Fokker-Planck equation approach to optical bistability in the bad-cavity limit

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In the general framework of the system size expansion of Van Kampen and Kubo, we consider the Fokker-Planck equation for a model of absorptive bistability in the bad-cavity limit. The physical system is described by the reduced atomic density operators after adiabatic elimination of the cavity field variables. Mapping of the master equation into c-number form according to the normal-ordering mapping scheme yields known results for the atomic fluctuations and correlation functions; however, it also leads to a Fokker-Planck equation with a non-positive-definite diffusion matrix. The symmetrical-order-mapping scheme eliminates this difficulty. The leading contribution to the system size expansion yields a Fokker-Planck equation for the symmetrical-ordered density function having a positive-definite diffusion matrix. The atomic expectation values and fluctuations previously derived from the quantum Langevin equations emerge naturally from this Fokker-Planck equation.

I. INTRODUCTION

The interesting properties of an optical cavity filled with resonant atomic systems and driven by an external coherent source have generated considerable interest.[1]

Detailed treatments have appeared which describe the bistable mode of operation of purely absorptive[2] and dispersive devices[3]; in the latter case the atomic transition frequency, the cavity mode of interest, and the carrier frequency of the driving field are allowed to take values that differ from one another. More recently a multimode theory has been formulated[4] and a striking new instability predicted.

In the case of absorptive bistability and in the limit of large-field relaxation rates, a quantum-mechanical master equation and the associated atomic Langevin equations have been discussed at some length[5] following a procedure based on the system size expansion of Van Kampen, Kubo, and others.[6]

Whenever possible, however, it is desirable to map operator master equations into c-number differential equations for suitably defined phase-space distribution functions.[7] Knowledge of these functions, for example, provides direct access to system expectation values and correlation functions without the need for introducing *ad hoc* factorization or decorrelation procedures.

The mapping method has been used by Gronchi and Lugiato[8] to transform a general master equation for optical bistability, which contains both atomic and field operators, into a c-number Fokker-Planck equation. This process is not devoid of difficulties and one mapping scheme may be found to be preferable to another. As shown in Ref. 8, and also in this paper, the normal-ordering mapping rule leads to a Fokker-Planck equation with a non-positive-definite diffusion matrix. The same difficulty has come up in connection with the treatment of the laser near threshold and it was eliminated by Haken[7] using physical arguments which unfortunately are not applicable in the present situation. The symmetrical-ordering rule, instead, leads to a Fokker-Planck equation with a well-behaved diffusion matrix.

In Sec. II we discuss the application of the normal-ordering technique following what appears to be the natural procedure. Upon using the system size expansion scheme we justify certain results concerning the atomic fluctuations which have previously been derived from the quantum-mechanical Langevin equations.[5] However, as mentioned, the c-number differential equation for the atomic quasiprobability distribution has a nonpositive diffusion matrix. Although formal techniques could be devised for dealing with the highly singular solutions occurring in such cases, we have found it analytically convenient as well as physically more appealing to describe the atomic evolution in terms of the new density function associated with the symmetrical-ordering convention. This ordering rule has been adopted independently by Gronchi and Lugiato[8] in connec-

tion with their study of a general bistability master equation containing both field and atomic operators. Their Fokker-Planck equation reduces to the one derived in this paper upon adiabatic elimination of the field variables (i.e., in the bad-cavity limit) and linearization around steady state. It must be mentioned, however, that the general structure of the Gronchi-Lugiato Fokker-Planck equation also allows consideration of the opposite limit (good-cavity limit) where the atomic variables are eliminated adiabatically.

We show explicitly in Sec. III that the symmetrized Fokker-Planck equation has a well-behaved structure. As in the case of the normal ordering, here also, the system size expansion paves the way to a rather simple method for deriving the results of physical interest after appropriate linearization. Appendix A contains some technical details of the mapping procedure while Appendix B analyzes the role of the single-atom decay term with regard to the existence of a bistable behavior.

II. ATOMIC FOKKER-PLANCK EQUATION: THE NORMAL-ORDERING SCHEME

The master equation which describes absorptive bistability in the limit when the cavity-field relaxation rate is much larger than the atomic dephasing and relaxation rates is assumed to be of the form

$$\frac{\partial W}{\partial t} = -iLW + \Lambda_s W + \Lambda_A W , \tag{2.1}$$

where

$$LW = \Omega[S^+ + S^-, W] , \tag{2.2}$$

$$\Lambda_s W = (-g^2/\kappa)(S^+S^-W - 2S^-WS^+ + WS^+S^-) , \tag{2.3}$$

$$\Lambda_A W = -\gamma_\perp \sum_i (s_i^+ s_i^- W - 2s_i^- Ws_i^+ + Ws_i^+ s_i^-) , \tag{2.4}$$

and where W is the reduced atomic density operator. The terms (2.2)–(2.4) describe the coherent atomic evolution under the action of the driving external field with a Rabi frequency Ω, the collective atomic emission into the resonant cavity mode,[9] and the incoherent decay due to spontaneous

relaxation, respectively. As usual, κ denotes the decay rate of the field out of the empty cavity, g is the coupling constant between the atoms and the resonant cavity mode, and $\gamma_\perp$ is the single-atom decay rate.

In an earlier paper[5] we have analyzed the master equation (2.1) and the associated Langevin equations. We have solved the Langevin equations using the method of system size expansion as developed by Van Kampen, Kubo, and others,[6] and suitably adapted for quantum systems; we have analyzed the behavior of the collective atomic expection values $\langle S^\pm \rangle$ and $\langle S^Z \rangle$ and of their moments and, finally, we have calculated explicit expressions for the atomic correlation functions and the spectrum of the fluctuations.

A more-comprehensive statistical picture of the atomic fluctuations can be best obtained by various types of mapping techniques which associate c-number distribution functions to the density operator W and correspondingly map the master equation into a differential equation for the distribution function.

In the following we consider the "normally ordered" characteristic function

$$C_N(\xi, \xi^*, \eta) = \langle \exp(i\xi S^+) \exp(i\eta S^Z) \exp(i\xi^* S^-) \rangle \tag{2.5}$$

and the "Weyl-ordered" (symmetric-ordered) characteristic function

$$C_W(\xi, \xi^*, \eta) = \langle \exp(i\xi S^+ + i\eta S^Z + i\xi^* S^-) \rangle . \tag{2.6}$$

Furthermore, we introduce the distribution functions P_N and P_W via

$$P_{\{^N_W\}}(Z, Z^*, M)$$
$$= \frac{1}{2\pi^3} \int \int d^2\xi \, d\eta \, C_{\{^N_W\}}(\xi, \xi^*, \eta)$$
$$\times \exp(-i\xi Z - i\xi^* Z^* - im\eta) . \tag{2.7}$$

Following the standard procedure (an outline of the calculation is given in Appendix A) it is easy to arrive at the following Fokker-Planck equation for the normally ordered distribution function P_N:

$$\frac{\partial P_N}{\partial t} = \gamma_\perp \left[\frac{\partial}{\partial Z} Z + \frac{\partial}{\partial Z^*} Z^* + \frac{\partial}{\partial m} 2\left(m + \frac{N}{2}\right) \right] P_N - \frac{2g^2}{\kappa} \left(\frac{\partial}{\partial Z} mZ + \frac{\partial}{\partial Z^*} mZ^* - \frac{\partial}{\partial m} ZZ^* \right) P_N$$

$$+ i\Omega \left(2\frac{\partial}{\partial Z} m - 2\frac{\partial}{\partial Z^*} m + \frac{\partial}{\partial m}(Z - Z^*) \right) P_N$$

$$+ \gamma_\perp \frac{\partial^2}{\partial m^2} \left[\left(m + \frac{N}{2}\right) P_N \right] + \frac{2g^2}{\kappa} \left(\frac{1}{2} \frac{\partial^2}{\partial Z^2} Z^2 + \frac{1}{2} \frac{\partial^2}{\partial Z^{*2}} Z^{*2} - \frac{1}{2} \frac{\partial^2}{\partial m^2} ZZ^* \right) P_N$$

$$- i\Omega \left(\frac{\partial^2}{\partial Z^2} Z - \frac{\partial^2}{\partial Z^{*2}} Z^* + \frac{1}{2} \frac{\partial^2}{\partial m^2}(Z - Z^*) \right) P_N + L_3 P_N . \tag{2.8}$$

The term $L_3 P_N$ involves derivatives of order higher than second:

$$L_3 P_N \equiv \sum_{n=3}^{\infty} \frac{1}{n!} \frac{\partial^n}{\partial m^n}\left[2\gamma_\perp\left(m + \frac{N}{2}\right) - i\Omega(-i)^n(Z - Z^*)\right.$$
$$\left. - 2\frac{g^2}{\kappa}(-1)^n ZZ^*\right]P_N. \qquad (2.9)$$

It is a typical drawback of the mapping induced by Eqs. (2.5)–(2.7) that higher-order derivatives of the distribution function appear in the differential equation. They are usually neglected for obvious computational reasons. The effect of this truncation, however, is not easy to assess. Mapping techniques based on the atomic coherent state representation,[10] on the other hand, are free from such difficulties because the order of differentiation is only as high as the number of atomic operators entering in each term of the master

equation (two, in the present case).[11] Unfortunately, we do not know how to map the single-atom decay term (2.4) using the atomic coherent state representation so that this procedure cannot be applied in the present case.

Equation (2.8) has the structure of a Fokker-Planck equation. Its drift and diffusion coefficients are nonlinear in the independent variables. However, this equation has interesting scaling properties.

If we introduce the following parameters and scaled variables:

$$\tau = \gamma_\perp t, \quad c = g^2 N/2\gamma_\perp \kappa, \quad y = \sqrt{2}\,\Omega/\gamma_\perp, \qquad (2.10)$$

$$Z = Nv, \quad Z^* = Nv^*, \quad m = Nv_3, \qquad (2.11)$$

$$P_N d^2Z\, dm = N^3 Q\, d^2v\, dv_3, \qquad (2.12)$$

the Fokker-Planck equation for Q takes the form

$$\frac{\partial Q}{\partial \tau} = \left(\frac{\partial}{\partial v}v + \frac{\partial}{\partial v^*}v^* + \frac{\partial}{\partial v_3}(2v_3 + 1)\right)Q - 4c\left(\frac{\partial}{\partial v}vv_3 + \frac{\partial}{\partial v^*}v^*v_3 - \frac{\partial}{\partial v_3}vv^*\right)Q + \frac{iy}{\sqrt{2}}\left(2\frac{\partial}{\partial v}v_3 - 2\frac{\partial}{\partial v^*}v_3 + \frac{\partial}{\partial v_3}(v - v^*)\right)Q$$

$$+ \frac{1}{N}\frac{\partial^2}{\partial v_3^2}\left(v_3 + \frac{1}{2}\right)Q + \frac{4c}{N}\left(\frac{1}{2}\frac{\partial^2}{\partial v^2}v^2 + \frac{1}{2}\frac{\partial^2}{\partial v^{*2}}v^{*2} - \frac{1}{2}\frac{\partial^2}{\partial v_3^2}vv^*\right)Q$$

$$- \frac{iy}{\sqrt{2}N}\left(\frac{\partial^2}{\partial v^2}v - \frac{\partial^2}{\partial v^{*2}}v^* + \frac{1}{2}\frac{\partial^2}{\partial v_3^2}(v - v^*)\right)Q + O\left(\frac{1}{N^2}\right). \qquad (2.13)$$

The terms of order $1/N^2$ arise from $L_3 P$ [Eq. (2.9)]. This provides some justification for neglecting these higher-order contributions.

In terms of the new scaled variables, the Fokker-Planck equation has diffusion coefficients which are of order $1/N$. This implies that, in general, the fluctuations of the variables v, v^*, v_3 are also of order $1/N$ and that the solution of Eq. (2.14) will be sharply peaked around the macroscopic values $v^{(0)}(t), v^{(0)*}(t), v_3^{(0)}(t)$ with a spread of order $1/\sqrt{N}$. We have qualified this remark with the clause "in general" because the above need not be true over the entire range of operating parameters. We feel that this point will have to be investigated more closely. For example, anomalous fluctuations have been predicted[2(b)] over a narrow range of values of the driving field. It is true that the physical situation described by Bonifacio and Lugiato is different from ours (their atomic relaxation rates are assumed to be the largest decay rates in the system). Still, it would not be surprising to find a similar behavior

even under the physical situation discussed in this paper.

The macroscopic atomic variables in this case satisfy the equations of motion

$$\dot{v} = -v + 4cvv_3 - \sqrt{2}\,iyv_3,$$
$$\dot{v}^* = -v^* + 4cv^*v_3 + \sqrt{2}\,iyv_3, \qquad (2.14)$$
$$\dot{v}_3 = -2(v_3 + \tfrac{1}{2}) - 4cvv^* - (iy/\sqrt{2})(v - v^*).$$

The fluctuations around the macroscopic values can be studied using the system size expansion. In the limit of large N and upon retaining the leading corrections of order $1/\sqrt{N}$ we can set

$$v = v^{(0)}(t) + N^{-1/2}v^{(1)}, \quad v_3 = v_3^{(0)}(t) + N^{-1/2}v_3^{(1)}. \qquad (2.15)$$

The Fokker-Planck equation for the probability distribution

$$\phi = \phi(v^{(1)}, v^{(1)*}, v_3^{(1)}, \tau)$$

follows readily from Eq. (2.13):

$$\frac{\partial \phi}{\partial \tau} = \frac{\partial}{\partial v^{(1)}}\{v^{(1)} + \sqrt{2}\,iyv_3^{(1)} - 4c[v^{(0)}(t)v_3^{(1)} + v_3^{(0)}(t)v^{(1)}]\}\phi + \frac{\partial}{\partial v^{(1)*}}\{v^{(1)*} - \sqrt{2}\,iyv_3^{(1)} - 4c[v^{(0)*}(t)v_3^{(1)} + v_3^{(0)}(t)v^{(1)*}]\}\phi$$

$$+ \frac{\partial}{\partial v_3^{(1)}}\left(2v_3^{(1)} + \frac{iy}{\sqrt{2}}(v^{(1)} - v^{(1)*}) + 4c[v^{(0)}(t)v^{(1)*} + v^{(1)}v^{(0)*}(t)]\right)\phi + \frac{\partial^2}{\partial v^{(1)2}}(D_{11}\phi) + \frac{\partial^2}{\partial v^{(1)*2}}(D_{11}^*\phi) + \frac{\partial^2}{\partial v_3^{(1)2}}(D_{33}\phi),$$

$$(2.16)$$

where the diffusion coefficients are given by

$$D_{11} = -(iy/\sqrt{2})v^{(0)}(t) + 2cv^{(0)}(t)^2 ,$$

$$D_{33} = \tfrac{1}{2} + v_3^{(0)}(t) - 2cv^{(0)}(t)v^{(0)}*(t) \qquad (2.17)$$

$$- (iy/2\sqrt{2})(v^{(0)}(t) - v^{(0)}*(t)) .$$

The Fokker-Planck equation (2.16) is characterized by the following properties: (i) the drift terms are linear in the variables $v^{(1)}$, $v^{(1)}*$, and $v_3^{(1)}$; (ii) the diffusion terms are independent of the variables $v^{(1)}$, $v^{(1)}*$, and $v_3^{(1)}$; (iii) both drift and diffusion terms are time dependent; and (iv) the time dependence is determined by the macroscopic equations (2.14). Note that the distribution function ϕ yields the normally ordered expectation values

$$\left\langle \left(\frac{S^+}{N} - v^{(0)}(t)\right)^\alpha \left(\frac{S^z}{N} - v_3^{(0)}(t)\right)^\beta \left(\frac{S^-}{N} - v^{(0)}*(t)\right)^\gamma \right\rangle = \int d^2 v^{(1)} \int dv_3 \, \phi(v^{(1)}, v^{(1)}*, v_3^{(1)}, \tau) \frac{v^{(1)\alpha} v_3^{(1)\beta} (v^{(1)}*)^\gamma}{N^{(\alpha+\beta+\gamma)/2}}$$

$$\equiv \langle v^{(1)\alpha} v_3^{(1)\beta} v^{(1)}*^\gamma \rangle / N^{(\alpha+\beta+\gamma)/2} . \qquad (2.18)$$

In the remainder of this section we discuss the properties of the atomic correlation functions and demonstrate the lack of positive definiteness of the diffusion matrix of the Fokker-Planck equation. The consequences of this fact and an alternative description will be discussed in Sec. III.

The correlation matrix is defined by

$$\sigma^{(N)} = \begin{pmatrix} \langle v^{(1)}v^{(1)}\rangle & \langle v^{(1)}v^{(1)}*\rangle & \langle v^{(1)}v_3^{(1)}\rangle \\ \langle v^{(1)}v^{(1)}*\rangle & \langle v^{(1)}*v^{(1)}*\rangle & \langle v^{(1)}*v^{(1)}\rangle \\ \langle v_3^{(1)}v^{(1)}\rangle & \langle v_3^{(1)}v^{(1)}*\rangle & \langle v_3^{(1)}v_3^{(1)}\rangle \end{pmatrix} , \qquad (2.19)$$

with the superscript (N) denoting the normal-ordering rule. From the Fokker-Planck equation (2.16) we can easily derive the equation of motion of $\sigma^{(N)}$ in the form[12]

$$\frac{\partial \sigma^{(N)}}{\partial t} = M(t)\sigma^{(N)} + \sigma^{(N)}\vec{M}(t) + 2D(t) . \qquad (2.20)$$

The matrix M is the time-dependent relaxation matrix

$$M(t) = \begin{pmatrix} 4cv_3^{(0)}(t) - 1 & 0 & 4cv^{(0)}(t) - \sqrt{2}iy \\ 0 & 4cv_3^{(0)}(t) - 1 & 4cv^{(0)}* + \sqrt{2}iy \\ -4cv^{(0)}*(t) - (iy/\sqrt{2}) & -4cv^{(0)}(t) + (iy/\sqrt{2}) & -2 \end{pmatrix} \qquad (2.21)$$

and $D(t)$ has the form

$$D(t) = \begin{pmatrix} D_{11}(t) & 0 & 0 \\ 0 & D_{11}^*(t) & 0 \\ 0 & 0 & D_{33}(t) \end{pmatrix} . \qquad (2.22)$$

Because of the special structure of the Fokker-Planck equation (2.16), the distribution function ϕ will be a Gaussian function of the independent variables provided the matrix of the diffusion coefficients is positive definite.[13]

In order to investigate this question with a minimum of algebraic complications we consider the structure of $\sigma^{(N)}$ in steady state.

In this limit, Eqs. (2.14) can be easily solved to yield the steady-state values

$$v^{(0)} = \frac{i}{\sqrt{2}} \frac{x}{1+x^2} , \quad v_3^{(0)} = -\frac{1}{2} \frac{1}{1+x^2} . \qquad (2.23)$$

The transmitted field amplitude x is related to the incident field y by the well-known state equation

$$y = x + 2cx/(1+x^2) . \qquad (2.24)$$

Furthermore, as $t \to \infty$, the matrices $D(t)$ and $\sigma^N(t)$ become, respectively,

$$D = \lim_{t\to\infty} D(t) = \frac{1}{2} \frac{x^2}{1+x^2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \qquad (2.25)$$

and

$$\sigma^{(N)} = \lim_{t\to\infty} \sigma^{(N)}(t) = \begin{pmatrix} \mu & \eta & i\delta \\ \eta & \mu & -i\delta \\ i\delta & -i\delta & \nu \end{pmatrix} . \qquad (2.26)$$

The parameters δ, η, μ, and ν appearing in (2.26) are given explicitly by

$$\delta = -\frac{1}{2\sqrt{2}}\frac{x^3}{1+x^2}\left(\frac{y}{x}+2x^2\Lambda\right)^{-1},$$

$$\eta = -\sqrt{2}\,x^2\delta/y,$$

$$\mu = \sqrt{2}\frac{x^2\delta}{y}+\frac{1}{2}\frac{x^3}{y(1+x^2)},$$ (2.27)

$$\nu = \sqrt{2}\,x\Lambda\delta+\frac{1}{2}\frac{x^2}{1+x^2},$$

with $\Lambda = 1 - y/2x$.

Finally the steady-state limit of the relaxation matrix is given by

$$M = \lim_{t\to\infty} M(t) = \begin{pmatrix} -y/x & 0 & -\sqrt{2}\,ix \\ 0 & -y/x & \sqrt{2}\,ix \\ -i\sqrt{2}\,x\Lambda & i\sqrt{2}\,x\Lambda & -2 \end{pmatrix}. \quad (2.28)$$

As a consequence of the above calculations, the steady-state atomic correlation functions derived on the basis of the system size expansion method are readily seen to satisfy the coupled linear equations

$$\frac{d}{d\tau}\begin{pmatrix} \langle v^{(1)}(\tau)y(0)\rangle \\ \langle v^{(1)*}(\tau)y(0)\rangle \\ \langle v_3^{(1)}(\tau)y(0)\rangle \end{pmatrix} = M\begin{pmatrix} \langle v^{(1)}(\tau)y(0)\rangle \\ \langle v^{(1)*}(\tau)y(0)\rangle \\ \langle v_3^{(1)}(\tau)y(0)\rangle \end{pmatrix},$$

(2.29)

where $y(0)$ denotes an arbitrary functional of the variables $v^{(1)}(0)$, $v^{(1)*}(0)$, and $v_3^{(1)}(0)$. All the explicit results (2.23)–(2.29) for the macroscopic mean values, correlation matrix, and steady-state correlation functions are in agreement with those derived in Ref. 5 using the quantum Langevin equation.

We may also point out that the above results do not depend explicitly on the analytic properties of the distribution function ϕ other than through the obvious requirement that the appropriate expectation values be defined, i.e., that the function ϕ be integrable at least in the sense of the theory distributions.

We are now in a position to examine the steady-state properties of the Fokker-Planck equation (2.16). To this purpose it will be convenient to introduce the Cartesian coordinates

$$v^{(1)} = v_1 + iv_2, \quad v^{(1)*} = v_1 - iv_2, \quad v_3^{(1)} = v_3 \quad (2.30)$$

and observe that the transformed equation

$$\frac{\partial\phi}{\partial\tau} = -\frac{\partial}{\partial v_1}(-\alpha v_1\phi) - \frac{\partial}{\partial v_2}(-\alpha v_2 - \beta v_3)\phi$$

$$-\frac{\partial}{\partial v_3}(-2v_3 + 2\gamma_0 v_2)\phi$$

$$+\tfrac{1}{2}q\left(\frac{\partial^2}{\partial v_1^2} - \frac{\partial^2}{\partial v_2^2}\right)\phi + 2q\frac{\partial^2\phi}{\partial v_3^2}, \quad (2.31)$$

with

$$\alpha = y/x, \quad \beta = \sqrt{2}\,x, \quad \gamma_0 = \frac{2x-y}{\sqrt{2}}, \quad q = \frac{1}{2}\frac{x^2}{1+x^2},$$

(2.32)

has a non-positive-definite diffusion matrix. Thus the solution of Eq. (2.31) (if we can even talk about one) develops a behavior which is more singular than that of an arbitrary derivative of a δ function.

Previous experience with c-number distributions associated with boson systems has shown that the distribution function corresponding to the normal-ordering mapping rule need not exist even in the sense of the theory of distributions.[14] By analogy we do not find the result, Eq. (2.31), too surprising. This difficulty will be removed in Sec. III by the use of the Weyl-ordering mapping rule.

III. FOKKER-PLANCK EQUATION FOR THE WEYL-ORDERED DISTRIBUTION FUNCTION

In this section we construct the equation of motion satisfied by the distribution function P_W defined by Eq. (2.7). The calculation will be carried out in the context of the system size expansion approach.

To this purpose we only need to observe that the characteristic functions C_N and C_W, defined by Eqs. (2.5) and (2.6), can be directly related to one another with the help of a limiting form of Baker-Haussdorf identity.[15]

Consider first the correspondence

$$v^{(1)} \to S^+ = \sqrt{N}\left(\frac{S^+}{N} - v^{(0)}\right),$$

$$v^{(1)*} \to S^- = \sqrt{N}\left(\frac{S^-}{N} - v^{(0)*}\right), \quad (3.1)$$

$$v_3^{(1)} \to S^z = \sqrt{N}\left(S^z/N - v_3^{(0)}\right),$$

between the c-number fluctuation variables $v^{(1)}, v^{(1)*}, v_3^{(1)}$ and the fluctuation operators $S^\pm, S^z$. These operators obey the commutation relations

$$[S^+, S^-] = 2(v_3^{(0)} + S^z/N),$$

$$[S^z, S^+] = v^{(0)} + S^+/N. \quad (3.2)$$

Therefore, to order $1/N$, these commutators are c numbers:

$$[S^+, S^-] \simeq 2v_3^{(0)},$$

$$[S^z, S^+] \simeq v^{(0)}. \quad (3.3)$$

To this order of approximation we can apply the Baker-Haussdorf identity to find

$$C_W \equiv \langle \exp(i\xi S^+ + i\eta S^z + i\xi^* S^-) \rangle$$

$$= C_N \exp[\tfrac{1}{2}(i\eta)(i\xi^*)v^{(0)}* + \tfrac{1}{2}(i\eta)(i\xi)v^{(0)}$$

$$- (i\xi)(i\xi^*)v_3^{(0)}], \tag{3.4}$$

where

$$C_N \equiv \langle e^{i\xi S^+} e^{i\eta S^z} e^{i\xi^* S^-} \rangle . \tag{3.5}$$

Equation (3.4) gives the approximate relation between the characteristic functions corresponding to the Weyl- and normal-ordering schemes. In turn, this equation implies that the distribution functions ϕ_N and ϕ_W (as well as P_N and P_W) are related by

$$\phi_W = \exp\left(\frac{1}{2}\frac{\partial^2}{\partial v_3^{(1)}\partial v^{(1)}*}v^{(0)}* + \frac{1}{2}\frac{\partial^2}{\partial v_3^{(1)}\partial v^{(1)}}v^{(0)} - \frac{\partial^2}{\partial v^{(1)}\partial v^{(1)}*}v_3^{(0)}\right)\phi_N$$

$$\approx \phi_N + \left(\frac{1}{2}\frac{\partial^2}{\partial v_3^{(1)}\partial v^{(1)}*}v^{(0)}* - \frac{1}{2}\frac{\partial^2}{\partial v^{(1)}\partial v^{(1)}*}v_3^{(0)} + \text{c.c.}\right)\phi_N . \tag{3.6}$$

Higher-order terms have been neglected consistently with the system size expansion approximation implied in Eq. (3.4).

It is now a simple matter to derive the Fokker-Planck equation for the Weyl distribution function ϕ_W from Eqs. (3.6) and (2.16). In steady state, the result of simple algebraic manipulations is

$$\frac{\partial \phi_W}{\partial \tau} = \frac{\partial}{\partial v^{(1)}}(v^{(1)} + \sqrt{2}\,iy\,v_3^{(1)} - 4c(v^{(0)}v_3^{(1)} + v_3^{(0)}v^{(1)}))\phi_W$$

$$+ \frac{\partial}{\partial v^{(1)}*}(v^{(1)} - \sqrt{2}\,iy\,v_3^{(1)} - 4c(v^{(0)}*v_3^{(1)} + v_3^{(0)}v^{(1)}*))\phi_W$$

$$+ \frac{\partial}{\partial v_3^{(1)}}\left(2v_3^{(1)} + \frac{iy}{\sqrt{2}}(v^{(1)} - v^{(1)}*) + 4c(v^{(0)}v^{(1)}* + v^{(1)}v^{(0)}*)\right)\phi_W + \frac{xy}{2(1+x^2)}\frac{\partial^2}{\partial v_3^{(1)2}}\phi_W$$

$$+ \frac{y/x+x^2}{1+x^2}\frac{\partial^2}{\partial v^{(1)}\partial v^{(1)}*}\phi_W + \frac{iy}{\sqrt{2}(1+x^2)}\frac{\partial^2}{\partial v_3^{(1)}\partial v^{(1)}}\phi_W - \frac{iy}{\sqrt{2}(1+x^2)}\frac{\partial^2}{\partial v_3^{(1)}\partial v^{(1)}*}\phi_W . \tag{3.7}$$

The diffusion equation (3.7) in terms of Cartesian coordinates takes the simple form

$$\frac{\partial \phi_W}{\partial \tau} = -\frac{\partial}{\partial v_1}(-\alpha v_1 \phi_W) - \frac{\partial}{\partial v_2}(-\alpha v_2 - \beta v_3)\phi_W - \frac{\partial}{\partial v_3}(-2v_3 + 2\gamma_0 v_2)\phi_W$$

$$+ \frac{1}{4(1+x^2)}\left(\frac{y}{x} + x^2\right)\left(\frac{\partial^2}{\partial v_1^2} + \frac{\partial^2}{\partial v_2^2}\right)\phi_W + \frac{xy}{2(1+x^2)}\frac{\partial^2}{\partial v_3^2}\phi_W + \frac{y}{\sqrt{2}(1+x^2)}\frac{\partial^2}{\partial v_2\partial v_3}\phi_W , \tag{3.8}$$

where the Cartesian variables and the parameters have been defined by Eqs. (2.30) and (2.33). The diffusion matrix associated with Eq. (3.8) is

$$D_W = \frac{1}{2(1+x^2)}\begin{pmatrix} \tfrac{1}{2}(y/x+x^2) & 0 & 0 \\ 0 & \tfrac{1}{2}(y/x+x^2) & y/\sqrt{2} \\ 0 & y/\sqrt{2} & xy \end{pmatrix} . \tag{3.9}$$

It is easily seen that all the conditions for positive definiteness of D_W are satisfied, i.e.,

$$\tfrac{1}{2}(y/x+x^2) > 0 , \tag{3.10a}$$

$$\det D_W = \frac{1}{32(1+x^2)^3}\left(\frac{y}{x}+x^2\right)x^3 y > 0 , \tag{3.10b}$$

$$\det\begin{pmatrix} \tfrac{1}{2}(y/x+x^2) & y/\sqrt{2} \\ y/\sqrt{2} & xy \end{pmatrix} = \tfrac{1}{2}x^3 y > 0 . \tag{3.10c}$$

It follows at once that the Fokker-Planck equation

(3.8) for the Weyl distribution has a well-behaved solution which is Gaussian in the variables

$$v_1 = \frac{v^{(1)} + v^{(1)}*}{2} , \quad v_2 = \frac{v^{(1)} - v^{(1)}*}{2i} , \quad v_3 = v_3^{(1)} .$$

The appropriate second moments of the Cartesian variables are

$$\langle v_1^2 \rangle_W = \tfrac{1}{2}\left(\mu + \eta + \frac{1}{2(1+x^2)}\right) ,$$

$$\langle v_2^2 \rangle_W = \tfrac{1}{2}\left(\eta + \frac{1}{2(1+x^2)} - \mu\right) ,$$

$$\langle v_3^2 \rangle_W = \nu , \tag{3.11}$$

$$\langle v_1 v_2 \rangle_W = \langle v_1 v_3 \rangle = 0 ,$$

$$\langle v_2 v_3 \rangle_W = \delta + \frac{x}{2\sqrt{3}(1+x^2)} ,$$

where the parameters μ, ν, η, and δ have been defined in Eq. (2.17).

One can also verify from the above that the

correlation matrix

$$\sigma^{(W)} = \begin{pmatrix} \langle v_1^2 \rangle_W & \langle v_1 v_2 \rangle_W & \langle v_1 v_3 \rangle_W \\ \langle v_2 v_1 \rangle_W & \langle v_2^2 \rangle_W & \langle v_2 v_3 \rangle_W \\ \langle v_3 v_1 \rangle_W & \langle v_3 v_2 \rangle_W & \langle v_3^2 \rangle_W \end{pmatrix} \qquad (3.12)$$

is also positive definite. Note that the various expectation values $\langle \cdots \rangle_W$ denote symmetrized averages, i.e.,

$$\langle v_1 v_3 \rangle = \tfrac{1}{2} \langle \{\tfrac{1}{2}(S^+ + S^-), S^z\} \rangle . \qquad (3.13)$$

From Eq. (3.13) and similar equations for the other second moments in (3.12), it follows that the characteristic function C_W can be written

$$C_W = \exp(-\tfrac{1}{2}\xi^2 \langle S^+ S^+ \rangle - \tfrac{1}{2}\xi^{*2} \langle S^- S^- \rangle$$
$$-\tfrac{1}{2}\eta^2 \langle S^z S^z \rangle - \xi \xi^* \tfrac{1}{2}\langle\{S^+, S^-\}\rangle$$
$$-\eta \xi \tfrac{1}{2}\langle\{S^+, S^z\}\rangle - \eta \xi^* \tfrac{1}{2}\langle\{S^-, S^z\}\rangle) . \qquad (3.14)$$

Thus, to first order in the system size expansion we have shown that the Weyl distribution function ϕ_W and its characteristic function C_W have a well-behaved Gaussian structure.

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APPENDIX A: OUTLINE OF THE DERIVATION OF EQ. (2.8)

The mapping of the master equation (2.1) into a c-number diffusion equation via the normal-ordered characteristic function (2.5) is a rather standard procedure which has been discussed in several publications (see, for example, Haken in Ref. 7). For convenience we summarize the main steps in this appendix.

Let

$$C_N(\xi, \xi^*, \eta, t) = \langle D \rangle \equiv \text{Tr}[w(t)D], \qquad (A1)$$

where

$$D \equiv e^{i\xi S^+} e^{i\eta S^z} e^{i\xi^* S^-} . \qquad (A2)$$

The time evolution of C_N follows at once from Eq. (A1) and the master equation (2.1):

$$\frac{\partial C_N}{\partial t} = \text{Tr}\left(\frac{\partial w}{\partial t}D\right) = -\gamma_\perp \sum_i \langle Ds_i^+ s_i^- - 2s_i^+ Ds_i^- + s_i^+ s_i^- D\rangle$$
$$- (g^2/\kappa)\langle DS^+ S^- - 2S^+ DS^- + S^+ S^- D\rangle$$
$$+ i\Omega\langle[S^+ + S^-, D]\rangle . \qquad (A3)$$

The main goal of the following calculation is to reduce the right-hand side of Eq. (A3) to a combination of terms involving on C_N and its derivatives with respect to the independent variables $\xi, \xi^*,$ and η.

To this purpose consider the following operator identities:

$$S^+ D = \frac{\partial}{\partial(i\xi)}D , \quad Ds^z = \frac{\partial}{\partial(i\eta)}D + (i\xi^*)\frac{\partial}{\partial(i\xi^*)}D , \qquad (A4)$$

$$DS^+ = e^{i\xi S^+} e^{i\eta S^z} S^+ e^{i\xi^* S^-} + e^{i\xi S^+} e^{i\eta S^z}[e^{i\xi^* S^-}, S^+]$$
$$= \frac{\partial}{\partial(i\xi)}D + (e^{i\eta} - 1)\frac{\partial}{\partial(i\xi)}D - 2i\xi^*\frac{\partial}{\partial(i\eta)}D$$
$$- (i\xi^*)^2 \frac{\partial}{\partial(i\xi^*)}D , \qquad (A5)$$

$$DS^- = \frac{\partial}{\partial(i\xi^*)}D . \qquad (A6)$$

The identity (A5) can be verified using the commutations

$$[e^{i\xi^* S^-}, S^+] = -(2i\xi^* S^z + (i\xi^*)^2 S^-)e^{i\xi^* S^-} , \qquad (A7)$$

$$[e^{i\eta S^z}, S^+] = (e^{i\eta} - 1)S^+ e^{i\eta S^z} . \qquad (A8)$$

The identities (A4) and (A6) are a trivial consequence of the definition of D.

Next we observe that the cooperative term can be cast into the form

$$\langle DS^+ S^- - 2S^+ DS^- + S^+ S^- D\rangle = \langle[D, S^+]S^- + hc\rangle . \qquad (A9)$$

Hence from Eqs. (A4)-(A6) we can easily verify that

$$\langle[D, S^+]S^-\rangle + c.c. = (e^{i\eta} - 1)\frac{\partial^2}{\partial(i\xi)\partial(i\xi^*)}C_N$$
$$- 2i\xi^* \frac{\partial^2}{\partial(i\eta)\partial(i\xi^*)}$$
$$+ (i\xi^*)^2 \frac{\partial^2}{\partial(i\xi^*)^2}C_N + c.c. \qquad (A10)$$

Similarly, the coherent driving term can be written in the form

$$\langle[S^+ + S^-, D]\rangle = \langle[S^+, D]\rangle + c.c.$$
$$= -(e^{i\eta} - 1)\frac{\partial}{\partial(i\xi)}C_N + 2i\xi^*\frac{\partial}{\partial(i\eta)}C_N$$
$$+ (i\xi^*)^2 \frac{\partial}{\partial(i\xi^*)}C_N + c.c. \qquad (A11)$$

The single-atom decay term requires more attention. We start by writing it in the more convenient form

$$\sum_i \langle Ds_i^+ s_i^- - 2s_i^+ Ds_i^- + s_i^+ s_i^- D\rangle$$
$$= \langle D(\tfrac{1}{2}N + S^z)\rangle + \langle(\tfrac{1}{2}N + S^z)D\rangle - 2\sum_i \langle s_i^+ Ds_i^-\rangle . \qquad (A12)$$

The first two terms on the right-hand side of Eq. (A12) can be handled with the help of the identity

$$DS^z = \frac{\partial}{\partial(i\eta)}D + i\xi^* \frac{\partial}{\partial(i\xi^*)}D, \qquad (A13)$$

which can be easily proved using the commutator

$$[e^{i\xi^* S^-}, S^z] = i\xi^* \frac{\partial}{\partial i\xi^*} e^{i\xi^* S^-}. \qquad (A14)$$

The remaining term will be written as follows:

$$\sum_i \langle s_i^+ D s_i^- \rangle = \sum_i \langle (s_i^+ e^{i\xi s_i^+} e^{i\eta s_i^z} e^{i\xi^* s_i^-} s_i^-) \prod_{j \neq i} D_j \rangle, \qquad (A15)$$

where D_j is the normal-ordered exponential operator containing s_i^+, s_i^-, and s_i^z instead of the collective operators $S^\pm, S^z$. Next, we note the identities

$$s_i^+ e^{i\eta s_i^z} = e^{i\eta s_i^z} s_i^+ e^{-i\eta}, \qquad (A16)$$

$$s_i^+ e^{i\eta s_i^z} s_i^- = e^{i\eta s_i^z} s_i^+ s_i^- e^{-i\eta}$$
$$= e^{-i\eta} e^{i\eta s_i^z}(\tfrac{1}{2} + s_i^z), \qquad (A17)$$

which hold for spin-$\frac{1}{2}$ operators. Finally, with the help of Eqs. (A16) and (A17) we arrive at

$$\sum_i \langle (s_i^+ e^{i\xi s_i^+} e^{i\eta^* s_i^z} e^{i\xi^* s_i^-} s_i^-) \prod_{j \neq i} D_j \rangle$$

$$= e^{-i\eta} \langle e^{i\xi S^+} (\tfrac{1}{2} N + S^z) e^{i\eta S^z} e^{i\xi^* S^-} \rangle$$

$$= e^{-i\eta} \left(\frac{N}{2} + \frac{\partial}{\partial(i\eta)} \right) C_N. \qquad (A18)$$

The density function $P_N(Z, Z^*, m)$ is defined in (2.7) as the Fourier transform of C_N. Equation (2.8) follows from the Fourier transform of the equation of motion for the characteristic function C_N after appropriate substitution of Eqs. (A10)–(A12) and (A18) into (A3).

APPENDIX B: ROLE OF THE SINGLE-ATOM DECAY TERM

The objective of this appendix is to analyze the role of the single-atom decay term in the bistability problem and to show explicitly that there is no optical bistability if $\gamma_\perp = 0$. An account of this analysis and some numerical solutions have been reported.[16] Here we examine this problem in the framework of the Fokker-Planck equation. In terms of the scaled variables

$$\tau = 2g^2 N t / \kappa, \quad z = N v,$$

$$\Omega_0 = \Omega \kappa / 2 g^2 N, \quad m = N v_z,$$

$$P d^2 z \, dm = Q \, d^2 v \, dv_z,$$

the diffusion equation for the atomic quasi-probability distribution takes the form

$$\frac{\partial Q}{\partial \tau} = - \left(\frac{\partial}{\partial v} v v_3 + \frac{\partial}{\partial v^*} v^* v_3 - \frac{\partial}{\partial v_3} v v^* \right) Q + i\Omega \left(2 \frac{\partial}{\partial v} v_3 - 2 \frac{\partial}{\partial v^*} v_3 + \frac{\partial}{\partial v_3}(v - v^*) \right) Q$$

$$+ \frac{1}{N} \left(\frac{1}{2} \frac{\partial^2}{\partial v^2} v^2 + \frac{1}{2} \frac{\partial^2}{\partial v^{*2}} v^{*2} - \frac{1}{2} \frac{\partial^2}{\partial v_3^2} v v^* \right) Q - \frac{i\Omega}{N} \left(\frac{\partial^2}{\partial v^2} v - \frac{\partial^2}{\partial v^{*2}} v^* + \frac{1}{2} \frac{\partial^2}{\partial v_3^2}(v - v^*) \right) Q + O\left(\frac{1}{N^2} \right). \qquad (B1)$$

The Fokker-Planck equation shows that the fluctuations are of order $1/N$ provided that the system has a well-defined macroscopic behavior. Hence, after setting

$$v = v^{(0)}(t) + \frac{1}{\sqrt{N}} v^{(1)}, \quad v_3 = v_3^{(0)}(t) + \frac{1}{\sqrt{N}} v_3^{(1)}, \qquad (B2)$$

we find that the macroscopic equations for $v^{(0)}, v^{(0)}*, v_3^{(0)}$ are

$$\dot{v}^{(0)} = v_3^{(0)} v^{(0)} - 2i\Omega_0 v_3^{(0)},$$

$$\dot{v}^{(0)}* = v_3^{(0)} v^{(0)}* + 2i\Omega_0 v_3^{(0)},$$

$$\dot{v}_3^{(0)} = -i\Omega_0 [v^{(0)} - v^{(0)}*] - v^{(0)} v^{(0)}*, \qquad (B3)$$

and that the equation of motion for the distribution function $\phi(v^{(1)}, v^{(1)}*, v_3^{(1)}, \tau)$ takes the form

$$\frac{\partial \phi}{\partial \tau} = - \frac{\partial}{\partial v^{(1)}} [(v_3^{(0)} v^{(1)} + v_3^{(1)} v^{(0)}) \Phi] - \frac{\partial}{\partial v^{(1)}*} [(v_3^{(0)} v^{(1)}* + v_3^{(1)} v^{(0)}*) \Phi]$$

$$+ \frac{\partial}{\partial v_3^{(1)}} [(v^{(0)} v^{(1)}* + v^{(1)} v^{(0)}*) \Phi] + i\Omega \left(2 \frac{\partial}{\partial v^{(1)}} v_3^{(1)} - 2 \frac{\partial}{\partial v^{(1)}*} v_3^{(1)}* + \frac{\partial}{\partial v_3^{(1)}} [v^{(1)} - v^{(1)}*] \right) \Phi$$

$$+ \left(\frac{1}{2} \frac{\partial^2}{\partial v^{(1)2}} [v^{(0)}(v^{(0)} - 2i\Omega) \Phi] - \frac{1}{4} \frac{\partial^2}{\partial v_3^{(1)2}} [v^{(0)} v^{(0)}* + i\Omega(v^{(0)} - v^{(0)}*)] \Phi + \text{c.c.} \right). \qquad (B4)$$

We first analyze the macroscopic equations (B3). They are characterized by the first integral of the motion

$$v^{(0)}(t)v^{(0)}*(t) + (v_3^{(0)}(t))^2 = \text{const}, \tag{B5}$$

which is the macroscopic analog of the conservation law $S^2 = \text{const}$. As it is usually done we set the constant in Eq. (B5) equal to $\frac{1}{4}$ and write

$$v^{(0)} = \tfrac{1}{2}(x_0 + iy_0), \quad v_3^{(0)}(t) = \tfrac{1}{2}z_0. \tag{B6}$$

The new dependent variables x_0, y_0, and z_0 satisfy the set of coupled equations

$$\dot{x}_0 = \tfrac{1}{2}z_0 x_0,$$
$$\dot{y}_0 = \tfrac{1}{2}z_0 y_0 - 2\Omega_0 z_0, \tag{B7}$$
$$\dot{z}_0 = \tfrac{1}{2}(z_0^2 - 1) + i\Omega_0 y_0,$$

which are identical to the neoclassical equations of Stroud and Jaynes.[17] In Ref. 17 both the time-dependent and time-independent solutions have been obtained. These solutions fall in two categories, depending on whether $\Omega_0 > \frac{1}{4}$ or $\Omega_0 < \frac{1}{4}$.

We will study these two cases separately.

A. $\Omega_0 < \frac{1}{4}$

In this case the solutions have a damped oscillatory character and approach a well-defined steady state. If we assume that the initial atomic dipole moment is zero, then in steady state we have

$$x_0 = 0, \quad y_0 = 4\Omega_0, \quad z_0 = -[1 - (4\Omega_0)^2]^{1/2}. \tag{B8}$$

This, in turn, implies

$$\langle S^+ \rangle = 2Ni\Omega_0,$$
$$\langle S^z \rangle = -\tfrac{1}{2}N[1 - (4\Omega_0)^2]^{1/2}. \tag{B9}$$

It is also clear from Eq. (B4) that in the transient regime the fluctuations are of the order $1/N$. For the steady-state fluctuations, we find from Eq. (B8) that all the diffusion coefficients in Eq. (B4) are identically zero. Furthermore, once the parameter Ω is fixed, the solutions are unique.

B. $\Omega_0 > \frac{1}{4}$

In this case the solutions are purely oscillatory. Stroud and Jaynes[17] have shown that a small amount of detuning causes the solutions to decay to a well-defined steady-state value. They have also presented numerical solutions for the detuned case.

Since the motion is purely oscillatory in this case, with a frequency of oscillation $[(4\Omega_0)^2 - 1]^{1/2}$, one may consider time-averaged solutions.[18] After

integrating the second of Eqs. (B7) it is clear that the time average of $z_0(t)$ is zero. The time average of y_0 can be calculated as follows. We write the second of Eqs. (A7) as

$$dt = \frac{dy_0}{(1 - y_0^2)^{1/2}(\tfrac{1}{2}y_0 - 2\Omega_0)}. \tag{B10}$$

If at $t = 0$, $y_0 = 0$ the solution of Eq. (B10) is

$$y_0(t) = \frac{1}{\beta}\left(1 - \frac{(1-\beta^2)^{1/2}}{1 - \beta\sin\{[(1-\beta^2)^{1/2}/\beta]t + \theta\}}\right), \tag{B11}$$

where

$$\beta = \tfrac{1}{4}\Omega,$$
$$\theta = \sin^{-1}\left(\frac{1 - (1-\beta^2)^{1/2}}{\beta}\right). \tag{B12}$$

The solution for z_0 is

$$z_0(t) = -[1 - y_0^2(t)]^{1/2}. \tag{B13}$$

Equation (A11) gives the explicit time dependence of $y_0(t)$. The leading term of the time average of y_0 is

$$\langle y_0 \rangle_{\text{av}}(t) \approx \tfrac{1}{2}\beta. \tag{B14}$$

Similarly, the leading term of the time average of $y_0^2(t)$ is

$$\langle y_0^2 \rangle_{\text{av}}(t) = \tfrac{1}{2}. \tag{B15}$$

Equation (A15), along with the conservation law (B5) implies that

$$\langle z_0^2 \rangle_{\text{av}}(t) = \tfrac{1}{2}. \tag{B16}$$

The fluctuations around the macroscopic values can be analyzed explicitly from the solutions of Eq. (B4).

In the presence of a small amount of detuning Δ, the steady-state solutions behave like $x_0 \sim O(\Delta)$, $y_0 \sim O(\Delta^2)$, $z_0 \sim O(\Delta^2)$, and hence in the steady state, the system does not really have a macroscopic behavior. If this is the case, then the conservation law

$$S^2 = \text{const} = \langle \tfrac{1}{2}(S^+S^- + S^-S^+) + S^zS^z \rangle$$

implies that the fluctuations should be important. This in a sense is also reflected by our time-averaged solutions

$$\langle y_0(t) \rangle_{\text{av}} = \langle z_0(t) \rangle_{\text{av}} = 0, \quad \langle y_0^2(t) \rangle_{\text{av}} = \langle z_0^2(t) \rangle_{\text{av}} = \tfrac{1}{2}.$$

In conclusion, we have seen that (a) for $\Omega_0 < \frac{1}{4}$, the system has a well-defined unique steady state (no bistability) with

$$\langle S^+ \rangle = 2i\Omega_0 N, \quad \langle S^z \rangle = -\tfrac{1}{2}N[1 - (4\Omega_0)^2]^{1/2}.$$

The fluctuations in steady state are even smaller than $O(1/N)$, i.e., they are at least of order $N^{-3/2}$; and (b) for $\Omega_0 > \frac{1}{4}$, the system exhibits oscillations with Eq. (B11) giving the explicit oscilla-tory solution. For $\Omega_0 > \frac{1}{4}$, the time-averaged values are $\langle y_0 \rangle_{av}(t) \approx \frac{1}{2}\beta$, $\langle y_0^2 \rangle_{av}(t) = \frac{1}{2}$, $\langle z_0^2 \rangle_{av}(t) = \frac{1}{2}$. Again the system does not exhibit bistability in agreement with our numerical calculations.[16,19]

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