Simultaneous diophantine approximation with quadratic and linear forms

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Dedicated to G.A. Margulis with admiration

Abstract

Let Q be a nondegenerate indefinite quadratic form on $\mathbb{R}^n$, $n \geq 3$, which is not a scalar multiple of a rational quadratic form, and let $C_Q = \{v \in \mathbb{R}^n \mid Q(v) = 0\}$. We show that given $v_1 \in C_Q$, for almost all $v \in C_Q \setminus \mathbb{R}v_1$ the following holds: for any $a \in \mathbb{R}$, any affine plane P parallel to the plane of v_1 and v, and $\epsilon > 0$ there exist primitive integral n-tuples x within ϵ distance of P for which $|Q(x) - a| < \epsilon$. An analogous result is also proved for almost all lines on C_Q.

Keywords: Values of quadratic forms, Diophantine approximation, flows on homogeneous spaces.

Mathematics Subject Classification: 11H55, 22E40.

1 Introduction

Margulis proved in the mid-nineteen-eighties, in response to a long-standing conjecture of A. Oppenheim, that given a nondegenerate indefinite real quadratic form Q on $\mathbb{R}^n$, $n \geq 3$, which is not a scalar multiple of a form with rational coefficients, the set $Q(\mathbb{Z}^n)$ of values of Q at integer points is a dense subset of $\mathbb{R}$, namely, for any $a \in \mathbb{R}$ and $\epsilon > 0$ there exists $x \in \mathbb{Z}^n$ such that $|Q(x) - a| < \epsilon$. There have been various strengthenings of the result and other developments around it since then, including some quantitative versions of the conjecture. A nice exposition of the developments on the theme until the mid-nineteen-nineties is given by Margulis in [9]; for some later results the reader may refer [1] and [2].

One question in this respect on which our understanding is still meager is whether we can specify location requirements for the integral solutions of the diophantine inequalities $|Q(x) - a| < \epsilon$ as above. For instance, given a proper non-zero subspace, or more generally an affine subspace W of $\mathbb{R}^n$ can the solution x of the inequality $|Q(x) - a| < \epsilon$ be chosen near W, say within distance ϵ? For an affine hyperplane the question is equivalent to whether the x can also be chosen simultaneously to be a solution of $|L(x) - b| < \epsilon$ for a given linear form L on $\mathbb{R}^n$ and $b \in \mathbb{R}$. A result from [4] shows that for $n = 3$ the answer is in the affirmative if the plane $\{v \in \mathbb{R}^n \mid L(v) = 0\}$ is tangential to the cone $\{v \in \mathbb{R}^n \mid Q(v) = 0\}$. On the other hand it was deduced in [1] from a result of Kleinbock and Margulis [8] that in absence of the additional condition, for a certain large class of L the two diophantine inequalities as above do not admit common integral solutions, for small enough ϵ. For $n \geq 4$, using Ratner's theorem on orbit closures, A. Gorodnik [6] obtained an affirmative answer for a large class of pairs (Q, L); (see also [7] for results for pairs of quadratic forms); it is stated in [6] that the result can be extended, with arguments similar to those involved in the proof there, to cover all pairs (Q, L) for $n = 4$; however for $n \geq 5$ the question remains to be resolved fully.

When the location requirement as above is specified in terms of subspaces or affine subspaces of codimension two or more, no significant result seems to be known. Our results here partially addresses the question for affine planes (2-dimensional affine subspaces) and also lines through the origin; see Theorem 1.1 and Corollaries 1.2 and 1.3 for details. The diophantine results are deduced from a result of Nimish Shah [10] on asymptotics of measures on homogeneous spaces under actions of sequences of diagonal elements; see §2.

Now let $V = \mathbb{R}^n$, $n \geq 3$; we consider it equipped with the usual inner product and the corresponding norm which we shall denote by $\|\cdot\|$. For any linearly independent pair of vectors $v_1, v_2 \in V$ we denote by $P(v_1, v_2)$ the plane (two-dimensional subspace) spanned by v_1 and v_2; in the sequel whenever we write $P(v_1, v_2)$ for vectors v_1 and v_2, they will be linearly independent, either by choice or construction, and we may omit specific mention of this. For a plane P in V we denote by $P^\perp$ the orthocomplement of P and by $\pi(x, P)$, $x \in V$, the orthogonal projection of x on $P^\perp$.

We recall that $x \in \mathbb{Z}^n$ is said to be *primitive* if $k^{-1}x \notin \mathbb{Z}^n$ for any integer $k \geq 2$. We denote by $\mathcal{P}(\mathbb{Z}^n)$ the set of all primitive elements in $\mathbb{Z}^n$. The solutions we obtain for various diophantine systems will be primitive.

A quadratic form Q on $\mathbb{R}^n$ is said to be rational if its coefficients with respect to the standard basis of $\mathbb{R}^n$ are rational.

Let Q be a nondegenerate quadratic form on V. We denote by C_Q the cone defined by $C_Q = \{v \in V \mid v \neq 0, Q(v) = 0\}$. Then C_Q is a differentiable submanifold of V. We shall consider C_Q equipped with a measure which on coordinate charts is equivalent to (has the same sets of measure zero as) the Lebesgue measure. If P is a plane in V such that for every $a \in \mathbb{R}$, $w \in P^\perp$ and

$\epsilon > 0$ there exists $x \in \mathbb{Z}^n$ such that $|Q(x) - t| < \epsilon$ and $\|\pi(x, P) - w\| < \epsilon$ then $P \cap C_Q$ is nonempty (see Remark 3.1 for details). It is therefore natural in addressing our question to consider only planes intersecting C_Q; furthermore while considering generic planes we may assume the plane to be spanned by two vectors in C_Q. We prove the following.

Theorem 1.1. *Let Q be a nondegenerate indefinite quadratic form on V which is not a scalar multiple of a rational quadratic form. Let $v_1 \in C_Q$ be given. Then for almost all $v \in C_Q \setminus \mathbb{R}v_1$ the following holds: for any $t \in \mathbb{R}$, $w \in P(v_1, v)^{\perp}$ and $\epsilon > 0$ there exists $x \in \mathcal{P}(\mathbb{Z}^n)$ such that*

$$|Q(x) - t| < \epsilon \quad \textit{and} \quad \|\pi(x, P(v_1, v)) - w\| < \epsilon.$$

The method of proof of Theorem 1.1, via study of flows on homogeneous spaces as we shall see below, is not applicable to one-dimensional subspaces, in the place of planes (higher dimensional subspaces are evidently taken care of, with any particular dimension). However, we deduce from Theorem 1.1, via elementary considerations, the following result dealing with the analogous question for lines (on C_Q, as would be natural to assume), with regard to values near 0, namely the "homogeneous" case of the diophantine inequalities; it turns out that generalisation to the inhomogeneous case is not to be expected in this case (see Remark 3.3). For $v \in V \backslash \{0\}$ let $\pi(x, v)$, $x \in V$, denote the projection of x on the orthocomplement of the subspace spanned by v. Then we have the following:

Corollary 1.2. *Let Q be a nondegenerate indefinite quadratic form on V which is not a scalar multiple of a rational quadratic form. Then for almost all $v \in C_Q$ the following holds: for any $\epsilon > 0$ there exists $x \in \mathcal{P}(\mathbb{Z}^n)$ such that*

$$|Q(x)| < \epsilon \quad \textit{and} \quad \|\pi(x, v)\| < \epsilon.$$

Theorem 1.1 and Corollary 1.2 can also be formulated as statements about simultaneous diophantine approximation involving a quadratic form together with $n-2$, or $n-1$ respectively, linearly independent linear forms. Let Q be a quadratic form as in Theorem 1.1 and for $l = 1, 2$ let $\mathcal{L}_Q^{n-l}$ be the manifold consisting of all $(n-l)$-tuples $(L_1, \ldots, L_{n-l})$ of linearly independent linear forms such that the restriction of Q to the l-dimensional subspace $\{v \in V \mid L_k(v) = 0 \text{ for all } k = 1, \ldots, n-l\}$ is a nondegenerate indefinite quadratic form. We consider $\mathcal{L}_Q^{n-l}$ equipped with a measure which on coordinate charts is equivalent to Lebesgue measure. Theorem 1.1 and Corollary 1.2 imply the following; see §3 for details.

Corollary 1.3. *i) For almost all* $(L_1, \dots, L_{n-2})$ *in* $\mathcal{L}_Q^{n-2}$ *the following holds: for all* $s_0, s_1, \dots, s_{n-2} \in \mathbb{R}$ *and* $\epsilon > 0$ *there exists* $x \in \mathcal{P}(\mathbb{Z}^n)$ *such that*

$$|Q(x) - s_0| < \epsilon \quad \textit{and} \quad |L_k(x) - s_k| < \epsilon \ \textit{ for all } k = 1, \dots, n-2.$$

ii) For almost all $(L_1, \dots, L_{n-1}) \in \mathcal{L}_Q^{n-1}$ *and* $\epsilon > 0$ *there exists* $x \in \mathcal{P}(\mathbb{Z}^n)$ *such that*

$$|Q(x)| < \epsilon \quad \textit{and} \quad |L_k(x)| < \epsilon \ \textit{ for all } k = 1, \dots, n-1.$$

The method of proof of Theorem 1.1 is analogous to Margulis's proof of Oppenheim conjecture and many subsequent results along the line, in that it involves considering orbits on the homogeneous space $SL(n,\mathbb{R})/SL(n,\mathbb{Z})$ under the action of the subgroup of $SL(n,\mathbb{R})$ leaving the diophantine system invariant. There is a major difference however that while in the earlier cases the subgroup concerned was generated by unipotent elements (up to finite index), in the case at hand the subgroup is a diagonalisable one-parameter subgroup, whose action does not have the rigidity properties of the actions of subgroups generated by unipotent elements. It is this aspect that constrains the results to "almost all" rather than all systems (with specified conditions).

In §2 we recall a special case of a theorem of Nimish Shah [10] (Theorem 2.1) and note a consequence of it (Corollary 2.2) to density of certain sequences of points on homogeneous spaces G/Γ of connected Lie groups by lattices. The proof of Theorem 1.1, which is described in §3, involves the case $G = SL(n,\mathbb{R})$ and $\Gamma = SL(n,\mathbb{Z})$, with the sequences arising as orbits of a cyclic subsemigroup of the diagonalisable one-parameter subgroup leaving invariant the diophantine system. Some related results based on a theorem from [5], which may be of indirect interest, are also noted.

Acknowledgement: I would like to thank G.A. Margulis for a discussion with him in 2000 at the Isaac Newton Institute, Cambridge, England, which has played a role on the way to arriving at the results presented here. Thanks are also due to D. Kleinbock, Nimish A. Shah and two anonymous referees for useful suggestions enabling improvements in the presentation of the results.

2 Dense sequences in G/Γ

We begin by recalling a result of Nimish Shah [10] from which Theorem 1.1 will be deduced.

Let G be a connected Lie group and Γ be a lattice in G, namely a discrete subgroup of G such that G/Γ carries a finite G-invariant measure; we shall denote by μ the G-invariant probability measure on G/Γ. We denote by e

the identity element in G. Let H be a connected semisimple real algebraic group realised as a closed subgroup of G. Let A be a maximal connected diagonalisable subgroup of H; i.e. A as in the Iwasawa decomposition of H. Let $\mathfrak{H}$ be the Lie algebra of H and $\mathfrak{H} = \oplus_{\alpha\in\Lambda}\mathfrak{H}_\alpha$ be the decomposition with respect to the adjoint action of A on $\mathfrak{H}$, Λ being the corresponding set of roots, consisting of all $\alpha : A \to \mathbb{R}^+ = \{t \in \mathbb{R} \mid t > 0\}$ for which there exists $\xi \in \mathfrak{H}$, $\xi \neq 0$, such that $(\mathrm{Ad}\, a)(\xi) = \alpha(a)\xi$ for all $a \in A$. Then Theorem 1.4 of [10] implies in particular the following:

Theorem 2.1. *Let the notation be as above. Let $\{a_i\}$ be a sequence in A such that for every $\alpha \in \Lambda$, either $\{\alpha(a_i)\}$ is relatively compact in $\mathbb{R}^+$, or $\alpha(a_i) \to 0$ or $\alpha(a_i) \to \infty$. Let $U = \{u \in H \mid a_i^{-1}ua_i \to e, \quad as\ i \to \infty\}$. Suppose that U is not contained in any proper closed normal subgroup of H. Let $x \in G/\Gamma$ be such that Hx is dense in G/Γ. Let λ be a Haar measure on U. Then for any Borel subset Θ of U and any bounded continuous function φ on G/Γ, as $i \to \infty$,*

$$\int_\Theta \varphi(a_iux)d\lambda(u) \longrightarrow \lambda(\Theta)\int_{G/\Gamma}\varphi d\mu.$$

Clearly it suffices to prove the assertion when $0 < \lambda(\Theta) < \infty$. For this we apply Theorem 1.4 of [10] with G and H as above in the place L and G respectively, in the notation of [10]; for the lattice Λ as in [10] we choose $g\Gamma g^{-1}$ with $g \in G$ such that $x = g\Gamma$, and the probability measure λ in the statement of the theorem in [10] is chosen to be the measure assigning the value $\lambda(E \cap \Theta)/\lambda(\Theta)$ for all Borel subsets E of U.

Corollary 2.2. *Let the notation be as in Theorem 2.1. Then for λ-almost all $u \in U$, $\{a_iux \mid i \in \mathbb{N}\}$ is dense in G/Γ.*

Proof. Let Ω be any nonempty open subset of G/Γ and let $E = \{u \in U \mid a_iux \notin \Omega \text{ for } i \in \mathbb{N}\}$. We shall show that $\lambda(E) = 0$; as G/Γ is second countable, this would imply the corollary. Let φ be a continuous nonnegative function on G/Γ, with compact support contained in Ω and $\int \varphi d\mu > 0$. By Theorem 2.1 $\int_E \varphi(a_iux)\, d\lambda(u) \to \lambda(E)\int \varphi d\mu$. For $u \in E$ we have $\varphi(a_iux) = 0$ for all i, so $\int_\Theta \varphi(a_iux)\, d\lambda(u) = 0$. Therefore $\lambda(E) = 0$. □

Let me also note here a property of the set of u for which the conclusion of Corollary 2.2 holds. The result, Corollary 2.3 below, does indeed have implications to assertions in Theorem 1.1 and Corollary 1.2. However their significance in terms of diophantine approximation seems limited from the present perspective, and hence the specifics are perhaps best left to the interested reader. The property is deduced from a result from [5], independently of Corollary 2.2, and may be of interest in other contexts. For the author it has been of interest en route to arriving at the results described above.

It can be seen that U as above is a simply connected nilpotent Lie group. Let $\mathfrak{U}$ be the Lie subalgebra corresponding to U. Then the exponential map $\exp : \mathfrak{U} \to U$ is a diffeomorphism onto U. Hence the theorem implies in particular, that for almost all $\xi \in \mathfrak{U}$, $\{a_i(\exp t\xi)x \mid i \in \mathbb{N}\}$ is dense in G/Γ for almost all $t \in \mathbb{R}$. In this respect we shall prove the following.

Corollary 2.3. *Let $\{a_i\}$, U and x be as in Corollary 2.2. Let $\xi \in \mathfrak{U}$ be such that there exists $t_0 \in \mathbb{R}$ for which $\{a_i(\exp t_0\xi)x \mid i \in \mathbb{N}\}$ is dense in G/Γ. Then $\{a_i(\exp t\xi)x \mid i \in \mathbb{N}\}$ is dense in G/Γ for almost all $t \in \mathbb{R}$.*

Corollary 2.3 is derived from the following theorem from [5], which is a "uniform version" of Ratner's uniform distribution theorem for unipotent one-parameter flows; we recall that $x = g\Gamma$, where $g \in G$, is said to be *generic* for the action of a unipotent one-parameter subgroup $\{u_t\}$ of G if there is no proper closed subgroup F of G containing $\{g^{-1}u_tg\}$ and such that $F \cap \Gamma$ is a lattice in F.

Theorem 2.4. *Let $\{u_t^{(i)}\}$ be a sequence of unipotent one-parameter subgroups of G converging to a one-parameter subgroup $\{u_t\}$ of G (i.e. $u_t^{(i)} \to u_t$ for all $t \in \mathbb{R}$, as $i \to \infty$). Let $\{x_i\}$ be a sequence of points in G/Γ converging to a point which is generic for the action of $\{u_t\}$. Let $\{T_i\}$ be a sequence of positive numbers such that $T_i \to \infty$ as $i \to \infty$. Then for any bounded continuous function φ on G/Γ, as $i \to \infty$,*

$$\frac{1}{T_i}\int_0^{T_i} \varphi(u_t^{(i)}x_i)dt \longrightarrow \int_{G/\Gamma} \varphi d\mu.$$

Corollary 2.5. *Let $\{a_i\}$, U and x be as before and let $\xi \in \mathfrak{U}$ be as in Corollary 2.3. Then for any Borel subset Θ of $\mathbb{R}$ and any bounded continuous function φ on G/Γ we have*

$$\int_\Theta \varphi(a_i(\exp t\xi)x)\, dt \to l(\Theta) \int_{G/\Gamma} \varphi d\mu, \quad \textit{as } i \to \infty,$$

where l is the Lebesgue measure on $\mathbb{R}$.

Proof. Recall that by hypothesis there exists $t_0 \in \mathbb{R}$ such that $\{a_i \exp(t_0\xi)x\}$ is dense in G/Γ. Considering $\exp(t_0\xi)x$ in the place of x we may assume that $t_0 = 0$, namely that $\{a_ix\}$ is dense in G/Γ. We note that by regularity of the Lebesgue measure it suffices to prove the assertion for Θ any interval, say (a, b), with $a < b$. For each i let $\alpha_i = \|\mathrm{Ad}\, a_i(\xi)\|$. Then $\alpha_i \to \infty$ as $i \to \infty$. Passing to a subsequence we may assume that $\{\alpha_i^{-1}\mathrm{Ad}\, a_i(\xi)\}$ converges, say

$\alpha_i^{-1}\mathrm{Ad}\, a_i(\xi) \to \theta \in \mathfrak{U}$. We note that $\|\theta\| = 1$, and in particular $\theta \neq 0$. Now for each i let $\xi_i = \alpha_i^{-1}\mathrm{Ad}\, a_i(\xi)$ and $u_t^{(i)} = \exp t\xi_i$ for all $t \in \mathbb{R}$. Also for all $t \in \mathbb{R}$ let $u_t = \exp t\theta$. Then $\xi_i \to \theta$, and $u_t^{(i)} \to u_t$ for all t. Since $\{a_i x\}$ is dense in G/Γ, passing to a subsequence we may assume that $\{a_i x\}$ converges to a point which is generic for the action of $\{u_t\}$. Now let φ be a bounded continuous function on G/Γ. Then by Theorem 2.4 we have

$$\frac{1}{\alpha_i(b-a)} \int_{\alpha_i a}^{\alpha_i b} \varphi(u_t^{(i)} a_i x)\, dt \longrightarrow \int_{G/\Gamma} \varphi d\mu.$$

We note that, for all $t \in \mathbb{R}$ and $i \in \mathbb{N}$, $u_t^{(i)} = \exp t\xi_i = \exp \alpha_i^{-1} t\mathrm{Ad}\, a_i(\xi) = a_i(\exp \alpha_i^{-1} t\xi) a_i^{-1}$. Therefore, for all i, we have

$$\int_{\alpha_i a}^{\alpha_i b} \varphi(u_t^{(i)} a_i x)\, dt = \int_{\alpha_i a}^{\alpha_i b} \varphi(a_i(\exp \alpha_i^{-1} t\xi)x)\, dt = \alpha_i \int_a^b \varphi(a_i(\exp t\xi)x)\, dt.$$

Substituting this in the above convergence we see that $\int_a^b \varphi(a_i(\exp t\xi)x)\, dt \to (b-a)\int \varphi d\mu$, as $i \to \infty$, as sought to be proved. $\square$

Proof of Corollary 2.3: follows from Corollary 2.5 in the same way as Corollary 2.2 from Theorem 2.1. We omit the details.

3 Diophantine approximation

We now prove Theorem 1.1 using Corollary 2.2, and deduce the corollaries described in §1.

Proof of Theorem 1.1: Let the notation be as in the hypothesis. Then, as $Q(v_1) = 0$, $\{v_1\}$ can be extended to a basis $\{v_1, \ldots, v_n\}$ of V such that for any $\xi_1, \ldots, \xi_n \in \mathbb{R}$,

$$Q(\Sigma_{j=1}^n \xi_j v_j) = 2\xi_1\xi_n - \Sigma_{j=2}^{n-1} \sigma_j \xi_j^2,$$

where $\sigma_j = \pm 1$ for all $j = 2, \ldots, n-1$.

Let $G = SL(V)$, the special linear group of $V = \mathbb{R}^n$, and Γ be the lattice in G consisting of all elements leaving invariant the lattice $\mathbb{Z}^n$ in $V = \mathbb{R}^n$; namely Γ is the subgroup $SL(n, \mathbb{Z})$ when $SL(V)$ is realised as $SL(n, \mathbb{R})$ with respect to the standard basis of $\mathbb{R}^n$. Let H be the connected component of the identity in $SO(Q) = \{g \in SL(V) \mid Q(gv) = Q(v) \text{ for all } v \in V\}$, the special orthogonal group corresponding to the quadratic form Q. Let $\alpha > 1$ and $a \in G = SL(V)$ be the element such that $a(v_1) = \alpha v_1$, $a(v_n) = \alpha^{-1} v_n$, and $a(v_j) = v_j$ for all $j = 2, \ldots, n-1$. Then clearly $a \in H$. Furthermore, the sequence $\{a^i\}$ satisfies the condition for the sequence $\{a_i\}$ as in Theorem 2.1.

Let $U = \{h \in H \mid a^{-i}ha^i \to e \text{ as } i \to \infty\}$. It can be seen that U consists of all transformations such that $v_1 \mapsto v_1$, $v_j \mapsto v_j + \sigma_j u_j v_1$ for $j = 2, \ldots, n-1$, and $v_n \mapsto v_n + \Sigma_{j=2}^{n-1} u_j v_j + \frac{1}{2}(\Sigma_{j=2}^{n-1} \sigma_j u_j^2) v_1$, with $u_2, \ldots, u_{n-1} \in \mathbb{R}$.

Let $x \in G/\Gamma$ be the identity coset Γ. Since Q is not a scalar multiple of a rational quadratic form it follows that Hx is dense in G/Γ; see [3]. Thus the conditions in Theorem 2.2 are satisfied, and we get that for almost all $u \in U$ the set $\{a^i ux \mid i \in \mathbb{N}\}$ is dense in G/Γ. Let U' be the set of all u in U such that $\{a^i ux \mid i \in \mathbb{N}\}$ is dense in G/Γ. Also, let λ be a Haar measure on U. We have $\lambda(U \setminus U') = 0$.

Consider any $u \in U'$. For $k = 2, 3, \ldots, n-1$ let L_k be the linear forms on $V = \mathbb{R}^n$ defined by $L_k(\sum_{j=1}^n \xi_j v_j) = \xi_k$, for all $\xi_1, \ldots, \xi_n \in \mathbb{R}$, where $\{v_1, v_2, \ldots, v_n\}$ is the basis of V as above. Let $f : V \to \mathbb{R}^{n-1}$ be the map defined by $f(v) = (Q(v), L_2(v), \ldots, L_{n-1}(v))$ for all $v \in V$. It can be seen, using the expression for Q in terms of the basis $\{v_1, \ldots, v_n\}$, that f is surjective. We note also that f is a-invariant; that is, $f(av) = f(v)$ for all $v \in V$.

Now let $v \in V \setminus \{0\}$ be arbitrary. We can write v as $g(p)$, with $g \in G$ and $p \in \mathcal{P}(\mathbb{Z}^n)$. Since $\{a^i ux \mid i \in \mathbb{N}\}$ is dense in G/Γ, there exists a sequence $\{k_i\}$ in $\mathbb{N}$ such that $a^{k_i}ux \to g\Gamma$, as $i \to \infty$. Hence there exists a sequence $\{\gamma_i\}$ in Γ such that $a^{k_i}u\gamma_i \to g$. Therefore $a^{k_i}u\gamma_i(p) \to g(p) = v$, and in turn $f(a^{k_i}u\gamma_i(p)) \to f(v)$ as $i \to \infty$. Since f is a-invariant we get that $f(u\gamma_i(p)) \to f(v)$. Therefore $f(v) \in \overline{f(u\mathcal{P}(\mathbb{Z}^n))}$ for any $v \in V \setminus \{0\}$. Since $f : V \to \mathbb{R}^{n-1}$ is surjective, it follows that $f(u\mathcal{P}(\mathbb{Z}^n))$ is dense in $\mathbb{R}^{n-1}$. Since Q is u-invariant, we get that $\{(Q(p), L_2(u(p)), \ldots, L_{n-1}(u(p))) \mid p \in \mathcal{P}(\mathbb{Z}^n)\}$ is dense in $\mathbb{R}^{n-1}$.

Let P_u be the plane spanned by v_1 and $u^{-1}(v_n)$, and let $\rho : P_u^\perp \to \mathbb{R}^{n-2}$ be linear the map defined by $\rho(v) = (L_2(u(v)), \ldots, L_{n-1}(u(v)))$ for all $v \in P_u^\perp$. Since P_u is contained in the kernel of each $L_k \circ u$, $k = 2, \ldots, n-1$, it follows that ρ is an isomorphism. Let $\nu = \|\rho^{-1}\|$, the norm as a linear transformation, with $\mathbb{R}^{n-2}$ considered equipped with the usual norm, and $P_u^\perp$ with the norm induced from $\mathbb{R}^n$.

Now let $t \in \mathbb{R}$, $w \in P_u^\perp$ and $\epsilon > 0$ be given. From what we have seen, there exists $p \in \mathcal{P}(\mathbb{Z}^n)$ such that $|Q(p) - t| < \epsilon$ and $\|(L_2(u(p)), \ldots, L_{n-1}(u(p))) - \rho(w)\| < \epsilon/\nu$. We note that $\rho(\pi(p, P_u)) = (L_2(u(p)), \ldots, L_{n-1}(u(p)))$. Therefore we have $\|\rho(\pi(p, P_u)) - \rho(w)\| < \epsilon/\nu$ and hence $\|\pi(p, P_u) - w\| < \epsilon$. Thus we have shown that for any $v \in C_Q \setminus \mathbb{R}v_1$ such that $P(v_1, v) = P_u$ for some $u \in U'$, the assertion as in the statement of the theorem holds.

From the description of the U-orbit of v_n noted earlier it can be seen that the orbit equals $\mathfrak{O} = \{v = \sum_{j=1}^{n-1} \xi_j v_j + v_n \mid \xi_1, \ldots, \xi_{n-1} \in \mathbb{R}, Q(v) = 0\}$; $\xi_2, \ldots, \xi_{n-1}$ can be chosen freely and then ξ_1 is precisely such that $Q(v) = 0$. Hence the set $\mathbb{R}^*\mathfrak{O}$ consisting of all nonzero points on the lines through points of $\mathfrak{O}$ contains all points of C_Q other than those on the hyperplane spanned by $\{v_1, \ldots, v_{n-1}\}$. We note that the intersection of C_Q with this hyperplane is a differentiable submanifold of codimension 1, and hence it is a set of measure 0 in C_Q. Now, since $\lambda(U \backslash U') = 0$, $\{tu^{-1}(v_n) \mid t \in \mathbb{R}^*, u \in U'\}$ is a set of full measure in $\mathbb{R}^*\mathfrak{O}$, and hence in C_Q. For all $v \in \mathbb{R}^*\mathfrak{O}$ the plane $P(v_1, v)$ has the form P_u, and we have shown that this holds for almost all $v \in C_Q \setminus \mathbb{R}v_1$. Hence in view of the conclusion in the preceding paragraph the assertion as in the theorem holds for almost all v in C_Q. This proves the theorem. □

Remark 3.1. Let the notation be as in Theorem 1.1. We note that if P is a plane in V such that $P \cap C_Q = \emptyset$ then the conclusion as in the theorem is not to be expected. In fact, if P is such a plane, for $t \in \mathbb{R}$ and $w \in P^\perp$ the inequalities $|Q(x) - t| < \epsilon$ and $\|\pi(x, P) - w\| < \epsilon$ admit common solutions $x \in \mathbb{Z}^n$ for all $\epsilon > 0$ only if there exists $x \in \mathbb{Z}^n$ such that $Q(x) = t$ and $\pi(x, P) = w$. This may be seen as follows: firstly, the restriction of Q to P is a definite quadratic form, which we may assume without loss of generality to be positive definite. Hence there exists a basis $\{v_1, \ldots, v_n\}$ of V such that P is the span of v_1 and v_2 and the form Q is given by $Q(\sum_{j=1}^n \xi_j v_j) = \sum_{j=1}^s \xi_j^2 - \sum_{j=s+1}^n \xi_j^2$, for all $\xi_1, \ldots, \xi_n \in \mathbb{R}$, where $2 \leq s \leq n-1$ is the signature of Q. It can be seen from this that for any $t \in \mathbb{R}$, $w \in P^\perp$ and $\alpha > 0$ the set $\{v \in V \mid |Q(v) - t| \leq \alpha, \|\pi(x, P) - w\| \leq \alpha\}$ is a compact subset, which in turn leads to the desired conclusion as above. In particular the observation shows that when $P \cap C_Q = \emptyset$ there exist only countably many pairs (t, w), with $t \in \mathbb{R}$ and $w \in P^\perp$ for which the inequalities as above hold.

Proof of Corollary 1.2: Let R be the set of points $v \in C_Q$ such that the assertion in the corollary holds, namely for every $\epsilon > 0$ there exists $x \in \mathcal{P}(\mathbb{Z}^n)$ such that $|Q(x)| < \epsilon$ and $\|\pi(x, v)\| < \epsilon$. Let $v_1 \in C_Q \backslash R$; (if such a v_1 does not exist then we are through already). Let $v \in C_Q \backslash \mathbb{R}v_1$ be such that the conclusion as in Theorem 1.1 holds, for this choice of v_1, and further such that the restriction of Q to the plane $P(v_1, v)$ is a nondegenerate quadratic form; the latter part of the condition holds for almost all $v \in C_Q \backslash \mathbb{R}v_1$ (and in fact it may be seen that the vectors for which the conclusion of Theorem 1.1 was upheld satisfy this condition). We shall show that $v \in R$; by Theorem 1.1 this would then imply the corollary.

Without loss of generality we may assume that $\|v_1\| = \|v\| = 1$. Let $v_n = v$. There exist $v_2, \ldots, v_{n-1} \in V$, such that $\{v_1, \ldots, v_n\}$ is a basis of V and the quadratic form Q is given, for any $\xi_1, \ldots, \xi_n \in \mathbb{R}$, by

$$Q(\Sigma_{j=1}^n \xi_j v_j) = 2\xi_1 \xi_n - \Sigma_{j=2}^{n-1} \sigma_j \xi_j^2,$$

where $\sigma_j = \pm 1$ for all $j = 2, \ldots, n-1$. Let P be the plane spanned by v_1 and v_n and W, W_1 and W_n be the subspaces spanned by $\{v_2, \ldots, v_{n-1}\}$,

$\{v_2, \ldots, v_n\}$, and $\{v_1, \ldots, v_{n-1}\}$ respectively. Let $\alpha > 1$ be such that the following conditions are satisfied; (existence of such an α follows from elementary considerations): (i) if $v = y + w$ with $y \in P$ and $w \in W$ then $\|w\| \leq \alpha\|\pi(v, P)\|$, (ii) if $v = \xi v_j + w$, with $j = 1$ or n, $\xi \in \mathbb{R}$ and $w \in W_j$, then $\|\pi(v, v_j)\| \leq \alpha\|w\|$, and (iii) $|Q(v)| \leq \alpha\|v\|^2$ for all $v \in V$.

Now let $0 < \epsilon < 1$ be given, and let $\delta = \epsilon/2\alpha^4$. Since $v = v_n$ is chosen so that the conclusion of Theorem 1.1 holds, there exists $x \in \mathcal{P}(\mathbb{Z}^n)$ such that $|Q(x)| < \delta^2$ and $\|\pi(x, P)\| < \delta$. Let $x = \Sigma_{j=1}^n \xi_j v_j$, $\xi_1, \ldots, \xi_n \in \mathbb{R}$, and let $w = \Sigma_{j=2}^{n-1} \xi_j v_j$. Since $\|\pi(x, P)\| < \delta$, by condition (i) we have $\|w\| \leq \alpha\delta$, and in turn by condition (iii) $|Q(w)| \leq \alpha^3\delta^2$. Now

$$|2\xi_1\xi_n| = |Q(x) - Q(w)| \leq |Q(x)| + |Q(w)| < \delta^2 + \alpha^3\delta^2 < 2\alpha^4\delta^2.$$

Hence at least one of ξ_1 or ξ_n is less than $\alpha^2\delta$. By condition (ii) above, $\|\pi(x, v_1)\| \leq \alpha\|\xi_n v_n + w\| \leq \alpha(|\xi_n| + \|w\|) < \alpha^2(|\xi_n| + \delta)$ and similarly $\|\pi(x, v_n)\| < \alpha^2(|\xi_1| + \delta)$. Hence the preceding conclusion implies that either $\|\pi(x, v_1)\|$ or $\|\pi(x, v_n)\|$ is less than $2\alpha^4\delta = \epsilon$. Thus for any $\epsilon > 0$ there exists $x \in \mathcal{P}(\mathbb{Z}^n)$ such that $|Q(x)| < \epsilon$, and either $\|\pi(x, v_1)\| < \epsilon$ or $\|\pi(x, v_n)\| < \epsilon$. Since $v_1 \notin R$, when ϵ is small enough the inequalities $|Q(x)| < \epsilon$ and $\|\pi(x, v_1)\| < \epsilon$ have no common solution $x \in \mathcal{P}(\mathbb{Z}^n)$. Therefore for all $\epsilon > 0$ there must exist $x \in \mathcal{P}(\mathbb{Z}^n)$ such that $|Q(x)| < \epsilon$ and $\|\pi(x, v_n)\| < \epsilon$. In other words, $v = v_n \in R$. As noted earlier this proves the corollary. □

Proof of Corollary 1.3. The set $\mathfrak{P}$ of planes P in V such that $P \cap C_Q$ is a pair of lines is an open submanifold of the Grassmannian manifold of planes in V; we consider it equipped with a measure which on coordinate charts is equivalent to the Lebesgue measure. Theorem 1.1 implies in particular that for almost all P in $\mathfrak{P}$, given $a \in \mathbb{R}$, $w \in P^\perp$ and $\epsilon > 0$ there exists $x \in \mathcal{P}(\mathbb{Z}^n)$ such that $|Q(x) - a| < \epsilon$ and $\|\pi(x, P) - w\| < \epsilon$. This may be seen to be equivalent to assertion (i) in the corollary. Assertion (ii) is an equivalent formulation of Corollary 1.2. □

As may be seen from its proof, in content assertion (i) in Corollary 1.3 is weaker than Theorem 1.1. We next describe an equivalent formulation of Theorem 1.1 in terms of linear forms. Let the notation be as before, and for any $v \in C_Q$ let $\mathcal{L}_{Q,v}^{n-2}$ denote the subset of $\mathcal{L}_Q^{n-2}$ consisting of all $(L_1, \ldots, L_{n-2})$ such that $L_k(v) = 0$ for all $k = 1, \ldots, n-2$. Every $(L_1, \ldots, L_{n-2})$ in $\mathcal{L}_Q^{n-2}$ belongs to $\mathcal{L}_{Q,v}^{n-2}$ for vectors v on two lines of C_Q. Then each $\mathcal{L}_{Q,v}^{n-2}$, $v \in V$, is a differentiable submanifold and carries a measure m_v which on coordinate charts is equivalent to Lebesgue measure. Then Theorem 1.1 may be seen to correspond to the following.

Theorem 3.2. *Let $v \in C_Q$ be given. Then for m_v-almost all $(L_1, \ldots, L_{n-2})$ in $\mathcal{L}_{Q,v}^{n-2}$ the following holds: for all $s_0, s_1, \ldots, s_{n-2} \in \mathbb{R}$ and $\epsilon > 0$ there exists $x \in \mathcal{P}(\mathbb{Z}^n)$ such that*

$$|Q(x) - s_0| < \epsilon \quad \text{and} \quad |L_k(x) - s_k| < \epsilon \quad \text{for all } k = 1, \ldots, n-2.$$

Remark 3.3. We note that inhomogeneous inequalities as in assertion (i) of Corollary 1.3 are not to be expected to hold in the case of systems of $n-1$ linear forms as in assertion (ii) in the Corollary. Let Q be as above, $(L_1, \ldots, L_{n-1}) \in \mathcal{L}_Q^{n-1}$, and consider the map $\psi : V \to \mathbb{R}^n$ defined by $\psi(v) = (Q(v), L_1(v), \ldots, L_{n-1}(v))$. It can be seen that on $V \backslash \psi^{-1}(\{0\})$, ψ is a nonsingular two-to-one map. Thus for every neighbourhood Ω of 0 in $\mathbb{R}^n$ the restriction of ψ to $V \backslash \psi^{-1}(\Omega)$ is a proper map. It follows that 0 is the only possible limit point of $\psi(\mathbb{Z}^n)$. Hence given a nonzero $(s_0, s_1, \ldots, s_{n-1})$ in $\mathbb{R}^n$ the system of inequalities $|Q(x) - s_0| < \epsilon$ and $|L_k(x) - s_k| < \epsilon$, $k = 1, \ldots, n-1$, admits an integral solution for every $\epsilon > 0$ only if there exists a $x \in \mathbb{Z}^n$ such that $Q(x) = s_0$ and $L_k(x) = s_k$ for all $k = 1, \ldots, n-1$.

Before concluding we note also that the statements of the diophantine results in §1 can not be improved from "almost all" to "all". For simplicity let us restrict the discussion to assertion (i) in Corollary 1.3; analogous comments apply to the other results as well. Consider first the case $n = 3$, and let Q be a quadratic form as in Corollary 1.3. Then firstly we need to exclude the linear forms L such that some nontrivial linear combination of Q and L^2 (with real coefficients) is a rational quadratic form, in which case an assertion as in the Corollary would not hold. However, there is actually a much larger class of such systems for which the conclusion does not hold; this was shown in [1] using a result of Kleinbock and Margulis [8] that for the action of a diagonalisable one-parameter subgroup on a homogeneous space there exists a "large" class of orbits whose closure is compact (in particular such orbits are not dense); the largeness is in terms of the Hausdorff dimension of the set of such points being equal to the dimension of the homogeneous space. The observations in [1] extend also to systems of the form $(Q, L_1, \ldots, L_{n-2})$ as above, for all $n \geq 3$, for which also, as noted earlier the invariance subgroup is, up to finite index, a diagonalisable one-parameter subgroup. The latter fact suggests also that it may not be feasible to identify precisely the class of systems for which the diophantine systems of inequalities as in Theorem 3.2 admit solutions. It would however be interesting to know specific sufficient conditions which would ensure that solutions exist.

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