DUALITIES IN THEORIES WITH 32 SUPERSYMMETRIES: A BEGINNER'S GUIDE

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This is an introductory review of dualities in theories with 32 supersymmetries. I describe the maximally supersymmetric theories in 11 and 10 dimensional space-time, their spectrum, symmetries and inter-relationships, and their toroidal compactifications. The emphasis is on presenting a few simple ideas explicitly and with clarity.

1 Introduction

The nature of classical supersymmetry has been under investigation for over two decades. The basic supersymmetry algebras in various dimensions were substantially classified, and the field contents of possible supersymmetric Lagrangians written down, quite some time ago[a].

Many of the new developments which will be reviewed in these notes are essentially quantum mechanical in nature. They are not, therefore, (as is sometimes suggested) re-statements of well-known old results from the days of classical supersymmetry. However, it is true that the understanding of classical supersymmetry, and more specifically the supersymmetry algebra – a precise classical statement that carries over directly into the quantum theory – are the essential scaffolding on which the structure of modern duality symmetries is erected.

As we will see, in various situations, conjectured duality symmetries can be thought of as evidence that a quantum theory really exists, when no other firm evidence is available. Some of these dualities involve statements about strongly coupled field and string theories, hence they cannot be actually demonstrated by any known technique. Some people would even say that these symmetries provide *definitions* of quantum theories, in which case they should not be thought of as conjectures at all.

What is certain is that the nonperturbative duality symmetries are extremely natural, and are supported by impressive evidence. The modern approach goes roughly like this: on the basis of some initial evidence, assume a duality symmetry. Find at least one nontrivial consequence of this assumption that is not already known to be true or false. Investigate it. If (as has usually

[a] For a comprehensive review, see Ref. [1].

been the case) it turns out to be true, we have one more reason to believe the duality. In practice, the community working on the subject reaches a consensus on the validity of a duality symmetry, when the amount of evidence crosses a critical value.

It has long been believed that classical supersymmetry, (together with other conventionally assumed symmetries), can only exist in spacetime dimensions less than or equal to 11. This requires an assumption that there is no tensor or spinor particle with spin greater than 2 (it has generally been believed that field theories with higher spins than this are inconsistent). In these notes I will start with this highest possible dimension and work downwards from there.

In this article, I have cited only a few books, review articles and seminal papers that might help the beginning student. Those, in turn, will contain references to the original literature that the student will need to access to gain a deeper understanding and to start on research.

2 Supersymmetry

If we require spins ≤ 2, then the maximal number of supersymmetries (in components) is 32. A rough argument for this goes as follows. On-shell, only half the components of a spinor are physical, so in this case we have 16 on-shell supersymmetry generators. Taking complex combinations, we can make 8 raising and 8 lowering operators out of these. Each of these raises or lowers the spin projection of a state by $\frac{1}{2}$. Starting with a state of spin projection $-s$ for some s, and assuming it is annihilated by all the lowering operators, we can raise it until we get a maximum spin projection of $-s + 4$. A supermultiplet must of course contain all spin projections from $-s$ to $+s$, so with this number of supersymmetries, $s = 2$. With more supersymmetries, s will necessarily be larger.

In 4 dimensions, 32 supersymmetries are organized as 8 Majorana supercharges with 4 (off-shell) components each, hence the corresponding theories are said to have $N = 8$ supersymmetry in that case. In 10 dimensions one has 2 Majorana-Weyl supercharges of 16 components each, hence $N = 2$ supersymmetry.

The maximal spacetime dimension which admits 32 supersymmetries is $d = 11$. A Majorana spinor has 32 off-shell components in this case, and the supersymmetry is called $N = 1$. Above 11 dimensions, a supercharge will have more than 32 components and hence, as explained above, we are forced to have undesirably large spins. Indeed, in 11 dimensions there is a unique supersymmetric Lagrangian of conventional type upto two-derivative order. It is generally known as $D = 11$ supergravity. A quantum theory ("M-theory") is now believed to exist, for which this supergravity is the effective low-energy Lagrangian for the massless particles. Below, we will encounter considerable evidence for the existence of M-theory.

In 10 dimensions there are two distinct supersymmetry algebras with 32 supersymmetries: the type IIA and IIB theories. Each is associated to a definite content of massless fields and a Lagrangian that is unique upto two-derivative order (there is a subtlety for the type IIB case, where a manifestly Lorentz-invariant Lagrangian cannot be written down, as we will see later). The type IIA field content is vectorlike (parity conserving), while the IIB field content is chiral (parity violating) [2].

One of the key features of all these theories is that their excitation spectrum contains a variety of different stable objects that occur as soliton solutions, some of which are pointlike ("particles") and others are extended along p space directions ("p-branes"). Conceptually this is no different from the situation with some physically relevant gauge theories in $3+1$ dimensions, in which the spectrum contains cosmic strings ("1-branes) and domain walls ("2-branes" or "membranes") in addition to the usual fundamental particles. But because the theories of interest here are higher dimensional, and supersymmetric, they admit p-branes for higher values of p, and the p-branes are endowed with many interesting and calculable properties. (For more details of solitonic branes, see the lectures of K. Stelle at this school, and references therein.)

We will say all that we can about these three theories: M, IIA, IIB, before going to lower dimensions. Here is a partial summary, in advance, of the interesting facts that we will encounter along the way:

(i) M-theory is a quantum theory with massless gravitons and other massless particles among its excitations. It also contains stable 2-branes and 5-branes in its spectrum, but no other stable branes. It is not a string theory, but (as far as we understand it) it is a consistent theory, incorporating quantum gravity, in 11 dimensions.

(ii) IIA and IIB theories can both be obtained as limits of M-theory. These are also consistent theories of quantum gravity, in 10 dimensions. In their spectrum they contain, besides massless particles, various kinds of stable p-branes. These include 1-branes or strings. Indeed, the IIA and IIB theories can be phrased as string theories, with a consistent perturbation series, in which the particles are just low-lying excitations of the strings. Thus these theories are better understood than M-theory, as their consistency is established at least in the perturbative regime and loop corrections can be calculated. Nonperturbatively they are best understood as limits of M-theory and this understanding is consequently as limited as that of M-theory.

3 Theories In Their Maximal Dimensions

3.1 M-theory

The unique supersymmetric classical field theory with fields of spin ≤ 2 in 11 dimensions, contains the following massless fields:

$$\begin{aligned} g_{MN}: &\quad \text{metric} \\ C_{MNP}: &\quad 3-\text{form potential} \\ \psi_{M\alpha}: &\quad \text{spin}-\frac{3}{2}\ \text{Majorana fermion} \end{aligned} \tag{1}$$

Instead of trying to prove that this is the right field content, we will argue that this is plausibly a supersymmetry multiplet. Let us count on-shell degrees of freedom. In light-cone gauge, where only on-shell degrees of freedom survive, each tensor index can be assigned a set of 9 (transverse) values, while each Majorana spinor index takes 16 values on-shell. Taking into account various symmetries for the bosonic fields, and remembering to project out a spin-$\frac{1}{2}$ field from the gravitino, we find:

$$\begin{aligned} g_{MN}: &\quad \frac{9\times 10}{2} - 1 = 44 \\ C_{MNP}: &\quad \frac{9\times 8\times 7}{6} = 84 \\ \psi_{M\alpha}: &\quad 9\times 16 - 1\times 16 = 128 \end{aligned} \tag{2}$$

Thus, the total on-shell degrees of freedom match between bosons and fermions.

To write the Lagrangian, let $e = \sqrt{-\mathrm{det} g}$ (this and all other conventions are as in Ref. [2]). Then the classical Lagrangian of 11-dimensional supergravity can be written

$$\mathcal{L} = \mathcal{L}_B + \mathcal{L}_F \tag{3}$$

where the bosonic part is

$$\begin{aligned} \mathcal{L}_B \quad = \quad & -\frac{1}{2\kappa^2} eR - \frac{1}{48} e G_{MNPQ} G^{MNPQ} \\ & -\frac{\sqrt{2}\kappa}{3456} \epsilon^{MNPQRSTUVWX} C_{MNP} G_{QRST} G_{UVWX} \end{aligned} \tag{4}$$

and the fermionic part is given by

$$\begin{aligned} \mathcal{L}_F \quad = \quad & -\frac{1}{2} e\bar{\psi}_M \Gamma^{MNP} D_N \psi_P \\ & -\frac{\sqrt{2}\kappa}{192} \left(\bar{\psi}_M \Gamma^{MNPQRS} \psi_S + 12 \bar{\psi}^N \Gamma^{PQ} \psi^R\right) G_{NPQR} \\ & + 4\ -\ \text{fermi terms} \end{aligned} \tag{5}$$

Here, $\Gamma^{M_1 \ldots M_k} \equiv \frac{1}{k!}(\Gamma^{M_1} \cdots \Gamma^{M_k} \pm (k!-1)$ terms). The 4th rank antisymmetric tensor G_{MNPQ} is defined as $\frac{1}{4}\left[\partial_M C_{NPQ} + 3 \text{ terms}\right]$.

The above action is invariant under the supersymmetry transformations

$$\begin{aligned}
\delta e_M^A &= \frac{\kappa}{2}\bar{\eta}\Gamma^A \psi_M \\
\delta C_{MNP} &= -\frac{\sqrt{2}}{8}\bar{\eta}\Gamma_{[MN}\psi_{P]} \\
\delta \psi_{M\alpha} &= \frac{1}{\kappa}(D_M \eta)_\alpha \\
&\quad + \frac{\sqrt{2}}{288}\left[(\Gamma_M^{PQRS} - 8\delta_M^P \Gamma^{QRS})\eta\right]_\alpha G_{PQRS} \\
&\quad + 3 - \text{ fermi terms}
\end{aligned} \tag{6}$$

A useful tip about dealing with complicated supergravity Lagrangians (this is one of the simplest!) is that one usually needs only: (i) the bosonic part $\mathcal{L}_B$ of the Lagrangian, (ii) the fermionic variation $\delta\psi_{M\alpha}$ in the supersymmetry transformation.

So far we have only written down a classical Lagrangian. It is not clear that there is a corresponding quantum theory, since by the usual criteria we are dealing with a highly non-renormalizable Lagrangian. Evidence for an underlying quantum theory will slowly energe.

Continuing with the classical theory, we look for stable solitonic solutions of the equations of motion. These are expected to be important if the theory can be quantized, as they would then correspond to nonperturbative, stable quantum states. Generically, solitons in quantum theory are stable only if they carry a quantized charge. The lightest soliton carrying a single unit of this charge cannot decay, by charge conservation.

Point particles naturally carry electric charge with respect to an electromagnetic field A_μ. One manifestation of this is an interaction $e \oint A_\mu dx^\mu$ on the world-line of the particle. A magnetic monopole in 4 dimensions will instead couple to the "dual photon" $\hat{A}_\mu$ via $\oint \hat{A}_\mu dx^\mu$ where $\hat{A}$ is defined by the duality transform

$$\partial_{[\mu}\hat{A}_{\nu]} = \frac{1}{2}\epsilon_{\mu\nu\lambda\rho}\partial^{[\lambda}A^{\rho]} \tag{7}$$

Note that the above transform interchanges the electric field $E_i = F_{0i}$ with the magnetic field $B_i = \frac{1}{2}\epsilon_{ijk}F_{jk}$ in 4 spacetime dimensions.

This concept generalizes to extended solitons and higher-rank antisymmetric tensor fields in arbitrary dimensions. The only antisymmetric tensor field in M-theory is the 3-form C_{MNP}. Hence we wish to look for a stable electrically charged object with respect to this 3-form, having an interaction

$\int C_{MNP}\, dx^M \wedge dx^N \wedge dx^P$ on its world-volume. Such an object must have a 3 spacetime-dimensional world-volume, so it is a 2-brane or membrane.

A stable magnetically charged object, on the other hand,will have a world-volume interaction $\int \hat{C}_{MNPQRS}\, dx^M \wedge \cdots \wedge dx^S$, where

$$\partial_{[T}\tilde{C}_{MNPQRS]} = \frac{1}{7!}\epsilon_{TMNPQRSABCD}\, \partial^{[A}C^{BCD]} \tag{8}$$

It follows that this object has a 6 spacetime-dimensional world-volume, so it is a 5-brane. Thus the potentially stable objects in M-theory are singly charged electric 2-branes and magnetic 5-branes. It is pleasing that the 3-form field of 11D supergravity acquires a useful role in this way. Supergravity theories must of course contain gravitons and gravitinos, by definition, but now it emerges that other fields are there to provide charges to stabilize branes.

The postulated branes actually do exist as classical solutions of the low-energy supergravity. We have a family of solutions, one for every integer k which labels the elctric/magnetic charge carried. From our discussion above, the truly stable ones have $k = 1$. The solutions are given by:

<u>2-brane</u>:

$$\begin{aligned} ds^2 &= \left(1+\frac{k}{r^6}\right)^{-2/3} dx^\mu dx^\nu \eta_{\mu\nu} \\ &\quad + \left(1+\frac{k}{r^6}\right)^{+1/3} dy^m dy^n \delta_{mn} \\ C_{\mu\nu\lambda} &= \epsilon_{\mu\nu\lambda}\left(1+\frac{k}{r^6}\right)^{-1} \end{aligned} \tag{9}$$

where $\mu,\nu = 0,1,2$; $m,n = 3,\ldots,10$; $r = \sqrt{y^m y^m}$, and the components of C not appearing above are zero along with the fermion fields.

<u>5-brane</u>:

$$\begin{aligned} ds^2 &= \left(1+\frac{k}{r^3}\right)^{-1/3} dx^\mu dx^\nu \eta_{\mu\nu} \\ &\quad + \left(1+\frac{k}{r^3}\right)^{2/3} dy^m dy^n \delta_{mn} \\ G_{mnpq} &= 3k\epsilon_{mnpqs}\frac{y^s}{r^5} \end{aligned} \tag{10}$$

where this time $\mu,\nu = 0,1,\ldots,5$; $m,n = 6,\ldots,10$; $r = \sqrt{y^m y^m}$ and again the components not appearing above, along with the fermions, are set to zero.

One is tempted to assume that there is a quantum theory with massless point particles corresponding to the fields in the classical Lagrangian, and also quantum states corresponding to stable 2-branes and 5-branes. This is the conjecture that M-theory exists. We will gradually uncover evidence for this conjecture.

One can show that the 2-brane and 5-brane solutions preserve half the supersymmetry of the underlying theory. The remaining half of the supersymmetries is broken in the presence of branes. Explicitly, of the 32 independent supersymmetry variations $\delta\psi_{\mu\alpha}$ (for 32 independent spinors η), 16 are zero when the brane solution for the metric and G_{MNPQ} given by Eq. 9 or Eq. 10 is inserted into the right hand side of Eq. 6.

This is closely related to the fact that the most general supersymmetry algebra in 11d is:

$$\begin{aligned}\{Q_\alpha, \bar{Q}_\beta\} \quad &= \quad (\Gamma^\mu)_{\alpha\beta} P_M + (\Gamma_{MN})_{\alpha\beta} Z^{MN}_{(2)} \\ &\quad + (\Gamma_{MNPQR})_{\alpha\beta} Z^{MNPQR}_{(5)} \end{aligned} \tag{11}$$

where $Z_{(2)}$ and $Z_{(5)}$ are "central charges" which are non-vanishing precisely on 2 and 5-branes respectively. The branes satisfy an important relation between their mass density (measured by P_0) and their charge density (measured by $Z_{(2)}$, $Z_{(5)}$).

Schematically, half the supercharges satisfy

$$\{Q^{(1)}_\alpha, Q^{(1)}_\beta\} \sim P + Z \tag{12}$$

while the other half satisfy

$$\{Q^{(2)}_\alpha, Q^{(2)}_\beta\} \sim P - Z \tag{13}$$

So, if the branes satisfy $P - Z = 0$ or $P + Z = 0$ then they can preserve half the supersymmetries, otherwise they break all. In the former case we have $|P| = |Z|$ on the branes while in the latter we evidently have the strict bound $|P| > |Z|$. This bound is called the Bogomolny-Prasad-Sommerfeld or BPS bound.

The stable 2-brane and 5-brane solutions written down in Eqs. 9,10 satisfy $|P| = |Z|$, hence they saturate the bound and are said to be "BPS-saturated".

Note that the total number of independent charges (including momenta) occurring on the RHS is 11 (momenta) + 55 ($Z^{MN}_{(2)}$) + 462 ($Z^{MNPQR}_{(5)}$) = 528.

This concludes our preliminary analysis of M-theory, but we will return to it soon. A nice review of M-theory can be found in Refs. [3], and there are surely others.

3.2 Type IIA Theory

The type IIA and IIB theories can be discovered in a rather analogous way. Just by studying the realizations of 32 supersymmetries in 10 dimensions, one finds that there are two *distinct* multiplets of massless fields with spins ≤ 2. One of these, leading to type IIA theory, is as follows:

$$g_{\mu\nu}, \phi, \tilde{A}_\mu, B_{\mu\nu}, \tilde{C}_{\mu\nu\rho} \text{ (Bose)} \tag{14}$$

$$\psi^{(1)}_{\mu\alpha}, \phi^{(2)}_{\mu\alpha'}, \lambda^{(1)}_\alpha, \lambda^{(2)}_{\alpha'} \text{ (Fermi)} \tag{15}$$

(the tilde on the 1-form and 3-form fields will be explained in the next subsection). The Fermi fields come in pairs of opposite chirality: α is the spinor index for $SO(9,1)$ and α' is the conjugate spinor.

These fields are related to those of M-theory in a very interesting way. Suppose M-theory is compactified on a circle (somewhat inconsistently with our earlier notation, we call this dimension 11). The resulting spectrum from the 10d point of view consists of some massless fields (coming from 11d fields independent of the 11 direction) and massive fields from Fourier modes excited in the 11 direction.

Consider first just the massless modes:

$$\begin{array}{ccl} \underline{d=11} & & \underline{d=10} \\ g_{MN} & \to & g_{\mu\nu}; \quad g_{\mu,11} = A_\mu; \quad g_{11,11} = \phi \\ C_{MNP} & \to & C_{\mu\nu\rho}; \quad C_{\mu\nu,11} = B_{\mu\nu} \end{array} \tag{16}$$

Thus the Bose fields of type IIA theory, listed in Eq. 14, are exactly reproduced. It is easily checked that the same is true for the Fermi fields of IIA theory: they arise by dimensional reduction of the the gravitino of M-theory.

What about the Lagrangian? It is clear that taking the M-theory Lagrangian to be independent of x^{11} will lead to a Lagrangian in 10 spacetime dimensions which necessarily has the right supersymmetries. The single Majorana supercharge in 11 dimensions splits into a pair of Majorana-Weyl spinors in 10 dimensions, one of each chirality. This happens because under dimensional reduction, local massless fields split in a non-chiral way and so we always get vectorlike theories. So dimensional reduction gives a valid way of actually deriving the type IIA Lagrangian.

It is illuminating to carry through this derivation. From now on, we will ignore numerical coefficients in front of individual terms in supergravity Lagrangians. Thus we write the bosonic 11d Lagrangian as:

$$\alpha^{(11)}_B = \frac{1}{\kappa^2}\left(eR + e|G|^2 + C \wedge G \wedge G\right) \tag{17}$$

(we have performed a redefinition $C \to \frac{C}{\kappa}$).

To compactify on a circle, we replace g_{MN} by a matrix like

$$g_{MN} \sim \begin{bmatrix} g_{\mu\nu} & \tilde{A}_\mu \\ \\ \tilde{A}_\nu & e^{\alpha\phi} \end{bmatrix} \tag{18}$$

where α is an arbitrary constant to be chosen later.

This form exhibits the Kaluza-Klein scalar ϕ and gauge field $\tilde{A}_\mu$. However, it is convenient to modify it since otherwise $\sqrt{g^{(11)}}$ will depend on $\tilde{A}_\mu$, leading to ugly formulae. A better ansatz is

$$g_{MN} \sim \begin{bmatrix} g_{\mu\nu} + e^{\alpha\phi}\tilde{A}_\mu \tilde{A}_\nu & e^{\alpha\phi}\tilde{A}_\mu \\ \\ e^{\alpha\phi}\tilde{A}_\nu & e^{\alpha\phi} \end{bmatrix} \tag{19}$$

so that $\sqrt{g^{(11)}} = e^{\frac{\alpha}{2}\phi}\sqrt{g^{(10)}}$.

If we treat x^{11} as an angle-valued coordinate (taking values from 0 to 2π), the radius of the compactification circle is $R_{11} = e^{\frac{\alpha}{2}\phi}$. Note that $\phi(x)$ is a scalar field, so we really mean $R_{11} = e^{\frac{\alpha}{2}\langle\phi(x)\rangle} = e^{\frac{\alpha}{2}\phi}$ where ϕ is the constant part, or VEV, of $\phi(x)$.

Thus we have

$$\begin{aligned} R^{(11)} &\equiv g^{MN} R^{(11)}_{MN} \\ &\sim g^{\mu\nu}\left[R^{(10)}_{\mu\nu} + \partial_\mu\phi\partial_\nu\phi + e^{-\alpha\phi}|e^{\alpha\phi}\tilde{F}|^2 + \cdots \right] \\ &= g^{\mu\nu}\left[R^{(10)}_{\mu\nu} + \partial_\mu\phi\partial_\nu\phi + e^{\alpha\phi}|\tilde{F}|^2 + \cdots \right] \end{aligned} \tag{20}$$

So

$$\sqrt{g^{(11)}}R^{(11)} \sim \sqrt{g^{(10)}}\left[e^{\frac{\alpha}{2}\phi}R^{(10)} e^{\frac{\alpha}{2}\phi}|d\phi|^2 + e^{\frac{3}{2}\alpha\phi}|\tilde{F}|^2 \right] \tag{21}$$

Similarly,

$$\begin{aligned} \sqrt{g^{(11)}}|\tilde{G}^{(11)}|^2 &\sim e^{\frac{\alpha\phi}{2}}\sqrt{g^{(10)}}|\tilde{G}^{(10)}|^2 + e^{\frac{\alpha}{2}\phi}\sqrt{g^{(10)}}e^{-\alpha\phi}|H^{(10)}|^2 \\ &= \sqrt{g^{(10)}}\left[e^{\frac{\alpha}{2}\phi}|G^{(10)}|^2 + e^{-\frac{\alpha}{2}\phi}|H^{(10)}|^2 \right] \end{aligned} \tag{22}$$

Collecting the above results, dropping the (10) superscript on the fields, and temporarily ignoring the $C \wedge G \wedge G$ term, the $10d$ bosonic Lagrangian is:

$$\mathcal{L}^{(10)} = \sqrt{g^{(10)}}\Bigg[e^{\frac{\alpha}{2}\phi}R + e^{\frac{\alpha}{2}\phi}|d\phi|^2$$

$$+e^{\frac{\alpha}{2}\phi}|\tilde{G}|^2 + e^{\frac{3}{2}\alpha\phi}|\tilde{F}|^2 + e^{-\frac{\alpha}{2}\phi}|H|^2\Bigg] \tag{23}$$

where $\tilde{F} = d\tilde{A}$, $H = dB$ and $\tilde{G} = d\tilde{C}$ are 2-form, 3-form and 4-form field strengths respectively.

It is useful to make a Weyl rescaling:

$$g^{(10)}_{\mu\nu} \to e^{\beta\phi} g^{(10)}_{\mu\nu} \tag{24}$$

where the constant β will be determined by our convenience. The various terms transform under this rescaling as follows, modulo higher-derivative terms which we are ignoring:

$$\begin{aligned}
\sqrt{g^{(10)}} &\to e^{5\beta\phi}\sqrt{g^{(10)}}, \qquad R_{\mu\nu} \to R_{\mu\nu}, \qquad R \to e^{-\beta\phi} R \\
|d\phi|^2 &\to e^{-\beta\phi}|d\phi|^2, \qquad |\tilde{F}|^2 \to e^{-2\beta\phi}|\tilde{F}|^2 \\
|H|^2 &\to e^{-3\beta\phi}|H|^2, \qquad |\tilde{G}|^2 \to e^{-4\beta\phi}|\tilde{G}|^2
\end{aligned} \tag{25}$$

Thus we arrive at the Weyl-rescaled Lagrangian

$$\begin{aligned}
\mathcal{L}^{(10)}_B = \sqrt{g^{(10)}}\Bigg[& e^{(4\beta+\frac{\alpha}{2})\phi} R \\
& + e^{(4\beta+\frac{\alpha}{2})\phi}|d\phi|^2 + e^{(3\beta+\frac{3}{2}\alpha)\phi}|\tilde{F}|^2 \\
& + e^{(2\beta-\frac{\alpha}{2})\phi}|H|^2 + e^{(\beta+\frac{\alpha}{2})\phi}|\tilde{G}|^2\Bigg]
\end{aligned} \tag{26}$$

Now notice that by taking $\beta + \frac{\alpha}{2} = 0$, we can make the e^{ϕ} terms disappear in front of *both* $|\tilde{F}|^2$ and $|\tilde{G}|^2$. Also, the factors in front of the other three terms: R, $|d\phi|^2$, $|H|^2$ become equal. Thus with this choice, and with the conventional choice $\alpha = \frac{4}{3}$, we finally get

$$\mathcal{L}^{(10)}_B = \sqrt{g}\left[e^{-2\phi}\left(R + |d\phi|^2 + |H|^2\right) + |\tilde{F}|^2 + |\tilde{G}|^2\right]. \tag{27}$$

The constant part of $e^{-2\phi}$ is like $\frac{1}{\lambda^2}$, where λ is the coupling constant for the metric, ϕ and B fields. The other two fields appear in a "nonstandard" way, with no coupling factors in front (this partly explains why we placed tildes over them). We have $\lambda = e^{\phi}$, while $R_{(11)} = e^{\frac{2}{3}\phi}$ so:

$$R_{(11)} = \lambda^{2/3}. \tag{28}$$

All this has a physical interpretation. In M-theory there is no scalar field, hence no parameter like $\langle\phi\rangle$ which can play the role of an adjustable, dimensionless coupling. M-theory only has a dimensionful constant κ, the 11-dimensional Planck scale (which we have set equal to 1 in the above discussion). Thus there is nothing analogous to a weak-coupling limit in M-theory.

In type IIA theory, obtained by compactification on a circle, there is such a scalar – the dilaton – and its VEV is related to the radius of the 11th dimension. Thus type IIA should admit a weak coupling expansion.

Before investigating this expansion, let us examine the spectrum of stable branes in the type IIA theory. The gauge fields and the associated charged branes are:

$$\begin{aligned}
\tilde{A}_\mu &\quad : \quad 0-\text{brane (electric)} \\
B_{\mu\nu} &\quad : \quad 1-\text{brane (electric)} \\
\tilde{C}_{\mu\nu\lambda} &\quad : \quad 2-\text{brane (electric)}
\end{aligned} \tag{29}$$

Let the 7-form $\hat{A}_{\mu\nu\lambda\rho\sigma\tau\alpha}$ be the dual of the 1-form $\tilde{A}_\mu$ (in 10 spacetime dimensions) via $d\hat{A} =^\star d\tilde{A}$. A 6-brane couples electrically to this, which is equivalent to saying that is carries magnetic charge under $\tilde{A}_\mu$. Similarly for the 2-form $B_{\mu\nu}$ and 3-form $\tilde{C}_{\mu\nu\lambda}$ above, there are dual 6-form and 5-form potentials $\hat{B}$ and $\hat{C}$ respectively. Thus we can also have the following magnetically charge branes:

$$\begin{aligned}
\tilde{A}_\mu &\quad : \quad 6-\text{brane (magnetic)} \\
B_{\mu\nu} &\quad : \quad 5-\text{brane (magnetic)} \\
\tilde{C}_{\mu\nu\lambda} &\quad : \quad 4-\text{brane (magnetic)}
\end{aligned} \tag{30}$$

To summarize, these simple arguments suggest that type IIA theory should have stable p-branes for $p = 0, 1, 2, 4, 5, 6$. All the corresponding soliton solutions have been determined, so these branes really do exist.

Since IIA theory arose by compactifying M-theory, all its branes should have explanations in M-theory. Indeed we can find them as follows. When M-theory is compactified on a circle, the resulting theory will have distinct quantum states in its Hilbert space corresponding to 2-branes that are independent of the circle and 2-branes that wrap the circle. Similarly, it has quantum states corresponding to 5-branes that are independent of, or wrapped on, the circle. In the limit of a small circle, when the correct description is as type IIA theory, these four kinds of states behave respectively as 2-branes, 1-branes, 5-branes and 4-branes.

We have used up all the basic branes in M-theory, but have not yet found an M-theoretic explanation for the 0-brane and 6-brane of type IIA theory. To understand these branes, note that the gauge field $\tilde{A}_\mu$, whose electric charge is carried by the 0-brane, is a Kaluza-Klein gauge field arising from the $11d$ metric. So the missing 0-brane should also arise in some way from the Kaluza-Klein mechanism. Indeed, this is the case.

On the circle x^{11}, we can make a mode expansion of the $11d$ massless fields:

$$g^{(11)}_{MN}(x^1 \ldots x^{10}; x^{11}) = \sum_{n\in\mathbb{Z}} e^{i\frac{x^{11}}{R}n} g^{(10)}_{MN}(x^1, \ldots, x^{10}) \tag{31}$$

The fields on the RHS are of course reducible when viewed as $10d$ tensors.

Now observe that a translation $x^{11} \to x^{11} + \epsilon R$ of the eleventh direction sends

$$g^{(10)(N)}_{MN} \to e^{i\frac{n}{R}\epsilon R} g^{(10)(n)}_{MN} \tag{32}$$

Hence the local version of this transformation, $x^{11} \to x^{11} + \epsilon(x^1, \cdots, x^{10})$ is just a local $U(1)$ gauge transformation for which $\tilde{A}_\mu$ is the gauge field. Thus the fields $g^{(10)(n)}_{MN}$ have charge n under $\tilde{A}_\mu$.

Moreover, except for $n = 0$, these fields are all *massive*: since the eleven-dimensional metric components satisfy a massless wave equation

$$\Box^{(11)} g^{(11)}_{MN} = 0 \tag{33}$$

it follows that the 10-dimensional fields satisfy

$$\Box^{(10)} g^{(10)(n)}_{MN} - \left(\frac{n}{R}\right)^2 g^{(10)(n)}_{MN} = 0 \tag{34}$$

Thus the Kaluza-Klein fields have equal mass and charge in some units. (Actually in string units, (mass) $= \frac{1}{\lambda}$(charge).) In modern language, the Kaluza-Klein mechanism automatically gives rise to BPS saturated states! Of course, it is the supersymmetry algebra which guarantees that configurations which are classically BPS saturated actually go over into BPS saturated states in the full quantum theory.

The field of lowest nonzero charge will correspond to a stable particle for reasons that we have already cited. So we can identify the multiplet of particles corresponding to $g^{(10)(1)}_{MN}$, $C^{(10)(1)}_{MNP}$ with the multiplet of states of a stable unit-charge 0-brane. Similarly, the "Kaluza-Klein monopole", magnetically charged under $\tilde{A}_\mu$, is the 6-brane of IIA theory.

So far, we have not needed any detailed information about either M-theory or IIA theory, especially about how to quantize them. Formulae arising from supersymmetry, such as the BPS formula, give us a lot of information once we merely *assume* that some quantum theory exists. At this stage, however, we can argue that a quantum type-IIA theory really does exist. Among the branes of the IIA theory is a 1-brane, or string. One can work out the world-sheet action for this string, and then quantize the resulting string theory.

It turns out that the string theory so obtained is perturbatively well-defined, consistent (even finite) and unitary. The perturbation series is given, in powers of $\lambda^2 = e^{2\phi}$, by an expansion in Riemann surfaces. The low-energy spectrum of the string reproduces the IIA supergravity fields, and string interactions reproduce the type of Lagrangian we have written down, with calculable corrections.

This shows that "IIA theory" (the theory of massless particles and various branes) does correspond to a well-defined quantum theory, at weak coupling. It is perfectly reasonable to assume that the theory exists also at strong coupling, in which case it follows that the strongly coupled theory is 11-dimensional and can be thought of as a definition of M-theory.

3.3 Neveu-Schwarz and Ramond sectors

At this point we digress a little from the main theme of these notes. Our point of view has been that classical supersymmetry completely dictates the structure of Lagrangians for massless fields in 11 and 10 dimensions, with 32 supersymmetries. Since these Lagrangians admit stable brane solutions, one must expect that upon quantization they lead not just to quantum field theories of point particles, but rather to quantum theories of particles and branes together.

In the 10-dimensional case, we argued that one such theory (type IIA) can be formulated as a quantum string theory (and we will soon see that this is true also of the other 10d theory, type IIB). This fact gives us a whole new perspective on these theories. To do full justice to this, we would need an entire course on string theory, but we will instead make a few brief remarks in this direction and establish some relevant facts for later use. Details can be found in Ref. [2].

Quantization of the 1-brane, or string, of type IIA theory is somewhat complicated (this is true for any relativistic string theory) by the presence of local symmetries and constraints on the world sheet. At the end of this story, which involves gauge-fixing and related issues, it emerges that the type IIA string has massless states in 10 dimensions which arise as follows. If σ labels points on the string and τ is the time parameter on the world-sheet, then the modes of this string, which we take to be closed, factorize into "left-movers" depending only on $(\sigma - \tau)$ and "right-movers" depending only on $(\sigma + \tau)$. The total Hilbert space is obtained by tensoring these two factorized spaces subject to some constraints. Massless particles in spacetime arise by keeping the lowest modes in the two sectors. For the type IIA string, the Lorentz quantum numbers of these modes (on-shell) are:

$$
\begin{aligned}
\text{left movers}: &\quad |i\rangle_L, |\alpha\rangle_L \\
\text{right movers}: &\quad |j\rangle_R, |\beta'\rangle_R
\end{aligned}
\tag{35}
$$

where $i, j = 1, \ldots, 8$ represent the transverse components of an $SO(9,1)$ vector, $\alpha = 1, \ldots, 8$ represents the on-shell components of an $SO(9,1)$ spinor, and $\beta' = 1, \ldots, 8$ represents a conjugate spinor of $SO(9,1)$. The spinor and conjugate-spinor states are, as one would expect, spacetime-fermionic. For historical reasons, the vector states are said to be in the "Neveu-Schwarz" (NS) sector, while the spinor states are in the "Ramond" (R) sector.

Some $SO(9,1)$ group theory then enables us to build up the massless particle states and hence work out the Lorentz covariant massless fields that must appear in the low-energy Lagrangian. Bosonic fields arise by combining $|i\rangle_L \otimes |j\rangle_R$ which decomposes into a symmetric traceless, an antisymmetric and a trace part. Th corresponding fields are a metric $g_{\mu\nu}$, a 2-form $B_{\mu\nu}$ and a scalar (dilaton) ϕ. Thus we have already recovered some of the massless fields of type IIA supergravity that were listed in Eq. 14. More bosonic fields arise from $|\alpha\rangle_L \otimes |\beta'\rangle_R$, and it is quite easy to work out that these correspond to the remaining fields $\tilde{A}_\mu$ and $\tilde{C}_{\mu\nu\lambda}$.

So we have re-discovered what we already knew, but with an extra insight. The fields g, B, ϕ arise by tensoring world-sheet modes that were bosonic separately in the left-moving and right-moving sectors, hence they are NS-NS fields. The other fields $\tilde{A}, \tilde{C}$ arise by combining spinor modes from the left- and right-moving sectors, so they are called R-R fields (this is what the tildes were supposed to denote). This distinction is apparent only from the point of view of string theory, and it turns out to be a very important one. The crucial difference between the two types of fields, which is a consequence of their distinct origins, is that perturbative states in the theory carry charge under the former and not under the latter. Indeed, it can be shown that couplings of perturbative states to the R-R gauge fields are through the *field strength tensors* rather than the gauge fields themselves, somewhat like magnetic-moment couplings in QED.

As one example, the string itself is charged under B. Therefore, if we compactify on a circle with coordinate x^i, the winding modes of the string on that circle will be charged under the 1-form gauge field $A^{(i)}_\mu \equiv B_{\mu i}$. Also, momentum modes of the string along that circle are charged under another

1-form gauge field $A'^{(i)}_\mu \equiv g_{\mu i}$. But no perturbative modes of the string are charged under the R-R gauge fields $\tilde{A}_\mu$ or $\tilde{C}_{\mu\nu\lambda}$ or their components after compactification. We will see, however, that solitonic branes exist that do carry R-R charge. This will prove important in what follows.

Before concluding this section, we note that the type IIB string differs in an apparently very small detail from the IIA string. Instead of the modes given in Eq. 35, it turns out that type IIB string gives rise to the modes

$$\begin{aligned} \text{left movers}: &\quad |i\rangle_L, |\alpha\rangle_L \\ \text{right movers}: &\quad |j\rangle_R, |\beta\rangle_R \end{aligned} \tag{36}$$

where the only difference is a missing prime on the spinor state $|\beta\rangle_R$. Thus, there are only spinors and no conjugate spinors of $SO(9,1)$. The NS-NS sector is identical to that of the type IIA theory, but the R-R sector is now quite different. It arises by decomposing $|\alpha\rangle_L \otimes |\beta\rangle_R$, which gives rise to the bosonic fields $\tilde{\phi}, \tilde{B}_{\mu\nu}, \tilde{D}^+_{\mu\nu\lambda\rho}$, namely a new scalar, a new 2-form and a 4-form potential whose 5-form field strength is self-dual in 10 dimensions. Thus the massless bosonic spectrum of type IIB supergravity, analogous to that listed in Eq. 14, is

$$g_{\mu\nu}, \phi, B_{\mu\nu}, \tilde{\phi}, \tilde{B}_{\mu\nu}, \tilde{D}^+_{\mu\nu\lambda\rho} \tag{37}$$

Evidently the IIB theory has a pair of scalars and a pair of 2-forms, each pair having one NS-NS and one R-R member.

3.4 Type IIB Theory

As we have seen above, type IIB theory has two 2-form gauge potentials $B_{\mu\nu}$ and $\tilde{B}_{\mu\nu}$, and a self-dual 4-form potential $D^+_{\mu\nu\lambda\rho}$. Following our previous arguments, we expect to find the following branes:

$$\begin{aligned} B_{\mu\nu} &: \quad 1-\text{brane (electric)} \\ B_{\mu\nu} &: \quad 5-\text{brane (magnetic)} \\ \tilde{B}_{\mu\nu} &: \quad 1-\text{brane (electric)} \\ \tilde{B}_{\mu\nu} &: \quad 5-\text{brane (magnetic)} \\ \tilde{D}^+_{\mu\nu\lambda\rho} &: \quad 3-\text{brane (self}-\text{dual)} \end{aligned} \tag{38}$$

Thus we should have two distinct (electric) 1-branes, or strings, and two distinct (magnetic) 5-branes. Also, because the 4-form is self-dual, the 3-brane can equally well be described as electric or magnetic, so we just call it "self-dual" [b]

As in the type IIA theory, here too the expected branes can be found as soliton solutions of the classical equations of motion. However, in this case there are some remarkable surprises. Let us write down the bosonic part of the type IIB supergravity action, upto two-derivative order:

$$\begin{aligned}\mathcal{L}^{(IIB)}_{\rm Bose} &= \sqrt{g}\Big[e^{-2\phi}(R+|d\phi|^2+|H|^2)\\ &\qquad +|d\tilde\phi|^2+|\tilde H-\tilde\phi H|^2+|I|^2\Big]\\ &\qquad +\tilde D^+\wedge H\wedge \tilde H \qquad (39)\end{aligned}$$

where $H = dB$, $\tilde H = d\tilde B$, $\tilde I = d\tilde D^+$.

Actually, self-duality implies that $|\tilde I|^2 = \tilde I \wedge^\star \tilde I = \tilde I \wedge \tilde I = 0$ since $\tilde I$ is a 5-form. So it is not really correct to write the action as above when $\tilde I \neq 0$. However, we adopt the procedure of relaxing the self-duality condition in the action, and then imposing it after obtaining equations of motion. This will be adequate for us, although a number of important subtleties are associated to this problem, because of which a covariant action for the fields of type IIB supergravity does not exist.

The above action has a global $SL(2,\mathbb{R})$ symmetry under which the two 2-form fields transform as a doublet. To exhibit this, it is convenient to make a Weyl transformation. Let $g_{\mu\nu} \to e^{\frac{1}{2}\phi} g_{\mu\nu}$. This implies that

$$\begin{aligned}\sqrt{g} &\to e^{\frac{5}{2}\phi}\sqrt{g}, \qquad & R &\to e^{-\frac{1}{2}\phi}\\ |d\phi|^2 &\to e^{-\frac{1}{2}\phi}|d\phi|^2, & |H|^2 &\to e^{-\frac{3}{2}\phi}|H|^2\\ |\tilde I|^2 &\to e^{-\frac{5}{2}\phi}|\tilde I|^2 & & \qquad (40)\end{aligned}$$

In this frame (the "Einstein frame") the Einstein-Hilbert action has no dilaton dependence. The action becomes

$$\begin{aligned}\mathcal{L}^{(IIB),E.F.}_{\rm Bose} &= \sqrt{g}\Big[R+(|d\phi|^2+e^{2\phi}|d\tilde\phi|^2)\\ &\qquad +e^{-\phi}|H|^2+|\tilde I|^2+e^{\phi}|\tilde H-\tilde\phi H|^2\Big]+D^+\wedge H\wedge\tilde H \qquad (41)\end{aligned}$$

[b] We will be hiding some "high branes": in addition to the above, type IIB theory has a 7-brane and a 9-brane, while type IIA has an 8-brane. Their roles are a bit more subtle, and we will not need them here.

Now defining the complex scalar field $\tau = \tilde{\phi} + ie^{-\phi}$, we can write

$$|d\phi|^2 + e^{2\phi}|d\tilde{\phi}|^2 = \frac{|d\tau|^2}{(Im\tau)^2} \tag{42}$$

It is easy to check that under $\tau \rightarrow \frac{a\tau+b}{c\tau+d}$ with $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{R})$, this term is invariant.

Also, if $H, \tilde{H}$ transform as

$$(H, -\tilde{H}) \rightarrow (H, -\tilde{H}) \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{43}$$

under $SL(2,\mathbb{R})$, then $e^{\phi}|\tilde{H} - \tilde{\phi}H|^2 + e^{-\phi}|H|^2$ is invariant.

Thus, the low-energy action of type IIB theory is invariant under a global $SL(2,\mathbb{R})$ symmetry, which in particular rotates the two 2-form gauge potentials into each other. One may ask whether this symmetry extends to the full type IIB theory, which includes branes in addition to the massless fields.

We have already predicted the existence of 1-branes, or strings, carrying unit charge under the potentials $B_{\mu\nu}$ and $\tilde{B}_{\mu\nu}$. Under the $SL(2,\mathbb{R})$ transformation described above, we would generate a continuous infinity of 1-branes carrying various (arbitrary real) charges under $B, \tilde{B}$. This conflicts with Dirac quantization for branes, unless the $SL(2,\mathbb{R})$ transformation matrix has all integer entries. This defines the infinite discrete subgroup $SL(2,\mathbb{Z})$. It follows that the largest subgroup of $SL(2,\mathbb{R})$ that can be a symmetry of the full type IIB theory is $SL(2,\mathbb{Z})$. Impressive evidence exists that this subgroup really does correspond to an exact symmetry. This symmetry is often called "S-duality".

One of the simplest pieces of evidence comes from re-examining soliton solutions. Let us label the string carrying unit charge under $B_{\mu\nu}$ as a $(1,0)$ string, and the one carrying unit charge under the R-R field $\tilde{B}_{\mu\nu}$ as a $(0,1)$ string. The latter can be obtained from the former by the particular $SL(2,\mathbb{Z})$ transformation:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{44}$$

Moreover, a general $SL(2,\mathbb{Z})$ transformation by the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ maps the $(1,0)$ string to a new object, an (a,c) string carrying charge a under B and simultaneously charge c under $\tilde{B}$. Note that since the matrix has integer entries with $ad - bc = 1$, it follows that a, c are coprime. Thus if this is to be a symmetry then type IIB theory must support strings with arbitrary integer charges (p, q) that are relatively prime. Indeed, it turns out that such soliton solutions do exist.

The action of the S-duality group $SL(2, \mathbb{Z})$ on the 5-brane coupling magnetically to $B_{\mu\nu}$ also generates a family of 5-branes in the obvious way. For the 3-brane, however, the situation is different. Since S-duality leaves the self-dual 4-form potential invariant, it does not generate new 3-branes, and there is only a single species of 3-brane in type IIB theory.

The $(1, 0)$ string, which is electrically charged under $B_{\mu\nu}$ is very similar to the string in type IIA theory. In particular, both are weakly coupled when the dilaton expectation value satisfies $e^{\langle\phi\rangle} \ll 1$. Quantizing this string leads to "type IIB string theory", of which type IIB supergravity is the effective low-energy theory for the massless fields.

Observe that $SL(2, \mathbb{Z})$ contains the transformation $\tau \to -\frac{1}{\tau}$ which converts a weakly coupled theory into a strongly coupled one. It is therefore a remarkably non-trivial duality symmetry, and has very powerful consequences. Clearly, the evidence that we have uncovered to support its existence does not amount to a proof, which would require an unimaginable level of control over the full nonperturbative dynamics of the theory.

In order to relate the type IIB theory to M-theory, we will need an important property of string theory, called "T-duality". The type IIA and type IIB theories are distinct in 10 dimensions, but after compactifying on a circle to 9 dimensions, they are actually equivalent. This is briefly described below. For more details see [4].

In the quantization of strings propagating on a circle, we find momentum modes quantized in units of $\frac{1}{R}$, just as for particles. Indeed, these modes correspond to particle-like motion of the string centre-of-mass, and the quantization arises for the usual reasons associated with a compact spactial direction. In addition, there are modes representing a string wound one or more times around the compact direction. These are quantized in units of R. The transformation $R \to \frac{1}{R}$ interchanges these two types of modes. These two types of modes appear symmetrically in various formulae, including those giving the spectrum and interactions of the theory.

Therefore, this transformation (called T-duality) would appear to be a symmetry of string theory – but for one subtlety. The operation $R \to \frac{1}{R}$ changes the spacetime chirality of half the fermions, so it interchanges the type IIA gravitinos $(\psi^{(1)}_{\mu\alpha}, \psi^{(2)}_{\mu\alpha'})$ with type IIB gravitinos $(\psi^{(1)}_{\mu\alpha}, \psi^{(2)}_{\mu\alpha})$, and similarly for the other fermions. The difference is in the spinor representations: type IIA fermions come in pairs of opposite chirality, namely a spinor and a conjugate spinor of $SO(9, 1)$, while type IIB fermions come in pairs with a common chirality, as we have seen above.

Indeed, it can be shown that under the circle T-duality $R \to \frac{1}{R}$, the type IIA and IIB strings are exchanged. As a result, type IIA theory on a circle of radius R is equivalent to type IIB theory on a circle of radius $1/R$, where the string winding modes of one map to the momentum modes of the other.

It follows that if we take type IIA theory in 10 dimensions, compactify on a circle of radius R and take the limit $R \to 0$, we recover type IIB theory in 10 dimensions! It is important to realize that from the type IIA point of view, as $R \to 0$ the *winding* modes along the x^{10} direction, together with the usual momentum modes in the remaining (noncompact) directions, assemble to give rise to 10-dimensional Lorentz invariance – a string miracle which would be hard to understand without knowing T-duality.

This appears to solve our problem about the origin of type IIB theory from M-theory. Recalling that type IIA theory is M-theory compactified on a circle of some radius R_{11}, it would appear that IIB theory is obtained by compactifying M-theory on a 2-torus (R_{11}, R_{10}) and taking $R_{10} \to 0$. This is not quite correct. Radii of circles as measured in M-theory and string theory are different because of the Weyl rescaling that we made. So we need to examine the issue more closely.

Compactify M-theory on a (rectangular) 2-torus and let R_{11}, R_{10} be the lengths of the basic cycles of the torus in the M-theory metric. We have already shown above that

$$R_{11} \sim \lambda^{2/3} \tag{45}$$

Now let $R_{10}^{(IIA)}$ be the radius of x^{10} in the type IIA metric. Since

$$g_{MN}^{(10)}(M) = e^{-\frac{1}{3}\phi} g_{MN}^{(10)}(IIA) \tag{46}$$

it follows that

$$\begin{aligned} R_{10} &= e^{-1/3\phi} R_{10}^{(IIA)} \\ &= \frac{R_{10}^{(IIA)}}{\lambda^{1/3}} \end{aligned} \tag{47}$$

Now T-duality is performed in 9 dimensions, and it keeps the 9-dimensional coupling invariant. For the type IIA theory in 9 dimensions, the coupling is

$$\frac{1}{\lambda_9^2} = \frac{R_{10}^{(IIA)}}{\lambda_A^2} = \frac{R_{10}^{(IIB)}}{\lambda_B^2} \tag{48}$$

Together with $R_{10}^{(IIB)} = \frac{1}{R_{10}^{(IIA)}}$, this implies that

$$\lambda_B = \frac{\lambda_A}{R_{10}^A} = \frac{R_{11}^{3/2}}{R_{10}\lambda^{1/3}} = \frac{R_{11}}{R_{10}}$$

$$R_{10}^{(IIB)} = \frac{1}{R_{10}^{(IIA)}} = \frac{1}{R_{10}\lambda^{1/3}} = \frac{1}{R_{10}R_{11}^{1/2}} \qquad (49)$$

We are interested in the limit $R_{10}^{(IIB)} \to \infty$ with λ_B fixed, so we must take R_{10}, R_{11} to zero *together*, with the ratio $\frac{R_{11}}{R_{10}}$ fixed. This is the limit in which M-theory compactified on a 2-torus gives rise to uncompactified, 10-dimensional type IIB theory. (It is important to keep in mind, however, that on a 2-torus with generic R_{11}, R_{10}, M-theory is equivalent to type IIB theory compactified on a circle.)

To the extent that M-theory is supposed to be the parent theory underlying the 10-dimensional theories, this should enable us to *deduce* the nonperturbative duality group $SL(2,\mathbb{Z})$ of type IIB theory that we discussed earlier, and indeed this is the case. The interchange $R_{10} \leftrightarrow R_{11}$ is a symmetry of M-theory (it is part of Lorentz invariance), hence the type IIB theory must have the symmetry $\lambda_B \to \frac{1}{\lambda_B}$.

One can repeat the above calculation for the case of a slanted 2-torus in the 10-11 directions. In this case one finds that $i\lambda_B$ is replaced by a complex quantity τ, which in M-theory is the modular parameter labelling possible complex structures of the torus, while in type IIB theory it is the complex scalar field $\tau = \tilde{\phi} + ie^{-\phi}$.

Thus the symmetry $\lambda_B \to \frac{1}{\lambda_B}$ is replaced by $\tau \to -\frac{1}{\tau}$. As is well known, this is just one element of the full global diffeomorphism group of a 2-torus, which is given by integer matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in $SL(2,\mathbb{Z})$, acting on the modular parameter τ as

$$\tau \to \frac{a\tau + b}{c\tau + d} \qquad (50)$$

Thus we conclude that, if M-theory has $11d$ Lorentz invariance, then type IIB theory necessarily has $SL(2,\mathbb{Z})$ symmetry! The mysterious, conjectural S-duality of type IIB theory thus gets geometrized into a completely natural symmetry of M-theory. The assumption that M-theory exists, is really all that is needed. For derivations and more details pertaining to $SL(2,\mathbb{Z})$ duality in 10d, see Ref. [5].

4 Moduli Space

To understand vacuum configurations, or backgrounds of the theories that we have been studying, it is crucial to introduce the concept of moduli space. This is basically the parameter space of the theory, modulo global identifications.

Moduli space can be assigned a topology and a metric. The idea is that an infinitesimal vector tangent to moduli space shifts a theory in one background to a theory in a neighbouring background. In string theory, a background is described perturbatively by a conformal field theory (CFT), and a deformation is described by a marginal operator in the CFT.

If the collection of all marginal operators is denoted $\Phi_i(\tau,\bar{\tau})$ then the CFT can be perturbed by shifting the 2-dimensional action:

$$S \to S + \sum_i g_i \int d^2\tau \; \Phi_i(\tau,\bar{\tau}) = S + \delta S \tag{51}$$

Now the correlation function on the 2-sphere, in the presence of the perturbation:

$$\langle \Phi_i(\tau,\bar{\tau})\Phi_j(w,\bar{w})e^{-\delta S}\rangle = \frac{g_{ij}(g_i)}{|z-w|^4} \tag{52}$$

defines a metric $g_{ij}(g)$ on the parameter space.

For type IIA theory in 10 dimensions, the moduli space is the space of vacuum expectation values of the dilaton, or more naturally the space of values of e^{ϕ}, which is the half-line $\mathbb{R}^+$. The moduli space of the IIB theory in 10 dimensions is more complicated. It is the space of vacuum expectation values of $\tau = \tilde{\phi} + ie^{-\phi}$ modulo the identification

$$\tau \to \frac{a\tau+b}{c\tau+d}, \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z}) \tag{53}$$

Since $\mathrm{Im}(\tau) > 0$, τ lies in the upper half plane (UHP), the moduli space is the quotient $(UHP)/SL(2,\mathbb{Z})$.

From the M-theory point of view, the picture is as follows. In 11 non-compact dimensions, M-theory has no moduli space since there is no scalar field to take an expectation value. After compactifying on a circle or 2-torus, one expects to find that the moduli space is the parameter space of a circle or 2-torus respectively. In the former case this is labelled by the compactification radius with no further identifications, hence it is $\mathbb{R}^+$, while in the latter case it is the moduli space of complex structures on the 2-torus, which is well-known to be $(UHP)/SL(2,\mathbb{Z})$. Thus the moduli spaces of the two 10-dimensional $N=2$ theories emerge naturally from M-theory.

5 Toroidal Compactification

In this section we discuss toroidal compactifications of M-theory. If we compactify M-theory on a $(d+1)$-dimensional torus for $d > 2$, we expect to obtain larger moduli spaces, but also larger analogues of the duality group $SL(2, \mathbb{Z})$ from geometrical symmetries of the torus. This indeed happens, but there are several interesting surprises.

First, let us look at things from the point of view of the type IIA/IIB theories. From our previous discussions, M-theory on a $(d+1)$-dimensional torus is equivalent to type IIA/IIB theory on a d-torus. Since the latter are string theories, they have a T-duality symmetry group which, on a torus T^d of $d > 1$, is quite nontrivial. This group is perturbatively visible from the point of view of string theory, so whatever happens, the full duality group must contain T-duality. Additionally, the full duality group must contain the $SL(2, \mathbb{Z})$ S-duality of type IIB theory.

To start with, T-duality in 9 dimensions is just $R \to \frac{1}{R}$ as we have seen. Thus the T-duality group is Z_2. S-duality commutes with this, so the full duality group in 9 dimensions is $SL(2, \mathbb{Z}) \times \mathbb{Z}_2$. For compactification on a d-torus T^d with $d > 1$, we have to first understand T-duality in some detail and then try to discover the maximal duality group, called "U-duality", which combines T and S dualities along with others.

We start by recalling a few standard formulae from the conformal field theory of massless compact scalars. These scalars are coordinates of the d-torus, which can be thought of as the quotient of d-dimensional Euclidean space by a lattice Γ. Let $\vec{e}_i,\ i = 1, \ldots, d$ be the generators of Γ. The generators of the dual lattice $\Gamma^\star$, denoted by $\vec{e}^{\,\star}_j$, are defined by $\vec{e}_i \cdot \vec{e}^{\,\star}_j = \delta_{ij}$. It follows that the inner product of a vector in Γ with a vector in the dual lattice $\Gamma^\star$ is necessarily an integer. This fact will be useful shortly.

The Hamiltonian of such a CFT is given by $L_0 + \bar{L}_0$ where

$$\begin{aligned} L_0 &= \frac{1}{2}(\vec{p}_L)^2 + \text{oscillators} \\ \bar{L}_0 &= \frac{1}{2}(\vec{p}_R)^2 + \text{oscillators.} \end{aligned} \tag{54}$$

where

$$\begin{aligned} \vec{p}_L &= \sum_{i=1}^{d} m_i \vec{e}_i + \sum_{i=1}^{d} n_i \vec{e}^{\,\star}_i \\ \vec{p}_R &= \sum_{i=1}^{d} m_i \vec{e}_i - \sum_{i=1}^{d} n_i \vec{e}^{\,\star}_i \end{aligned} \tag{55}$$

This all-important formula follows from the fact that on a torus, the string has momentum modes which are integer multiples of $\vec{e}^{\,\star}_i$, and winding modes which are integer multiples of $\vec{e}_i$.

Now the vectors $(\vec{e}_i, \vec{0})$ and $(\vec{0}, \vec{e}_i^\star)$ together generate a $2d$-dimensional lattice $\Gamma \oplus \Gamma^\star$. A general vector in $\Gamma \oplus \Gamma^\star$ is $(\vec{v}, \vec{w})$ with $\vec{v} \in \Gamma$, $\vec{w} \in \Gamma^\star$. We define a norm on this lattice as follows:

$$\|(\vec{v}, \vec{w})\|^2 = \begin{pmatrix} \vec{v} & \vec{w} \end{pmatrix} \begin{pmatrix} \mathbf{0} & \mathbf{1} \\ \mathbf{1} & \mathbf{0} \end{pmatrix} \begin{pmatrix} \vec{v} \\ \vec{w} \end{pmatrix} = 2\vec{v} \cdot \vec{w} \tag{56}$$

where $\mathbf{1}$ denotes the $d \times d$ identity matrix. Since $\vec{v} \cdot \vec{w}$ is integer for $\vec{v} \in \Gamma$ and $\vec{w} \in \Gamma^\star$, the norm of any vector according to the above definition is an *even* integer. Hence the lattice $\Gamma \oplus \Gamma^\star$ is said to be an even lattice.

This lattice is also self-dual, since $(\Gamma \oplus \Gamma^\star)^\star = \Gamma^\star \oplus \Gamma$ which is isomorphic. Note that the norm defined above, using the matrix $\begin{pmatrix} \mathbf{0} & \mathbf{1} \\ \mathbf{1} & \mathbf{0} \end{pmatrix}$ is a Minkowski norm, since this matrix has d eigenvalues equal to $+1$ and d eigenvalues equal to -1.

We will be interested in continuous deformations of this even, self-dual lattice which preserve the norm. These are given by all $2d \times 2d$ matrices $\mathbf{M}$ acting as

$$\begin{pmatrix} \vec{v} \\ \vec{w} \end{pmatrix} \to \mathbf{M} \begin{pmatrix} \vec{v} \\ \vec{w} \end{pmatrix} \tag{57}$$

and satisfying

$$\mathbf{M}^T \begin{pmatrix} \mathbf{0} & \mathbf{1} \\ \mathbf{1} & \mathbf{0} \end{pmatrix} \mathbf{M} = \begin{pmatrix} \mathbf{0} & \mathbf{1} \\ \mathbf{1} & \mathbf{0} \end{pmatrix} \tag{58}$$

By definition, such matrices $\mathbf{M}$ parametrize the orthogonal group $O(d, d; \mathbb{R})$ which is like the Lorentz group for a spacetime with d space and d time directions. Of course, this strange signature is in no way related to the signature of the physical spacetime on which our string theory lives. Rather, it arises from the mathematics of the lattice $\Gamma \oplus \Gamma^\star$ on which this Minkowski norm is natural and leads to the property of being even and self-dual.

This action of $O(d, d; \mathbb{R})$ generates a whole family of lattices starting from a given $\Gamma \oplus \Gamma^\star$. (Not all the lattices so generated have a direct-sum form, however.) All the lattices in this family are equally valid backgrounds for compactification, and in the 2d CFT sense, one interpolates continuously among them by marginal deformations. Thus the matrices $\mathbf{M}$ generate a moduli space: a family of backgrounds for string theory.

By an obvious change of basis, we can define the group $O(d, d; \mathbb{R})$ as the collection of all real matrices satisfying

$$\mathbf{M}^T \begin{pmatrix} \mathbf{1} & \\ & -\mathbf{1} \end{pmatrix} \mathbf{M} = \begin{pmatrix} \mathbf{1} & \\ & -\mathbf{1} \end{pmatrix} \tag{59}$$

However, some of these matrices $\mathbf{M}$ do not generate physically new string backgrounds, but only spatial rotations of the old ones. These correspond to matrices $\mathbf{M}$ which in the above basis are block diagonal and of the form:

$$\mathbf{M} = \begin{pmatrix} \mathbf{M_1} & 0 \\ 0 & \mathbf{M_2} \end{pmatrix} \tag{60}$$

Here $\mathbf{M_1}, \mathbf{M_2}$ are independent $d \times d$ matrices, which form an $O(d, \mathbb{R}) \times O(d, \mathbb{R})$ subgroup of $O(d, d, \mathbb{R})$. This subgroup consists of independent rotations of the world-sheet left-movers and right-movers, a symmetry of the 2d CFT. Indeed, one can check that L_0 and $\bar{L}_0$ are separately invariant under this subgroup.

As a result, the moduli space is not the full parameter space of $O(d, d; \mathbb{R})$ but must be quotiented by these symmetries. We write the quotient

$$O(d, d; \mathbb{R}) / \left(O(d, \mathbb{R}) \times O(d, \mathbb{R})\right) \tag{61}$$

The dimension of this quotient space is the difference between the dimensions of the groups in the numerator and the denominator, and comes out to be d^2. Indeed, it has been shown in general that deformations of the lattice $\Gamma \oplus \Gamma^\star$ are parametrized by the scalar fields g_{ij}, B_{ij} corresponding to components of the metric and the NS-NS 2-form along the internal directions. As g is symmetric and B is antisymmetric, these scalars are d^2 in number as expected.

This is still not the end of the story, however. There are discrete global identifications on the moduli space, arising from the group of T-dualities. These are precisely the lattice autmorphisms: the discrete transformations that leave the lattice $\Gamma \oplus \Gamma^\star$ invariant. Clearly, all linear transformations of the lattice with integer entries will map the lattice into a sublattice of itself. The ones which preserve the inner product will map it *onto* itself. Thus all $2d \times 2d$ matrices $\mathbf{M}$ with *integer entries* and satisfying Eq. 59 correspond to automorphisms of the lattice, and they define the infinite discrete group $O(d, d; \mathbb{Z})$. Thus the moduli space in Eq. 61 needs to be further quotiented by this discrete group.

Before doing that, let us consider a simple example. For compactification on a circle of radius R, the lattice $\Gamma \oplus \Gamma^\star$ is generated by $(R, 0)$ and $(0, \frac{1}{R})$. The only 2×2 integer matrix, besides the identity, satisfying

$$\mathbf{M}^T \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \mathbf{M} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{62}$$

is $\pm \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ itself. This sends

$$\begin{pmatrix} mR \\ \frac{n}{R} \end{pmatrix} \rightarrow \begin{pmatrix} \frac{n}{R} \\ mR \end{pmatrix} \tag{63}$$

so it acts as $R \to \frac{1}{R}$, a fact which was used earlier.

From the above considerations, one may expect that the duality group of type II theories compactified on a d-torus should be $O(d, d; \mathbb{Z})$ and the moduli space should be

$$O(d, d; \mathbb{Z}) \backslash O(d, d; \mathbb{R}) / \left(O(d, \mathbb{R}) \times O(d, \mathbb{R})\right) \tag{64}$$

(we follow the now-standard convention of quotienting by continuous groups from the right and discrete groups from the left).

This structure of the moduli space is all that can be deduced purely from T-duality. However, the actual duality group and moduli space are larger, because we have not yet taken into account S-duality. This is most evident from the type IIB point of view. Even before compactification, type IIB theory had an S-duality group $SL(2, \mathbb{Z})$ and a nontrivial moduli space $SL(2, \mathbb{Z}) \backslash (UHP)$. So the full duality group must contain both S-duality and T-duality, possibly along with other dualities.

The upper half-plane can be thought of as the quotient of the group manifold of $SL(2, \mathbb{R})$ by $U(1)$. As $SL(2, \mathbb{R})$ is the same as $SO(2, 1; \mathbb{R})$ and $U(1)$ is the same as $SO(2)$, we can equivalently write the duality group of type IIB in 10 dimensions as $SO(2, 1; \mathbb{Z})$ and its duality group as

$$SO(2, 1; \mathbb{Z}) \backslash SO(2, 1; \mathbb{R}) / SO(2, \mathbb{R}) \tag{65}$$

which is structurally quite similar to the space in Eq. 64. The issue is now to find the right duality group that combines both T-duality and S-duality (as mentioned before, this is called the U-duality group), and the right moduli space that contains the structure of both Eq. 64 and Eq. 65.

These considerations are not enough to fix the maximal U-duality group. A further input is the fact that compactification of type II theories on T^d is equivalent to an M-theory compactification on T^{d+1}, hence must possess the geometric symmetries of the latter torus. Finally, the classical supergravity Lagrangian with 32 supersymmetries, in various dimensions, has associated continuous duality groups analogous to $SL(2, \mathbb{R})$ in 10 dimensions. One expects these to be broken to discrete subgroups, also by analogy with 10 dimensions.

Thus, the answer for the U-duality group and associated moduli space has to be found on a case-by-case basis. Let us start with compactification to 8 dimensions on a 2-torus. Based on T-duality alone, we would expect the moduli space to be as in Eq. 64, namely

$$O(2, 2; \mathbb{Z}) \backslash O(2, 2; \mathbb{R}) / \left(O(2, \mathbb{R}) \times O(2, \mathbb{R})\right) \tag{66}$$

of dimension 4. The massless scalar fields in the spacetime Lagrangian that parametrize this moduli space are $g_{99}, g_{10,10}, g_{9,10}, B_{9,10}$.

However, there are three more massless scalars in this 8-dimensional theory, which from the type IIB point of view are $\phi, \tilde{\phi}$ and $\tilde{B}_{9,10}$. Thus the moduli space must have dimension 7 rather than 4. At this point, the supergravity action in 8 dimensions with 32 supersymmetries provides a clue. The scalar fields in that Lagrangian are known to parametrize a coset space $(SL(3,\mathbb{R})/S0(3,\mathbb{R})) \times (SL(2,\mathbb{R})/SO(2,\mathbb{R}))$. The two factors are 5 and 2 dimensional respectively, so this space is 7-dimensional. We can therefore assume that this must be the moduli space, upto global identifications, which in turn must include the T-duality and S-duality groups.

The *U-duality conjecture* says that the global identifications, or U-duality groups, are integer subgroups of the continuous group G appearing in the coset $H\backslash G$. According to this conjecture, the 8-dimensional theory therefore has U-duality group $SL(3,\mathbb{Z}) \times SL(2,\mathbb{Z})$, and its moduli space is

$$(SL(3,\mathbb{Z}) \times SL(2,\mathbb{Z})) \setminus (SL(3,\mathbb{R}) \times SL(2,\mathbb{R})) \,/\, (SO(3,\mathbb{R}) \times SO(2,\mathbb{R}) \tag{67}$$

As very important evidence for this conjecture, we note that parts of this structure can be seen naturally in the various different formulations. From M-theory, one expects one factor of the U-duality group to be $SL(3,\mathbb{Z})$, the mapping-class group of the 3-torus. Indeed from these considerations the moduli space would be

$$SL(3,\mathbb{Z})\backslash SL(3,\mathbb{R})/SO(3,\mathbb{R}) \tag{68}$$

which is 5-dimensional, and is parametrized by the 6 scalars $g_{ij}, i,j = 1,\ldots,3$ with the constraint that $|g|$ is fixed.

This incidentally shows that the $SL(2,\mathbb{Z})$ S-duality of IIB is contained in $SL(3,\mathbb{Z})$ and is not the second factor $SL(2,\mathbb{Z})$ in the product! But $SL(3,\mathbb{Z})$ also includes part of the T-duality group of IIB. So U-duality mixes, and hence unifies, S- and T-duality.

In 7 dimensions, the U-duality group is similarly predicted to be $SL(5,\mathbb{Z})$. The full moduli space is

$$SL(5,\mathbb{Z})\backslash SL(5,R)/SO(5,R) \tag{69}$$

Of this, an $SL(4,\mathbb{Z})$ part is obvious from 11-dimensional M-theory. Indeed, it looks rather as though a 12-dimensional theory were responsible for this moduli space, since it is the geometric moduli space for a 5-torus. But it is presently not clear what such a 12-dimensional theory should be. A different $SL(4,\mathbb{Z})$ subgroup comes from T-duality of IIB string theory, due to the coincidence that $SO(3,3;\mathbb{Z}) = SL(4,\mathbb{Z})$, the former being the T-duality group for strings on a 3-torus. Seven dimensions is the first case where the U-duality group is *simple*.

It is instructive to see how the gauge fields transform under U-duality in 7d. From the type IIA point of view, the gauge fields are:

$$\begin{array}{rcl} g_{\mu i} & \to & 3A_\mu \\ B_{\mu i} & \to & 3A_\mu \\ \tilde{C}_{\mu ij} & \to & 3\tilde{A}_\mu \\ \tilde{A}_\mu & \to & \tilde{A}_\mu \end{array} \tag{70}$$

In particular there are 6 NS-NS gauge fields, in the **6** of $O(3,3;\mathbb{Z}) \sim SL(4,\mathbb{Z})$. And there are 4 R-R gauge fields transforming in the **4** (spinor of $O(3,3)$, or fundamental of $SL(4)$). With respect to $SL(5)$ they transform together, irreducibly, in the **10** of $SL(5)$ (the 2nd rank antisymmetric tensor representation).

As for the scalars, there are 10 of them (g_{ij}, B_{ij}, ϕ) in the NS-NS sector, and 4 more $(\tilde{C}_{ijk}, \tilde{A}_i)$ in the RR sector. The first 9 parametrize $SL(4,\mathbb{R})/SO(4,\mathbb{R})$ (the usual situation with T-duality), while all 14 together parametrize the coset $SL(5,\mathbb{R})/SO(5,\mathbb{R})$.

By similar arguments, the moduli spaces for toroidal compactification to 6,5,4,3 dimensions are conjectured to be:

$$\begin{array}{rcl} \underline{6d} & : & SO(5,5;\mathbb{Z})\backslash SO(5,5;\mathbb{R})/\left(SO(5,\mathbb{R}) \times SO(5,R)\right) \\ \underline{5d} & : & E_{6,6}(\mathbb{Z})\backslash E_{6(6)}(\mathbb{R})/USp(8) \\ \underline{4d} & : & E_{7,7}(\mathbb{Z})\backslash E_{7(7)}(\mathbb{R})/SU(8) \\ \underline{3d} & : & E_{8,8}(\mathbb{Z})\backslash E_{8(8)}(\mathbb{R})/S0(10,\mathbb{R}) \end{array} \tag{71}$$

Here, $E_{n(n)}$ denote certain non-compact versions of the exceptional groups.

By definition, the U-dualities give the complete set of dualities for theories with 32 supersymmetries. Details of what was described above can be found in the literature, see in particular Refs. [6,7].

6 Conclusions

We have seen that a logical picture of the symmetries and even some of the dynamics of maximally supersymmetric theories can be assembled from the knowledge of T-, S- and U-duality groups. String theory is not an essential starting postulate. Maximally supersymmetric theories contain extended objects (branes) that we are forced to consider as they play an important role, and these include in particular strings. Central charges and the BPS bound, both originally field-theoretic notions, play a crucial part in the study of branes and their dynamics.

Enormous progress has been made beyond what is presented here, but that could not be discussed in this short and very introductory course of lectures. Other review lectures, in this school and elsewhere, discuss developments like stringy solitons, D-branes, F-theory, and M(atrix) theory that are now part of the standard lore on the subject.

References

1. Abdus Salam and E. Sezgin, *"Supergravities in Diverse Dimensions"*, Vol. 1 and 2; World Scientific, Singapore (1989).
2. M.B. Green, J.H. Schwarz and E. Witten, *"Superstring Theory"*, Vol. 1 and 2; Cambridge University Press, Cambridge (1987).
3. P.K. Townsend, *"Four Lectures on M Theory"*; (Lectures given at ICTP); hep-th/9612121.
4. A. Giveon, M. Porrati and E. Rabinovici, *"Target Space Duality in String Theory"*; Physics Reports **244** (1994), 77-202, hep-th/9401139.
5. J.H. Schwarz, *"Lectures on Superstring and M Theory Dualities"* (Lectures given at ICTP and TASI); Nucl. Phys. Proc. Suppl. **55B** (1997), 1-32; hep-th/9607201
6. C. Hull and P. Townsend, *"Unity of Superstring Dualities"*, Nucl. Phys. **B438** (1995) 109-137; hep-th/9410167.
7. E. Witten, *"String Theory Dynamics in Various Dimensions"*, Nucl. Phys. **B443** (1995), 85-126; hep-th/9503124.