTS AND POSSIBLE INTERNAL SPACES

For a structureless relativistic point particle, configuration space coincides with Minkowski space-time M with coordinates x^μ. The canonical formalism based on the phase space $T^* M$ uses the conjugate momentum p_μ and the orbital angular momentum $L_{\mu\nu} = x_\mu p_\nu - x_\nu p_\mu$ as generators for the Poincaré group P. Absence of internal structure is reflected in a vanishing Pauli-Lubanski vector: $W_\mu = \frac{1}{2}\varepsilon_{\mu\nu\rho\sigma} p^\nu L^{\rho\sigma} = 0$.

To accommodate internal structure we need an enlarged configuration space $M \times Q$ with Q the space of internal variables q^r. Irreducibility or indecomposability of Q is expressed by the property that any two points of Q can be connected by some element of the homogeneous Lorentz group SO(3, 1). In order to accommodate spinorial variables as well as with a view towards eventual quantization we will use $G = SL(2, C)$ in place of SO(3, 1). Thus G must act transitively on Q. With respect to the development of a canonical formalism starting from some Lagrangian, two qualitatively different possibilities arise depending on the nature of Q. In case Q supports a G-invariant symplectic structure, such that furthermore the associated two-form is exact, we can restrict the Lagrangian $\mathscr{L}$ to be of first order with respect to the q^r, but of course not necessarily so with respect to x^μ. Then the canonical formalism uses $T^* M \times Q$, rather than $T^*(M \times Q) = T^* M \times T^* Q$, as phase space. In all other cases $\mathscr{L}$ cannot be so restricted and the final phase space has to be $T^* M \times T^* Q$. We refer to these two types as first order and second order internal spaces respectively; the former are necessarily of even dimension. The Poincare algebra is realised by functions and Poisson Brackets (PB) on $T^*M \times Q$ or $T^*M \times T^*Q$ as the case may be. The generators of the homogeneous Lorentz group will have the form $J_{\mu\nu} = L_{\mu\nu} + S_{\mu\nu}$, with $S_{\mu\nu}$ related to the action of G on Q. Nontrivial internal structure signals a non-zero Pauli-Lubanski vector $W_\mu = \frac{1}{2}\varepsilon_{\mu\nu\rho\sigma} p^\nu S^{\rho\sigma}$.

The physical principles underlying the analysis are: (a) the already stated transitive G action on Q; (b) manifest Poincare invariance of the action; (c) reparametrization invariance of the action, implying that $\mathscr{L}$ be homogeneous of degree one in the velocities $\dot{x}^\mu$ and $\dot{q}^r$. Property (c) allows full freedom in the choice of th evolution parameter, such as physical time, proper time etc. Not all spaces Q allowed by (a) are physically relevant. For a nontrivial Lagrangian to exist, we must be able to find sets of functions on Q or TQ, depending on whether we have a first order or a second order internal space, transforming according to the vector, symmetric second rank tensor, . . ., representation of SO (3, 1), so that on contraction with $\dot{x}^\mu$, $\dot{x}^\mu \dot{x}^\nu$, . . . we may be able to form Lorentz scalars on which L can depend. Another physically motivated criterion that may distinguish some choices of Q compared to others is whether at the end of the constraint analysis the Dirac Brackets (DB) of the x^μ among themselves vanish or not.

We now briefly describe the internal spaces Q permitted by the minimality condition (a) above. It is well known that any space Q carrying a transitive action of a group G is essentially the coset space G/H for some subgroup H in G. We are only interested in connected manifolds Q as possible internal spaces, so we restrict H to be a closed continuous subgroup of G. All such nontrivial H are known upto conjugation[6]: there are eleven distinct possibilities plus two one-parameter families. In each case, the dimension of $Q = G/H$ is the difference of the dimensions of G and H. In detail we find: there are three possible distinct choices plus a one-parameter family of H's of dimension one, (Q's of dimension five); there are three H's of dimension two, (Q's of dimension four); there are four distinct choices plus a one-parameter family of three-dimensional H's (three-dimensional Q's); and finally there is just one H of dimension four, (Q of dimension two). In this list we have not mentioned the case where H is the trival identity subgroup of G. This is certainly an allowed choice leading to the internal space Q being G itself. If G had been defined as SO (3, 1) rather than as SL (2, C), this would then correspond to the Hanson-Regge model where the internal variable is an element of SO (3, 1).

In the context of linear relativistic quantum mechanical wave equations Finkelstein[7] had made a similar classification of possible internal structures. His enumeration was however incomplete as only eleven possibilities were listed.

FIRST ORDER INTERNAL SPACES[4]

The physical idea behind the definition of a first order Q is that in such a case there is no need to introduce new variables canonically conjugate to the internal variables q^r, but that G-invariant PB's can be directly defined among the q^r themselves. The dimension of such a Q must be six, four or two. The case of dimension six corresponds to Q being G itself. If this were indeed a first order internal space, it would mean that the Hanson-Regge model permits an essential simplification. However, since G = SL (2, C) is a semi-simple Lie group, we can make use of the following mathematical result[8]: the only coset spaces G/H admitting a G-invariant symplectic structure are orbits in the Lie algebra **G** of G under the adjoint action of G, or covering spaces of such orbits. Now every non-trivial orbit in **G** is known to be dimension 4. It follows that the Q's of dimension six or two, corresponding to H of dimension zero or four, are not first order internal spaces. This applies in particular to the Hanson-Regge model. Among the three distinct Q's of dimension four, only two turn out to be first order spaces, and the third is neither an orbit in **G** nor a covering space of an orbit. We proceed to describe the two distinct first order Q's.

The linear space **G** can be identified with the space of all real second rank antisymmetric tensors $\xi_{\mu\nu} = -\xi_{\nu\mu}$, and is of dimension six. Under the adjoint action by G, $\xi_{\mu\nu}$ transforms as stated, and there are two independent invariants; in terms of the three-dimensional break-up $\xi_{oj} = \eta_j$ and $\xi_{jk} = \varepsilon_{jkl}\,\xi^l$, these are

$$\mathscr{C}_1 = \tfrac{1}{2}\xi^{\mu\nu}\xi_{\mu\nu} = |\xi|^2 - |\eta|^2,$$
$$\mathscr{C}_2 = -\tfrac{1}{8}\varepsilon_{\mu\nu\rho\sigma}\xi^{\mu\nu}\xi^{\rho\sigma} = \xi\cdot\eta \quad (1)$$

We omit from discussion the case $\xi_{\mu\nu} = 0$ which has $\mathscr{C}_1 = \mathscr{C}_2 = 0$ and forms an orbit all by itself. For specified values $\mathscr{C}_1 = a^2 - b^2$, $\mathscr{C}_2 = ab$ of the two invariants, the set of all allowed $\xi_{\mu\nu}$ constitute the orbit $\mathscr{O}_{a,b}$. If $\mathscr{C}_2 \neq 0$, it suffices to take $a > 0$, $b \neq 0$; while if $\mathscr{C}_2 = 0$, we must allow the three possibilities ($a > 0$, $b = 0$), ($a = 0$, $b = 0$) and ($a = 0$, $b > 0$). It now turns out that, viewed as coset spaces, the orbit $\mathscr{O}_{0,0}$ is quite unique and exceptional, while *all* other orbits $\mathscr{O}_{a,b}$ are really essentially the same. Since SL (2, C) and SO (3, 1) share the same Lie algebra, we may if we wish describe these orbits in the language of the latter group. It then turns out that the exceptional orbit $\theta_{0,0}$ is essentially the coset space, SO (3, 1)/N, where N is the abelian two-dimensional subgroup generated by, say, $J_1 - K_2$ and $J_2 + K_1$; while every other orbit is diffeomorphic to the coset space SO (3, 1)/SO (2) × SO (1, 1), the subgroups in question being generated by J_3 and K_3. (Here we use the standard physics notation **J**, **K** for the generators of SL(2, C) or SO(3, 1)). Thus as possible candidates for first order internal spaces we need only consider $Q = \theta_{0,0}$ and $Q = \theta_{0,b}$ for $b > 0$; the latter choice is made for simplicity and later convenience.

The abelian subgroup N appears in the Iwasawa decomposition $G = KAN$: K is the maximal compact subgroup SU (2), A is abelian and is generated by K_3. The coset space G/N is the internal space for the spinorial model of reference (3); it is a two-fold covering of $\theta_{0,0}$.

On either orbit $\theta_{0,0}$, $\theta_{0,b}$ the definition of the G-invariant symplectic structure is contained in the more physical PB relations

$$\{\xi_{\mu\nu}, \xi_{\rho\sigma}\} = g_{\mu\rho}\xi_{\nu\sigma} - g_{\nu\rho}\xi_{\mu\sigma} + g_{\mu\sigma}\xi_{\rho\nu} - g_{\nu\sigma}\xi_{\rho\mu}, \qquad (2)$$

which are of course consistent with the constancy of $\mathscr{C}_1$, $\mathscr{C}_2$ over an orbit. The internal contribution $S_{\mu\nu}$ to the total homogeneous Lorentz generator $J_{\mu\nu}$ is in both cases $\xi_{\mu\nu}$ itself. Using the (four independent components of) $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$ as local coordinates on an orbit, one can explicitly reconstruct the closed nondegenerate two-form ω "belonging to" the above PB's. For both choices of Q, it then turns out that ω is exact: we have the globally valid expressions

$$\omega = \mathrm{d}\varphi,\ \varphi = \frac{1}{|\eta|^2}\,\xi\cdot\eta\wedge \mathrm{d}\eta. \qquad (3)$$

Both on $\theta_{0,0}$ and $\theta_{0,b}$ for $b > 0$, $\eta = 0$ is disallowed, so φ is indeed well-defined. Thus on these grounds, both the coset spaces SO (3, 1)/N and SO (3, 1)/SO (2) × SO (1, 1) qualify as possible first order internal spaces. We next describe briefly the kinds of nontrivial Lagrangians that can be built in the two cases.

(1) $Q = \theta_{0,0}$

Taking into account the structure of the one-form φ in (3), we write the Lagrangian in the form

$$\mathscr{L}(x, \dot{x}; \xi, \dot{\xi}) = \frac{1}{|\eta|^2}\,\xi\ \eta\wedge\dot{\eta} + \mathscr{L}'(x, x; \xi). \qquad (4)$$

For an isolated system, translational invariance implies that $\mathscr{L}'$ cannot depend on x. Beyond this, $\mathscr{L}'$ must not depend on ξ, must be explicitly Lorentz-invariant, and must be homogeneous of degree one in $\dot{x}$. The leading term arising from φ happens to be explicitly Lorentz invariant by itself; it is also obviously homogeneous of degree one in velocities.

To be able to construct a nontrivial and interesting $\mathscr{L}'$, we ask if we can build functions on Q, *i.e.* functions of $\xi_{\mu\nu}$, transforming as a four-vector, or as a symmetric second-rank tensor, ... It turns out that one can form a lightlike four-vector V_μ as follows:

$$V_0 = (\xi^2)^{1/2} = (\eta^2)^{1/2};\ V = \eta\wedge\xi/(\xi^2)^{1/2}.$$

This is basic in the sense that all higher order symmetric tensors $D^{(j,j)}$ that one might try to form on Q are essentially polynomials in V_μ. Here $D^{(j,j)}$ is the symmetric traceless tensor representation of SL(2, C) of rank $2j$. Thus the most general $\mathscr{L}'$ in this case involves one arbitrary function and has the form

$$\mathscr{L}' = (-\dot{x}^2)^{1/2} f(\lambda),\ \lambda = \dot{x}^\mu V_\mu/(-\dot{x}^2)^{1/2}. \qquad (5)$$

The dynamics and the constraint and trajectory structure, *i.e.* mass-spin relation, for this model are described elewhere[3]. Here it suffices to remark that the space-time motion is similar in all respects to the spinor model of reference (3).

(2) $Q = \theta_{0,b}$, $b > 0$

On this orbit the one-form φ is explicitly invariant under SO (3) but changes by an exact piece under pure Lorentz transformations, so that d φ remains SO (3, 1) invariant. In fact under the infinitesimal boost $\delta\xi = \alpha\wedge\eta$, $\delta\eta = -\alpha\wedge\xi$, $|\alpha| \ll 1$, one finds

$$\delta\varphi = \mathrm{d}(-b^2\alpha\cdot\eta/\eta^2). \qquad (6)$$

Turning to the construction of a term like $\mathscr{L}'$in (4), we now find that it is not possible to form a four-vector on Q. In fact the simplest tensor of type $D^{(j,j)}$ that one can form is a second rank symmetric tensor $t_{\mu\nu}$ belonging to the representation $D^{(1,1)}$:

$$t_{\mu\nu} = \xi_{\mu\rho}\xi^\rho_\nu + \frac{b^2}{2}g_{\mu\nu}. \qquad (7)$$

So the simplest Lorentz scalar variable coupling internal and space-time variables is in this case $x^\mu x^\nu\xi_{\mu\rho}\xi^\rho_\nu$ and we have for $\mathscr{L}'$ the general form

$$\mathscr{L}' = (-\dot{x}^2)^{1/2} f(\lambda),\ \lambda = \dot{x}^\mu\dot{x}^\nu\xi_{\mu\rho}\,\xi^\rho_\nu/(-x^2). \qquad (8)$$

The dynamics and other aspects of this model are analysed in reference (9).

SECOND-ORDER INTERNAL SPACES[5]

The number of possible distinct candidates is now quite large: ten individual H's plus two one-parameter families. However, whereas with both possible first order internal spaces it was possible to couple $\dot{x}^\mu$ and Q to construct nontrivial Lagrangians, we now find the very interesting result that for certain choices of H, or correspondingly Q, no interesting coupling is possible.

We are now concerned with constructing Lagrangians on $TM \times TQ$, *i.e.* as functions of $\dot{x}$, q and $\dot{q}$, with the requisite properties. Under the (generally nonlinear) action of G on the q^r, the $\dot{q}^r$ transform linearly, with functions of q as coefficients. We try to form sets of functions of q and $\dot{q}$, homogeneous of some given degree in the $\dot{q}$'s, furnishing one of the representations $D^{(j,j)}$ of G; if we succeed coupling of TQ to TM is possible. First we consider expressions of the first degree in the $\dot{q}$, with functions of q^r as coefficients. At the point on $Q = G/H$ corresponding to the identity coset, the $\dot{q}^r$ on the fibre of TQ transform according to a linear representation $D(H)$ of H called the linear isotopy representation associated with the G action on Q. A necessary and sufficient condition that we be able to construct quantities linear in $\dot{q}$ belonging to the representation $D^{(j,j)}$ of G is that the restriction of this representation to H must contain, upon reduction, the isotopy representation $D(H)$ at least once. A similar necessary and sufficient condition exists for second, third, . . . degree expressions in $\dot{q}$: we replace $D(H)$ by its symmetrised Kronecker products $D(H) \times D(H)_{\text{symm.}}$, $D(H) \times D(H) \times D(H)_{\text{symm.}}$ . . . . For expressions independent of the $\dot{q}$'s, we use the identity representation of H instead of $D(H)$.

It can now be shown by using this criterion that for certain choices of H, hence of Q, no coupling of TQ and TM is possible, because we are unable to form functions on TQ furnishing any of the representations $D^{(j,j)}$ of G. This is the case when $H = AN$ corresponding to $Q = K = \mathrm{SU}(2)$; $H = \mathrm{U}(1)\, AN$ corresponding to $Q = S^2$; and when H is the three-parameter group generated by $J_1 - K_2$, $J_2 + K_1$ and $J_3 \sin\beta + K_3 \cos\beta$. The fact that these three otherwise allowed internal spaces are not physically viable is a somewhat unexpected result.

Even if no such nontrivial coupling of TM to TQ exists, it might be possible to construct quantities $s(q, \dot{q})$, which are scalars under $SL(2, C)$, and homogeneous of some degree n in the $\dot{q}$'s. We can then set up a Lagrangian

$$\mathscr{L} = (-\dot{x}^2)^{1/2} f\,(s(q, \dot{q})/(-\dot{x}^2)^{n/2} \qquad (9)$$

which may be regarded as trivial. Further analysis of such possibilities is given in reference (5).

The remaining second-order internal spaces can be classified according to the lowest rank $(2j)$ symmetric tensor $D^{(j,j)}$ that can be formed in each case, the number of these, and so on in order of increasing complexity. Thus there are eight cases of spaces $Q = G/H$ such that vectors can be constructed on TQ, and one case where nothing simpler than a tensor $D^{(1,1)}$ can be built. The Hanson-Regge model, with $H = e$ and $Q = G$, provides us with an example where four linearly independent vectors are already available on Q.

As an example of a nontrivial second order internal space we metion briefly the case $H = 0(1, 1)$. The five-dimensional manifold $Q = G/H$ is conveniently visualized as the set of all pairs of mutually orthogonal unit spacelike vectors (a^μ, b^μ): $a^2 = b^2 = 1$, $a\cdot b = 0$. A natural "internal symmetry group" arising in this case is the group SO(2) mixing a^μ and b^μ. The most general Lorentz, SO(2) and reparametrization invariant Lagrangian which furthermore leads to the two natural primary constraints $P\cdot a \approx 0$, $P\cdot b \approx 0$ turns out to be

$$\mathscr{L} = (-\xi_1)^{1/2} f\left(\frac{\xi_5(\xi_1+\xi_2) - 2\xi_4}{(-\xi_1)^{3/2}}, \frac{\xi_3 - 2\xi_5\xi_4}{\xi_5(-\xi_1)^{3/2}}, \frac{\xi_6}{\xi_1}, \frac{\xi_7}{\xi_1^2}\right), \qquad (10)$$

where f is an arbitrary function of four arguments and the complete independent set of Lorentz and SO(2) invariants ξ are:

$\xi_1 = \dot{x}^2$, $\xi_2 = (\dot{x}.a)^2 + (\dot{x}.b)^2$, $\xi_3 = (\dot{x}.a)^2 + (\dot{x}.\dot{b})^2$,
$\xi_4 = (\dot{x}.\dot{a})\,(\dot{x}.b) - (\dot{x}.\dot{b})\,(\dot{x}.a)$, $\xi_5 = a.\dot{b}$, $\xi_6 = \dot{a}^2 + \dot{b}^2$,
$\xi_7 = (\dot{a}.\dot{b})^2 - \dot{a}^2\dot{b}^2$.

This model is in some respects like the Hanson-Regge model in that imposition of an internal symmetry simplifies the problem; its constraint structure is however more like the vector model of reference (3), in that both $P\cdot a$ and $P\cdot b$ involve conjugate momenta only linearly. Further analysis of this model is given in reference (9).

CONCLUDING REMARKS

Classical relativistic particles with minimal internal structure and their dynamics have been shown to be related to coset spaces of SL(2, C) and Lagrangian functions on them. First-order internal spaces, of which there are just two examples, are related to orbits in the Lie algebra of G under the adjoint action. In both cases we have exhibited the general Lagrangians that can be constructed on $TM \times TQ$, with the property of being linear inhomogeneous in the $\dot{q}$. Second-order internal spaces, of which there are several, have to be handled via the concept of the isotopy representation, which determines whether and which representations $D^{(j,\bar{j})}$ of G can be realised on TQ.

Various models studied by others can be recognised in our general formalism. The original Hanson-Regge model has $H = e, Q = G$, and the Lagrangian is defined on $TM \times T\,SO(3, 1)$ and has an internal SO (3, 1) symmetry. It is a second order theory in our nomenclature. The choice $H = N$ leads to a first-order internal space which is the spinor model of reference (3): the latter is a "two-fold covering" of the exceptional orbit $\theta_{0,0}$. The case $H = SU(1, 1)$ gives a second order theory identical to the vector model of reference (3). The model of Barducci and Lusanna[10] uses an internal space $SO(3, 1) \times R^3$ which is not transitively acted upon by G: in our terminology this is therefore not indecomposable or minimal. The particle with a dipole moment studied by Cognola *et al*[11] uses a Lagrangian which is of first order in the internal variables, but the corresponding internal space is again non-minimal. With additional kinematical restrictions, their Lagrangian can be rewritten in a second order form on an indecomposible internal space. This again corresponds to $H = SU(1, 1)$. Further connections to available models in the literature are given in reference (5).

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