PHYSICAL REVIEW A **67**, 032103 (2003)

Minimum-correlation mixed quantum states

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(Received 21 October 2002; published 13 March 2003)

We consider states leading to the equality sign in the uncertainty inequalities associated with correlations in open quantum systems which have been recently derived by Ponomarenko and Wolf [Phys. Rev. A **63**, 062106 (2001)]. The new inequalities involve fluctuations defined in terms of the square of the density operator that characterizes mixed states. We find the minimum-correlation states associated with the quadratures of single-mode and two-mode electromagnetic fields in a cavity and for the angular momentum operators which can describe atomic degrees of freedom. We show that while in the case of single-mode quadratures the functional form of the minimum-correlation state is uniquely specified, this is not so for the other pairs of noncommuting operators. In general, the states with the least amount of correlations are mixed and they exhibit squeezing.

DOI: 10.1103/PhysRevA.67.032103 PACS number(s): 03.65.Ta, 42.50.Dv

I. INTRODUCTION AND GENERALIZED UNCERTAINTY INEQUALITIES FOR MIXED STATES

The Heisenberg uncertainty principle [1] has played an important role in the development of modern quantum theory of measurement [2] and more recently, in research on quantum computation and quantum information [3]. The mathematical formulation of such a principle was first given by Robertson who has derived the now standard uncertainty relation (UR) [4] in the form [5]

$$\langle(\Delta A)^2\rangle\langle(\Delta B)^2\rangle \geqslant 1/4|\langle[A,B]_-\rangle|^2. \quad (1)$$

Here the angle brackets $\langle\cdots\rangle \equiv \mathrm{Tr}(\rho\cdots)$ denote the average over an ensemble of quantum systems that are prepared in the same state with the density operator ρ; $[\,,]_-$ denotes the commutator of a pair of operators, $[A,B]_- \equiv AB-BA$, and $\langle(\Delta A)^2\rangle = \langle(A-\langle A\rangle)^2\rangle$ is the variance of the operator A.

The states that lead to the equality sign in inequality (1) are the minimum-uncertainty states that can be prepared under ideal circumstances when the system is isolated from the outside world. The minimum-uncertainty states have played a significant role in quantum optics. In fact, all known coherent and squeezed states in quantum optics have been shown to be solutions of a certain eigenvalue problem that arises when a state corresponding to the equality sign in the Heisenberg uncertainty relation is sought (see, for example, Ref. [6]). It should be pointed out, however, that, in general, any preparation of a quantum system involves interaction of the system with the environment, resulting in the system being prepared in a mixed state. Such a mixed state clearly cannot lead to the equality sign in inequality (1) because of the additional noise caused by the interaction of the system with the environment. In this connection, we recall that mixed states are characterized by the fact that the square of the density operator is not equal to the density operator itself. It is therefore instructive to explore possible uncertainty inequalities involving the square of the density operator.

To date, several approaches to this problem have been proposed [7–10]. One approach [8] consists in deriving uncertainty inequalities associated with the following nonnegative quantities:

$$\langle(\Delta A)^2\rangle_+ \equiv \mathrm{Tr}([\Delta A,\rho]_+^2) \quad (2a)$$

and

$$\langle(\Delta A)^2\rangle_- \equiv -\mathrm{Tr}([\Delta A,\rho]_-^2), \quad (2b)$$

where $[A,B]_+ \equiv AB+BA$ is an anticommutator of a pair of operators A and B. The corresponding uncertainty inequalities for a pair of noncommuting operators A and B can be shown [8] to take the form

$$\langle(\Delta A)^2\rangle_-\langle(\Delta B)^2\rangle_+ \geqslant |\mathrm{Tr}([A,B]_-\rho^2)|^2, \quad (3a)$$

$$\langle(\Delta B)^2\rangle_-\langle(\Delta A)^2\rangle_+ \geqslant |\mathrm{Tr}([A,B]_-\rho^2)|^2. \quad (3b)$$

It follows at once from Eq. (2b) that in the basis of the eigenstates of A, $A|a\rangle = a|a\rangle$, the generalized measure of uncertainty $\langle(\Delta A)^2\rangle_-$ is given by the expression [11]

$$\langle(\Delta A)^2\rangle_- = \sum_{a,a'} (a-a')^2|\langle a|\rho|a'\rangle|^2. \quad (4)$$

It is seen from Eq. (4) that such a generalized measure of uncertainty characterizes the rms width of correlations between pairs of eigenvalues of A in the mixed state ρ. It can also be deduced from the definitions (2) that in the case of a pure state, the generalized uncertainties reduce to usual variances apart from a numerical factor. The squares of the correlation widths $\langle(\Delta A)^2\rangle_-$ and $\langle(\Delta B)^2\rangle_-$ attain their lower bounds for any state whose density operator satisfies the pair of equations [8]

$$[\Delta B,\rho]_+ + i\lambda[\Delta A,\rho]_- = 0, \quad (5a)$$

$$[\Delta A,\rho]_+ + i\mu[\Delta B,\rho]_- = 0. \quad (5b)$$

Here λ and μ are arbitrary real constants.

It should be noted that not only does the pair of inequalities (3) provide the information about the structure of a mixed state [12] which is, in general, *different* from the information contained in the Heisenberg-Robertson inequality, but the new URs have also a different physical meaning. Inequalities (3) specify lower bounds for the widths of correlations of a pair of noncommuting operators measured on different replicas of an ensemble of the systems prepared in the same mixed state. In this connection, a natural question arises regarding the structure of the *minimum-correlation* states, i.e., the states that satisfy Eqs. (5) for different pairs on noncommuting operators.

In this work, we address this question by considering minimum-correlation states for single- and two-mode quadratures of the electromagnetic field and for the angular momentum operators. In the case of the single-mode field, it was shown in Ref. [8] that *either* of inequalities (3) becomes an equality for a squeezed thermal state. In this paper, we demonstrate that a squeezed Gaussian mixed state is the most *general* state for which the correlation widths of the quadratures *simultaneously* attain their lower bounds. We also demonstrate that there is a whole family of minimum-correlation states associated with different two-mode quadratures of the field. We then determine minimum-correlation states for the angular momentum operators, which are often encountered in a theoretical description of the atomic squeezed states [13]. We show that such states are, in general, also not unique and that in a certain range of parameters, they exhibit nonclassical features.

II. MINIMUM-CORRELATION MIXED STATES FOR A SINGLE-MODE ELECTROMAGNETIC FIELD

In this case, coordinate x and momentum p operators of a simple harmonic oscillator may serve as the quadrature operators. Assuming, for simplicity, that $\langle x\rangle=\langle p\rangle=0$, we can express Eqs. (5) for coordinate and momentum operators in the form

$$[x,\rho]_+ + i\lambda[p,\rho]_- = 0, \tag{6a}$$

$$[p,\rho]_+ + i\mu[x,\rho]_- = 0. \tag{6b}$$

On transforming to coordinate representation in Eqs. (6), we obtain the pair of equations

$$\left(\frac{\partial}{\partial x_1}+\frac{\partial}{\partial x_2}+\frac{x_1+x_2}{\lambda}\right)\langle x_2|\rho|x_1\rangle = 0, \tag{7a}$$

$$\left[\frac{\partial}{\partial x_1}-\frac{\partial}{\partial x_2}+\mu(x_1-x_2)\right]\langle x_2|\rho|x_1\rangle = 0. \tag{7b}$$

Next, on introducing the variables $\xi = x_1 - x_2$ and $\eta=(x_1+x_2)/2$, we can transform Eqs. (7) to

$$\left(\frac{\partial}{\partial \eta}+\frac{2\eta}{\lambda}\right)\rho(\eta,\xi) = 0, \tag{8a}$$

$$\left(\frac{\partial}{\partial \xi}+\frac{\mu\xi}{2}\right)\rho(\eta,\xi) = 0. \tag{8b}$$

The solution of Eq. (8a) is readily obtained in the form

$$\rho(\eta,\xi) = f(\xi)e^{-\eta^2/\lambda}, \tag{9}$$

and that of Eq. (8b) takes the form

$$\rho(\eta,\xi) = g(\eta)e^{-\mu\xi^2/4}, \tag{10}$$

where $f(\xi)$ and $g(\eta)$ are arbitrary functions. On combining Eq. (9) and Eq. (10), we obtain for the density matrix of the most general minimum-correlation state for the x,p pair the expression

$$\rho(x_1,x_2) \propto \exp\left[-\frac{(x_1+x_2)^2}{4\lambda}-\mu\frac{(x_1-x_2)^2}{4}\right]. \tag{11}$$

It follows at once from Eq. (11) that, since the density matrix as a function of x_1 and x_2 does not factorize, the state with the minimal correlation widths of x and of p must be mixed. The Wigner function $W(x,p)$, of a quantum state is defined as [14]

$$W(x,p) \equiv \int_{-\infty}^{\infty}\frac{dx'}{2\pi}\rho(x-x'/2,\, x+x'/2)\, e^{ipx'}. \tag{12}$$

On substituting from Eq. (11) into Eq. (12) and on performing the straightforward integration, we obtain for the Wigner function of the minimum-correlation state the expression

$$W(x,p) \propto \exp\left(-\frac{x^2}{4\lambda}-\frac{p^2}{\mu}\right). \tag{13}$$

It can be concluded from Eq. (13) that as long as $4\lambda \neq \mu$ and $\lambda\mu \geqslant 1/4$, a general minimum-correlation state is a squeezed Gaussian state. Particular cases include squeezed vacuum and squeezed thermal states. The latter states can be produced, for example, by degenerate four-wave mixing in a cavity coupled to a thermal reservoir in order to model relaxation processes [15].

III. MIXED STATE WITH MINIMUM CORRELATIONS FOR A TWO-MODE FIELD

Let us introduce the two-mode quadratures X_1 and X_2 of the electromagnetic field by the expressions

$$X_1 = a + b^\dagger + a^\dagger + b, \tag{14a}$$

$$X_2 = (a + b^\dagger - a^\dagger - b)/i. \tag{14b}$$

The corresponding minimum-correlation state can be found by solving the pair of equations

$$[X_1,\rho]_+ + i\lambda[X_2,\rho]_- = 0, \tag{15a}$$

$$[X_2,\rho]_+ + i\mu[X_1,\rho]_- = 0, \tag{15b}$$

where we have assumed that $\langle X_1\rangle=0$ and $\langle X_2\rangle=0$. Introducing the two-mode Husimi Q function, namely, $Q(\alpha,\beta)\equiv 1/\pi^2\langle\beta,\alpha|\rho|\alpha,\beta\rangle$, and using the properties of coherent states $|\alpha,\beta\rangle$, we can convert Eqs. (15) to the pair of differential equations

$$(1-\lambda)\alpha Q+(1+\lambda)\frac{\partial Q}{\partial\alpha}+(1+\lambda)\beta Q+(1-\lambda)\frac{\partial Q}{\partial\beta}+\text{c.c.}=0, \tag{16a}$$

$$(1+\mu)\alpha Q-(1-\mu)\frac{\partial Q}{\partial\alpha}-(1-\mu)\beta Q+(1+\mu)\frac{\partial Q}{\partial\beta}-\text{c.c.}=0. \tag{16b}$$

The analysis of Eqs. (16) leads to the conclusion that these equations have no unique solution. Any particular solution depends on a functional form of Q on characteristics.One can then consider a Gaussian solution. It should be noticed, however, that although mathematically a general Gaussian function may satisfy Eqs. (16), a physically acceptable solution is restricted to the classes that can be realized in experiment. Specifically, any Gaussian solution can be obtained using an appropriate nonlinear process in the undepleted pump approximation. Since all such processes are very sensitive to phase-matching conditions, only one can play a significant role in a particular experimental situation. This circumstance will dictate a specific realization of a Gaussian state. We will now consider some Gaussian solutions of this kind which can be generated experimentally.

To this end, let us look for a solution to Eqs. (16) in the form

$$Q_a(\alpha,\beta)\propto\exp[-A|\alpha|^2-B|\beta|^2+C\alpha\beta+\text{c.c.}], \tag{17}$$

where $\mathrm{Re}\,A\geqslant 0$, $\mathrm{Re}\,B\geqslant 0$, and C are yet undetermined parameters. On substituting from Eq. (17) into Eqs. (16), we obtain for the Q function after some algebra, the expression

$$Q_a(\alpha,\beta)\propto\exp\{-(1-C)[\tanh(r)|\alpha|^2+\coth(r)|\beta|^2]\}\times\exp[-C(\alpha\beta+\alpha^*\beta^*)]. \tag{18}$$

Here C is a real constant, and Eqs. (16a) and (16b) are satisfied simultaneously provided that $\lambda=-\mu$. We have also introduced the squeezing parameter r such that $\lambda=e^{-2r}$. The requirement that Q function converge for large values of $|\alpha|$ and $|\beta|$ gives the condition $C\leqslant 1/2$.

The state specified by the Q function in Eq. (18) is readily recognized as a two-mode squeezed thermal state [16], which is a "noisy version" of the two-mode squeezed state introduced in Ref. [17]. Such a state can be shown to be produced in the process of nondegenerate parametric amplification (see Chap. 5 of Ref. [18]) in a cavity coupled to thermal bath. The Hamiltonian describing this process is given by

$$H_a=\hbar(a^\dagger a+b^\dagger b)+\hbar g(ab+a^\dagger b^\dagger), \tag{19}$$

where g is a coupling constant.

Another interesting minimum-correlation state is associated with the choice of the photon number difference Y_1 and the phase difference Y_2 of the two modes as new quadratures that are defined as

$$Y_1=a^\dagger a-b^\dagger b \tag{20a}$$

and

$$Y_2=(a^\dagger b-b^\dagger a)/i. \tag{20b}$$

Repeating the analysis similar to the one that led to Eq. (18), we obtain for the Q function of the minimum-correlation state associated with Y_1 and Y_2 the expression

$$Q_{up}(\alpha,\beta)\propto\exp[-A(|\alpha|^2+|\beta|^2)+A(\alpha^*\beta+\alpha\beta^*)]. \tag{21}$$

Here $A\geqslant 0$ is an arbitrary real constant, and the state with the Q function given by Eq. (21) satisfies both equations of the pair (16) providing the condition $\lambda=-\mu=\pm 1$. Such a state is known to have no nonclassical properties (see Chap. 5 of Ref. [18]), and it can be generated experimentally in the process of parametric up-conversion with the Hamiltonian

$$H_{up}=\hbar(a^\dagger a+b^\dagger b)+\hbar\kappa(ab^\dagger+a^\dagger b). \tag{22}$$

Here κ is a coupling constant.

IV. MINIMUM-CORRELATION MIXED STATES FOR SYSTEMS DESCRIBED BY THE ANGULAR MOMENTUM OPERATORS

In the case of projections, J_x and J_y, $[J_x,J_y]_-=iJ_z$, say, of the angular momentum operator J, Eqs. (5) for the minimum-correlation state take the form

$$[J_x,\rho]_+ + i\lambda[J_y,\rho]_-=0, \tag{23a}$$

$$[J_y,\rho]_+ + i\mu[J_x,\rho]_-=0. \tag{23b}$$

Here we have assumed, for simplicity, that $\langle J_x\rangle=0$ and $\langle J_y\rangle=0$. On introducing an auxiliary operator R_z by the expression

$$R_z=J_x\cosh\theta-iJ_y\sinh\theta, \tag{24}$$

where

$$\cosh\theta\equiv\frac{1}{\sqrt{1-\lambda^2}}\quad\text{and}\quad\sinh\theta\equiv-\frac{\lambda}{\sqrt{1-\lambda^2}}, \tag{25}$$

and $\lambda\neq\pm 1$, we can transform Eq. (23a) to

$$R_z^\dagger\rho+\rho R_z=0. \tag{26}$$

Let us now take a matrix element of Eq. (26) in the basis of eigenstates of R_z. It was shown in Ref. [19] that an eigenstate $|\psi_m\rangle$ of the operator R_z with the eigenvalue m, $-j\leqslant m\leqslant j$, such that

$$R_z|\psi_m\rangle=m|\psi_m\rangle, \tag{27a}$$

$$\langle \psi_m | R_z^\dagger = \langle \psi_m | m \tag{27b}$$

is given by the expression

$$|\psi_m\rangle = \exp(\theta J_z)\exp(-i\pi/2J_y)|jm\rangle. \tag{28}$$

On substituting from Eqs. (27) into Eq. (26) and on introducing the notation $\rho_{m_1,m_2} \equiv \langle \psi_{m_2}|\rho|\psi_{m_1}\rangle$, we obtain the equation

$$\rho_{m_1,m_2}(m_1+m_2)=0. \tag{29}$$

It readily follows from Eq. (29) that the density operator of the state with the minimal widths of angular momentum correlations is given by the expression

$$\rho = \sum_{m=-j}^{j} \rho_{m,-m}|\psi_m\rangle\langle\psi_{-m}| + \text{H.c.} \tag{30}$$

Here $\rho_{m,-m}$ is an arbitrary matrix subject only to the Hermiticity constraint, $\rho_{m,-m}^* = \rho_{-m,m}$, and the constraint $\langle J_x\rangle = \langle J_y\rangle = 0$, which we have imposed earlier. The analysis of Eq. (30), together with the expressions for the first moments of J_x and of J_y, indicates that the latter expressions are equal to zero in the minimum-correlation state for any $\rho_{m,-m}$. It can also be shown by a direct substitution that the density operator given by Eq. (30) satisfies Eq. (23b) with $\mu=0$. Hence Eq. (30) specifies the density operator of the minimum-correlation state for both projections J_x and J_y of the angular momentum.

In order to study nonclassical properties of the minimum-correlation state, we calculate the variances of J_x and of J_y, defined by the expressions

$$\langle \Delta J_x^2\rangle = \text{Tr}(\Delta J_x^2 \rho), \tag{31a}$$

$$\langle \Delta J_y^2\rangle = \text{Tr}(\Delta J_y^2 \rho). \tag{31b}$$

For this purpose, we substitute from Eq. (30) into Eqs. (31) and use the commutation relation

$$[J'_+, J'_-]_- = -2J_z \sinh\theta\cosh\theta. \tag{32}$$

We then obtain for the variances the expressions

$$\langle \Delta J_x^2\rangle = \langle J_z\rangle \tanh(\theta)/2, \tag{33a}$$

$$\langle \Delta J_y^2\rangle = \langle J_z\rangle \coth(\theta)/2 - c(j)/\sinh^2\theta. \tag{33b}$$

Here the function $c(j)$ is defined by the expression

$$c(j) \equiv \frac{\sum_{m=-j}^{j} m^2 \rho_{m,-m}\langle\psi_{-m}|\psi_m\rangle + \text{H.c.}}{\sum_{m=-j}^{j} \rho_{m,-m}\langle\psi_{-m}|\psi_m\rangle + \text{H.c.}}. \tag{34}$$

It is seen from Eqs. (33) that since for real values of θ $\tanh\theta \leqslant 1$, the x component of the angular momentum is squeezed.

Two points are worth making in connection with Eqs. (33) and (34). First of all, for any j equal to an integer number, a minimum-correlation state can be either pure or mixed. In particular, the pure state is realized for $\rho_{m,-m} = \rho_{00}\delta_{m0}$, where δ_{mn} is the Kronecker delta. The density operator of such a state is given by the expression

$$\rho \propto |\psi_0\rangle\langle\psi_0|. \tag{35}$$

In this case $c(j)=0$, and it follows at once from the Heisenberg-Robertson inequality (1) and from Eqs. (33) that such a minimum-correlation pure state is a minimum-uncertainty state as well. Such a state was shown to be a steady state for the system of an even number N of two-level atoms interacting with a broadband squeezed bath [13]. However, if j is equal to a half-integer, the minimum-correlation state must be mixed. We also remark that in order for the variances of J_x and of J_y evaluated in a minimum-correlation state to satisfy inequality (1), the function $c(j)$ has to take on values such that $c(j)/\sinh^2\theta \leqslant 0$.

It should be noticed that the previous approach fails in the degenerate case corresponding to $\lambda = \pm 1$. In this case, Eq. (23a) can be written as

$$J_+\rho + \rho J_- = 0, \tag{36}$$

where $J_\pm = J_x \pm iJ_y$ are the usual raising and lowering operators. One can readily take the matrix elements of Eq. (36) in the basis of the Wigner states $|jm\rangle$ resulting in the equation

$$\sqrt{(j+m_2)(j-m_2+1)}\rho_{m_1,m_2-1} + \sqrt{(j+m_1)(j-m_1+1)}\rho_{m_1-1,m_2} = 0, \tag{37}$$

Let us recall the assumption

$$\text{Tr}(J_x\rho) = \text{Tr}(J_y\rho) = 0, \tag{38}$$

which we have made before. Equations (38) can be rewritten in the basis of the Wigner states, leading to the result

$$\sum_{m=-j}^{j} \rho_{m,m-1} = 0. \tag{39}$$

Equations (37) and (39) specify the density operator of a minimum-correlation state in the degenerate case, $\lambda = 1$.

Since it is difficult to obtain an explicit solution for any value of j, we only consider the cases of $j=1/2$ and $j=1$. In the first case, the solution of Eqs. (37) and (39) is

$$\rho = 1/2[I + \sigma_z \tanh(\beta/2)]. \tag{40}$$

Here I is a unit matrix, σ_z is a Pauli matrix, and β is defined by $e^{-\beta} = \rho_-/\rho_+$, where $\rho_+(\rho_-)$ is a probability of finding the system with the spin up (down). Such a state does not display any nonclassical features and it can be realized experimentally, for instance, by placing a spin-(1/2) particle in a magnetic field and by letting the system reach equilibrium with a thermal bath at temperature T [20].In the case of $j=1$, the density operator of the minimum-correlation state is found to be given by the expression

$$\rho \propto \begin{pmatrix} p_1 & ia & -1 \\ -ia & 1 & -ia \\ -1 & ia & p_{-1} \end{pmatrix}. \tag{41}$$

Here a is an arbitrary real constant, and p_1 and p_{-1} are non-negative real numbers proportional to the probabilities of finding the system in the eigenstates with $J_z = 1$ and with $J_z = -1$, respectively. The calculation of the variances of J_x and of J_y in such a state yields

$$\langle \Delta J_x^2 \rangle = \frac{1}{2} - \frac{1/2}{1+p_1+p_{-1}}, \tag{42a}$$

$$\langle \Delta J_y^2 \rangle = \frac{1}{2} + \frac{3/2}{1+p_1+p_{-1}}. \tag{42b}$$

It can be shown with the help of Eqs. (42) that in this particular state, $\langle \Delta J_x^2 \rangle \geqslant 1/2 |\langle J_z \rangle|$, and hence neither of the components of the angular momentum is squeezed.

V. CONCLUSIONS

We conclude by saying that we have determined the states with the minimal widths of correlations of a pair of noncommuting operators for the cases of single- and two-mode quadratures of the electromagnetic field and for the case of the angular momentum operators. Only in the single-mode case has the minimum-correlation state been found to have a unique functional form of a squeezed Gaussian. The two-mode case features a variety of minimum-correlation states, including a two-mode squeezed thermal state. The nonuniqueness of minimum-correlation states in the two-mode case is due to the fact that we have used only one set of rotated quadratures rather than two sets. The density operator would become, of course, more and more specified as more and more additional constraints are imposed. The structure of the minimum-correlation state associated with the angular momentum operators has been shown to depend on the value of the angular momentum j. For the integer values of j, the minimum-correlation state can be either pure or mixed, whereas for the half-integer values of j, the minimum-correlation state has to be mixed. In both instances, however, the minimum-correlation state exhibits squeezing in one of the components of the angular momentum. The minimum-correlation states may be expected to be useful in studies of decoherence in open quantum systems where the interaction with the environment causes a decay of correlations that are represented by the off-diagonal elements of the density operators of such quantum systems [21].

ACKNOWLEDGMENTS

The authors thank Professor Emil Wolf for a critical reading of the manuscript. This work was supported by the U.S. Air Force Office of Scientific Research under Grant No. F49260-96-1-0400, and by the Engineering Research Program of the Office of Basic Energy Sciences at the U.S. Department of Energy under Grant No. DE-Fg02-90 ER 14119.

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