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Structure of electromagnetic fields in spatially dispersive media of arbitrary geometry*

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The nature of the electromagnetic field in a spatially dispersive medium, occupying an arbitrary domain V is investigated, under conditions when spatial dispersion effects arise from the presence of an isolated exciton transition band. It is shown that the electric field at frequency ω close to the exciton transition frequency may, in general, be expressed in the form $\vec{E}(\vec{r},\omega)=\vec{E}_t^{(1)}(\vec{r},\omega)+\vec{E}_t^{(2)}(\vec{r},\omega)+\vec{E}_l(\vec{r},\omega)$, where $\vec{E}_t^{(j)}(\vec{r},\omega)$ $(j=1,2)$ are transverse fields and $\vec{E}_l(\vec{r},\omega)$ is a longitudinal field; and that each of these three fields satisfies a Helmholtz equation. The wave numbers occurring in the three Helmholtz equations are the roots of the dispersion relations appropriate to the medium. It is further shown that the three fields are coupled by a linear relation, which is shown to imply a recently derived nonlocal boundary condition on the nonlocal polarization, expressed in the form of an extinction theorem. These results are generalizations of certain results obtained not long ago by Sein, Birman and Sein, Agarwal, Pattanayak, and Wolf, and Maradudin and Mills.

I. INTRODUCTION

We have recently discussed the structure of the electromagnetic field in a spatially dispersive model medium forming a plane-parallel slab.[1(a),1(b)] The medium was characterized by a dielectric response function appropriate to the neighborhood of an isolated exciton transition frequency, viz.,

$$\epsilon(\vec{k},\omega)=\epsilon_0+\frac{\chi}{\vec{k}^2-\mu^2(\omega)}, \tag{1.1}$$

where

$$\chi=4\pi\alpha m_e^*\omega_e/\hbar, \quad \mu^2(\omega)=(m_e^*/\hbar\omega_e)(\omega^2-\omega_e^2+i\omega\Gamma). \tag{1.2}$$

Here ϵ_0 is a background dielectric constant which takes into account all the transitions except those due to an exciton band at frequency $\omega=\omega_e$, m_e^* is the effective exciton mass (assumed to be nonnegative), α is essentially the oscillator strength, and Γ characterizes the lifetime of the excited states of the atoms. We applied the results to the problem of refraction and reflection on a spatially dispersive half space and showed that our analysis leads to a resolution of a long-standing controversy about the nature of the so-called additional boundary conditions needed in solving problems of this kind.

In this paper we will present a generalization of some of our results to monochromatic electromagnetic fields in spatially dispersive dielectric media characterized by the dielectric response function (1.1), occupying a finite volume V of arbitrary geometry. We show that the electric field in the medium may, in general, be expressed as the sum of two transverse and one longitudinal field. Each of these partial fields satisfies a Helmholtz equation, whose wave numbers are the roots of the dispersion relations appropriate to a plane wave that can be propagated in a medium with the dielectric response given by (1.1), occupying the whole space. The three partial fields are found to be coupled by a linear relation that is a generalization of certain relations found by Sein,[2(a)-2(c)] Birman and Sein,[3] Agarwal, Pattanayak, and Wolf,[1(a),1(b)] and Maradudin and Mills.[4] This relation is found to imply a nonlocal boundary condition on the nonlocal polarization, which is identical with the recently derived extinction theorem for the nonlocal polarization.[1(a)]

The main results derived in this paper are summarized in a theorem given in Sec. VI and are illustrated by determining the mode expansion of the electric field in a medium characterized by the dielectric response function of the form given by Eq. (1.1), occupying a half space. The results agree with those obtained for this special case in Ref. 1(a) by a different method.

II. BASIC EQUATIONS

Consider an electromagnetic field in a dielectric medium occupying a volume V, bounded by a closed surface S. Let $\vec{E}(\vec{r},t)$ and $\vec{D}(\vec{r},t)$ represent the electric field and the electric displacement vector and let $\vec{H}(\vec{r},t)$ and $\vec{B}(\vec{r},t)$ represent the magnetic field and the magnetic induction vector, respectively, at a point $\vec{r}$, at time t. We assume that the medium is nonmagnetic, so that, in the Gaussian system of units (used throughout this paper), $\vec{B}(\vec{r},t)=\vec{H}(\vec{r},t)$. We denote by $\vec{E}(\vec{r},\omega)$ the Fourier time transform of $\vec{E}(\vec{r},t)$ defined as

$$\vec{E}(\vec{r}, \omega) = \int_{-\infty}^{+\infty} \vec{E}(\vec{r}, t) e^{i\omega t} dt , \tag{2.1}$$

with similar definitions for the Fourier time transforms of the other fields. Maxwell's equations in the medium, written in frequency space, are

$$\nabla \times \vec{H}(\vec{r}, \omega) + ik_0 \vec{E}(\vec{r}, \omega) = -4\pi i k_0 \vec{P}(\vec{r}, \omega) , \tag{2.2a}$$

$$\nabla \times \vec{E}(\vec{r}, \omega) - ik_0 \vec{H}(\vec{r}, \omega) = 0 , \tag{2.2b}$$

$$\nabla \cdot \vec{E}(\vec{r}, \omega) = -4\pi \nabla \cdot \vec{P}(\vec{r}, \omega) , \tag{2.2c}$$

$$\nabla \cdot \vec{H}(\vec{r}, \omega) = 0 , \tag{2.2d}$$

where

$$k_0 = \omega / c , \tag{2.3}$$

c being the vacuum speed of light, and $\vec{P}(\vec{r}, \omega)$ is the polarization vector,

$$\vec{P}(\vec{r}, \omega) = (1/4\pi)[\vec{D}(\vec{r}, \omega) - \vec{E}(\vec{r}, \omega)] . \tag{2.4}$$

Let us now assume that the dielectric response function of the medium is given by Eq. (1.1). Then, as shown in Eq. (2.6) of Ref. 1(a), the polarization field is the sum of two terms,

$$\vec{P}(\vec{r}, \omega) = \vec{P}_L(\vec{r}, \omega) + \vec{P}_{NL}(\vec{r}, \omega) , \tag{2.5}$$

$\vec{P}_L$ depending locally on $\vec{E}$ and $\vec{P}_{NL}$ depending on $\vec{E}$ in a nonlocal manner. More specifically,[5]

$$\vec{P}_L(\vec{r}, \omega) = \frac{\epsilon_0 - 1}{4\pi} \vec{E}(\vec{r}, \omega) , \tag{2.6a}$$

$$\vec{P}_{NL}(\vec{r}, \omega) = \frac{\chi}{(4\pi)^2} \int_V G_\mu(|\vec{r} - \vec{r}'|) \vec{E}(\vec{r}', \omega) d^3 r' , \tag{2.6b}$$

where

$$G_\mu(|\vec{r} - \vec{r}'|) = e^{i\mu|\vec{r} - \vec{r}'|} / |\vec{r} - \vec{r}'| . \tag{2.7a}$$

In (2.7) μ is the root of the second expression in (1.2), for which

$$\mathrm{Im}\mu > 0 , \tag{2.7b}$$

consistent with our Fourier kernel $e^{i\omega t}$ in (2.1). We note for the purpose of later discussion, that G_μ satisfies the equation

$$(\nabla^2 + \mu^2) G_\mu(|\vec{r} - \vec{r}'|) = -4\pi \delta(\vec{r} - \vec{r}') , \tag{2.8}$$

where δ is the three-dimensional Dirac δ function.

We now have all the information necessary to derive the basic equations from which the general structure of the electromagnetic field in the spatially dispersive medium may readily be determined. We obtain these equations as follows: First we eliminate the magnetic field $\vec{H}$ between Eqs. (2.2a) and (2.2b) and make use of Eqs. (2.5) and (2.6a). We then find that

$$\nabla \times [\nabla \times \vec{E}(\vec{r}, \omega)] - \epsilon_0 k_0^2 \vec{E}(\vec{r}, \omega) = 4\pi k_0^2 \vec{P}_{NL}(\vec{r}, \omega) . \tag{2.9}$$

Next we apply the operator $(\nabla^2 + \mu^2)$ to both sides of (2.6b) and use (2.8). We then obtain the equation

$$(\nabla^2 + \mu^2) \vec{P}_{NL}(\vec{r}, \omega) = -(\chi/4\pi) \vec{E}(\vec{r}, \omega) . \tag{2.10}$$

Finally on substituting from (2.5) into (2.2c) and on making use of (2.6a) we deduce that

$$\epsilon_0 \nabla \cdot \vec{E}(\vec{r}, \omega) = -4\pi \nabla \cdot \vec{P}_{NL}(\vec{r}, \omega) . \tag{2.11}$$

If we substitute into Eq. (2.9) for $\vec{P}_{NL}(\vec{r}, \omega)$ the expression (2.6b) [which implies Eq. (2.10)], we obtain the following integro-differential equation for the electric field $\vec{E}(\vec{r}, \omega)$ inside the volume V:

$$\nabla \times [\nabla \times \vec{E}(\vec{r}, \omega)] - \epsilon_0 k_0^2 \vec{E}(\vec{r}, \omega) = \frac{\chi k_0^2}{4\pi} \int_V G_\mu(|\vec{r} - \vec{r}'|) \vec{E}(\vec{r}', \omega) d^3 r' . \tag{2.12}$$

This equation was taken as the starting point of the investigation of Ref. 1(a) [Eq. (2.10) of that reference]. However, for the purpose of the main part of the present investigation it is advantageous to deal instead with the two coupled differential equations (2.9) and (2.10), together with the subsidiary relation (2.11). We note in passing that (2.11) follows trivially from Eq. (2.9) and also from the pair of equations (2.12) and (2.6b), provided that $k_0 \neq 0$.

III. SEPARATION OF THE ELECTRIC FIELD INTO TRANSVERSE AND LONGITUDINAL PARTS

We will now show that Eqs. (2.9) and (2.10) lead to a certain decomposition of the electric field into a sum of a transverse and a longitudinal part. For this purpose we first make use of the vector identity

$$\nabla^2 \vec{F} \equiv \nabla(\nabla \cdot \vec{F}) - \nabla \times (\nabla \times \vec{F}) \tag{3.1}$$

that applies to any well-behaved vector field $\vec{F}$. In particular, taking $\vec{F}$ to be the nonlocal polarization field $\vec{P}_{NL}$, Eq. (2.10) may be expressed in the form

$$\vec{P}_{NL}(\vec{r}, \omega) = -\frac{\chi}{4\pi\mu^2} \vec{E}(\vec{r}, \omega) - \frac{1}{\mu^2} \nabla[\nabla \cdot \vec{P}_{NL}(\vec{r}, \omega)] + \frac{1}{\mu^2} \nabla \times [\nabla \times \vec{P}_{NL}(\vec{r}, \omega)] . \tag{3.2}$$

Next we substitute from (3.2) into the right-hand side of Eq. (2.9) and obtain the relation

$$\nabla \times [\nabla \times \vec{E}(\vec{r}, \omega)] - \epsilon_0 k_0^2 \vec{E}(\vec{r}, \omega) = -\frac{\chi k_0^2}{\mu^2} \vec{E}(\vec{r}, \omega) - \frac{4\pi k_0^2}{\mu^2} \nabla[\nabla \cdot \vec{P}_{NL}(\vec{r}, \omega)]$$

$$+\frac{4\pi k_0^2}{\mu^2}\nabla\times[\nabla\times\vec{P}_{NL}(\vec{r},\omega)]. \quad (3.3)$$

This relation may be rewritten in the form

$$\vec{E}(\vec{r},\omega)=\vec{E}_t(\vec{r},\omega)+\vec{E}_l(\vec{r},\omega), \quad (3.4)$$

where

$$\vec{E}_t(\vec{r},\omega)=\frac{1}{k_0^2(\epsilon_0-\chi/\mu^2)} \times\nabla\times\left[\nabla\times\left(\vec{E}(\vec{r},\omega)-\frac{4\pi k_0^2}{\mu^2}\vec{P}_{NL}(\vec{r},\omega)\right)\right] \quad (3.4a)$$

and

$$\vec{E}_l(\vec{r},\omega)=\frac{4\pi}{\epsilon_0\mu^2-\chi}\nabla[\nabla\cdot\vec{P}_{NL}(\vec{r},\omega)]. \quad (3.4b)$$

In view of the vector identities $\nabla\cdot(\nabla\times\vec{F})\equiv 0$ and $\nabla\times(\nabla f)\equiv 0$, where $\vec{F}$ is an arbitrary vector field and f is an arbitrary scalar field, it follows that

$$\nabla\cdot\vec{E}_t(\vec{r},\omega)=0 \quad (3.5a)$$

and

$$\nabla\times\vec{E}_l(\vec{r},\omega)=0, \quad (3.5b)$$

so that the vector field $\vec{E}_t(\vec{r},\omega)$ defined by (3.4a) is transverse and the vector field $\vec{E}_l(\vec{r},\omega)$ defined by (3.4b) is longitudinal. Thus Eq. (3.4) expresses the electric field $\vec{E}(\vec{r},\omega)$ in our spatially dispersive medium as the sum of a transverse part and a longitudinal part.

IV. PARTIAL DIFFERENTIAL EQUATIONS SATISFIED BY THE TRANSVERSE AND THE LONGITUDINAL PARTS OF THE ELECTRIC FIELD

We will now derive differential equations satisfied by the transverse part $\vec{E}_t$ and the longitudinal part $\vec{E}_l$ of the electric field in our spatially dispersive medium.

We substitute from Eq. (3.4) into Eq. (2.9) and make use of the relation (3.5b), viz., $\nabla\times\vec{E}_l=0$. We then obtain the equation

$$\nabla\times[\nabla\times\vec{E}_t(\vec{r},\omega)]-\epsilon_0k_0^2\vec{E}_t(\vec{r},\omega)-\epsilon_0k_0^2\vec{E}_l(\vec{r},\omega) =4\pi k_0^2\vec{P}_{NL}(\vec{r},\omega). \quad (4.1)$$

Next we use again the vector identity (3.1), with $\vec{F}=\vec{E}_t$, and also the transversality condition (3.5a), viz., $\nabla\cdot\vec{E}_t=0$. Equation (4.1) then becomes

$$(\nabla^2+\epsilon_0k_0^2)\vec{E}_t(\vec{r},\omega)+\epsilon_0k_0^2\vec{E}_l(\vec{r},\omega) =-4\pi k_0^2\vec{P}_{NL}(\vec{r},\omega). \quad (4.2)$$

Next we apply to both sides of (4.2) the operator $(\nabla^2+\mu^2)$. We eliminate the term $(\nabla^2+\mu^2)\vec{P}_{NL}$ between the resulting equation and Eq. (2.10) and make use of Eq. (3.4). After rearranging the terms we obtain the equation

$$k_0^2\epsilon_0[\nabla^2+(\mu^2-\chi/\epsilon_0)]\vec{E}_l(\vec{r},\omega) =-[(\nabla^2+\epsilon_0k_0^2)(\nabla^2+\mu^2)-\chi k_0^2]\vec{E}_t(\vec{r},\omega). \quad (4.3)$$

We will rewrite (4.3) in the form

$$k_0^2\epsilon_0(\nabla^2+\vec{k}_l^2)\vec{E}_l(\vec{r},\omega) =-[\nabla^2+(\vec{k}_t^{(1)})^2][\nabla^2+(\vec{k}_t^{(2)})^2]\vec{E}_t(\vec{r},\omega), \quad (4.4)$$

where

$$\vec{k}_l^2=\mu^2-\chi/\epsilon_0 \quad (4.5)$$

and $(\vec{k}_t^{(1)})^2$ and $(\vec{k}_t^{(2)})^2$ are defined by the equation

$$[\nabla^2+(\vec{k}_t^{(1)})^2][\nabla^2+(\vec{k}_t^{(2)})^2]=(\nabla^2+\epsilon_0k_0^2)(\nabla^2+\mu^2)-\chi k_0^2. \quad (4.6)$$

It can readily be seen from (4.5) and (1.1) that $\vec{k}_l$ satisfies the equation

$$\epsilon(\vec{k}_l,\omega)=0, \quad (4.7)$$

which is nothing but the *longitudinal dispersion relation* appropriate for the propagation of plane monochromatic waves in a spatially dispersive medium of dielectric response function $\epsilon(\vec{k},\omega)$ given by Eqs. (1.1), occupying the whole space. Furthermore as we show in Appendix B, the roots $\vec{k}_t^{(1)}$ and $\vec{k}_t^{(2)}$ satisfy the corresponding *transverse dispersion relation*

$$\epsilon(\vec{k}_t^{(j)},\omega)=(\vec{k}_t^{(j)}/k_0)^2. \quad (4.8)$$

Equation (4.4) may be simplified by making use of an additional result that follows from our basic relations. To derive it we eliminate the term $\nabla\cdot\vec{P}_{NL}$ between Eqs. (2.11) and (3.4b) and use the relation (4.5). We then find that $\vec{E}_l=-(1/\vec{k}_l^2)\times\nabla[\nabla\cdot\vec{E}(\vec{r},\omega)]$, which, if Eqs. (3.4) and (3.5a) are used, implies that the longitudinal part of the electric field obeys the Helmholtz equation

$$(\nabla^2+\vec{k}_l^2)\vec{E}_l(\vec{r},\omega)=0. \quad (4.9)$$

Moreover, on using (4.9) in (4.4), we see that the transverse part of the electric field obeys the equation

$$[\nabla^2+(\vec{k}_t^{(1)})^2][\nabla^2+(\vec{k}_t^{(2)})^2]\vec{E}_t(\vec{r},\omega)=0. \quad (4.10)$$

Now according to a recently established theorem,[6] a solution of the differential equation (4.10) [with $(\vec{k}_t^{(1)})^2\neq(\vec{k}_t^{(2)})^2$], in a three-dimensional domain V, subject to appropriate boundary conditions specified on the surface S bounding the volume V, may be expressed in the form

$$\vec{E}_t(\vec{r},\omega)=\vec{E}_t^{(1)}(\vec{r},\omega)+\vec{E}_t^{(2)}(\vec{r},\omega), \quad (4.11)$$

where $\vec{E}_t^{(1)}$ and $\vec{E}_t^{(2)}$ are solutions of the Helmholtz equations

$$[\nabla^2+(\vec{k}_t^{(j)})^2]\vec{E}_t^{(j)}(\vec{r},\omega)=0 \quad (j=1,2). \quad (4.12)$$

We note that since each of the operators $[\nabla^2 + (\vec{k}_t^{(j)})^2]$ and the operator $\nabla\cdot$ commute,

$$[\nabla^2 + (\vec{k}_t^{(1)})^2]\nabla\cdot\vec{E}_t^{(1)}(\vec{r},\omega) = 0\,, \tag{4.13a}$$

$$[\nabla^2 + (\vec{k}_t^{(2)})^2]\nabla\cdot\vec{E}_t^{(2)}(\vec{r},\omega) = 0\,. \tag{4.13b}$$

On adding these two equations and using (3.5a) and (4.11) we readily deduce that

$$[(\vec{k}_t^{(1)})^2 - (\vec{k}_t^{(2)})^2]\nabla\cdot\vec{E}_t^{(j)}(\vec{r},\omega) = 0 \quad (j=1,2)\,. \tag{4.14}$$

Hence, provided that $(k_t^{(1)})^2 \neq (k_t^{(2)})^2$, i.e., provided that the transverse dispersion relation (4.8) has four distinct roots and not two double roots, as in general will be the case, we must necessarily have

$$\nabla\cdot\vec{E}_t^{(j)}(\vec{r},\omega) = 0 \quad (j=1,2)\,; \tag{4.15}$$

i.e., the partial fields $\vec{E}_t^{(1)}$ and $\vec{E}_t^{(2)}$ are *transverse*.

We have thus shown that the longitudinal part $\vec{E}_l$ of the electric field in our spatially dispersive medium satisfies the Helmholtz equation (4.9) and that in general the transverse part $\vec{E}_t$ is expressible as the sum of two transverse fields $\vec{E}_t^{(1)}$ and $\vec{E}_t^{(2)}$ each also satisfying a Helmholtz equation. The squares of the (generally complex) wave numbers appearing in these three Helmholtz equations are precisely the squares of the wave vectors obeying the dispersion relations for propagation of plane waves in a spatially dispersive medium of dielectric response function $\epsilon(\vec{k},\omega)$, given by Eqs. (1.1), occupying the whole space.

V. COUPLING BETWEEN THE LONGITUDINAL AND TRANSVERSE PARTS OF THE ELECTRIC FIELD, AND THE EXTINCTION THEOREM FOR NONLOCAL POLARIZATION

We have now expressed the electric field, which satisfies the coupled pair of differential equations (2.9) and (2.10), in the form

$$\vec{E}(\vec{r},\omega) = \vec{E}_t^{(1)}(\vec{r},\omega) + \vec{E}_t^{(2)}(\vec{r},\omega) + \vec{E}_l(\vec{r},\omega)\,. \tag{5.1}$$

Now in the process of deriving (5.1), we applied to equation (2.9) the operator $(\nabla^2+\mu^2)$. Moreover, in deriving (2.10) we have applied to (2.6b) this operator also. Hence we raised the order of the differential expressions that appear in the original equations and we must, therefore, expect that (5.1) represents a broader class of fields than the class of solutions of the basic integro-differential equation (2.12), viz.,

$$\nabla\times[\nabla\times\vec{E}(\vec{r},\omega)] - \epsilon_0 k_0^2\vec{E}(\vec{r},\omega) = \frac{\chi k_0^2}{4\pi}\int_V G_\mu(|\vec{r}-\vec{r}\,'|)\vec{E}(\vec{r}\,',\omega)d^3r'\,. \tag{5.2}$$

We will now examine the additional constraints (apart from those noted in Secs. III and IV) that must be imposed on the three partial fields appearing on the right-hand side of Eq. (5.1) in order that (5.1) is a solution of Eq. (5.2). To determine these constraints we only need to substitute from (5.1) into (5.2) and determine the conditions under which the resulting equation will be satisfied.

On substituting from (5.1) into (5.2) and on using the fact that $\nabla\times(\nabla\times\vec{E}_l)=0$ we obtain the relation

$$\sum_{j=1}^{2}\{\nabla\times[\nabla\times\vec{E}_t^{(j)}(\vec{r},\omega)] - \epsilon_0 k_0^2\vec{E}_t^{(j)}(\vec{r},\omega)\} - \epsilon_0 k_0^2\vec{E}_l(\vec{r},\omega) = \frac{\chi k_0^2}{4\pi}\sum_{j=1}^{2}\int_V G_\mu(|\vec{r}-\vec{r}\,'|)\vec{E}_t^{(j)}(\vec{r}\,',\omega) + \frac{\chi k_0^2}{4\pi}\int_V G_\mu(|\vec{r}-\vec{r}\,'|)\vec{E}_l(\vec{r}\,',\omega)d^3r'\,. \tag{5.3}$$

The three integrals appearing on the right-hand side of (5.3) may readily be simplified. Consider, for example, the integral involving $\vec{E}_l$. We have, according to (4.9),

$$(\nabla^2+\vec{k}_l^2)\vec{E}_l(\vec{r},\omega)=0\,. \tag{5.4}$$

Moreover, according to (2.8), the Green's function G_μ satisfies the equation

$$(\nabla^2+\mu^2)G_\mu(|\vec{r}-\vec{r}\,'|) = -4\pi\delta(\vec{r}-\vec{r}\,')\,. \tag{5.5}$$

On multiplying Eq. (5.4) by G_μ and Eq. (5.5) by $\vec{E}_l$, subtracting the resulting equations, and on integrating over the volume V, we find that for any point $\vec{r}$ in V,

$$\int_V G_\mu(|\vec{r}-\vec{r}\,'|)\vec{E}_l(\vec{r}\,',\omega)d^3r' = \frac{4\pi}{\vec{k}_l^2-\mu^2}\vec{E}_l(\vec{r},\omega) - \frac{1}{\vec{k}_l^2-\mu^2}\times\int_V[G_\mu(|\vec{r}-\vec{r}\,'|)\nabla^2\vec{E}_l(\vec{r}\,',\omega) - \vec{E}_l(\vec{r}\,',\omega)\nabla^2 G_\mu(|\vec{r}-\vec{r}\,'|)]d^3r'\,, \tag{5.6}$$

or on using Green's theorem,

$$\int_V G_\mu(|\vec{r}-\vec{r}\,'|)\vec{E}_l(\vec{r}\,',\omega)d^3r' = \frac{4\pi}{\vec{k}_l^2-\mu^2}\vec{E}_l(\vec{r},\omega) + \frac{1}{\vec{k}_l^2-\mu^2}\vec{S}_l(\vec{r},\omega)\,, \tag{5.7}$$

where

$$\vec{S}_l(\vec{r},\omega) = \int_S\left(\vec{E}_l(\vec{r}\,',\omega)\frac{\partial G_\mu(|\vec{r}-\vec{r}\,'|)}{\partial n} - G_\mu(|\vec{r}-\vec{r}\,'|)\frac{\partial\vec{E}_l(\vec{r}\,',\omega)}{\partial n}\right)dS\,. \tag{5.8}$$

The integration on the right-hand side of Eq. (5.8) extends over the surface S bounding the volume V occupied by our spatially dispersive medium and $\partial/\partial n$ denotes differentiation along the outward normal. In a strictly similar manner we obtain, if we use in place of (5.4) the corresponding equations (4.12), the following expressions for the two other integrals appearing on the right-hand side of Eq. (5.3), valid with $j=1$ and $j=2$:

$$\int_V G_\mu(|\vec{r}-\vec{r}'|)\vec{E}_t^{(j)}(\vec{r}',\omega)d^3r' = \frac{4\pi}{(\vec{k}_t^{(j)})^2-\mu^2}\vec{E}_t^{(j)}(\vec{r},\omega)+\frac{1}{(\vec{k}_t^{(j)})^2-\mu^2}\vec{S}_t^{(j)}(\vec{r},\omega), \tag{5.9}$$

where

$$\vec{S}_t^{(j)}=\int_S\left(\vec{E}_t^{(j)}(\vec{r}',\omega)\frac{\partial G_\mu(|\vec{r}-\vec{r}'|)}{\partial n} - G_\mu(|\vec{r}-\vec{r}'|)\frac{\partial \vec{E}_t^{(j)}(\vec{r}',\omega)}{\partial n}\right)dS. \tag{5.10}$$

We note in passing that since the $\vec{r}$ dependence of the expressions (5.10) and (5.8) comes entirely from the argument of the Green's function $G_\mu(|\vec{r}-\vec{r}'|)$ under the integral signs, and since $\vec{r}'$ is confined to the boundary S of the volume V, the integrals $\vec{S}_t^{(j)}$ and $\vec{S}_l$ satisfy, in view of (2.8), the Helmholtz equation with the wave number μ, i.e.,

$$(\nabla^2+\mu^2)\vec{S}_t^{(j)}(\vec{r},\omega)=0, \tag{5.11a}$$

$$(\nabla^2+\mu^2)\vec{S}_l(\vec{r},\omega)=0, \tag{5.11b}$$

at each point $\vec{r}$, inside V.

We now substitute the expressions (5.7) and (5.9) for the three volume integrals in (5.3) and obtain the relation

$$\sum_{j=1}^{2}\{\nabla\times[\nabla\times\vec{E}_t^{(j)}(\vec{r},\omega)] - \epsilon_0k_0^2\vec{E}_t^{(j)}(\vec{r},\omega)\} - \epsilon_0k_0^2\vec{E}_l(\vec{r},\omega) = \chi k_0^2\sum_{j=1}^{2}\left(\frac{1}{(\vec{k}_t^{(j)})^2-\mu^2}\vec{E}_t^{(j)}(\vec{r},\omega)+\frac{1}{4\pi}\vec{S}_t^{(j)}(\vec{r},\omega)\right) + \chi k_0^2\left(\frac{1}{\vec{k}_l^2-\mu^2}\vec{E}_l(\vec{r},\omega)+\frac{1}{4\pi}\vec{S}_l(\vec{r},\omega)\right), \tag{5.12}$$

where $\vec{S}_t^{(j)}$ and $\vec{S}_l$ are defined by Eqs. (5.10) and (5.8), respectively. Now if we make use of the vector identity $\nabla\times(\nabla\times\vec{E}_t^{(j)})=\nabla(\nabla\cdot\vec{E}_t^{(j)})-\nabla^2\vec{E}_t^{(j)}$ and use the transversality conditions (4.15), we may rewrite the Helmholtz equations (4.12) as

$$\nabla\times[\nabla\times\vec{E}_t^{(j)}(\vec{r},\omega)]=(\vec{k}_t^{(j)})^2\vec{E}_t^{(j)}(\vec{r},\omega), \tag{5.13}$$

so that the relation (5.12) may be expressed in the form

$$\sum_{j=1}^{2}c_t^{(j)}\vec{E}_t^{(j)}(\vec{r},\omega)+c_l\vec{E}_l(\vec{r},\omega) = \frac{\chi k_0^2}{4\pi}\left(\sum_{j=1}^{2}\frac{1}{(\vec{k}_t^{(j)})^2-\mu^2}\vec{S}_t^{(j)}(\vec{r},\omega)+\frac{1}{\vec{k}_l^2-\mu^2}\vec{S}_l(\vec{r},\omega)\right), \tag{5.14}$$

where

$$c_t^{(j)}=(\vec{k}_t^{(j)})^2-\epsilon_0k_0^2-\frac{\chi k_0^2}{(\vec{k}_t^{(j)})^2-\mu^2}, \tag{5.15a}$$

$$c_l=-\epsilon_0k_0^2-\frac{\chi k_0^2}{\vec{k}_l^{\,2}-\mu^2}. \tag{5.15b}$$

Now if we recall the definition (1.1) of the dielectric response function $\tilde{\epsilon}(\vec{k},\omega)$, it follows at once that, since $\vec{k}_t^j$ satisfies the transverse dispersion relation (4.8),

$$c_t^{(j)}\equiv 0,\quad (j=1,2), \tag{5.16a}$$

and since $\vec{k}_l^2$ satisfies the longitudinal dispersion relation (4.7),

$$c_l\equiv 0. \tag{5.16b}$$

Hence (5.14) reduces to

$$\frac{1}{(\vec{k}_t^{(1)})^2-\mu^2}\vec{S}_t^{(1)}(\vec{r},\omega)+\frac{1}{(\vec{k}_t^{(2)})^2-\mu^2}\vec{S}_t^{(2)}(\vec{r},\omega)+\frac{1}{\vec{k}_l^2-\mu^2}\vec{S}_l(\vec{r},\omega)=0, \tag{5.17}$$

where, we recall once again, $\vec{S}_t^{(1)}$, $\vec{S}_t^{(2)}$, and $\vec{S}_l$ are defined by Eqs. (5.10) and (5.8). Equation (5.17), which must be satisfied at every point $\vec{r}$ inside V, is the required constraint we have been seeking, and it ensures that the field (5.1) (with $\vec{E}_t^{(1)}$, $\vec{E}_t^{(2)}$, and $\vec{E}_l$ satisfying also the constraints derived in Secs. III and IV), is a solution of the integro-differential equation (5.2). In Sec. VI we will summarize all the constraints and will present a general expression for the full electromagnetic field inside a spatially dispersive medium.

Before doing so we will show that the relation (5.17) which couples the three partial fields $\vec{E}_t^{(1)}$, $\vec{E}_t^{(2)}$, and $\vec{E}_l$ imposes an interesting constraint on the nonlocal polarization $\vec{P}_{NL}$, defined by Eq. (2.6b), viz.,

$$\vec{P}_{NL}(\vec{r},\omega)=\frac{\chi}{(4\pi)^2}\int_V G_\mu(|\vec{r}-\vec{r}'|)\vec{E}(\vec{r}',\omega)d^3r'. \tag{5.18}$$

To see this we express the $\vec{E}$ field on the right-hand side of (5.18) as a sum of the three partial fields, in accordance with (5.1), and make use of the expressions (5.7) and (5.9) for the three volume integrals. If next we use the constraint (5.17), we find at once that

$$\vec{P}_{NL}(\vec{r},\omega)=\frac{\chi}{4\pi}\left(\sum_{j=1}^{2}\frac{1}{(\vec{k}_t^{(j)})^2-\mu^2}\vec{E}_t^{(j)}(\vec{r},\omega)+\frac{1}{\vec{k}_l^2-\mu^2}\vec{E}_l(\vec{r},\omega)\right). \tag{5.19}$$

Next we rewrite (5.17) by substituting for $\vec{S}_t^{(j)}$ and $\vec{S}_l$ the explicit expressions (5.10) and (5.8) and interchange the orders of summation and integrations. If in the resulting equation we make use of (5.19), we find that the nonlocal polarization must be such that the following relation holds at every point $\vec{r}$ in V:

$$\int_S\left(\vec{P}_{NL}(\vec{r}',\omega)\frac{\partial G_\mu(|\vec{r}-\vec{r}'|)}{\partial n}-G_\mu(|\vec{r}-\vec{r}'|)\frac{\partial\vec{P}_{NL}(\vec{r}',\omega)}{\partial n}\right)dS=0. \tag{5.20}$$

Equation (5.20) expresses a nonlocal boundary condition on the nonlocal polarization $\vec{P}_{NL}$. It is precisely the *extinction theorem for the nonlocal polarization* $\vec{P}_{NL}(\vec{r},\omega)$ [Ref. 1(a), Eq. (5.17)], whose importance for the theory of electrodynamics of spatially dispersive media occupying an arbitrary volume has recently become recognized [cf. Ref. 3 and Ref. 1(a), Sec. V].

VI. SUMMARY, DISCUSSION, AND AN EXAMPLE

The main results derived in this paper may be summarized in the following theorem.

Theorem: The monochromatic electric field inside a spatially dispersive medium occupying a volume V bounded by a closed surface S, whose dielectric response function is of the form

$$\epsilon(\vec{k},\omega)=\epsilon_0+\frac{\chi}{\vec{k}^2-\mu^2(\omega)}\ , \tag{1.1}$$

may be expressed within the accuracy of the present theory[5] in the form

$$\vec{E}(\vec{r},\omega)=\vec{E}_t^{(1)}(\vec{r},\omega)+\vec{E}_t^{(2)}(\vec{r},\omega)+\vec{E}_l(\vec{r},\omega), \tag{5.1}$$

where

(i) $\vec{E}_t^{(j)}$, ($j=1,2$), are transverse fields, i.e.,

$$\nabla\cdot\vec{E}_t^{(j)}(\vec{r},\omega)=0\ , \tag{4.15}$$

obeying the Helmholtz equations

$$[\nabla^2+(\vec{k}_t^{(j)})^2]\vec{E}_t^{(j)}(\vec{r},\omega)=0\ , \tag{4.12}$$

with $(\vec{k}_t^{(j)})^2$ being the two roots, assumed to be distinct, of the transverse dispersion equation

$$\epsilon(\vec{k}_t^{(j)},\omega)=(\vec{k}_t^{(j)}/k_0)^2; \tag{4.8}$$

(ii) $\vec{E}_l$ is a longitudinal field, i.e.,

$$\nabla\times\vec{E}_l(\vec{r},\omega)=0\ , \tag{3.5b}$$

obeying the Helmholtz equation

$$(\nabla^2+\vec{k}_l^2)\vec{E}_l(\vec{r},\omega)=0\ , \tag{4.9}$$

with $\vec{k}_l^2$ being the root of the longitudinal dispersion equation

$$\epsilon(\vec{k}_l,\omega)=0\ . \tag{4.7}$$

(iii) The partial fields $\vec{E}_t^{(1)}$, $\vec{E}_t^{(2)}$, and $\vec{E}_l$ are coupled by the relation

$$\frac{1}{(\vec{k}_t^{(1)})^2-\mu^2}\vec{S}_t^{(1)}(\vec{r},\omega)+\frac{1}{(\vec{k}_t^{(2)})^2-\mu^2}\vec{S}_t^{(2)}(\vec{r},\omega)+\frac{1}{\vec{k}_l^2-\mu^2}\vec{S}_l(\vec{r},\omega)=0\ , \tag{5.17}$$

where

$$\vec{S}_t^{(j)}(\vec{r},\omega)=\int_S\left(\vec{E}_t^{(j)}(\vec{r}',\omega)\frac{\partial G_\mu(|\vec{r}-\vec{r}'|)}{\partial n}-G_\mu(|\vec{r}-\vec{r}'|)\frac{\partial\vec{E}_t^{(j)}(\vec{r}',\omega)}{\partial n}\right)dS\ , \tag{5.10}$$

and

$$\vec{S}_l(\vec{r},\omega)=\int_S\left(\vec{E}_l(\vec{r}',\omega)\frac{\partial G_\mu(|\vec{r}-\vec{r}'|)}{\partial n}-G_\mu(|\vec{r}-\vec{r}'|)\frac{\partial\vec{E}_l(\vec{r}',\omega)}{\partial n}\right)dS\ , \tag{5.8}$$

with G_μ being the Green's function, defined by Eqs. (2.7).

We have also shown that the relation (5.17) implies that the nonlocal polarization $\vec{P}_{NL}(\vec{r},\omega)$, established in the medium and defined by Eq. (2.6b), must obey certain nonlocal boundary conditions, expressed in the form of the extinction theorem

$$\int_S\left(\vec{P}_{NL}(\vec{r},\omega)\frac{\partial G_\mu(|\vec{r}-\vec{r}'|)}{\partial n}-G_\mu(|\vec{r}-\vec{r}'|)\frac{\partial\vec{P}_{NL}(\vec{r},\omega)}{\partial n}\right)dS=0 \tag{5.20}$$

that is valid at every point $\vec{r}$ inside the medium.

In connection with the above theorem we wish to mention that Sein[2] and Birman and Sein[3] gave, not long ago, on the basis of a different analysis, a somewhat similar representation of the field inside the spatially dispersive medium. They assumed that the partial fields (corresponding to our waves $\vec{E}_t^{(1)}$, $\vec{E}_t^{(2)}$, and $\vec{E}_l$) can each be expressed as a superposition of plane waves. However, this assumption cannot be justified except in the special case when the volume V occupied by the spatially dispersive dielectric is a plane-parallel slab or a half space. In the case when the volume V is a sphere,[7] for example, the appropriate expansion of the three partial fields is in terms of vector spherical harmonics rather than in terms of plane waves.

For the sake of completeness we also present expressions for the magnetic field $\vec{H}(\vec{r}, \omega)$, the magnetic induction field $\vec{B}(\vec{r}, \omega)$, and the electric displacement vector $\vec{D}(\vec{r}, \omega)$ inside the volume V. We readily find from Maxwell's equations, if for $\vec{E}(\vec{r}, \omega)$ we substitute the representation (5.1), recall that the medium was assumed to be nonmagnetic, and use also Eqs. (4.15), (3.5b), and (4.12) that

$$\vec{H}(\vec{r}, \omega) \equiv \vec{B}(\vec{r}, \omega) = \frac{1}{ik_0} \sum_{j=1}^{2} \nabla \times \vec{E}_t^{(j)}(\vec{r}, \omega) , \tag{6.1}$$

$$\vec{D}(\vec{r}, \omega) = \sum_{j=1}^{2} \left(\frac{\vec{k}_t^{(j)}}{k_0^2}\right)^2 \vec{E}_t^{(j)}(\vec{r}, \omega) . \tag{6.2}$$

We note that only the transverse partial fields $\vec{E}_t^{(1)}$ and $\vec{E}_t^{(2)}$ contribute to the magnetic field $\vec{H}$, the magnetic induction field $\vec{B}$, and the dielectric displacement field $\vec{D}$; or, to put it differently, the longitudinal partial field $\vec{E}_l$ only contributes to the total electric field $\vec{E}$.

We will now illustrate our main results by determining the mode expansion for the electric field in a spatially dispersive medium occupying a half-space, assuming again, of course, that the dielectric response function of the medium is given by Eq. (1.1). We will see that the results are in agreement with those obtained not long ago by a different method.[1(a)]

According to Eq. (5.1), the electric field in the medium, assumed now to occupy the half-space $z \geq 0$, may be expressed in the form

$$\vec{E}(\vec{r}, \omega) = \vec{E}_t^{(1)}(\vec{r}, \omega) + \vec{E}_t^{(2)}(\vec{r}, \omega) + \vec{E}_l(\vec{r}, \omega) \tag{6.3}$$

subject to the properties of the three partial fields summarized in the theorem given at the beginning of this section.

Consider the partial fields $\vec{E}_t^{(j)}$ $(j = 1, 2)$. Each of them obeys the Helmholtz equation (4.12). Now it is well known that the general solution in the half-space $z > 0$ with $\vec{k}_t^{(j)}$ being complex (as is the case in the present problem) may be expressed in the form of an angular spectrum of plane waves,[8] i.e.,

$$\vec{E}_t^{(j)}(\vec{r}, \omega) = \iint_{-\infty}^{+\infty} \vec{A}_t^{(j)}(u, v; \omega) \times e^{i(ux + vy + w_j z)} \, du \, dv , \tag{6.4}$$

where

$$w_j = [(\vec{k}_t^{(j)})^2 - u^2 - v^2]^{1/2} \quad (\text{Im} w_j > 0) , \tag{6.5}$$

and $\vec{A}_t^{(j)}$ is an arbitrary vector function of the parameters u and v. In Eq. (6.5), $(\vec{k}_t^{(j)})^2$ is, of course, a root of the transverse dispersion relation (4.8). According to Eq. (4.15), the partial fields $\vec{E}_t^{(j)}$ are transverse. Hence it follows from (6.4) on taking the divergence and formally interchanging the orders of integration and of the "div" operation, that

$$\iint_{-\infty}^{+\infty} \vec{k}_t^{(j)} \cdot \vec{A}_t^{(j)}(u, v; \omega) \times e^{i(ux + vy + w_j z)} \, du dv = 0 , \tag{6.6}$$

where

$$\vec{k}_t^{(j)} \equiv (u, v, w_j) . \tag{6.7}$$

Next, setting $z = 0$ in (6.6) and taking the Fourier inverse, it follows that

$$\vec{k}_t^{(j)} \cdot \vec{A}_t^{(j)}(u, v; \omega) = 0 \tag{6.8}$$

i.e., all the plane waves in the angular spectrum representation (6.4) are *transverse*.

In a strictly similar manner we may also represent the partial field $\vec{E}_l(\vec{r}, \omega)$ as an angular spectrum of plane waves, viz.,

$$\vec{E}_l(\vec{r}, \omega) = \iint_{-\infty}^{+\infty} \vec{A}_l(u, v; \omega) e^{i(ux + vy + w_l z)} \, du \, dv, \tag{6.9}$$

where

$$w_l = (\vec{k}_l^{\,2} - u^2 - v^2)^{1/2} \quad (\text{Im} w_l > 0) , \tag{6.10}$$

$\vec{k}_l^2$ being the root of the longitudinal dispersion relation (4.7) and $\vec{A}_l$ being an arbitrary vector function of the parameters u and v. Moreover, it follows from (6.9) and the longitudinality condition (3.5b), by an argument similar to that leading to (6.8), that

$$\vec{k}_l \times \vec{A}_l(u, v; \omega) = 0 , \tag{6.11}$$

where

$$\vec{k}_l \equiv (u, v, w_l) . \tag{6.12}$$

Equation (6.11) implies that all the plane waves forming the angular spectrum (6.9) are *longitudinal*.

The two transverse fields (6.4) and the longitudinal field (6.9) are coupled by the relation (5.17). This relation involves three surface integrals, defined by Eqs. (5.8) and (5.10), which extend over the closed boundary of the medium. In the present case, when the medium occupies the half space $z = 0$

a portion of the closed boundary is the plane $z=0$. We may take as the rest of the boundary a hemisphere in the half space $z>0$, of limitingly large radius $R\to\infty$, centered at the origin. Now, as noted in footnote 8, the angular spectrum representations (6.4) and (6.9) contain only plane waves that are propagated into the half space $z>0$ and which decay exponentially with increasing values of z. Under these circumstances the contributions from the hemisphere may readily be shown to vanish, so that the three integrals $\vec{S}_t^{(1)}$, $\vec{S}_t^{(2)}$, and $\vec{S}_l$ defined by (5.10) and (5.8) now extend only over the plane $z=0$:

$$\vec{S}_t^{(j)}=\iint_{-\infty}^{+\infty}\left(\vec{E}_t^{(j)}(\vec{r}',\omega)\frac{\partial G_\mu(|\vec{r}-\vec{r}'|)}{\partial z'}-G_\mu(|\vec{r}-\vec{r}'|)\frac{\partial\vec{E}_t^{(j)}(\vec{r}',\omega)}{\partial z'}\right)_{z'=0}dx'\,dy'\ , \tag{6.13}$$

$$\vec{S}_l=\iint_{-\infty}^{+\infty}\left(\vec{E}_l(\vec{r}',\omega)\frac{\partial G_\mu(|\vec{r}-\vec{r}'|)}{\partial z'}-G_\mu(|\vec{r}-\vec{r}'|)\frac{\partial\vec{E}_l(\vec{r}',\omega)}{\partial z'}\right)_{z'=0}dx'\,dy'\ . \tag{6.14}$$

Now from (6.4), we have

$$\vec{E}_t^{(j)}(\vec{r}',\omega)\Big|_{z'=0}=\iint_{-\infty}^{+\infty}\vec{A}_t^{(j)}(u,v,\omega)\times e^{i(ux'+vy')}\,du\,dv\ , \tag{6.14a}$$

$$\frac{\partial\vec{E}_t^{(j)}(\vec{r}',\omega)}{\partial z'}\Big|_{z'=0}=\iint_{-\infty}^{+\infty}(iw_j)\vec{A}_t^{(j)}(u,v,\omega)\times e^{i(ux'+vy')}\,du\,dv\ . \tag{6.14b}$$

Now the Green's function $G_\mu(|\vec{r}-\vec{r}'|)$ that appears in (6.13) and (6.14) admits the angular spectrum representation [Ref. 1(a), Eqs. (3.3) and (3.4)]

$$G_\mu(|\vec{r}-\vec{r}'|)=\frac{i}{2\pi}\iint_{-\infty}^{+\infty}\frac{1}{w_\mu}\times e^{i[u(x-x')+v(y-y')+w_\mu|z-z'|]}\,du\,dv, \tag{6.15}$$

where

$$w_\mu=(\mu^2-u^2-v^2)^{1/2}\ , \tag{6.16a}$$

with

$$\mathrm{Re}\,w_\mu>0\ ,\quad \mathrm{Im}\,w_\mu>0\ . \tag{6.16b}$$

Hence, for points $\vec{r}$ in the half space $z>0$,

$$G_\mu(|\vec{r}-\vec{r}'|)\Big|_{z'=0}=\frac{i}{2\pi}\iint_{-\infty}^{+\infty}\frac{1}{w_\mu}\times e^{i[u(x-x')+v(y-y')+w_\mu|z|]}\,du\,dv, \tag{6.17a}$$

$$\frac{\partial G_\mu(|\vec{r}-\vec{r}'|)}{\partial z'}\Big|_{z'=0}=\frac{1}{2\pi}\iint_{-\infty}^{+\infty}e^{i[u(x-x')+v(y-y')+w_\mu|z|]}\,du\,dv\ . \tag{6.17b}$$

On substituting from (6.14a), (6.14b), (6.17a), and (6.17b) into (6.13) and interchanging the orders of integration we obtain

$$\vec{S}_t^{(j)}=\frac{1}{2\pi}\iint_{-\infty}^{+\infty}du\,dv\iint_{-\infty}^{+\infty}du'\,dv'\,\vec{A}_t^{(j)}(u',v',w)\times\left(1+\frac{w_j(u',v')}{w_\mu(u,v)}\right)e^{i(ux+vy+w_\mu z)}\times\iint_{-\infty}^{+\infty}dx'dy'\,e^{i[x'(u-u')+y'(v-v')]}\ . \tag{6.18}$$

The last integral has the value $(2\pi)^2\delta(u-u')\delta(v-v')$, where δ is the Dirac δ function. Hence (6.18) simplifies to

$$\vec{S}_t^{(j)}=2\pi\iint_{-\infty}^{+\infty}\vec{A}_t^{(j)}(u,v,w)\times e^{i(ux+vy+w_\mu z)}\left(1+\frac{w_j}{w_\mu}\right)du\,dv\ . \tag{6.19}$$

In a strictly similar way we find that

$$\vec{S}_l=2\pi\iint_{-\infty}^{+\infty}\vec{A}_l(u,v,w)\times\left(1+\frac{w_l}{w_\mu}\right)e^{i(ux+vy+w_\mu z)}\,du\,dv\ . \tag{6.20}$$

Next we substitute from (6.19) and (6.20) into the relation (5.17) and find that

$$\iint_{-\infty}^{+\infty}\vec{Q}(u,v)e^{i(ux+vy+w_\mu z)}\,du\,dv=0\ , \tag{6.21}$$

where

$$\vec{Q}(u,v)=\sum_{j=1}^{2}\frac{1+w_j/w_\mu}{(\vec{k}_t^{(j)})^2-\mu^2}\vec{A}_t^{(j)}+\frac{1+w_l/w_\mu}{\vec{k}_l^{\,2}-\mu^2}\vec{A}_l\ . \tag{6.22}$$

If we let $z\to0$ in (6.21) and take then the Fourier inverse we find at once that $\vec{Q}(u,v)=0$, i.e.,

$$\sum_{j=1}^{2} \frac{1+w_j/w_\mu}{(\vec{k}_t^{(j)})^2-\mu^2}\vec{A}_t^{(j)}+\frac{1+w_l/w_\mu}{\vec{k}_l^2-\mu^2}\vec{A}_l=0\ , \quad (6.23)$$

for all values of the pair of parameters u and v. Now from Eq. (6.5) and (6.16a) it follows that

$$(\vec{k}_t^{(1)})^2-\mu^2=w_j^2-w_\mu^2\ , \quad (6.24)$$

and from (6.10) and (6.16a) it follows that

$$\vec{k}_l^2-\mu^2=w_l^2-w_\mu^2\ . \quad (6.25)$$

With the help of Eqs. (6.24) and (6.25), Eq. (6.23) is readily simplified and we then obtain the relation

$$\sum_{j=1}^{2}\frac{\vec{A}_t^{(j)}(u,v;\omega)}{w_j-w_\mu}+\frac{\vec{A}_l(u,v;\omega)}{w_l-w_\mu}=0\ . \quad (6.26)$$

Thus we conclude that the general mode expansion for the electric field in the spatially dispersive medium occupying the half space $z>0$ is given by the sum of the three partial fields $\vec{E}_t^{(1)}$, $\vec{E}_t^{(2)}$, and $\vec{E}_l$, where $\vec{E}_t^{(1)}$ and $\vec{E}_t^{(2)}$ are superpositions of transverse plane waves [Eq. (6.4) and (6.8)], and $\vec{E}_l$ is a superposition of longitudinal plane waves [Eq. (6.9) and (6.11)]. The wave vectors of the transverse waves obey the transverse dispersion relation (4.8) and of the longitudinal waves obey the longitudinal dispersion relation (4.7). The amplitudes $\vec{A}_t^{(1)}(u,v;\omega)$, $\vec{A}_t^{(2)}(u,v;\omega)$, and $\vec{A}_l(u,v;\omega)$ of the three plane waves forming a typical $(u,v;\omega)$ mode are coupled by the linear relation (6.26). This mode representation is identical with the representation derived for this problem in Ref. 1(a) by a different method.

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APPENDIX A: ALTERNATIVE PROCEDURE

Some authors take Eq. (2.10) rather than our Eq. (2.6b) as expressing the nonlocal polarization in terms of the electric field. If we adopt this alternative standpoint, we have, in place of (2.6b),

$$\vec{P}_{NL}(\vec{r},\omega)=\frac{\chi}{(4\pi)^2}\int_V \tilde{G}_\mu(\vec{r},\vec{r}\,')\vec{E}(\vec{r}\,',\omega)d^3r' -\frac{1}{4\pi}\int_S\left(\vec{P}_{NL}(\vec{r}\,',\omega)\frac{\partial\tilde{G}_\mu(\vec{r},\vec{r}\,')}{\partial n} -\tilde{G}_\mu(\vec{r},\vec{r}\,')\frac{\partial\vec{P}_{NL}(\vec{r}\,',\omega)}{\partial n}\right)dS\ , \quad (A1)$$

where

$$(\nabla^2+\mu^2)\tilde{G}_\mu(\vec{r},\vec{r}\,')=-4\pi\delta(\vec{r}-\vec{r}\,') \quad (A2)$$

and $\partial/\partial n$ denotes differentiation along the outward normal to the surface S bounding the volume V.

Of course, neither Eq. (2.10) nor Eq. (A1) specify $\vec{P}_{NL}$ uniquely in terms of $\vec{E}$, without the knowledge of boundary conditions. Suppose that one assumes that the nonlocal polarization satisfies linear homogeneous boundary conditions, i.e., one assumes that for each point $\vec{r}$ of the boundary surface S

$$\vec{P}_{NL}(\vec{r},\omega)+\Lambda(\omega)\frac{\partial\vec{P}_{NL}(\vec{r},\omega)}{\partial n}=0\ , \quad (A3)$$

where $\Lambda(\omega)$ is some function of ω. If we choose as the Green's function $\tilde{G}_\mu$ in (A1) that solution of (A2) which satisfies the same boundary conditions as $\vec{P}_{NL}$, i.e.,

$$\tilde{G}_\mu(\vec{r},\omega)+\Lambda(\omega)\frac{\partial\tilde{G}_\mu(\vec{r},\omega)}{\partial n}=0 \quad (A4)$$

at each point $\vec{r}$ of S, Eq. (A1) may readily be shown to reduce to

$$\vec{P}_{NL}(\vec{r},\omega)=\frac{\chi}{(4\pi)^2}\int_V \tilde{G}_\mu(\vec{r},\vec{r}\,')\times\vec{E}(\vec{r}\,',\omega)d^3r'\ . \quad (A5)$$

Equation (A5) is seen to be of the same form as our Eq. (2.6b), the only difference being that the Green's function $G_\mu(|\vec{r}-\vec{r}\,'|)$ has been replaced by the Green's function $\tilde{G}_\mu(\vec{r},\vec{r}\,')$.

It may readily be verified that if Eq. (2.10) together with Eq. (A3) were employed in place of our Eq. (2.6b), all results of the present paper (except those derived in connection with a specific problem treated in Sec. VI) remain valid, provided that G_μ is replaced everywhere by $\tilde{G}_\mu$. However, it should be borne in mind that it is not possible to determine on the basis of the macroscopic Maxwell theory whether or not the choice of boundary conditions of the form (A3) is appropriate and if it is, what is the value of the parameter $\Lambda(\omega)$.

APPENDIX B: PROOF THAT EQ. (4.6) IMPLIES THAT $(\vec{k}_t^{(j)})^2$ ARE ROOTS OF THE TRANSVERSE DISPERSION RELATION (4.8)

Equation (4.6) may be rewritten

$$\nabla^4+[(\vec{k}_t^{(1)})^2+(\vec{k}_t^{(2)})^2]\nabla^2+(\vec{k}_t^{(1)})^2(\vec{k}_t^{(2)})^2 =\nabla^4+(\epsilon_0k_0^2+\mu^2)\nabla^2+k_0^2(\epsilon_0\mu^2-\chi)\ . \quad (B1)$$

Clearly (B1) can only be true if

$$(\vec{k}_t^{(1)})^2+(\vec{k}_t^{(2)})^2=\epsilon_0k_0^2+\mu^2 \quad (B2)$$

and

$$(\vec{k}_t^{(1)})^2(\vec{k}_t^{(2)})^2=k_0^2(\epsilon_0\mu^2-\chi)\ . \quad (B3)$$

Elimination of either $(\vec{k}_t^{(1)})^2$ or $(\vec{k}_t^{(2)})^2$ between equations (B2) and (B3) gives

$$(\vec{k}_t^{(j)})^4 - (\epsilon_0 k_0^2 + \mu^2)(\vec{k}_t^{(j)})^2 + k_0^2(\epsilon_0\mu^2 - \chi) = 0 \quad (j=1,2). \quad \text{(B4)}$$

Equation (B4) may be rewritten in the form

$$(\vec{k}_t^{(j)})^2[(\vec{k}_t^{(j)})^2 - \mu^2] = \epsilon_0 k_0^2[(\vec{k}_t^{(j)})^2 - \mu^2] + \chi k_0^2 ,$$

i.e.,

$$\left(\frac{\vec{k}_t^{(j)}}{k_0}\right)^2 = \left(\epsilon_0 + \frac{\chi}{(\vec{k}_t^{(j)})^2 - \mu^2}\right) . \quad \text{(B5)}$$

But according to Eq. (1.1), the right-hand side of (B5) is precisely $\epsilon(\vec{k}_t^{(j)}, \omega)$. Hence

$$(\vec{k}_t^{(j)}/k_0)^2 = \epsilon(\vec{k}_t^{(j)}, \omega) \quad (j=1,2) , \quad \text{(B6)}$$

showing that $(\vec{k}_t^{(j)})^2$ are roots of the transverse dispersion relation (4.8).

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[5]In the microscopic derivation of Eq. (1.1) the medium is assumed to fill the whole space, so that the integral (2.6b), defining the nonlocal polarization, also extends over the whole space. The formula (2.6b) applied to a bounded domain thus involved an approximation. For the discussion of this point see footnote 18 of Ref. 1(a) and the articles referred to therein. The approximation implicit in the use of Eq. (2.6b) for a bounded domain is the only approximation made throughout the present paper. An alternative procedure, which does not involve the use of this particular approximation, is indicated in Appendix A.

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[7]The mode expansion for the electromagnetic field inside a spatially dispersive sphere and scattering from such a sphere has recently been discussed on the basis of the present theory by J. T. Foley and D. N. Pattanayak, Opt. Commun. 12, 113 (1974).

[8]This representation may be derived, for example, by the same argument as given in the Appendix of a paper by E. Wolf, Proc. Phys. Soc. Lond. 74, 269 (1959), in connection with the special case when the wave number is real; however, in the present case we discard the wave which would be propagated into the half space $z < 0$, since their amplitude diverges as $z \rightarrow \infty$.