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Optical bistability: A self-consistent analysis of fluctuations and the spectrum of scattered light

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The main purpose of this paper is to study the behavior of the atomic fluctuations and the spectrum of the light transmitted by an absorptive bistable device. To this end we develop an approximation scheme based on the so-called system-size expansion and apply it to the quantum-mechanical Langevin equations for the atomic fluctuation operators. The Bonifacio-Lugiato mean-field equations for bistability are derived from the lowest-order approximation to the system-size expansion, while the atomic correlation functions result from the next-higher-order expansion. The calculated spectrum of the transmitted light exhibits line narrowing near the bistable thresholds, discontinuous formation of sidebands along the high-transmission branch of the device, and hysteresis effects.

I. INTRODUCTION

The recent observation of optical bistability[1,2] has stimulated considerable interest in the experimental[3-7] and theoretical[8-12] implications of the effect. In part, the present excitement is owing to the possibility that bistable optical devices may eventually find applications as miniaturized memory units, amplifiers with high differential gain, and pulse shapers, limiters, and clippers.[1] In addition, a bistable system, such as a Fabry-Perot cavity containing an absorbing medium, is of theoretical interest as an example of an open system,[13] driven to a stationary nonequilibrium state by an external source. It is known that, when an open system is driven sufficiently far from thermodynamic equilibrium, its state may "jump" from the thermodynamic branch to some other steady-state configuration. These new steady states have been called "dissipative structures."[14]

In the case of optical bistability, discontinuous transitions can be induced from the so-called cooperative branch (thermodynamic branch) to the single-atom branch (dissipative structure) and vice versa by varying the strength of the external driving field.[9]

This behavior, which is the consequence of the competition between cooperative and uncorrelated atomic motion in the absorbing medium, bears a close resemblance to ordinary first-order phase transitions in thermodynamic-equilibrium systems.[10] The similarity is far from accidental; indeed, the existence of a 1-1 mapping has been established between the equilibrium and nonequilibrium equations of state for systems that exhibit absorptive bistability. In fact, both equilibrium configuration and nonequilibrium steady states have been identified with the cusp catastrophe manifold.[15]

The first detailed study of absorptive bistability was proposed by Bonifacio and Lugiato[9] in terms of a simple mean-field quantum-mechanical model. In the semiclassical approximation, the model is exactly soluble and leads to a precise analytic description of the bistable operation, threshold conditions, and hysteresis properties. It also provides a unified description of single-mode superfluorescence,[16] bistability, and resonance fluorescence[17-19] indicating how the interplay of collective and single-atom dynamics can give rise to quite different transient and steady-state behaviors.[20] In addition, new and striking predictions have emerged concerning the relaxation properties of the atomic system and the behavior of the spectrum of the transmitted and of the fluorescence light.

The predictions concerning the fluorescence spectrum are especially interesting because they stand in sharp contrast with the well-known behavior of the single-atom resonance-fluorescence spectrum: in the case of ordinary resonance fluorescence[17] the scattered spectrum consists of a single broadened line when the driving field strength is below a certain threshold value; above threshold, the spectrum develops continuously a symmetric pair of sidebands which are displaced from the central component by an amount proportional to the driving Rabi frequency.[21,22]

On the contrary, in the case of a bistable system, Bonifacio and Lugiato have predicted line narrowing in the vicinity of the bistability

threshold, the sudden appearance of sidebands above threshold and hysteresis properties, as the strength of the driving mechanism is changed quasistatically in an appropriate way.

The semiclassical description of the steady-state condition of the atoms, coupled to a reasonable regression hypothesis, is sufficient to describe the relaxation properties of the system. If, on the other hand, one is interested in the shape of the spectrum of the fluctuations, the behavior of appropriate atomic correlation functions must be calculated explicitly.

In this paper we address ourselves to this problem with the main goal of deriving the shape of the spectrum of the transmitted light for bistable optical systems.

Our approach stems from the following considerations: bistable action results from the competition between the external coherent pump, the collective-atomic-decay process into the resonant-cavity mode, and the incoherent atomic relaxation owing to spontaneous emission.

Each of these three processes is modeled by a distinct contribution to a master equation[16,18] that describes the evolution of the reduced atomic-density operator. The master equation is constructed in the Born and Markov approximations, and upon adiabatic elimination of the cavity-field operators. Our calculations are appropriate if the cavity field relaxes over a time scale which is short compared to both the collective and incoherent atomic relaxation times. If the field relaxation time should exceed the atomic-decay times, a different formulation is required as pointed out by Bonifacio and Lugiato[23] and, independently, by Willis.[24]

In Sec. II of this paper, we discuss the atomic master equation and review the known steady-state solutions for the atomic and field expectation values.

In Sec. III we develop an approximation method based on the system-size expansion[25-29] and apply it to the quantum-mechanical Langevin equations. We calculate the atomic-diffusion coefficients from the generalized Einstein relations and construct a set of coupled equations for the atomic correlation functions of interest.

The mean-field equations of Bonifacio and Lugiato originate from the lowest-order term of the system-size expansion, while the correlation functions result from the next-higher-order correction. These equations are shown to agree with those derived in Ref. 11 using the regression theorem and a Gaussian decoupling approximation.

In Sec. IV, we derive the spectrum of the transmitted light and analyze the behavior of its three components as the system is driven across the bistability thresholds.

The spectral narrowing exhibited near threshold is shown to be the consequence of the onset of a soft mode.[9] The mathematical origin of the soft mode is reviewed in Appendix A, where we analyze the eigenvalues of the relaxation matrix that governs the system's approach to steady state from a slightly perturbed configuration. We also show explicitly the effect of the soft mode and the accompanying critical slowing down on the approach to steady state of the atomic variables.

In Appendix B, we present calculations to show that the system-size expansion is consistent with the positive definiteness of the density matrix. Appendix C is devoted to the calculations based on the Gaussian decoupling schemes. Finally, Appendix D deals with a model describing the resonance fluorescence from a collective atomic system, and discussed the relation of this model with the one presented in the main text of the present paper.

II. ATOMIC MASTER EQUATION

The interaction of an external source of radiation with an absorbing medium causes attenuation of the incident light and excitation of some of the atoms: the atoms, in turn, decay spontaneously and isotropically.

If the absorbing medium is placed inside a resonant cavity, one of the cavity modes may be made to assume a privileged role. Under appropriate conditions strong collective emission into one of the cavity modes will occur.

In describing the atomic evolution under the action of an external driving field, we consider explicitly the following physical processes: (a) the interaction of the atoms with the external field; (b) the incoherent decay of each atom by spontaneous emission; (c) the cooperative atomic interaction and the collective emission into a single mode of the cavity.

In our model, we assume perfect resonance between the driving field, one of the empty cavity modes, and the atomic absorption line. This idealized physical system has been referred to as an absorptive bistable device.[9] The partially reflecting end mirrors of the cavity cause leakage of the internal field to the outside at a rate given by $\kappa = c(1-R)/L$, where R is the reflectivity of the mirrors and L the cavity length. We assume the field damping rate κ to be much larger than the collective and single-atom relaxation rates. This enables us to eliminate the internal-field operators adiabatically.[16]

In the Born and Markov approximations, and in

the unitary frame rotating at the angular frequency of the driving field, the master equation for the reduced atomic-density operator becomes

$$\frac{dW}{dt} = -i\Omega_I[S^+ + S^-, W] + \Lambda_S W + \Lambda_A W . \tag{2.1}$$

The first term on the right-hand side of Eq. (2.1) which is proportional to the Rabi frequency Ω_I of the incident field describes the reversible coherent interaction of the atoms with the external classical field. The other two terms describe the collective and single-atom irreversible decay, respectively. The collective decay term is given by

$$\Lambda_S W = (2g^2/\kappa)(S^- W S^+ - \tfrac{1}{2} W S^+ S^- - \tfrac{1}{2} S^+ S^- W), \tag{2.2}$$

where S^+ and S^- are the collective atomic-polarization operators, g is the atom-field coupling constant, and κ is the inverse photon lifetime in the cavity. The single-atom decay term is

$$\Lambda_A W = \sum_{i=1}^{N} \gamma_\perp([s_i^-, W s_i^+] + [s_i^- W, s_i^+]) , \tag{2.3}$$

where s_i^+ and s_i^- are the individual atomic-polarization operators and $\gamma_\perp$ is the coherent relaxation rate (for simplicity we take the longitudinal relaxation rate $\gamma_\parallel$ to be twice the transverse relaxation rate $\gamma_\perp$).

For the purpose of carrying out an accurate analysis of the approach to steady state and of the atomic fluctuations, it would be desirable to map Eq. (2.1) into a c-number differential equation for a quasiprobability distribution. This procedure has been adopted successfully in the past.[30] Here the simultaneous presence of collective and single-atom operators in the master equation makes the mapping process less straightforward, especially if one wishes to use the coherent-atomic-state representation.[31] Attempts along this line are in progress.[32]

As a first cut to the problem we consider instead the equations of motion for the atomic expectation values. The derivation of these equations from Eq. (2.1) is straightforward. The result is

$$\begin{aligned}
\frac{d}{dt}\langle S^+\rangle &= -\gamma_\perp\langle S^+\rangle - 2i\Omega_I\langle S_3\rangle + \frac{2g^2}{\kappa}\langle S^+ S_3\rangle , \\
\frac{d}{dt}\langle S_3\rangle &= -2\gamma_\perp\left(\langle S_3\rangle + \frac{N}{2}\right) - i\Omega_I\langle S^+\rangle + i\Omega_I\langle S^-\rangle , \\
&\quad - (2g^2/\kappa)\langle S^+ S^-\rangle , \\
\langle S^-\rangle &= \langle S^+\rangle^* .
\end{aligned} \tag{2.4}$$

Of course, Eqs. (2.4) do not form a closed set: they can be closed by various decorrelation schemes, one of the most frequently used being the straight factorization ansatz (e.g., $\langle S^+ S_3\rangle = \langle S^+\rangle\langle S_3\rangle$). The resulting nonlinear equations have been discussed at length in Ref. 9. Here, we limit ourselves to a brief summary of the conclusion.

In view of the initial conditions $\langle S^+(0)\rangle = \langle S^-(0)\rangle = 0$, and $\langle S_3(0)\rangle = -\frac{1}{2}N$ it is clear that $\langle S^+(t)\rangle$, $\langle S^-(t)\rangle$ are purely imaginary at all times. Thus, Eqs. (2.4) are equivalent to

$$\begin{aligned}
\frac{dS}{dt} &= -S - \sqrt{2}\, y S_3 + \frac{4c}{N} S S_3 , \\
\frac{dS_3}{dt} &= -2\left(S_3 + \frac{N}{2}\right) + \sqrt{2}\, y S - \frac{4c}{N} S^2 ,
\end{aligned} \tag{2.5}$$

where $S = -i\langle S^+\rangle = i\langle S^-\rangle$ and $S_3 = \langle S_3\rangle$. The dimensionless time τ is defined as $\gamma_\perp t$; the parameter $y = \sqrt{2}\,\Omega_I/\gamma_\perp$ is proportional to the driving field amplitude and $c = g^2 N/2\kappa\gamma_\perp$ provides a measure of the atomic density in the cavity. The internal field is adiabatically linked to the atomic polarization as follows:

$$a \equiv \langle a\rangle = -(g/\kappa)S . \tag{2.6}$$

The steady-state properties of the bistable system can be summarized in terms of the long-time expectation values

$$\begin{aligned}
S_3(\infty) &= -\tfrac{1}{2}N[1/(1+x^2)] , \\
S(\infty) &= (N/\sqrt{2})[x/(1+x^2)] ,
\end{aligned} \tag{2.7}$$

where $x = \sqrt{2}\,(\Omega_I + ga)/\gamma_\perp$ is proportional to the transmitted field amplitude, and of the cubic equation

$$y = x + 2cx/(1+x^2) \tag{2.8}$$

which relates the transmitted field x to the incident field y, and the atomic density c.

It is apparent from Fig. 1, that x is a single-valued function of y for $c < 4$, and that instead, for $c > 4$ and $y_{\min} < y < y_{\max}$, the transmitted field amplitude is a multivalued function of y. The steady-state values x_1 and x_3 are stable solutions, while x_2 is unstable and cannot be realized.

When $c > 4$, the steady-state properties of the atomic system can be summarized as follows.

(a) Along the low transmission branch of the state equation $y = y(x)$, the atomic expectation values are not extensive functions of N. This is a direct manifestation of collective behavior.

(b) Along the high transmission branch, and for large values of the driving field y, the atomic expectation values are directly proportional to the number of atoms in the cavity (single-atom behavior dominates).

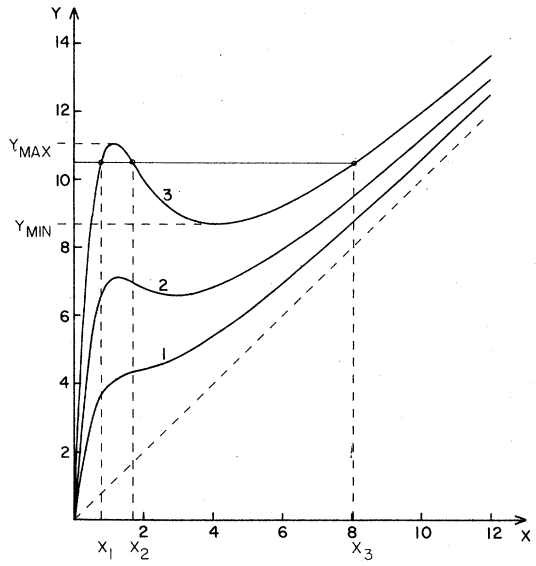

FIG. 1. Set of state equations $y = y(x)$ for (1) $c = 3$, (2) $c = 6$, (3) $c = 10$. The steady-state values of x corresponding to those segments of the state equation characterized by negative slope ($dy/dx < 0$) are unstable against perturbations. For the interesting case $c \gg 1$ it has become customary to refer to the low and high transmission branches as the cooperative and single-atom branches, respectively.

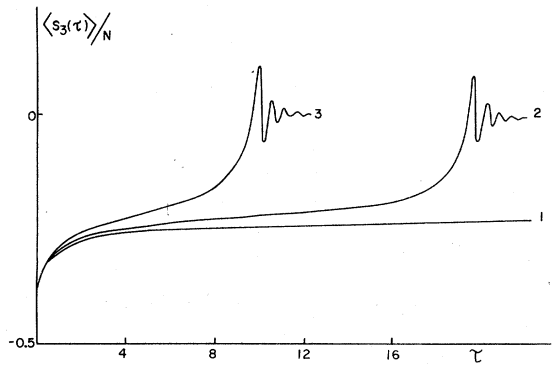

FIG. 2. Transient behavior of the atomic population $\langle S_3(\tau)\rangle/N$ for different values of the driving field amplitude y. In curve (1), y is just below threshold for bistable switching and the approach to steady states is monotonic. In curve (2), y is just above the switching threshold; the atomic population approaches a new steady-state value in an oscillating fashion. In curve (3), y is somewhat higher than the threshold value; the approach to steady state is qualitatively the same as in case (2) but the evolution occurs over a significantly shorter time scale.

(c) As y is increased beyond y_{max} from below, or decreased past y_{min} from above, the transmitted field undergoes a sharp discontinuity. Correspondingly, the atomic expectation values undergo a discontinuous transition.

(d) The hysteresis of the transmitted field and of the atomic expectation values is a consequence of the so-called delay convention.[33] In an exact description, we guess that the two possible steady-state values of x for a fixed y in the range $y_{min} < y < y_{max}$ should correspond to a pair of local maxima of the atomic quasiprobability distribution. In the mean-field-theory approximation, the gradual transition from one configuration to the other is replaced by a discontinuous jump at the bistability thresholds y_{max} and y_{min}.

An interesting visualization of the drastic difference between the approach to equilibrium along the cooperative and the single-atom branches is provided by the numerical integration of Eqs. (2.5). In Fig. 2 we show the approach to steady state of the atomic population difference for different values of the driving fields. Just below the upper threshold (curve 1), the population difference approaches steady state monotonically, while, just above threshold (curve 2), the approach to steady state is oscillatory. Furthermore, the steady-state values corresponding to curves 1 and 2 are very different, with almost complete saturation being predicted in the latter case.

An interesting feature of this solution is the length of time required to reach steady state, approximately 20 atomic lifetimes for the chosen value of y in curve 1. Upon increasing y further up above threshold, no additional qualitative changes are observed in the approach to steady state except for a considerable reduction of the overall time scale.

III. ATOMIC FLUCTUATIONS AROUND THE MEAN

In this section, we analyze the fluctuation and relaxation properties of the atomic system in the neighborhood of the steady state. Our approach is based on the system-size expansion as developed by Kubo, Van Kampen, and others in the context of the classical theory of stochastic processess.[25-29] In this problem, special care must be exercised with the handling of the quantum aspects, because quantum correlation effects contribute correction terms of the same order of magnitude ($1/N$, where N is the number of atoms) as the ordinary fluctuations around the mean.

In what follows we apply the method of the system-size expansion to the quantum-mechanical Langevin equations.[34] We define the scaled operators $x^{(\pm)} = (1/N)S^{(\pm)}$, $x^{(3)} = (1/N)S_3$. From the master equation (2.1) one can construct the following Langevin equations:

$$\frac{dx^{(+)}}{d\tau} = -x^{(+)} + 4cx^{(+)}x^{(3)} - \sqrt{2}iyx^{(3)} + \frac{1}{\sqrt{N}}F^{(+)},$$

$$\frac{dx^{(-)}}{d\tau} = -x^{(-)} + 4cx^{(3)}x^{(-)} + \sqrt{2}iyx^{(3)} + \frac{1}{\sqrt{N}}F^{(-)},$$

$$\frac{dx^{(3)}}{d\tau} = -2\left(x^{(3)} + \frac{1}{2}\right) - 4cx^{(+)}x^{(-)} \qquad (3.1)$$

$$-(iy/\sqrt{2})(x^{(+)} - x^{(-)}) + \frac{1}{\sqrt{N}}F^{(3)},$$

where $F^{(\pm)}$ and $F^{(3)}$ are δ-correlated random operators. Their stochastic properties can be summarized as follows:

$$\langle F^{(\pm)}(t)\rangle = \langle F^{(3)}(t)\rangle = 0, \qquad (3.2)$$

$$\begin{aligned}
&\langle F^{(+)}(t)F^{(-)}(t')\rangle = 2\langle D_{+-}(t)\rangle\delta(t-t'),\\
&\langle F^{(3)}(t)F^{(-)}(t')\rangle = 2\langle D_{3-}(t)\rangle\delta(t-t'),\\
&\langle F^{(+)}(t)F^{(3)}(t')\rangle = 2\langle D_{3-}(t)\rangle\delta(t-t'),\\
&\langle F^{(-)}(t)F^{(-)}(t')\rangle = 2\langle D_{--}(t)\rangle\delta(t-t').
\end{aligned} \qquad (3.3)$$

Clearly, $F^{(\pm)}$ and $F^{(3)}$ do not commute with one another; for example, we have

$$\langle F^{(-)}(t)F^{(+)}(t')\rangle = 2\langle D_{-+}(t)\rangle\delta(t-t'), \qquad (3.4)$$

where

$$\langle D_{-+}(t)\rangle \neq \langle D_{+-}(t)\rangle. \qquad (3.5)$$

In general, the random operators are not Gaussian in nature. However, since the behavior of the system is Markovian, it follows that

$$\langle A(0)F^{\pm}(t)B(0)\rangle = 0, \quad t>0.$$

The values of the diffusion constants are obtained from the generalized Einstein relation[34]

$$2\langle D_{MN}(t)\rangle = \frac{d}{dt}\langle M(t)N(t)\rangle - \langle A_M(t)N(t)\rangle - \langle M(t)A_N(t)\rangle, \qquad (3.6)$$

where A_M and A_N are the Langevin forces

$$\dot{M} = A_M + F_M \qquad (3.7)$$

and M is one of the system operators.

From the master equation (2.1) and from Eq. (3.6) one arrives at the following diffusion coefficients:

$$\begin{aligned}
&\langle D_{++}(t)\rangle = \langle D_{--}(t)\rangle^* = \langle D_{+3}(t)\rangle = \langle D_{3-}(t)\rangle\\
&\quad = \langle D_{+-}\rangle = 0,\\
&\langle D_{33}(t)\rangle = \langle \tfrac{1}{2} + x^{(3)}(t)\rangle + 2c\langle x^{(+)}(t)x^{(-)}(t)\rangle.
\end{aligned} \qquad (3.8a)$$

We note that if one adopts the normal ordering convention, i.e.,

$$\langle :F_i(t)F_j(t'):\rangle = 2\langle \mathfrak{D}_{ij}(t)\rangle\delta(t-t'), \qquad (3.8b)$$

one finds

$$\begin{aligned}
&\langle \mathfrak{D}_{++}(t)\rangle = \langle \mathfrak{D}_{--}(t)\rangle^* = 2c\langle x^{(+)}(t)x^{(-)}(t)\rangle\\
&\qquad - (iy/\sqrt{2})\langle x^{(+)}(t)\rangle\\
&\langle \mathfrak{D}_{33}(t)\rangle = \langle \tfrac{1}{2} + x^{(3)}(t)\rangle - 2c\langle x^{(+)}(t)x^{(-)}(t)\rangle\\
&\qquad - (iy/2\sqrt{2})\langle x^{(+)}(t) - x^{(-)}(t)\rangle,\\
&\langle \mathfrak{D}_{+3}(t)\rangle = \langle \mathfrak{D}_{3-}(t)\rangle = \langle \mathfrak{D}_{+-}(t)\rangle = 0.
\end{aligned} \qquad (3.8c)$$

The normal ordering convention is not really needed, except when calculating the equations for the normally ordered diffusion matrix [see Eq. (3.19) below]. It should also be noted that no approximations have been carried out thus far (except of course those which are required in the derivation of the master equation).

It is also important to observe that the fluctuating-operator forces are weighted by the parameter $1/\sqrt{N}$ and that, as a consequence, the relative fluctuations of the observable quantities are of order $1/N$. This fact enables us to carry out the above-mentioned expansion in powers of $1/\sqrt{N}$.

To this purpose, we introduce the fluctuation operators $y^{(\pm)}$ and $y^{(3)}$ defined by

$$x^{(\pm)} = x_0^{(\pm)} + (1/\sqrt{N})y^{(\pm)}, \quad x^{(3)} = x_0^{(3)} + (1/\sqrt{N})y^{(3)}, \qquad (3.9)$$

where $x_0^{(\pm)}$ and $x_0^{(3)}$ are c numbers, and expand $F^{(\pm)}$ and $F^{(3)}$ in inverse powers of $\sqrt{N}$. To lowest order in $N^{-1/2}$ the Langevin equations (3.1) take the form

$$\begin{aligned}
\dot{x}_0^{(+)} &= -x_0^{(+)} + 4cx_0^{(+)}x_0^{(3)} - \sqrt{2}iyx_0^{(3)},\\
\dot{x}_0^{(-)} &= -x_0^{(-)} + 4cx_0^{(3)}x_0^{(-)} + \sqrt{2}iyx_0^{(3)},\\
\dot{x}_0^{(3)} &= -2\left(\frac{1}{2} + x_0^{(3)}\right) - 4cx_0^{(+)}x_0^{(-)} - \frac{iy}{\sqrt{2}}(x_0^{(+)} - x_0^{(-)}).
\end{aligned} \qquad (3.10)$$

The first-order correction to the Langevin equations (3.1) can be cast in the compact form

$$\dot{y}_0 = M(t)y_0 + F_0(t), \quad y_0 = \begin{pmatrix} y_0^{(+)}\\ y_0^{(-)}\\ y_0^{(3)}\end{pmatrix}, \quad F_0 = \begin{pmatrix} F_0^{(+)}\\ F_0^{(-)}\\ F_0^{(3)}\end{pmatrix},$$

$$M(t) = \begin{pmatrix} -1 + 4cx_0^{(3)}(t) & 0 & 4cx_0^{(+)} - \sqrt{2}iy\\ 0 & -1 + 4cx_0^{(3)}(t) & 4cx_0^{(-)} + \sqrt{2}iy\\ -iy/\sqrt{2} - 4cx_0^{(-)}(t) & iy/\sqrt{2} - 4cx_0^{(+)}(t) & -2\end{pmatrix}, \qquad (3.11)$$

where $y_0^{(\pm)}$ is the lowest-order term in $y^{(\pm)}$. The new random operators $F_0^{(\pm)}$ and $F_0^{(3)}$ are again δ correlated and have zero expectation values. The diffusion constants are given by relations of the type (3.3) with $\mathfrak{D}$ replaced by $\mathfrak{D}^{(0)}$, and

$$\begin{aligned}
&\langle \mathfrak{D}_{++}^{(0)}(t)\rangle = 2c(x_0^{(+)}(t))^2 - (iy/\sqrt{2})x_0^{(+)}(t) = \langle \mathfrak{D}_{--}^{(0)}(t)\rangle^*,\\
&\langle \mathfrak{D}_{33}^{(0)}(t)\rangle = (\tfrac{1}{2} + x_0^{(3)}(t)) - 2cx_0^{(+)}(t)x_0^{(-)}(t)\\
&\qquad - (iy/2\sqrt{2})[x_0^{(+)}(t) - x_0^{(-)}(t)],\\
&\langle \mathfrak{D}_{+3}^{(0)}(t)\rangle = \langle \mathfrak{D}_{3-}^{(0)}(t)\rangle = \langle \mathfrak{D}_{+-}^{(0)}(t)\rangle = 0,
\end{aligned} \tag{3.12}$$

and similar equations for $\langle \mathfrak{D}_{ij}(t)\rangle$. It is also clear from the way the system-size expansion has been carried out that $\langle x^{(\pm)}\rangle = x_0^{(\pm)}$ + (terms of order $1/N$).

We note here several features of the above quantum-mechanical Langevin equations.

(a) Equations (3.11) imply that $\langle y_0^{(\pm)}(t)\rangle = \langle y_0^{(3)}(t)\rangle = 0$ if, at time $t=0$, $\langle y_0^{(\pm)}\rangle = \langle y_0^{(3)}\rangle = 0$.

Hence, in this case the expectation values of the operators $x^{(\pm)}$ and $x^{(3)}$ will be given by

$$\langle x^{(\pm)}(t)\rangle = x_0^{(\pm)}(t), \quad \langle x^{(3)}(t)\rangle = x_0^{(3)}(t), \tag{3.13}$$

where $x_0^{(\pm)}$ and $x_0^{(3)}$ are solutions of the system of Eq. (3.10). The identity of Eqs. (3.10) and of the mean-field equation (2.5) provides a justification of the system-size expansion.

(b) From Eq. (3.9) it is clear that

$$\langle x^{(\alpha)}(t)x^{(\beta)}(t')\rangle = x_0^{(\alpha)}(t)x_0^{(\beta)}(t') + (1/N)\langle y_0^{(\alpha)}(t)y_0^{(\beta)}(t')\rangle .$$

or

$$\langle [x^{(\alpha)}(t) - x_0^{(\alpha)}(t)][x^{(\beta)}(t') - x_0^{(\beta)}(t')]\rangle = (1/N)\langle y_0^{(\alpha)}(t)y_0^{(\beta)}(t')\rangle .$$

Hence the fluctuations around the mean are determined by the correlation between the operators y_0. In view of the Markovian character of our system, we can derive immediately from Eqs. (3.1) the following equations for the correlation function:

$$\frac{d}{dt}C(t,t') = M(t)C(t,t'), \quad t>t'$$

$$C(t,t') = \begin{pmatrix} \langle y_0^{(+)}(t)y_0^{(-)}(t')\rangle \\ \langle y_0^{(-)}(t)y_0^{(-)}(t')\rangle \\ \langle y_0^{(3)}(t)y_0^{(-)}(t')\rangle \end{pmatrix}. \tag{3.14}$$

(c) The matrix $M(t)$ in Eq. (3.14) is time dependent for any initial-value problem except in steady state, where Eq. (3.14) reduces to

$$\frac{d}{dt}C(t) = MC(t), \quad C(t) = \lim C(t+t',t'),$$

$$M = \begin{pmatrix} -1+4cx_0^{(3)} & 0 & 4cx_0^{(+)} - \sqrt{2}iy \\ 0 & -1+4cx_0^{(3)} & 4cx_0^{(-)} + \sqrt{2}iy \\ -(i/\sqrt{2})y - 4cx_0^{(-)} & iy/\sqrt{2} - 4cx_0^{(+)} & -2 \end{pmatrix}, \tag{3.15}$$

where $x_0^{(\pm)}$ and $x_0^{(3)}$ are the steady-state solutions of Eqs. (3.10):

$$x_0^{(\pm)} = \pm(i/\sqrt{2})[x/(1+x^2)], \quad x_0^{(3)} = -\tfrac{1}{2}[1/(1+x^2)]. \tag{3.16}$$

Upon substituting Eq. (3.16) into (3.15) we obtain the same set of equations which was derived in Ref. 11 using the regression theorem and a Gaussian decoupling approximation.

(d) We need equal-time expectation values of the type $\langle y_0^{(+)}(t)y_0^{(-)}(t)\rangle$. To this purpose we define the normal ordered correlation matrix $\sigma^{(N)}(t)$ as follows:

$$\sigma^{(N)}(t) = \begin{pmatrix} \langle y_0^{(+)}(t)y_0^{(+)}(t)\rangle & \langle y_0^{(+)}(t)y_0^{(-)}(t)\rangle & \langle y_0^{(+)}(t)y_0^{(3)}(t)\rangle \\ \langle y_0^{(+)}(t)y_0^{(-)}(t)\rangle & \langle y_0^{(-)}(t)y_0^{(-)}(t)\rangle & \langle y_0^{(3)}(t)y_0^{(-)}(t)\rangle \\ \langle y_0^{(+)}(t)y_0^{(3)}(t)\rangle & \langle y_0^{(3)}(t)y_0^{(-)}(t)\rangle & \langle y_0^{(3)}(t)y_0^{(3)}(t)\rangle \end{pmatrix} \tag{3.17}$$

$$\equiv \langle :y_0(t)\tilde{y}_0(t):\rangle . \tag{3.18}$$

In Eq. (3.18) the tilde denotes the transpose operator and : : implies normal ordering ($y^{(+)}$ operators at the extreme left, $y^{(-)}$ operators at the right, and $y^{(3)}$ in between).

The equation of motion for the correlation matrix $\sigma^{(N)}$ follows readily from Eq. (3.11). The result is

$$\dot{\sigma}^{(N)}(t) = M(t)\sigma^{(N)} + \sigma^{(N)}\tilde{M}(t) + 2\mathfrak{D} \tag{3.19}$$

where

$$\mathfrak{D} = \begin{pmatrix} \langle \mathfrak{D}_{++}^{(0)}(t)\rangle & 0 & 0 \\ 0 & \langle \mathfrak{D}_{--}^{(0)}(t)\rangle & 0 \\ 0 & 0 & \langle \mathfrak{D}_{33}^{(0)}(t)\rangle \end{pmatrix}. \tag{3.20}$$

In deriving Eq. (3.19) it does not matter whether

one uses the normal ordering convention or not. If one uses the diffusion coefficients (3.8a), the equation of motion for the diffusion matrix (3.19) still holds. This is because the commutators

$$[x^{(+)}, x^{(-)}] = (2/N)x^{(3)}, \quad [x^{(3)}, x^{(+)}] = (1/N)x^{(+)}$$

are of order $1/N$. Hence, the right-hand side of the commutation relations can be replaced by $x_0^{(3)}$ and $x_0^{(*)}$.

The steady-state value of $\sigma^{(N)}$ is given by the solution of the linear matrix equation

$$M\sigma^{(N)} + \sigma^{(N)}\tilde{M} + 2\mathfrak{D} = 0, \tag{3.21}$$

where now

$$M = \begin{bmatrix} -A & 0 & -\sqrt{2}ix \\ 0 & -A & \sqrt{2}ix \\ -i\sqrt{2}x\Lambda & i\sqrt{2}x\Lambda & -2 \end{bmatrix},$$

$$\mathfrak{D} = \frac{1}{2}\frac{x^2}{1+x^2}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}, \tag{3.22}$$

and

$$A = 1 + 2c/(1+x^2) = y/x, \quad \Lambda = 1 - y/2x.$$

The solution of the matrix equation (3.21) is

$$\sigma^{(N)} = \begin{pmatrix} \sqrt{2}x^2\delta/y + \frac{1}{2}x^3/y(1+x^2) & -\sqrt{2}x^2\delta/y & i\delta \\ -\sqrt{2}x^2\delta/y & \sqrt{2}x^2\delta/y + \frac{1}{2}x^3/y(1+x^2) & -i\delta \\ i\delta & -i\delta & \sqrt{2}x\Lambda\delta + \frac{1}{2}x^2/(1+x^2) \end{pmatrix}, \tag{3.23}$$

where

$$\delta = -\frac{1}{2\sqrt{2}}\frac{x^3}{1+x^2}\left(\frac{y}{x} + 2x^2\Lambda\right)^{-1}. \tag{3.24}$$

Equation (3.23) contains the stationary expectation values that will be needed for the calculation of the spectrum of the transmitted light.

The analysis of the steady-state solution summarized in Appendix A shows that the stationary state will be stable if

$$y/x + 2x^2(1 - y/2x) > 0. \tag{3.25}$$

Because the density matrix ρ is a positive definite operator, we must show that the results of our approximate calculations are consistent with the positivity property of ρ. In Appendix B, we prove that this is indeed the case.

Finally we point out that the diffusion matrix $\sigma^{(N)}$, as calculated, contains single-particle contributions and corrections owing to two-particle effects. For example, in the framework of the single-particle approximation the element $\sigma_{12}^{(N)}$ is given by

$$\begin{aligned}\sigma_{12}^{(N)} &= (1/N^2)(\langle S^+S^-\rangle - \langle S^+\rangle\langle S^-\rangle) \\ &= \frac{1}{N^2}\left(\sum_i \langle s_i^+ s_i^-\rangle + \sum_{i\neq j}\langle s_i^+ s_j\rangle - \sum_{ij}\langle s_i^+\rangle\langle s_j^-\rangle\right) \\ &\approx \frac{1}{N^2}\left(\sum_i \langle s_i^+ s_i^-\rangle + \sum_{i\neq j}\langle s_i^+\rangle\langle s_j^-\rangle - \sum_{ij}\langle s_i^+\rangle\langle s_j^-\rangle\right) \\ &= (1/N^2)(\tfrac{1}{2}N + \langle S^3\rangle - \langle S^+\rangle\langle S^-\rangle/N) \\ &= (1/N)\tfrac{1}{2}x^4/(1+x^2)^2. \end{aligned} \tag{3.26}$$

It is clear that $\sigma_{12}^{(N)}$ contains contributions of order $1/N$ owing to the fact that $s_i^{\pm}$ and s_i^3 are spin-$\frac{1}{2}$ operators, and that the contribution (3.26) arises purely from the quantum-mechanical properties of the system.

IV. SPECTRUM OF THE TRANSMITTED LIGHT

In Sec. III we have carried out a systematic analysis of the atomic fluctuations. We are now in a position to calculate the correlation function $\chi_1(t) = \langle \delta S^+(t)\delta S^-(0)\rangle$ which is related to the incoherent part of the spectrum of the transmitted light.[35]

The correlation function $\langle S^+(t)S^-(0)\rangle$ in steady state is given by

$$\begin{aligned}\langle S^+(t)S^-(0)\rangle &= N^2\langle x^{(+)}(t)x^{(-)}(0)\rangle \\ &= N^2|x_0^{(+)}|^2 + N\langle y_0^{(+)}(t)y_0^{(-)}(0)\rangle. \end{aligned} \tag{4.1}$$

It is clear from Eq. (4.1) that the coherent part of the spectrum is given by

$$I_{\text{coh}} \propto (\gamma_1^2/2g^2)x^2 \tag{4.2}$$

The total intensity of the incoherent part, instead, is proportional to $\langle y_0^{(+)}y_0^{(-)}\rangle$:

$$I_{\text{incoh}} \propto \frac{Nx^5}{2y(1+x^2)}\left(\frac{y}{x} + 2x^2\Lambda\right)^{-1}. \tag{4.3}$$

I_{incoh} is also positive definite along the stable branches in view of Eq. (3.25). A plot of I_{incoh} is given in Fig. 3 for various values of y.
From Eq. (3.15) we find

$$\begin{aligned}\chi_1(t) = N[&(e^{Mt})_{11}\langle y_0^{(+)}y_0^{(-)}\rangle + (e^{Mt})_{12}\langle y_0^{(-)}y_0^{(-)}\rangle \\ &+ (e^{Mt})_{13}\langle y_0^{(3)}y_0^{(-)}\rangle]. \end{aligned} \tag{4.4}$$

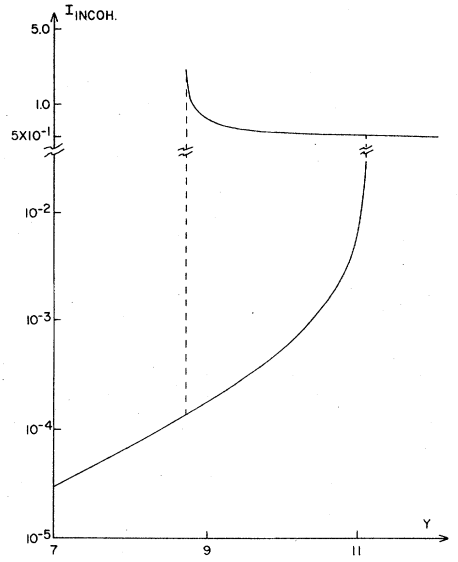

FIG. 3. Total transmitted incoherent intensity as a function of the applied field strength for $c = 10$.

Hence, the Laplace transform of $\chi_1(t)$ is

$$\tilde{\chi}_1(z) = \frac{N}{2}\frac{x^4}{(1+x^2)^2}\left(\frac{x}{y}\right)\left(1+\frac{(y/x-1)(1-x^2)}{1+x^2}\right)^{-1} P^{-1}(z) \times\left[\left(z+\frac{y}{x}\right)\left(z+2+\frac{y}{x}\right) +\left(2-\frac{y}{x}\right)\left(\frac{y}{x}+2x^2-xy\right)\right] \tag{4.5}$$

where

$$P(z) = (z+y/x)[(z+2)(z+y/x)+4x^2\Lambda]. \tag{4.6}$$

We note first that, in the limit of vanishing atomic density ($c \to 0$, $y \to x$, $\Lambda \to \frac{1}{2}$), Eq. (4.5) reduces to

$$\chi_1(z) \underset{c\to 0}{\longrightarrow} \frac{N}{2}\frac{y^4}{(1+y^2)^2}\frac{(z+1)(z+3)+y^2+1}{(z+1)[(z+1)(z+2)+2y^2]} \tag{4.7}$$

which is the well-known result for the single-atom spectrum of resonance fluorescence.[17-19]

The spectrum of the incoherent radiation is related directly to $\tilde{\chi}_1(z)$ by

$$S_{\text{incoh}}(\omega) = \text{Re}\,\tilde{\chi}_1(z)|_{z=i\omega}. \tag{4.8}$$

We have evaluated S_{incoh} numerically for several values of the driving field amplitude y in the case of interest $c > 4$. As y increases from 0 to $y_{\max}$ the atomic system goes through a series of steady-state configurations along the collective branch. The spectrum of the transmitted light is made up of a central component riding on top of a wide background. As $y \to y_{\max}$ the central component becomes sharply peaked; this corresponds to one of the relaxation modes becoming soft. Finally, above $y_{\max}$ the spectrum broadens discontinuously and displays sidebands which are removed from the central component by an amount approximately equal to the Rabi frequency of the driving field (Figs. 4 and 5). If the driving field strength is now reduced with the system moving quasistatically along the single-atom branch, the sidebands gradually merge into the central peak. As $y \to y_{\min}$, this peak becomes higher and higher, corresponding, again, to one of the modes becoming soft. Below the lower bistability threshold, the spectrum again broadens out discontinuously as the atomic system "jumps" back on the cooperative branch.

It is interesting to analyze the relative weights of the relaxation modes which are responsible for the spectral shapes illustrated in Figs. 4 and 5. To this purpose we calculate the inverse Laplace transform of Eq. (4.5):

$$\chi_1(t) = I_{\text{incoh}}(W_1 e^{\lambda_1 \tau} + W_2 e^{\lambda_2 \tau} + W_3 e^{\lambda_3 \tau}), \tag{4.9}$$

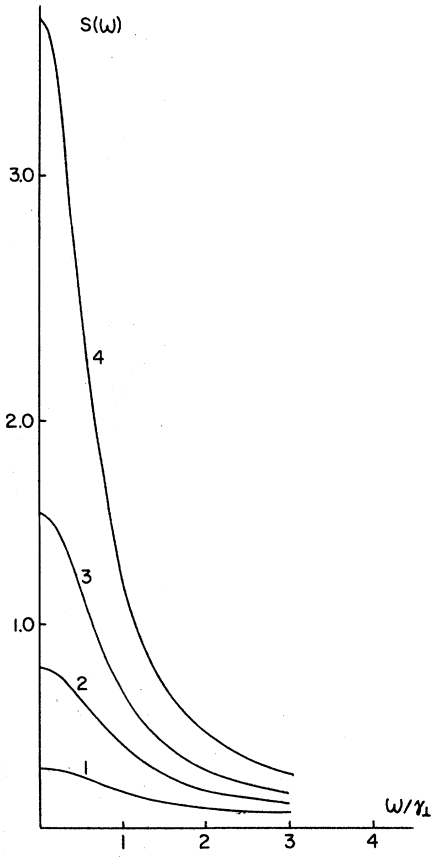

FIG. 4. Spectrum of the transmitted intensity along the cooperative branch. The values of y for curves (1)–(4) are increasingly closer to the switching threshold. The spectra are characterized by a central peak with a monotonically decreasing half width.

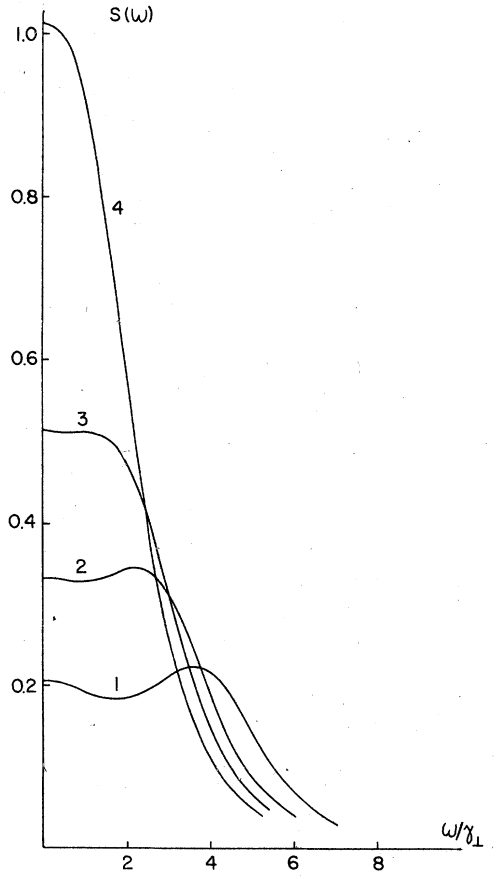

FIG. 5. Spectrum of the transmitted intensity along the single-atom branch. The values of y for curves (1)–(4) are gradually decreasing toward the lower bistability threshold. The sidebands merge into the central peak which grows increasingly higher and narrows as the lower threshold is approached.

In Eq. (4.9), I_{incoh} is the total intensity of the incoherent spectrum and λ_i are the eigenvalues of the relaxation matrix M (see Appendix A). The weights W_i are given by

$$W_1 = (y/x + 2x^2\Lambda)/2x^2, \tag{4.10}$$

$$W_2 = [\beta + \Lambda(1 - 2x^2)]/(\beta - \Lambda)2\beta, \tag{4.11}$$

$$W_3 = [-\beta + \Lambda(1 - 2x^2)]/(\beta + \Lambda)2\beta, \tag{4.12}$$

where

$$\beta = \tfrac{1}{2}[(2 + y/x)^2 - \delta(y/x - xy + 2x^2)]^{1/2}. \tag{4.13}$$

In the vicintiy of the upper bistability threshold we have $x \sim 1$, hence the weight factors approach

$$W_1 \to 0,$$
$$W_2 \to 2(1 - x^2\Lambda)/[(y/x)(2 + y/x)], \tag{4.14}$$
$$W_3 \to 0.$$

For large values of c, the weights W_2 and W_3 are very small compared with W_1. However, as $\lambda \to 0$ at threshold, the spectral component associated with the soft mode dominates. This is clearly borne out by Fig. 4.

In the vicinity of the lower bistability threshold one finds approximately

$$W_1 \sim 0, \quad W_2 \sim x/y, \quad W_3 \sim 0. \tag{4.15}$$

Here, unlike the previous case, it is the soft mode that carries the largest weight near threshold. This explains the qualitatively different structure of the spectrum in the two cases even after the sidebands have merged into the central peak.

For small values of the driving field, the cooperative spectrum is very broad. This can be seen from Eqs. (4.10)–(4.13) if we approximate the state equation with $y \sim x(1 + 2c)$ and, in addition, assume that c is sufficiently large. In this case we find

$$W_1 \to 4c^3/y^2, \quad W_2 \to 1/8c^2, \quad W_3 \to -\tfrac{1}{2}. \tag{4.16}$$

Thus, for small values of y, the mode λ_2 (the one that becomes soft near threshold) is not at all important. The mode λ_1 is dominant and the linewidth of the spectrum is essentially $\lambda_1 = 1 + 2c \approx 2c$ which compares very well with our numerical computations.

The spectra of the fluorescence and of the transmitted light have fairly similar shapes, except for the following main differences.

(i) On the cooperative branch the peak heights of the resonance-fluorescence spectra range over an order of magnitude from $y = 3$ (small driving field amplitude) to $y = 9.7$, while the peak heights of the spectra calculated in this paper vary over four orders of magnitude over a similar range of driving field strengths.

(ii) On the single-atom branch, both spectra converge to the single-atom limit for large values of y, although in the present case the convergence is slower than in the case of Ref. 10. For decreasing values of y, the sidebands disappear. However, in the present paper we find that the sidebands may become higher than the central peak before the merge is completed.

We have restricted our calculations to the steady-state spectrum. The transient spectrum can also be analyzed by using the Eqs. (3.14) and (3.19). Using the system-size expansion, we have also calculated the absorption spectrum of a collective system as well as the intensity correlations of the scattered light. We hope to discuss these aspects in a future publication.

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APPENDIX A: EIGENVALUES OF THE RELAXATION MATRIX

The following discussion contains a useful summary of some properties of the eigenvalues of the relaxation matrix M which are of interest in this work. A detailed analysis can be found in Refs. 9 and 23.

The linearized mean-field limit of Eq. (2.4) for the atomic expectation values and Eq. (3.15) for the atomic correlation function in steady state can be cast into the form

$$\frac{d\vec{R}}{dt} = M\vec{R}, \tag{A1}$$

where the relaxation matrix M is given by

$$M = \begin{pmatrix} -A & 0 & -i\sqrt{2}\,x \\ 0 & -A & i\sqrt{2}\,x \\ (i/\sqrt{2})(y-2x) & -(i/\sqrt{2})(y-2x) & -2 \end{pmatrix},$$

$$A = 1 + 2c/(1+x^2). \tag{A2}$$

It is a simple matter to show that the eigenvalues are given by

$$\lambda_1 = -[1 + 2c/(1+x^2)]$$

$$\lambda_{2,3} = \frac{-(2+A) \pm \{(2+A)^2 - 8[A - x(y-2x)]\}^{1/2}}{2}. \tag{A3}$$

The first eigenvalue is always real and negative. The eigenvalues λ_2 and λ_3 can be complex. Their real part will be negative provided

$$A - x(y-2x) > 0. \tag{A4}$$

An elementary calculation shows that Eq. (A4) coincides with the requirement

$$\frac{dy}{dx} > 0, \tag{A5}$$

where y is given by the state equation (2.8). For a given value of $c > 4$ the cubic equation $y(x)$ shown in Fig. 1 consists of three branches: a stable low-transmission cooperative branch ($0 < y < y_{max}$), a stable high-transmission single-atom branch ($y_{min} < y < \infty$) and an unstable branch joining the maximum and the minimum. It is clear both from Eqs. (A4) and (A5), and by continuity considerations that one of the eigenvalues of the matrix M vanishes at both the upper and lower bistability thresholds. The immediate consequence is that one of the normal modes of relaxation becomes very long lived. This is the reason why the spectrum of the atomic fluctuations, and the incoherent part of the spectrum of the transmitted light develop a dominating narrow central component in the vicinity of both upper and lower bistability thresholds (Figs. 4 and 5).

Another way to show the onset of the critical slowing down near threshold is to perturb the steady-state configuration of the atoms by suddenly changing the strength of the driving field, and to observe the relaxation of the system to the new steady state.[36]

In Fig. 6, we show the relaxation to steady state of the atomic population difference $\langle S_3\rangle/N$. We require the atoms to be initially in a cooperative steady-state corresponding to a given value of the driving field. A $t = 0$ the external field is increased suddenly to a slightly larger value. It is clear that the evolution of $\langle S_3\rangle/N$ is always monotonic

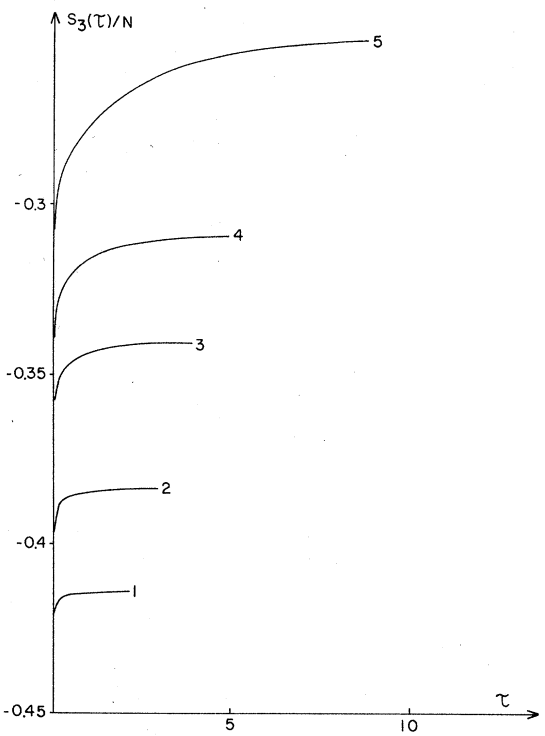

FIG. 6. Critical slowing down of the atomic parameters in the neighborhood of the upper bistability threshold. In curves (1)–(5) the atomic population is initially in a steady-state configuration characterized by increasingly larger values of y along the cooperative branch. The strength of the driving field is suddenly increased by a small amount in each curve, and the system relaxes to new steady-state values with longer and longer relaxation times.

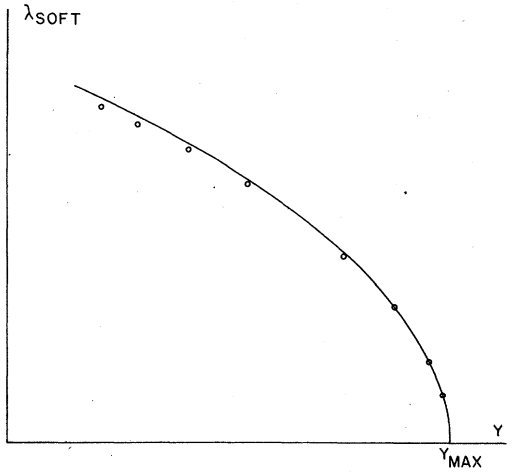

FIG. 7. Behavior of the relaxation rate of the soft mode in the neighborhood of the upper bistability threshold. The solid curve represents the $\frac{1}{2}$-power law $\lambda_{\text{soft}} \approx (y_{\text{th}} - y)^{1/2}$.

(the eigenvalues of M are all real). The approach to the new steady state is characterized by a decay time that becomes increasingly longer the closer the system is to the threshold for bistable switching.

It is also interesting to observe, although this is not apparent from Eq. (A3), that for decreasing values of y along the single-atom branch, the imaginary parts of the roots λ_2 and λ_3 vanish before y reaches $y_{\min}$.

Over a small range of values above $y_{\min}$, the roots λ_2 and λ_3 are real and, as already mentioned, one of the roots approaches zero as y approaches $y_{\min}$ from above. This explains why the sidebands in Fig. 5 merge continuously into the central peak before the onset of the lower instability.

Finally, it is of interest to observe that the soft eigenvalue approaches zero at the bistability threshold with a power law of the form

$$\lambda_{\text{soft}} = (\text{const}) | y - y |^{1/2} . \tag{A6}$$

This result was first demonstrated analytically for large values of c by Bonifacio and Lugiato. In Fig. 7, we compare the numerical values of λ_{soft} with a $\frac{1}{2}$-power law in the vicinity of $y_{\max}$.

APPENDIX B: POSITIVE DEFINITENESS OF THE CORRELATION MATRIX

In Sec. III, we have carried out an analysis of fluctuations around the mean values. Now we want to verify that our approximation (system-size expansion) is consistent with the requirement of positive definiteness of the density operator. It is clear that the relations

$$\mathrm{Tr}(WA^{\dagger}A) \geq 0, \quad \mathrm{Tr}(WAA^{\dagger}) \geq 0 \tag{B1}$$

must be true for any operator A. If we choose

$$A = \sum \alpha_i y^{(i)}, \quad i = +, -, 3 \tag{B2}$$

then Eq. (B1) implies that

$$\sum_{ij} \alpha_i \alpha_j^{*} \langle \{ y^{(i)}, y^{(i)+} \} \rangle \geq 0 . \tag{B3}$$

Therefore, the matrix

$$\sigma^{(W)} = \begin{pmatrix} \frac{1}{2}\langle \{ y_0^{(+)}, y_0^{(-)} \} \rangle & \langle y_0^{(+)} y_0^{(+)} \rangle & \frac{1}{2}\langle \{ y_0^{(+)}, y_0^{(3)} \} \rangle \\ \langle y_0^{(-)} y_0^{(-)} \rangle & \frac{1}{2}\langle \{ y_0^{(+)}, y_0^{(-)} \} \rangle & \frac{1}{2}\langle \{ y_0^{(-)}, y_0^{(3)} \} \rangle \\ \frac{1}{2}\langle \{ y_0^{(3)}, y_0^{(+)} \} \rangle & \frac{1}{2}\langle \{ y_0^{(3)}, y_0^{(-)} \} \rangle & \langle y_0^{(3)} y_0^{(3)} \rangle \end{pmatrix} \tag{B4}$$

should be positive definite. Our analysis of Sec. III will be consistent if we can show that the matrix $\sigma^{(W)}$ constructed in the approximation of Sec. IV is positive definite.

On using Eq. (3.23) and the commutation relations of the operators $y^{(\pm)}$, $y^{(3)}$, the steady-state limit of $\sigma^{(W)}$ takes the form

$$\sigma^{(W)} = \begin{pmatrix} -\sqrt{2}\,x^2\delta/y + i[1/2(1+x^2)] & \sqrt{2}\,x^2\delta/y + \frac{1}{2}x^3/y(1+x^2) & i\delta + ix/2\sqrt{2}\,(1+x^2) \\ \sqrt{2}\,x^2\delta/y + \frac{1}{2}x^3/y(1+x^2) & -\sqrt{2}\,x^2\delta/y + \frac{1}{2}1/(1+x^2) & -i\delta - ix/2\sqrt{2}\,(1+x^2) \\ -i\delta - ix/2\sqrt{2}\,(1+x^2) & i\delta + ix/2\sqrt{2}\,(1+x^2) & \sqrt{2}\,x\Lambda\delta + \frac{1}{2}x^2/(1+x^2) \end{pmatrix} . \tag{B5}$$

In order that $\sigma^{(W)}$ be positive definite, it is necessary and sufficient that the following relations be satisfied.

$$-\sqrt{2}\,x^2\delta/y + 1/2(1+x^2) > 0 , \tag{B6}$$

$$\det\sigma'^{(W)} \geq 0 , \tag{B7}$$

$$\det\sigma^{(W)} > 0 . \tag{B8}$$

The matrix $\sigma'^{(W)}$ is defined

$$\sigma'^{(W)} = \begin{pmatrix} -\sqrt{2}x^2\delta/y + 1/2(1+x^2) & \sqrt{2}x^2\delta/y + \frac{1}{2}x^3/y(1+x^2) \\ \sqrt{2}x^2\delta/y + x^3/2(1+x^2)y & -\sqrt{2}x^2\delta/y + \frac{1}{2}[1/(1+x^2)] \end{pmatrix} .$$

We have already shown that the eigenvalues of the matrix M have a negative real part when Eq. (3.26) is satisfied. If this is so, then Eq. (B6) automatically holds. Equation (B7) can be cast into the form

$$\det\sigma'^{(W)} = \frac{1}{4(1+x^2)^2}\left(1+\frac{x^3}{y}\right)\left(\frac{y}{x}+x(2x-y)\right)^{-1} \times\left(x^2(1+x^2)+\frac{y}{x}(1-x^2)\right) \tag{B9}$$

and Eq. (B8) leads to

$$\det\sigma^{(W)} = \frac{x^4}{8(1+x^2)^3}\left(1+\frac{x^3}{y}\right)\left(x^2+\frac{y}{x}\right) \times\left(\frac{y}{x}+x(2x-y)\right)^{-1} . \tag{B10}$$

It is obvious that Eq. (B8) is satisfied if Eq. (3.26) holds.

From Eqs. (B9) and (3.26) it is clear that (B7) is satisfied for $x<1$. If $x>1$, we rewrite Eq. (B9) as

$$\det\sigma'^{(W)} = \frac{1}{4(1+x^2)^2}\left(1+\frac{x^3}{y}\right) \times\left(1+\frac{x^2(x^2-1)}{y/x+x(2x-y)}\right) \tag{B11}$$

and Eq. (B7) is again satisfied.

We have proved that the matrix $\sigma^{(W)}$ calculated by using the analysis of Sec. III is positive definite whenever the eigenvalues of the relaxation matrix M have a negative real part.

APPENDIX C: ANALYSIS OF THE FLUCTUATIONS IN TERMS OF A GAUSSIAN DECORRELATION

In this appendix we show how a certain factorization ansatz[11] can be used to calculate one-time expectation values such as $\langle S^{\pm}\rangle$ and $\langle S^+S^-\rangle - \langle S^+\rangle\langle S^-\rangle$, and correlation functions of the type $\langle S^+(t)S^-(t')\rangle$. The results obtained by this technique are consistent with those obtained by the self-consistent system-size expansion method.

One can easily show from the master equation (2.1) that the one-time expectation values satisfy the following hierarchy of equations:

$$\frac{d}{d\tau}\langle S^+\rangle = -\langle S^+\rangle - \sqrt{2}iy\langle S_3\rangle + \frac{4c}{N}\langle S^+S_3\rangle . \tag{C1}$$

$$\frac{d}{d\tau}\langle S^-\rangle = -\langle S^-\rangle + \sqrt{2}iy\langle S_3\rangle + \frac{4c}{N}\langle S_3S^-\rangle , \tag{C2}$$

$$\frac{d}{d\tau}\langle S_3\rangle = -\frac{iy}{\sqrt{2}}\langle S^+\rangle + \frac{iy}{\sqrt{2}}\langle S^-\rangle - \frac{4c}{N}\langle S^+S^-\rangle - 2\left\langle S_3+\frac{N}{2}\right\rangle , \tag{C3}$$

$$\frac{d}{d\tau}\langle S^+S_3\rangle = -N\langle S^+\rangle - 3\langle S^+S_3\rangle - \frac{4c}{N}\langle S^+S^+S^-\rangle + \frac{4c}{N}\langle S^+S_3S_3\rangle - \frac{iy}{\sqrt{2}}\langle S^+S^+\rangle + \frac{iy}{\sqrt{2}}\langle S^+S^-\rangle - \sqrt{2}iy\langle S_3S_3\rangle , \tag{C4}$$

$$\frac{d}{d\tau}\langle S_3S^-\rangle = -N\langle S^+\rangle - 3\langle S_3S^-\rangle - \frac{4c}{N}\langle S^+S^-S^-\rangle + \frac{4c}{N}\langle S_3S_3S^-\rangle + \frac{iy}{\sqrt{2}}\langle S^-S^-\rangle - \frac{iy}{\sqrt{2}}\langle S^+S^-\rangle + \sqrt{2}iy\langle S_3S_3\rangle , \tag{C5}$$

$$\frac{d}{d\tau}\langle S^+S^-\rangle = -2\langle S^+S^-\rangle + \sqrt{2}iy\langle S^+S_3\rangle - \sqrt{2}iy\langle S_3S^-\rangle + \frac{8c}{N}\langle S^+S_3S^-\rangle , \tag{C6}$$

$$\frac{d}{d\tau}\langle S^+S^+\rangle = -2\langle S^+S^+\rangle - 2\sqrt{2}iy\langle S^+S_3\rangle - \sqrt{2}iy\langle S^+\rangle + \frac{4c}{N}\langle S^+S^+\rangle + \frac{8c}{N}\langle S^+S^+S_3\rangle , \tag{C7}$$

$$\frac{d}{d\tau}\langle S^-S^-\rangle = -2\langle S^-S^-\rangle + 2\sqrt{2}iy\langle S_3S^-\rangle + \sqrt{2}iy\langle S^-\rangle + \frac{4c}{N}\langle S^-S^-\rangle + \frac{8c}{N}\langle S_3S^-S^-\rangle , \tag{C8}$$

$$\frac{d}{d\tau}\langle S_3S_3\rangle = -2(N-1)\langle S_3\rangle - 4\left\langle S_3S_3-\frac{N}{4}\right\rangle - \sqrt{2}iy\langle S^+S_3\rangle + \sqrt{2}iy\langle S_3S^-\rangle - \frac{iy}{\sqrt{2}}\langle S^+\rangle + \frac{iy}{\sqrt{2}}\langle S^-\rangle - \frac{8c}{N}\langle S^+S_3S^-\rangle - \frac{4c}{N}\langle S^+S^-\rangle . \tag{C9}$$

It should be noted that we have put the operators in the normally ordered form. The above set of equations (12 in number after including the complex-conjugate expectation values of non-Hermitian operators) can be closed after decoupling the mean values involving the product of three operators. For example, if we use the Gaussian assumption, then cumulants of the type[37]

$$\langle ABC\rangle_c = 0 \tag{C10}$$

should be zero. From the definition of the cumulants, we immediately find that

$$\langle ABC\rangle = \langle AB\rangle\langle C\rangle + \langle BC\rangle\langle A\rangle + \langle AC\rangle\langle B\rangle - 2\langle A\rangle\langle B\rangle\langle C\rangle . \tag{C11}$$

Upon adopting the decorrelation (C11), the set of Eqs. (C1)–(C9) is closed. We have solved the resulting set of equations numerically. We find that the solutions for $\langle S^{\pm}\rangle$ and $\langle S^3\rangle$ are only slightly different from the ones given by (2.7). This can easily be understood from our system-size expansion discussed in Sec. III, where we have shown that the corrections to (2.7) are of order $1/N$. Our numerical solutions also yield fluctuations such as $\langle S^+S^-\rangle - \langle S^+\rangle\langle S^-\rangle$. We find again that the numerical solutions obtained from (C11) are in agreement with those given by (3.23). We also solved analytically the set of Eqs. (C1)–(C9) under the factorization ansatz (C11) and found the same diffusion matrix as given by (3.23).

To obtain the correlation functions let us consider the fluctuations around the mean values, i.e., for any two operators let us consider the correlation function

$$\langle \delta A(t)\delta B(0)\rangle = \langle A(t)B(0)\rangle - \langle A(t)\rangle\langle B(0)\rangle = \langle A(t)B(0)\rangle - \langle A(0)\rangle\langle B(0)\rangle . \tag{C12}$$

In steady state it is easy to prove that

$$\frac{d}{dt}\langle \delta A(t)\delta B(0)\rangle = \frac{d}{dt}\langle A(t)B(0)\rangle . \tag{C13}$$

Hence, from (C1)–(C3) and from the regression theorem,[34] it follows that

$$\frac{d}{dt}\chi = \begin{pmatrix} -\gamma_\perp & 0 & -2i\Omega_I \\ 0 & -\gamma_\perp & +2i\Omega_I \\ -i\Omega_I & +i\Omega_I & -2\gamma_\perp \end{pmatrix}\chi + \frac{2g^2}{\kappa}\begin{pmatrix} \langle \delta(S^+(t)S_3(t))\delta S^-(0)\rangle \\ \langle \delta(S_3(t)S^-(t))\delta S^-(0)\rangle \\ -\langle \delta(S^+(t)S^-(t))\delta S^-(0)\rangle \end{pmatrix}, \tag{C14}$$

where χ is a column matrix:

$$\chi_1 = \langle \delta S^+(t)\delta S^-(0)\rangle ,$$
$$\chi_2 = \langle \delta S^-(t)\delta S^-(0)\rangle , \tag{C15}$$
$$\chi_3 = \langle \delta S^3(t)\delta S^-(0)\rangle ,$$

and where $\delta(S^+(t)S_3(t))$ stands for the operator

$$\delta(S^+(t)S_3(t)) = S^+(t)S_3(t) - \langle S^+(t)S_3(t)\rangle = S^+(t)S_3(t) - \langle S^+(0)S_3(0)\rangle . \tag{C16}$$

Equations (C14) can be linearized by using the Gaussian decoupling scheme

$$\langle \delta(S^+(t)S_3(t))\delta S^-(0)\rangle = \langle \delta S^+(t)\delta S^-(0)\rangle\langle S_3(0)\rangle + \langle S^+(0)\rangle\langle \delta S_3(t)\delta S^-(0)\rangle ,$$
$$\langle \delta(S_3(t)S^-(t))\delta S^-(0)\rangle = \langle \delta S_3(t)\delta S^-(0)\rangle\langle S^-(0)\rangle + \langle \delta S^-(t)\delta S^-(0)\rangle\langle S_3(0)\rangle ,$$
$$\langle \delta(S^+(t)S^-(t))\delta S^-(0)\rangle = \langle \delta S^+(t)\delta S^-(0)\rangle\langle S^-(0)\rangle + \langle S^+(0)\rangle\langle \delta S^-(t)\delta S^-(0)\rangle . \tag{C17}$$

On using (C17), Eqs. (C14) reduce to

$$\frac{d\chi}{d\tau} = M\chi , \tag{C18}$$

where

$$M = \begin{pmatrix} -A & 0 & -i\sqrt{2}\,x \\ 0 & -A & i\sqrt{2}\,x \\ \frac{i}{\sqrt{2}}(y-2x) & -\frac{i}{\sqrt{2}}(y+2x) & -2 \end{pmatrix},$$
$$A = \frac{y}{x} \tag{C19}$$

where the time variable and the parameters y, x, and c are defined as before. We thus see that the Gaussian decoupling scheme leads to the same results as the ones obtained from the self-consistent analysis based on the system-size expansion. One, of course, has to study the validity of the decoupling scheme and the justification for such a decoupling scheme is provided by the system-size expansion of Sec. III.

APPENDIX D: RESONANCE FLUORESCENCE FROM A COLLECTION OF ATOMS

The resonance fluorescence from a collection of atoms undergoing correlated motion has been described in the past[20] by the master equation

$$\frac{\partial\rho}{\partial t} = -\gamma(S^+S^-\rho - 2S^-\rho S^+ + \rho S^+S^-) - i\Omega[S^+ + S^-, \rho], \tag{D1}$$

where 2γ is the Einstein A coefficient and Ω is the Rabi frequency of the driving laser field. Except

for the absence of the single-atom relaxation term, Eq. (D1) is identical to the master equation (2.1) which has been discussed in the main text.

There is, of course, a major difference between the atomic evolution described by Eq. (D1) and the one described by Eq. (2.1). In the former case, if a system is initially prepared in an eigenstate of the cooperation-number operator (formally, the total angular momentum S^2) it will evolve in a fixed manifold of constant cooperation number. Thus the evolution described by Eq. (D1) is more "rigid" than the one described by the master equation (2.1), where the total cooperation number is not conserved owing to the incoherent single-atom relaxation term.

In some sense the expectation-value equations that follow from Eq. (D1) lead to a formally identical bistable behavior as discussed in the main text. Namely, the equations of motion for the expectation values $\langle S^+\rangle$ and $\langle S_3\rangle$ from (D1) take the form

$$\langle \dot{S}^+\rangle = -\gamma\langle S^+\rangle - 2i\Omega\langle S^3\rangle + 2\gamma\sum_{i\neq j}\langle s_i^+ s_j^3\rangle , \tag{D2}$$

$$\langle \dot{S}^3\rangle = -2\gamma\langle S^3 + \tfrac{1}{2}N\rangle - i\Omega\langle S^+\rangle + i\Omega\langle S^-\rangle - 2\gamma\sum_{i\neq j}\langle s_i^+ s_j^-\rangle . \tag{D3}$$

If one uses the factorization ansatz which ignores the two-particle correlation effects, one has

$$\langle s_i^+ s_j^3\rangle \approx \langle s_i^+\rangle\langle s_j^3\rangle = (1/N^2)\langle s^+\rangle\langle s^3\rangle ,$$
$$\langle s_i^+ s_j^-\rangle \approx (1/N^2)\langle S^+\rangle\langle S^-\rangle . \tag{D4}$$

Hence, Eqs. (D2) and (D3) reduce to

$$\frac{\partial}{\partial\tau}\langle S^+\rangle = -\langle S^+\rangle - \sqrt{2}\,iy\langle S^3\rangle + 2\left(1-\frac{1}{N}\right)\langle S^+\rangle\langle S^3\rangle ,$$

$$\frac{\partial}{\partial\tau}\langle S^3\rangle = -2\left\langle S^3 + \frac{N}{2}\right\rangle - \frac{iy}{\sqrt{2}}\langle S^+\rangle + \frac{iy}{\sqrt{2}}\langle S^-\rangle - 2\left(1-\frac{1}{N}\right)\langle S^+\rangle\langle S^-\rangle , \tag{D5}$$

where y and τ are defined as before.

As observed in Ref. 12, Eqs. (D15) have the same structure as the mean-field equations for bistability if one makes the identification $c = \frac{1}{2}N$. Thus the results discussed in the text remain valid for the present model also. In particular, the onset of bistability from a collective system will take place provided $c > 4$ (or $N > 8$). In an earlier paper[20] we analyzed exactly the resonance fluorescence from a collective system of two and three atoms and found no bistability, in apparent agreement with threshold condition $N > 8$.

It is questionable to us whether the bistable behavior exhibited by Eqs. (D5) is intrinsic to the master equation (D1) and not a consequence, instead, of the factorization ansatz (D4). We are studying at present the exact evolution of the atomic observables according to the master equation (D1) for $N > 8$. We will discuss this question in greater depth in a subsequent publication.

Note added in proof. After submission of the manuscript we have learned that Bonifacio and Lugiato have derived a quantum Fokker-Planck equation for the atomic quasiprobability distribution. Their linearized Fokker-Planck equation leads to the same atomic correlation function discussed in the present paper.[38] One of us (L.M.N.) is grateful to Professor Lugiato for an illuminating correspondence.

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