Strong subadditivity inequality for quantum entropies and four-particle entanglement

Asoka Biswas and G. S. Agarwal
Physical Research Laboratory, Navrangpura, Ahmedabad - 380 009, India

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Strong subadditivity inequality for a three-particle composite system is an important inequality in quantum information theory which can be studied via a four-particle entangled state. We use two three-level atoms in Λ configuration interacting with a two-mode cavity and the Raman adiabatic passage technique for the production of the four-particle entangled state. Using this four-particle entanglement, we study for the first time various aspects of the strong subadditivity inequality.

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I. INTRODUCTION

Entanglement [1] in a composite system refers to certain implicit correlation between the subsystems arising from their interaction. It is the key resource of quantum computation and quantum information processing [2]. Due to recent advances in this field, entanglement has generated renewed interest. There have been different approaches to understand and to quantify entanglement [3]. But so far the entanglement, only in a bipartite pure state has been investigated very extensively. The von Neumann entropy [4] of either of the subsystems provides a good measure of entanglement in this case [5]. This is the quantum partner of the Shannon's entropy [6] in classical information theory and is defined as [7]

$$S(j) = -\mathrm{Tr}_j(\rho_j \log_2 \rho_j) \ , \tag{1}$$

where $j = A, B$. Here, ρ_j is the reduced density operator of the subsystem j and is given by

$$\rho_j = \mathrm{Tr}_l \rho_{AB} \ , \tag{2}$$

where ρ_{AB} is the density operator of the composite system under consideration and $j, l = A, B$, $j \neq l$. In general, the quantities $S(j)$ satisfy the following inequality (due to Araki and Lieb) [8]:

$$|S(A) - S(B)| \leq S(A, B) \leq S(A) + S(B) \ , \tag{3}$$

where $S(A, B)$ is the joint entropy of the composite system comprising A and B. The second part of the above inequality is known as subadditivity inequality [9]. For a pure state, $S(A, B) = 0$ and thus $S(A) = S(B)$. The equality sign in the above relation holds good if and only if the composite density matrix ρ_{AB} can be written as a tensor product of its two reduced density matrices ρ_A and ρ_B, i.e., for a disentangled state. One can define the index of correlation I_c given by the expression $S(A) + S(B) - S(A, B)$ [10], which can also be interpreted as information entropy in quantum information point of view. We note that Kim *et al.* have calculated the entropies of different kinds of pure states including two-mode Fock states and squeezed states [11]. Further, the above relation for entropy has been studied in the context of entangled Gaussian states [12].

So far we have discussed about the measurement of entanglement in a bipartite pure state. If the composite system is in a mixed state (defined by the density operator ρ), the entanglement of formation E_F can be defined in terms of the average von Neumann entropies of the pure states of the decompositions [5]. Wootters has shown the quantity E_F to be an explicit function of ρ [13]. He has introduced the notion of concurrence in this context.

We further notice that from the Schmidt decomposition of a pure bipartite state, one can properly identify the entanglement present in the state [14]. This is also very useful to study bipartite continuous systems [15]. On the other hand, for a mixed state ρ, the separability criterion has been proposed [16] to study entanglement. This is based on positive partial transpose mapping of ρ. Thus the negativity (entanglement monotone) of the eigenvalues of the partial transpose of ρ could be a measure of entanglement in a mixed bipartite system [17]. The concept of negativity as an entanglement measure has been used in context of interaction of atoms with thermal field [18]. The separability criterion has been extended to continuous systems [19] also.

Despite many approaches to define entanglement for a bipartite system, there have been only a few approaches to quantify entanglement in the composite systems of three or more particles [13, 20, 21]. We note that a generalization of Schmidt decomposition in multipartite systems in pure states has been introduced [22]. Coffman *et al.* [23] proposed a measurement of entanglement in a tripartite system in terms of concurrences of the pairs of subsystems. This measure is invariant under permutations of the subsystems. An average entanglement in a four-partite entangled state has been defined in terms of von Neumann entropies of the pairs of subsystems [24]. Very recently, Yukalov has addressed the question more generally and quantified multipartite entanglement [25] in terms of the ratio of norms of an entangling operator and of a disentangling operator in the relevant disentangled Hilbert space.

In this paper, we put forward a possible measurement of entanglement of a four-particle system by studying the entropy of the reduced three-particle system. As mentioned above, the von Neumann entropy is a good measure for entanglement in a bipartite system. For a tripartite composite state, this entropy satisfies a strong subadditivity inequality (SSI) [2], which has many important implications in the subject of quantum information theory. In this paper, we study the properties of a four-particle entangled state through the three-particle entropy and the SSI.

The structure of the paper is as follows. In Sec. II, we provide a brief discussion on strong subadditivity inequality from the quantum information point of view. In Sec. III, we describe a physical model and show the preparation of a four-particle entangled state. In Sec. IV, we study the validity of the SSI in the present context and provide a physical explanation of the results. We conclude this paper by proposing a measurement of the corresponding four-particle entanglement.

II. STRONG SUBADDITIVITY INEQUALITY

We have already mentioned that for a bipartite composite system of two particles A and B, the joint entropy $S(A,B)$ satisfies the subadditivity inequality (3). For a composite system of three particles A, B, and C, this inequality can be extended to the following form [26]:

$$S(A,B,C)+S(B)\leq S(A,B)+S(B,C)\ . \tag{4}$$

This inequality is known as strong subadditivity inequality. The most obvious situation that the equality sign holds in (4) is when the composite density matrix ρ_{ABC} can be written as the tensor product of its three reduced density matrices as $\rho_A\otimes\rho_B\otimes\rho_C$, i.e., when the system is in a disentangled state. However, the more stringent condition for this reads as [9]

$$\log_2(\rho_{ABC})-\log_2(\rho_{AB})=\log_2(\rho_{BC})-\log_2(\rho_B)\ . \tag{5}$$

There have been numerous implications of the above inequality (4) in quantum information theory [2]. Firstly, it refers to the fact that the conditioning on the subsystem always reduces the entropy, i.e., $S(A|B,C)\leq S(A|B)$, where, $S(A|B)=S(A,B)-S(B)$ is the entropy of A conditional on knowing the state of B. Secondly, the above inequality implies that discarding a quantum system never increases mutual information, i.e., $S(A:B)\leq S(A:B,C)$, where, $S(A:B)=S(A)+S(B)-S(A,B)$ is the mutual information of the subsystems A and B. Thirdly, quantum operations never increase mutual information of two subsystems. This means that if the mutual information of the two subsystems A and B becomes $S'(A:B)$ after trace-preserving operation on B, then $S'(A:B)\leq S(A:B)$. Further, this inequality (4) implies that the conditional entropy of the subsystems A, B, and C is also subadditive, i.e., $S(A,B|C)\leq S(A|C)+S(B|C)$.

To verify SSI, one needs to calculate the entropies like $S(A,B,C)$ which clearly requires a three-particle mixed state which we can produce using a pure four-particle entangled state [27]. In the next section, we discuss how one can prepare a pure four-particle entangled state so that we can study SSI for the first time for a system realizable using cavity QED methods.

III. PREPARATION OF FOUR-PARTICLE ENTANGLED STATE

We consider two three-level atoms (A and B) with relevant energy levels in Λ-configuration (see Fig. 1) interacting with a two-mode high quality optical cavity. The specified annihilation operators for the cavity modes are a and b. The atoms are interacting with the cavity mode a in $|e\rangle\leftrightarrow|g\rangle$ transition and with the mode b in $|e\rangle\leftrightarrow|f\rangle$ transition.

The Hamiltonian for the system under rotating wave approximation can be written as

$$H=\hbar(\omega_1 a^\dagger a+\omega_2 b^\dagger b)+\hbar\sum_{k=A,B}\left[\omega_{e_kg_k}|e_k\rangle\langle e_k|+\omega_{f_kg_k}|f_k\rangle\langle f_k|+\{g_{1k}|e_k\rangle\langle g_k|a+g_{2k}|e_k\rangle\langle f_k|b+\text{h.c.}\}\right]\ , \tag{6}$$

where, $\omega_{l_km_k}$ is the atomic transition frequency between the levels $|l_k\rangle$ and $|m_k\rangle$, ω_j $(j=1,2)$ is the respective frequency of the cavity modes a and b, g_{jk} $(j=1,2)$ provides the atom-cavity coupling. We assume g_{jk}'s to be real and function of time.

We start with the initial state $|\psi_i\rangle=|g_A,g_B,n,\mu\rangle$, where n and μ are the initial numbers of photons in the cavity modes a and b, respectively and the two atoms are in $|g\rangle$ state. The state of the system can be expanded in terms of the relevant basis states in the following way:

$$\begin{aligned}|\psi(t)\rangle\ =\ & c_1|g_A,g_B,n,\mu\rangle+c_2|g_A,e_B,n-1,\mu\rangle+c_3|g_A,f_B,n-1,\mu+1\rangle\\ &+c_4|e_A,f_B,n-2,\mu+1\rangle+c_5|e_A,e_B,n-2,\mu\rangle+c_6|f_A,f_B,n-2,\mu+2\rangle\\ &+c_7|f_A,e_B,n-2,\mu+1\rangle+c_8|f_A,g_B,n-1,\mu+1\rangle+c_9|e_A,g_B,n-1,\mu\rangle\ .\end{aligned} \tag{7}$$

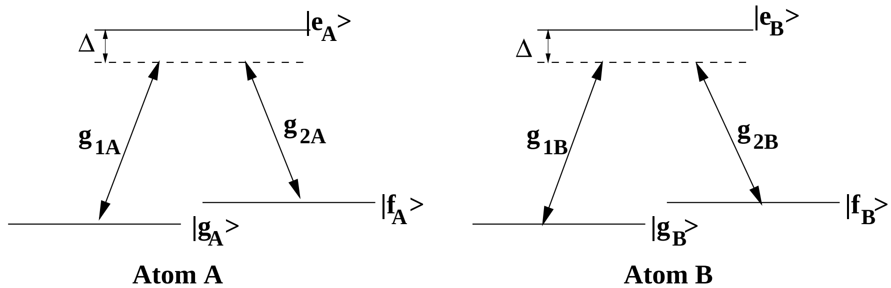

FIG. 1: Level diagram of two three-level atoms in Λ-configuration, interacting with two cavity modes defined by annihilation operators a and b. g_{1k} and g_{2k} (k = A,B) are the atom-cavity coupling terms for the k-th atom. Δ is the common one-photon detuning of the fields.

From the Schrödinger equation we find the following equations of the corresponding probability amplitudes:

$$\begin{aligned}
\dot{d}_1 &= -i(\sqrt{n}g_{1B}d_2 + \sqrt{n}g_{1A}d_9)\ ,\\
\dot{d}_2 &= -i(\sqrt{n}g_{1B}d_1 + \Delta d_2 + \sqrt{\mu+1}g_{2B}d_3 + \sqrt{n-1}g_{1A}d_5)\ ,\\
\dot{d}_3 &= -i(\sqrt{\mu+1}g_{2B}d_2 + \sqrt{n-1}g_{1A}d_4)\ ,\\
\dot{d}_4 &= -i(\sqrt{n-1}g_{1A}d_3 + \Delta d_4 + \sqrt{\mu+1}g_{2B}d_5 + \sqrt{\mu+2}g_{2A}d_6)\ ,\\
\dot{d}_5 &= -i[\sqrt{n-1}(g_{1A}d_2 + g_{1B}d_9) + \sqrt{\mu+1}(g_{2B}d_4 + g_{2A}d_7) + 2\Delta d_5]\ ,\\
\dot{d}_6 &= -i(\sqrt{\mu+2}g_{2A}d_4 + \sqrt{\mu+2}g_{2B}d_7)\ ,\\
\dot{d}_7 &= -i(\sqrt{\mu+1}g_{2A}d_5 + \sqrt{\mu+2}g_{2B}d_6 + \Delta d_7 + \sqrt{n-1}g_{1B}d_8)\ ,\\
\dot{d}_8 &= -i(\sqrt{n-1}g_{1B}d_7 + \sqrt{\mu+1}g_{2A}d_9)\ ,\\
\dot{d}_9 &= -i(\sqrt{n}g_{1A}d_1 + \sqrt{n-1}g_{1B}d_5 + \sqrt{\mu+1}g_{2A}d_8 + \Delta d_9)\ ,
\end{aligned} \tag{8}$$

where, we have used the following transformations:

$$\begin{aligned}
&c_1 = d_1\ ,\quad c_2e^{-i\Delta t} = d_2\ ,\quad c_3 = d_3\ ,\quad c_4e^{-i\Delta t} = d_4\ ,\quad c_5e^{-2i\Delta t} = d_5\ ,\\
&c_6 = d_6\ ,\quad c_7e^{-i\Delta t} = d_7\ ,\quad c_8 = d_8\ ,\quad c_9e^{-i\Delta t} = d_9,\quad \Delta_k = \omega_{e_kl_k} - \omega_{1,2}\ ,
\end{aligned} \tag{9}$$

where, $l_k = g_k, f_k$, Δ_k is the one-photon detuning of the cavity modes for the k-th atom. Here we have assumed that the cavity modes are in two-photon resonance and $\Delta_A = \Delta_B = \Delta$.

Writing these equations (8) in the matrix form $[\dot{d}_i] = -i[M][d_i]$, we find that one of the eigenvalues of the matrix $[M]$ is zero. The corresponding eigenstate is

$$|\psi_0\rangle = \frac{1}{P}\left[\alpha|g_A, g_B, n, \mu\rangle + \beta|f_A, f_B, n-2, \mu+2\rangle - \gamma|g_A, f_B, n-1, \mu+1\rangle - \delta|f_A, g_B, n-1, \mu+1\rangle)\right]\ , \tag{10}$$

where

$$\begin{aligned}
\alpha &= g_{2A}g_{2B}\sqrt{(\mu+1)(\mu+2)}\ ,\ \beta = g_{1A}g_{1B}\sqrt{n(n-1)}\ ,\\
\gamma &= g_{1B}g_{2A}\sqrt{n(\mu+2)}\ ,\ \delta = g_{1A}g_{2B}\sqrt{n(\mu+2)}\ ,\\
P &= \sqrt{\alpha^2+\beta^2+\gamma^2+\delta^2}\ .
\end{aligned} \tag{11}$$

Clearly, this state is an entangled state of four particles, namely, the atoms A and B, and the two modes a and b. Using appropriate time-dependence of the pulses, the four-particle system can be prepared in this state, as discussed in the next section. Recently, there have a few experimental demonstrations of preparation of four-particle entangled state [28] and performance of a C-NOT gate [29]. Interestingly, the state $|\psi_0\rangle$ is a two-atom two-mode multipartite coherent population trapping (CPT) state which is a counterpart of the well-known CPT state for a single atom interacting with two coherent fields [30].

IV. STUDY OF STRONG SUBADDITIVITY INEQUALITY

We next discuss how the state $|\psi_0\rangle$ can be prepared by using Raman adiabatic passage technique. We assume that both the atoms are initially in $|g\rangle$ state. We further assume the time-dependence of the Rabi frequencies g_{jk}

of the two modes as

$$g_{1A} = g_{1B} = g_{10} \exp\left(-(t-T)^2/\tau^2\right) ;$$
$$g_{2A} = g_{2B} = g_{20} \exp\left(-t^2/\tau^2\right) . \quad (12)$$

Here, g_{j0} ($j = 1, 2$) is the amplitude of the respective pulse, τ and T are the width and time-separation respectively, of the two pulses. Note that the pulses are applied in counterintuitive sequence. Under this condition, the atom-cavity system follows the evolution of the state $|\psi_0\rangle$ adiabatically. This state is a zero eigenvalue eigenstate (adiabatic state) of the Hamiltonian (6). During this process, known as stimulated Raman adiabatic technique (STIRAP) [31], the atom-cavity system remains in this state for all times. In the present case, at the end of the evolution, the population of both the atoms are simultaneously transferred to the state $|f\rangle$. However, if the atoms are not in one-photon resonance, i.e., if $\Delta\tau \neq 0$, then this transfer process is not complete. This happens because the system does not remain confined in the null adiabatic state $|\psi_0\rangle$ for $\Delta\tau \neq 0$ [31].

We now investigate the validity of SSI for any trio of quantum systems in the present process. We can express this inequality for any three particles, namely, atom A, atom B, and cavity mode a with number n of photons out of the four-particle system under consideration as

$$E = S(A,B) + S(A,n) - S(A,B,n) - S(A) \geq 0 . \quad (13)$$

Here, S defines the joint von Neumann entropy of the relevant subsystems [see Eq. (1)]. This can be calculated from the state (7) by tracing over the other subsystems, e.g.,

$$S(A,B) = -\mathrm{Tr}_{AB}(\rho_{AB} \log_2 \rho_{AB}) , \quad (14)$$

where, ρ_{AB} is the reduced density matrix of the atoms A and B and is given by

$$\rho_{AB} = \mathrm{Tr}_{n,\mu}(|\psi(t)\rangle\langle\psi(t)|) . \quad (15)$$

We show the time variation of E in Fig. 2. Clearly, $E(t)$ never becomes negative during the evolution and thus the SSI (13) holds for all times.

From Fig. 2, one clearly sees that for $\Delta\tau = 0$, in long time limit, E becomes zero. This means that the subsystems (A, B, and the mode a with photon number n) become disentangled. This happens because of complete adiabatic transfer of population to the level $|f\rangle$ of both the atoms at long time limit. The entire process can be written as

$$|g_A, g_B, n, \mu\rangle \longrightarrow |f_A, f_B, n-2, \mu+2\rangle . \quad (16)$$

We have shown the time-variation of the coefficients α/P, β/P, γ/P, and δ/P [see Eq. (11)] in Fig. 3. This figure reveals the above evolution according to the state $|\psi_0\rangle$ under the action of the pulses (12). But for $\Delta\tau = 60$, since complete population transfer does not occur, the system remains entangled in the state $|\psi\rangle$ at long time limit. This is clear from the dashed curve of Fig. 2, as the equality $E = 0$ no longer holds at this time limit.

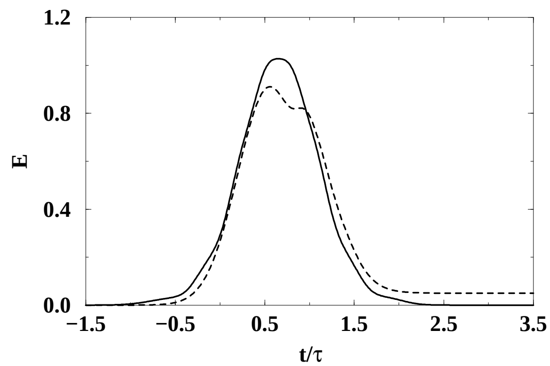

FIG. 2: Time evolution of the parameter E for $\Delta\tau = 0$ (solid curve) and $\Delta\tau = 60$ (dashed curve). This clearly shows that the strong subadditivity inequality remains valid in the present physical situation. The parameters chosen here are $n = 2$, $\mu = 0$, $g_{j0}\tau = 15$ ($j = 1, 2$), $T = 4\tau/3$.

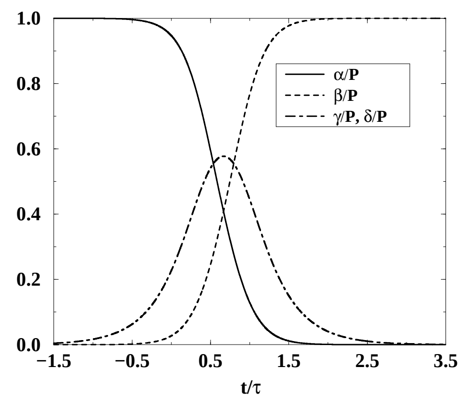

FIG. 3: Variation of the coefficients α/P, β/P, γ/P, and δ/P with time. The parameters chosen here are $n = 2$, $\mu = 0$, $g_{j0}\tau = 15$, $T = 4\tau/3$, and $\Delta\tau = 0$.

Thus we can recognize the expression E [see Eq. (13)] as a measure of four-particle entanglement in the present process. Precisely, $E \geq 0$, where the equality sign holds good for the disentangled states. An increase in value of E refers to increase in entanglement. Thus, during the evolution, the system gets more entangled for $\Delta\tau = 0$ than for $\Delta\tau = 60$. However, at the end of the evolution, the entanglement persists for nonzero $\Delta\tau$. We must emphasize here that, the present definition of entanglement measurement satisfies all the relevant criteria, namely, (a) it is semipositive, i.e., $E \geq 0$, (b) $E = 0$ for an disentangled state, and (c) the function $E(t)$ is continuous in time domain.

Using our four-particle entanglement, we can also study the inequality (3) involving the entropies of two particles, say, atoms A and B. They remain strongly correlated during the evolution, as I_c remains much larger than zero. At the end of the evolution, I_c becomes zero for $\Delta\tau = 0$ which means that the subsystems become uncorrelated. In other words, the entanglement between them vanishes.

We note in passing that for a four-particle GHZ state defined by

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|0000\rangle + |1111\rangle) \tag{17}$$

of four qubits A, B, C, and D, one is led to a three-particle mixed state ρ_{ABC} defined by

$$\rho_{ABC} = \frac{1}{2}(|000\rangle\langle 000| + |111\rangle\langle 111|) \,. \tag{18}$$

In this case, $S(A, B, C) = S(A) = S(A, B) = S(B, C) = \log_2 2 = 1$. Therefore, the parameter E in this case becomes zero, as from Eq. (13). So we have a counter-example, in which the equality sign in (4) holds for an entangled state, too. However, we note that the above state (18) satisfies the condition (5) and thus the said equality.

V. CONCLUSIONS

In conclusion, we have shown for the first time the role of strong subadditivity inequality for entropies in a four-particle composite system. The stimulated Raman adiabatic technique has been used to prepare the four-particle entangled state using two three-level atoms initially in their ground states in a two-mode cavity. We further show that the parameter E could serve as a possible measurement of entanglement in the four-particle entangled state under consideration.

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