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THE THEORY OF THE FLUCTUATIONS IN BRIGHTNESS OF THE MILKY WAY. III

S. CHANDRASEKHAR AND G. MÜNCH
Yerkes Observatory
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ABSTRACT

In this paper the integral equation (derived in Paper I) governing the fluctuations in brightness of the Milky Way is explicitly solved for the case in which the system extends to a finite distance in the direction of the line of sight and when all the clouds are equally transparent. The derived theoretical distributions are illustrated.

1. *Introduction.*—In the two earlier papers[1] of this series we pointed out that the theory of the fluctuations in brightness of the Milky Way leads one to consider the probability distribution of the quantity

$$u = \int_0^{\xi} \prod_{i=1}^{i=n(r)} q_i d r , \tag{1}$$

where $q_i \leqslant 1$ is a chance variable occurring with a known frequency $\psi(q)$ and the number of factors—"clouds"—$n(r)$ is also a chance variable governed by the Poisson distribution

$$\frac{e^{-r}}{n!} r^n \qquad (n = 0, 1, \ldots) . \tag{2}$$

Further, in equation (1), ξ is some fixed positive constant. If $f(u, \xi)$ denotes the probability that u exceeds the specified value, then, as we have shown in Paper I, $f(u, \xi)$ as a function of the two variables u and ξ satisfies the partial integrodifferential equation

$$f + \frac{\partial f}{\partial u} + \frac{\partial f}{\partial \xi} = \int_0^1 f\left(\frac{u}{q'}, \xi\right) \psi(q') \, d q' . \tag{3}$$

From the definition of u as an integral over a quantity which must always be less than 1, it follows that u can never exceed the value ξ. But it can take the value ξ itself with *exactly* the probability $e^{-\xi}$ (cf. the remarks in Paper I following eq. [18]). The conditions on f resulting from this fact are

$$f(u, \xi) = 0 \qquad (u > \xi)$$

and

$$\lim_{u \to \xi - 0} f(u, \xi) = e^{-\xi} . \tag{4}$$

Equation (3) should therefore be strictly written in the form

$$f + \frac{\partial f}{\partial u} + \frac{\partial f}{\partial \xi} = \int_{u/\xi}^1 f\left(\frac{u}{q'}, \xi\right) \psi(q') \, d q' . \tag{5}$$

Further, the definition of f as a probability distribution requires that

$$f(0, \xi) \equiv 1 . \tag{6}$$

This is the normalization condition.

[1] *Ap. J.*, **112**, 380 and 393, 1950. These papers will be referred to as "Paper I" and "Paper II," respectively.

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If, as in Paper I, $g(u, \xi)$ describes the frequency distribution of u, then, in accordance with conditions (4), we should define

$$f(u,\xi) = \int_u^\xi g(u,\xi)\, du\,. \tag{7}$$

The discontinuity of f at $u = \xi$ now implies that $g(u, \xi)$ has at this point a singularity of the nature of a δ-function (with "amplitude" $e^{-\xi}$).

The solution of equation (5) with boundary conditions (4) and (6) presents a problem of considerable mathematical interest. In this paper we shall show how the complete distribution of u can be obtained for the case when all the clouds are equally transparent, i.e., when

$$\psi(q') = \delta(q - q') \qquad (q = \text{constant} < 1)\,. \tag{8}$$

In this case the equation governing f reduces to

$$f + \frac{\partial f}{\partial u} + \frac{\partial f}{\partial \xi} = f\left(\frac{u}{q}, \xi\right). \tag{9}$$

2. *Preliminary remarks on the structure of equation* (9).—Since u/q exceeds ξ when $q\xi < u \leqslant \xi$ and $f = 0$ for $u > \xi$, we conclude from equation (9) that

$$f + \frac{\partial f}{\partial u} + \frac{\partial f}{\partial \xi} = 0 \qquad (q\xi < u \leqslant \xi)\,. \tag{10}$$

The general solution of this equation is

$$f(u,\xi) = e^{-\xi}\phi(\xi - u)\,, \tag{11}$$

where ϕ is an arbitrary function of the argument. If ϕ can be specified in some way, then the solution can be continued into the region $q^2\xi < u \leqslant q\xi$; for the left-hand side of equation (9) in the domain $q^2\xi < u \leqslant q\xi$ depends only on f in the domain $q\xi < u \leqslant \xi$. And if the solution can be extended to the domain $q^2\xi < u \leqslant q\xi$, then it can also be extended to the next domain $q^3\xi < u \leqslant q^2\xi$, and so on. It thus appears that we must consider equation (9), successively, in the domains

$$q^n\xi < u \leqslant q^{n-1}\xi \qquad (n = 1, 2, \ldots)\,. \tag{12}$$

The completion of the solution of equation (9) in the manner we have indicated requires that we know the solution in the first of the domains (12), namely,

$$q\xi < u \leqslant \xi\,. \tag{13}$$

And, to obtain the solution in this first domain, we must go back to the original definitions and appeal to the problem which gave rise to equation (9).

A further consequence which may be derived directly from equation (9) may be noted here. Since $f(0, \xi) \equiv 1$ for all ξ, it follows, successively, that all the derivatives of f with respect to u must vanish at $u = 0$:

$$\left(\frac{\partial^n f}{\partial u^n}\right) \equiv 0 \quad \text{for} \quad u = 0 \quad \text{and} \quad n = 1, 2, \ldots\,. \tag{14}$$

3. *The determination of $f(u, \xi)$ and $g(u, \xi)$ in the domain $q\xi < u \leqslant \xi$ and the enumeration of all their discontinuities.*—We inferred that $f(u, \xi) \to e^{-\xi}$ as $u \to \xi - 0$ and that $f(u, \xi) = 0$ for $u > \xi$ by appealing to the physical problem, in particular, to the fact that the probability that no cloud occurs in the interval $(0, \xi)$, is $e^{-\xi}$. The question now arises

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whether further information concerning $f(u, \xi)$ cannot be similarly obtained by considering situations in which one or more specified numbers of clouds occur in the interval $(0, \xi)$. Actually, we shall see that this provides the key to the entire problem. Let

$$f_n(u, \xi) \quad \text{and} \quad g_n(u, \xi) \qquad (n = 0, 1, \ldots) \tag{15}$$

govern the absolute probability distribution of u when it is known that there are n clouds in the interval $(0, \xi)$ (but it is not known *where* they occur in the interval). More precisely, for every $n = 0, 1, 2, \ldots$, $f_n(u, \xi)$ represents the probability that there will be exactly n clouds in the interval $(0, \xi)$ and that the random variable u will exceed the preassigned quantity u. Further, let $g_n(u, \xi)$ represent the probability density of u, i.e., the derivative of $f_n(u, \xi)$ with respect to u at such points where this derivative exists. The results concerning f and g obtained from the consideration of the case when there are no clouds in $(0, \xi)$ can now be expressed in the form

$$f_0(u, \xi) = e^{-\xi} \quad \text{and} \quad g_0(u, \xi) = e^{-\xi}\delta(\xi - u). \tag{16}$$

With f_n and g_n defined in the foregoing manner, we can write

$$f(u, \xi) = \sum_{n=0}^{\infty} f_n(u, \xi) \quad \text{and} \quad g(u, \xi) = \sum_{n=0}^{\infty} g_n(u, \xi). \tag{17}$$

Now the probability that there are n clouds somewhere in the interval $(0, \xi)$ is $e^{-\xi}\xi^n/n!$. Therefore, the probability that there are n clouds in this interval and that they occur between $(\xi_1, \xi_1 + d\xi_1)$, $(\xi_2, \xi_2 + d\xi_2)$, $\ldots$, and $(\xi_n, \xi_n + d\xi_n)$, where

$$0 \leqslant \xi_1 \leqslant \xi_2 \leqslant \xi_3 \ldots \leqslant \xi_{n-1} \leqslant \xi_n \leqslant \xi\,, \tag{18}$$

is

$$e^{-\xi} d\xi_1 d\xi_2 \ldots d\xi_n\,. \tag{19}$$

The inequalities (18) define a simplex[2] in the n-dimensional Euclidean space $(\xi_1, \xi_2, \ldots, \xi_n)$, whose vertices are at

$$\begin{gathered}(\xi, \xi, \ldots, \xi, \xi), (0, \xi, \ldots, \xi, \xi), \ldots, (0, 0, \ldots, 0, \xi, \xi), \\ (0, 0, \ldots, 0, \xi), \quad \text{and} \quad (0, 0, \ldots, 0, 0).\end{gathered} \tag{20}$$

We shall call this the "fundamental simplex." Its volume is $\xi^n/n!$.

Since the total probability of occurrence of n clouds in the interval $(0, \xi)$ is $e^{-\xi}\xi^n/n!$, it follows that in the $(\xi_1, \xi_2, \ldots, \xi_n)$-space we have a uniform distribution of a priori probability in the fundamental simplex with weight $e^{-\xi}$. This is, of course, in accordance with (19).

The value of u which would result from a distribution of n clouds specified by inequality (18), is

$$u = \xi_1 + q(\xi_2 - \xi_1) + q^2(\xi_3 - \xi_2) + \ldots + q^{n-1}(\xi_n - \xi_{n-1}) + q^n(\xi - \xi_n). \tag{21}$$

According to this equation, the smallest value which u can take, when it is known that there are n clouds in the interval $(0, \xi)$, is $q^n\xi$; this happens when $\xi_1 = \xi_2 = \ldots = \xi_n = 0$. And the largest value which u can take is ξ; this happens when $\xi_1 = \xi_2 = \ldots = \xi_n = \xi$. It therefore follows that $f_n(u, \xi)$ is defined only in the domain $q^n\xi \leqslant u \leqslant \xi$ and that

$$\begin{aligned} f_n(u, \xi) &= 0 && \text{for} \quad u > \xi \\ &= e^{-\xi}\xi^n/n! && \text{for} \quad u \leqslant q^n\xi\,. \end{aligned} \tag{22}$$

Since the total probability that n clouds occur in the interval $(0, \xi)$ is $e^{-\xi}\xi^n/n!$, we mus[t] also have

$$\int_{q^n\xi}^{\xi} g_n(u, \xi)\,du = \frac{e^{-\xi}}{n!}\xi^n\,. \tag{23}$$

[2] A simplex is the n-dimensional analogue of the three-dimensional tetrahedron.

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From our earlier remark concerning the distribution of a priori probability in the $(\xi_1, \xi_2, \ldots, \xi_n)$-space, it follows that *the probability that u exceeds a specified value, when it is known that n clouds occur in the interval* $(0, \xi)$ *is $e^{-\xi}$ times the volume of the space in which inequalities* (18) *and the further inequality,*

$$\xi_1 + q(\xi_2 - \xi_1) + q^2(\xi_3 - \xi_2) + \ldots + q^n(\xi - \xi_n) \geqslant u\,, \tag{24}$$

are simultaneously satisfied.

Now the shape of the solid into which the fundamental simplex is truncated by hyperplane (21) depends on the specified value of u. In fact, we must distinguish the n cases

$$q^l \xi < u \leqslant q^{l-1}\xi \qquad (l = 1, \ldots, n)\,. \tag{25}$$

Except in the two cases

$$q\xi < u \leqslant \xi \tag{26}$$

and

$$q^n \xi < u \leqslant q^{n-1}\xi\,, \tag{27}$$

it is not simple to evaluate the volume of the truncated solid. But in the two cases (26) and (27), one of the two solids into which the hyperplane (21) cuts the fundamental simplex is again a simplex. Hence in these two cases the volume of the required solid may be found simply.

Considering, first, case (26), we observe that the volume delimited by inequalities (18) and (24) is a simplex, the co-ordinates of whose vertices are given by

$$\xi_1 = \xi_2 = \ldots = \xi_l\,, \qquad \xi_{l+1} = \xi_{l+2} = \ldots = \xi_n = \xi \qquad (l = 1, \ldots, n) \tag{28}$$

and

$$\xi_1 = \xi_2 = \xi_3 = \ldots = \xi_n = \xi \tag{29}$$

Using equation (21), we readily find that the n vertices defined by equations (28) have the co-ordinates

$$\xi_1 = \xi_2 = \ldots = \xi_l = \frac{u - q^l \xi}{1 - q^l}\,, \qquad \xi_{l+1} = \xi_{l+2} = \ldots = \xi_n = \xi\,. \tag{30}$$

The volume of the simplex whose vertices are given by equations (29) and (30) is[3]

$$\frac{1}{n!}\begin{vmatrix} 1 & 1 & \cdots & 1 & 1 & 1 \\ \frac{u-q^n\xi}{1-q^n} & \xi & \cdots & \xi & \xi & \xi \\ \frac{u-q^n\xi}{1-q^n} & \frac{u-q^{n-1}\xi}{1-q^{n-1}} & \cdots & \xi & \xi & \xi \\ \cdot & \cdot & \cdots & \cdot & \cdot & \cdot \\ \frac{u-q^n\xi}{1-q^n} & \frac{u-q^{n-1}\xi}{1-q^{n-1}} & \cdots & \frac{u-q^2\xi}{1-q^2} & \xi & \xi \\ \frac{u-q^n\xi}{1-q^n} & \frac{u-q^{n-1}\xi}{1-q^{n-1}} & \cdots & \frac{u-q^2\xi}{1-q^2} & \frac{u-q\xi}{1-q} & \xi \end{vmatrix} \tag{31}$$

[3] Cf. D. M. Y. Sommerville, *Geometry of n Dimensions* (London: Methuen, 1929), p. 124.

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By subtracting from each column of this determinant the column immediately preceding it, we can make all the elements above the principal diagonal zero and leave along the principal diagonal the elements

$$1\,,\ \frac{\xi-u}{1-q^{n}}\,,\quad \frac{\xi-u}{1-q^{n-1}}\,,\ \dots,\ \frac{\xi-u}{1-q^{2}}\,,\quad \text{and}\quad \frac{\xi-u}{1-q}\,. \tag{32}$$

The volume of the simplex is, therefore,

$$\frac{1}{n!}\,\frac{(\xi-u)^{n}}{\prod_{j=1}^{n}(1-q^{j})}\,. \tag{33}$$

The probability that u exceeds a specified value in the interval $q\xi \leqslant u \leqslant \xi$ when it is known that there are n clouds in $(0, \xi)$ is therefore given by

$$f_n(u,\xi)=\frac{e^{-\xi}}{n!}\,\frac{(\xi-u)^{n}}{\prod_{j=1}^{n}(1-q^{j})}\qquad (q\xi\leqslant u\leqslant \xi;\quad n=1,2,\dots)\,. \tag{34}$$

The corresponding expression for $g_n(u, \xi)$ is

$$g_n(u,\xi)=\frac{e^{-\xi}}{(n-1)!}\,\frac{(\xi-u)^{n-1}}{\prod_{j=1}^{n}(1-q^{j})}\qquad (q\xi< u< \xi;\quad n=1,2,\dots)\,; \tag{35}$$

in particular,

$$g_1(u,\xi)=\frac{e^{-\xi}}{1-q}\qquad (q\xi<u<\xi) \tag{36}$$ [4]

and

$$g_2(u,\xi)=\frac{e^{-\xi}(\xi-u)}{(1-q)(1-q^{2})}\qquad (q\xi<u\leqslant\xi)\,. \tag{37}$$

Considering, next, case (27), we observe that now the solid cut *off* from the fundamental simplex is a simplex. The co-ordinates of its $(n+1)$ vertices are given by

$$\xi_1=\xi_2=\dots=\xi_l=0\quad \text{and}\quad \xi_{l+1}=\xi_{l+2}=\dots=\xi_n\quad (l=0,1,\dots,n-1) \tag{38}$$

and

$$(0,0,\dots,0,0)\,. \tag{39}$$

From equations (21) and (38) we readily find that the co-ordinates of the n vertices defined by equations (38) are

$$\xi_1=\xi_2=\dots=\xi_l=0\quad \text{and}\quad \xi_{l+1}=\xi_{l+2}=\dots=\xi_n=\frac{u-q^{n}\xi}{q^{l}(1-q^{n-l})}\qquad (l=0,\dots,n-1)\,. \tag{40}$$

[4] Actually, in this case, $g_1(u, \xi) = 0$ for $u < q\xi$.

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The volume of this simplex is clearly

$$\frac{1}{n!}\frac{(u-q^n\xi)^n}{\prod_{l=0}^{n-1} q^l(1-q^{n-l})}=\frac{(u-q^n\xi)^n}{n!\,q^{n(n-1)/2}\prod_{j=1}^{n}(1-q^j)}. \tag{41}$$

The probability that u exceeds a specified value in the interval $q^n\xi \leqslant u \leqslant q^{n-1}\xi$ is therefore given by

$$f_n(u,\xi)=\frac{e^{-\xi}}{n!}\left[\xi^n-\frac{(u-q^n\xi)^n}{q^{n(n-1)/2}\prod_{j=1}^{n}(1-q^j)}\right]\qquad(q^n\xi\leqslant u\leqslant q^{n-1}\xi;\quad n=1,\ldots). \tag{42}$$

The corresponding expression for $g_n(u, \xi)$ is

$$g_n(u,\xi)=\frac{e^{-\xi}}{(n-1)!}\frac{(u-q^n\xi)^{n-1}}{q^{n(n-1)/2}\prod_{j=1}^{n}(1-q^j)}\qquad(q^n\xi<u<q^{n-1}\xi;\quad n=1,\ldots)\cdot \tag{43}$$

In particular,

$$g_2(u,\xi)=e^{-\xi}\frac{u-q^2\xi}{q(1-q)(1-q^2)}\qquad(q^2\xi\leqslant u\leqslant q\xi). \tag{44}$$ [5]

According to equation (43),

$$\begin{aligned}\frac{\partial^{n-1}}{\partial u^{n-1}}g_n(u,\xi)&=\frac{e^{-\xi}}{q^{n(n-1)/2}\prod_{j=1}^{n}(1-q^j)}\\&=\text{constant}\qquad(q^n\xi<u<q^{n-1}\xi;\qquad n=1,\ldots).\end{aligned} \tag{45}$$

It follows from this last equation that at $u = q^n\xi(n = 1, 2, \ldots)$ the complete frequency function $g(u, \xi)$ must have a jump in its $(n - 1)$th derivative of amount given by the right-hand side of equation (45); the function and all its lower-order derivatives (if they exist) must, however, be continuous at $u = q^n\xi$. Thus, at $u = q\xi$, the function itself suffers a jump of amount $e^{-\xi}/(1 - q)$; at $u = q^2\xi$ there is a jump in the first derivative of amount $e^{-\xi}/q(1 - q)(1 - q^2)$, but the function itself is continuous here; at $u = q^3\xi$ there occurs a discontinuity in the second derivative, but the function and its first derivative are continuous, and so on.

Returning to equation (34), we can now write down, in accordance with equations (17), the complete probability distribution of u in the domain (13). Thus, combining the results expressed by equations (16) and (34), we have

$$f(u,\xi)=e^{-\xi}\left[1+\sum_{n=1}^{\infty}\frac{(\xi-u)^n}{n!\prod_{j=1}^{n}(1-q^j)}\right] \tag{46}$$

[5] It is readily verified that the solutions given by eqs. (37) and (44) are continuous at $u = q\xi$; but there is sudden change of slope. Also, in this case, $g_2(u, \xi) = 0$ for $u < q^2\xi$.

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and

$$g(u, \xi) = e^{-\xi}\left[\delta(\xi - u) + \sum_{n=1}^{\infty} \frac{(\xi - u)^{n-1}}{(n-1)!\prod_{j=1}^{n}(1-q^j)}\right], \tag{47}$$

for $q\xi < u \leqslant \xi$.

4. *An alternative form of $f(u, \xi)$ in the domain $q\xi < u \leqslant \xi$.*—Solution (46) for $f(u, \xi)$ in the domain included between the straight lines $u = \xi$ and $u = q\xi$ in the (u, ξ)-plane can be rewritten in the form

$$f(u, \xi) = Ke^{-\xi}\sum_{n=0}^{\infty}\frac{(\xi - u)^n}{n!}\prod_{j=1}^{\infty}(1 - q^{j+n}) \qquad (q\xi \leqslant u < \xi), \tag{48}$$

where

$$\frac{1}{K} = \prod_{j=1}^{\infty}(1 - q^j). \tag{49}$$

Equation (48) can be transformed into a more convenient form by making use of the identity

$$\prod_{j=1}^{\infty}(1 - q^{n+j}) = \sum_{k=0}^{\infty}(-1)^k\prod_{j=1}^{k}\frac{q^{j+n}}{1-q^j}, \tag{50}$$

established in Paper II (eq. [18]). Letting

$$Q_0 = 1, \qquad Q_k = -\frac{q^k}{1-q^k}Q_{k-1} = (-1)^k\prod_{j=1}^{k}\frac{q^j}{1-q^j} \qquad (k = 1, \ldots), \tag{51}$$

we can rewrite identity (50) in the form

$$\prod_{j=1}^{\infty}(1 - q^{n+j}) = \sum_{k=0}^{\infty}Q_k q^{kn}. \tag{52}$$

Two particular results which follow from equation (52) and which we shall find useful in the subsequent work are

$$\sum_{k=0}^{\infty}Q_k = \frac{1}{K} \qquad \text{and} \qquad \sum_{k=0}^{\infty}\frac{Q_k}{q^{lk}} = 0 \qquad (l = 1, 2, \ldots). \tag{53}$$

Combining equations (48) and (52), we can now write

$$f(u, \xi) = Ke^{-\xi}\sum_{n=0}^{\infty}\frac{(\xi - u)^n}{n!}\sum_{k=0}^{\infty}Q_k q^{kn}; \tag{54}$$

or, inverting the order of the summations, we have

$$f(u, \xi) = Ke^{-\xi}\sum_{k=0}^{\infty}Q_k\sum_{n=0}^{\infty}\frac{1}{n!}[(\xi - u)q^k]^n. \tag{55}$$

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Hence

$$f(u, \xi) = K e^{-\xi} \sum_{k=0}^{\infty} Q_k \exp [q^k (\xi - u)] \qquad (q\xi \leqslant u < \xi). \quad (56)$$

This is the required form of the solution.

5. *The completion of the solution for f.*—With $f(u, \xi)$ now known in the domain $q\xi \leqslant u < \xi$, the solution of equation (9) can be completed in the manner suggested in § 2.

First, it is convenient to use

$$F(u, \xi) = e^{\xi} f(u, \xi), \quad (57)$$

instead of f. The equation satisfied by F is

$$\frac{\partial F}{\partial u} + \frac{\partial F}{\partial \xi} = F\left(\frac{u}{q}, \xi\right). \quad (58)$$

As we have already explained in § 2, the solution of equation (58) must be carried out, successively, in the domains

$$q^n \xi \leqslant u < q^{n-1} \xi \qquad (n = 1, 2, \ldots). \quad (59)$$

We shall refer to the region included between the straight lines $u = q^{n-1}\xi$ and $u = q^n \xi$ as the nth domain; let $F_n(u, \xi)$ denote the required solution of equation (9) in this domain.

According to equation (58), the equation governing F_{n+1} in the $(n + 1)$th domain is

$$\left(\frac{\partial}{\partial u} + \frac{\partial}{\partial \xi}\right) F_{n+1}(u, \xi) = F_n\left(\frac{u}{q}, \xi\right); \quad (60)$$

for, when (u, ξ) lies in the $(n + 1)$th domain, $(u/q, \xi)$ lies in the nth domain.

Now, in the first domain included between the straight lines $u = \xi$ and $u = q\xi$ (see Fig. 1), the solution is (cf. eq. [56])

$$F_1(u, \xi) = K \sum_{k=0}^{\infty} Q_k \exp [q^k (\xi - u)]. \quad (61)$$

We shall now show in detail how this knowledge of the solution in the first domain can be used to extend the solution into the second domain included between the straight lines $u = q\xi$ and $u = q^2\xi$.

Consider a point P, $(u = q\xi_0;\ \xi = \xi_0)$, on the straight line $u = q\xi$. Since $(\partial/\partial u + \partial/\partial\xi)$ is proportional to the directional derivative along a straight line of unit (positive) slope, we draw through P a line PP' inclined at 45° to the co-ordinate axes. The co-ordinates (u, ξ) of a point S on PP', which is at a distance s from P, are

$$u = q\xi_0 - \frac{s}{\sqrt{2}} \quad \text{and} \quad \xi = \xi_0 - \frac{s}{\sqrt{2}}. \quad (62)$$

Conversely,

$$\left(\frac{1}{q} - 1\right)\frac{s}{\sqrt{2}} = \xi - \frac{u}{q} \quad \text{and} \quad \xi_0 (1 - q) = \xi - u. \quad (63)$$

In terms of the variable s, the equation governing F_2 can be expressed in the form

$$\sqrt{2}\,\frac{dF_2}{ds} = -F_1\left(\xi_0 - \frac{s}{q\sqrt{2}},\quad \xi_0 - \frac{s}{\sqrt{2}}\right); \quad (64)$$

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or, substituting for F_1 according to equations (61), we have

$$\sqrt{2}\,\frac{dF_2}{ds}=-K\sum_{k=0}^{\infty}Q_k\exp\left[q^k\left(\frac{1}{q}-1\right)\frac{s}{\sqrt{2}}\right]. \tag{65}$$

Integrating this equation from 0 to s and remembering that the solution must be continuous along the lines $u = q^n \xi$, $(n = 1, 2, \ldots)$, we obtain

$$F_2(s;\xi_0)=F_1(q\xi_0,\xi_0)-K\frac{q}{1-q}\sum_{k=0}^{\infty}\frac{Q_k}{q^k}\left\{\exp\left[q^k\left(\frac{1}{q}-1\right)\frac{s}{\sqrt{2}}\right]-1\right\}. \tag{66}$$

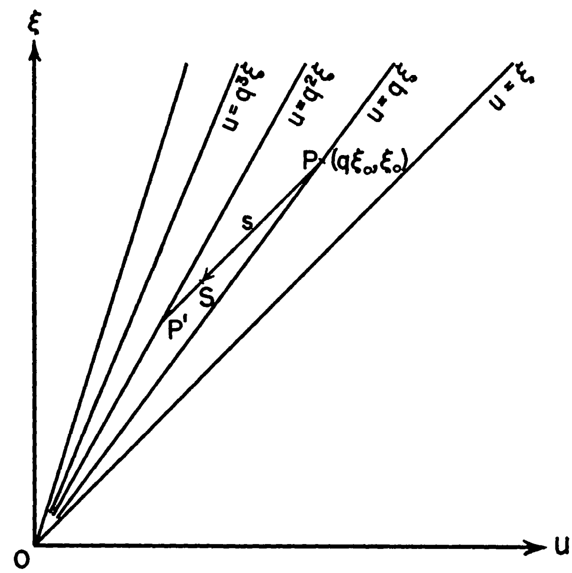

FIG. 1.—Illustrating the manner of integrating equation (60), successively, in the domains $q^n\xi \leqslant u \leqslant q^{n-1}\xi (n = 1, 2, \ldots)$.

The term

$$K\frac{q}{1-q}\sum_{k=0}^{\infty}\frac{Q_k}{q^k}, \tag{67}$$

which occurs on the right-hand side of equation (66), vanishes in virtue of the second of the identities in (53). Equation (66), therefore, reduces to (cf. eqs. [51])

$$F_2(s;\xi_0)=K\left\{Q_0\sum_{k=0}^{\infty}Q_k\exp\left[q^k\xi_0(1-q)\right]+Q_1\sum_{k=0}^{\infty}\frac{Q_k}{q^k}\exp\left[q^k\left(\frac{1}{q}-1\right)\frac{s}{\sqrt{2}}\right]\right\}. \tag{68}$$

Reverting to the variables (u, ξ) in accordance with equations (63), we can rewrite the foregoing solution in the form

$$F_2(u,\xi)=K\left\{Q_0\sum_{k=0}^{\infty}Q_k\exp\left[q^k(\xi-u)\right]+Q_1\sum_{k=0}^{\infty}\frac{Q_k}{q^k}\exp\left[q^k\left(\xi-\frac{u}{q}\right)\right]\right\}. \tag{69}$$

This is the required solution for F in the second domain.

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We shall now complete the solution by proving by induction that, quite generally, the solution $F_n(u, \xi)$ in the domain included between the straight lines $u = q^n\xi$ and $u = q^{n-1}\xi$ is given by

$$F_n(u, \xi) = K\sum_{l=0}^{n-1} Q_l \sum_{k=0}^{\infty} \frac{Q_k}{q^{lk}} \exp\left[q^k\left(\xi - \frac{u}{q^l}\right)\right] \quad (n = 1, 2, \ldots). \tag{70}$$

Solutions (61) and (69), valid in domains 1 and 2, verify the truth of equation (70) for $n = 1$ and $n = 2$; to establish its general validity, we must prove that if the truth of equation (70) be assumed for a general value of n, then its truth for $(n + 1)$ can be derived. For this latter purpose, we shall integrate equation (60) along a straight line of unit slope in the domain $(n + 1)$ and passing through a given point $(q^n\xi_0, \xi_0)$ on the line $u = q^n\xi$. The co-ordinates (u, ξ) of a point on the line of unit slope distant s from $(q^n\xi_0, \xi_0)$ are

$$u = q^n\xi_0 - \frac{s}{\sqrt{2}} \quad \text{and} \quad \xi = \xi_0 - \frac{s}{\sqrt{2}}. \tag{71}$$

In terms of s, equation (60) becomes

$$\sqrt{2}\frac{dF_{n+1}}{ds} = -F_n\left(q^{n-1}\xi_0 - \frac{s}{q\sqrt{2}}, \quad \xi_0 - \frac{s}{\sqrt{2}}\right). \tag{72}$$

Substituting for F_n according to equation (70), we have

$$\sqrt{2}\,\frac{dF_{n+1}}{ds} = -K\sum_{l=0}^{n-1} Q_l \sum_{k=0}^{\infty} \frac{Q_k}{q^{lk}} \exp\left[q^k\left\{\xi_0(1 - q^{n-l-1}) + \frac{s}{\sqrt{2}}\left(\frac{1}{q^{l+1}} - 1\right)\right\}\right]. \tag{73}$$

Integrating this last equation from 0 to s and remembering that $F_{n+1}(0; s) = F_n(u = q^n\xi_0; \xi = \xi_0)$, we obtain

$$\begin{aligned} F_{n+1}(s; \xi_0) = {} & K\sum_{l=0}^{n-1} Q_l \sum_{k=0}^{\infty} \frac{Q_k}{q^{lk}} \exp\left[q^k\xi_0(1 - q^{n-l})\right] \\ & - K\sum_{l=0}^{n-1} \frac{q^{l+1}Q_l}{1 - q^{l+1}} \sum_{k=0}^{\infty} \frac{Q_k}{q^{(l+1)k}} \Big\{ \exp\left[q^k\xi_0(1 - q^{n-l-1})\right] \\ & - \exp\left[q^k\xi_0(1 - q^{n-l-1}) + \frac{s}{\sqrt{2}}\,q^k\left(\frac{1}{q^{l+1}} - 1\right)\right]\Big\}. \end{aligned} \tag{74}$$

After some further reductions, in which we make use of equations (51) and (53), we find that we can simplify equation (74) to the form

$$F_{n+1}(s; \xi_0) = K\sum_{l=0}^{n} Q_l \sum_{k=0}^{\infty} \frac{Q_k}{q^{lk}} \exp\left[q^k\left\{\xi_0(1 - q^{n-l}) + \frac{s}{\sqrt{2}}\left(\frac{1}{q^l} - 1\right)\right\}\right]. \tag{75}$$

Now, reverting to the variables (u, ξ) in accordance with the transformation formulae (71), we have

$$F_{n+1}(u, \xi) = K\sum_{l=0}^{n} Q_l \sum_{k=0}^{\infty} \frac{Q_k}{q^{lk}} \exp\left[q^k\left(\xi - \frac{u}{q^l}\right)\right]. \tag{76}$$

This agrees with equation (70) for $(n + 1)$. This completes the proof of equation (70).

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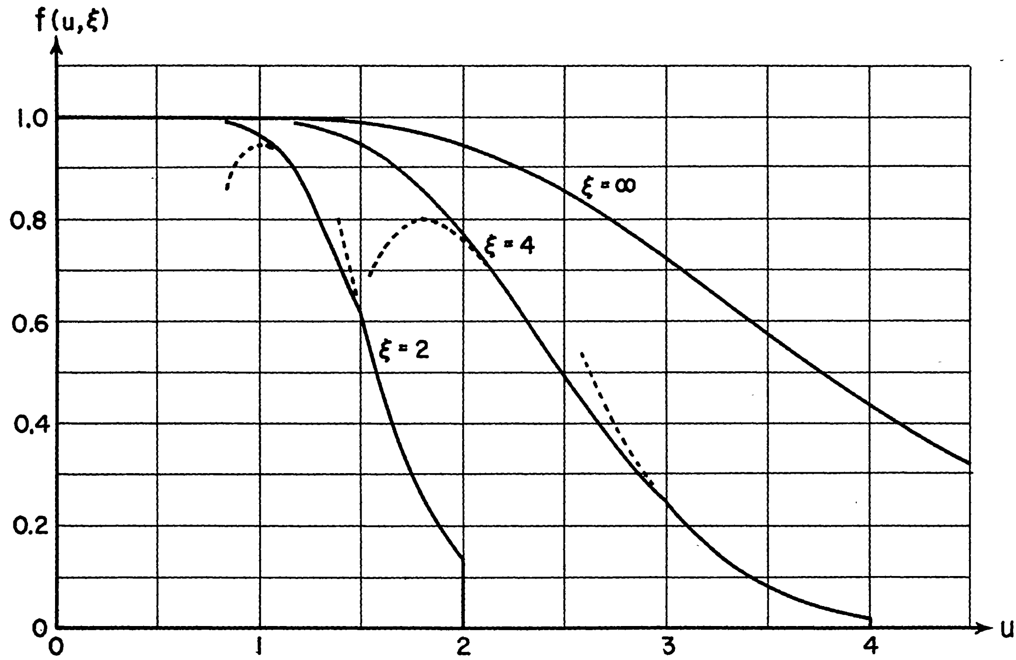

FIG. 2.—The probability distribution $f(u, \xi)$ for $q = 0.75$ and $\xi = 2, 4$, and ∞ (*full-line curves*). The solution for $f(u, \xi)$ in domains $q^n\xi \leqslant u < q^{n-1}\xi$ ($n = 1, 2, \ldots$) has been extended to $u < q^n\xi$ (*dashed curves*) to illustrate the discontinuities of the function and/or its derivatives at $u = q^n\xi$.

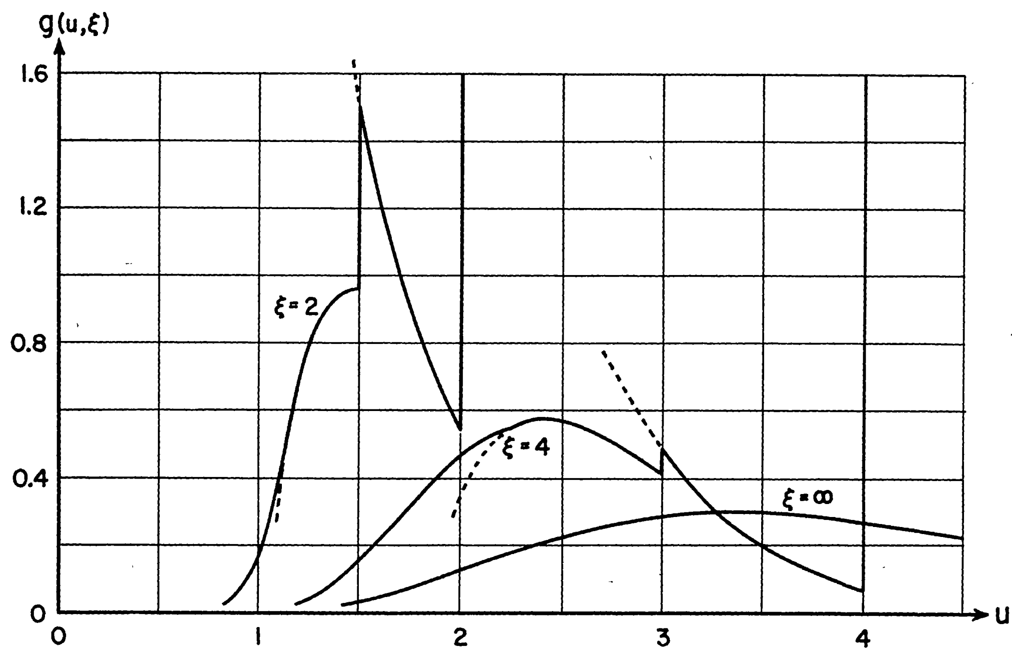

FIG. 3.—The frequency function $g(u, \xi)$ for $q = 0.75$ and $\xi = 2, 4$, and ∞ (*full-line curves*). The solu tion for $f(u, \xi)$ in domains $q^n\xi < u \leqslant q^{n-1}\xi$ ($n = 1, 2, \ldots$) has been extended to $u < q^n\xi$ (*dashed curve* to illustrate the discontinuities of the function and/or its derivatives at $u = q^n\xi$.

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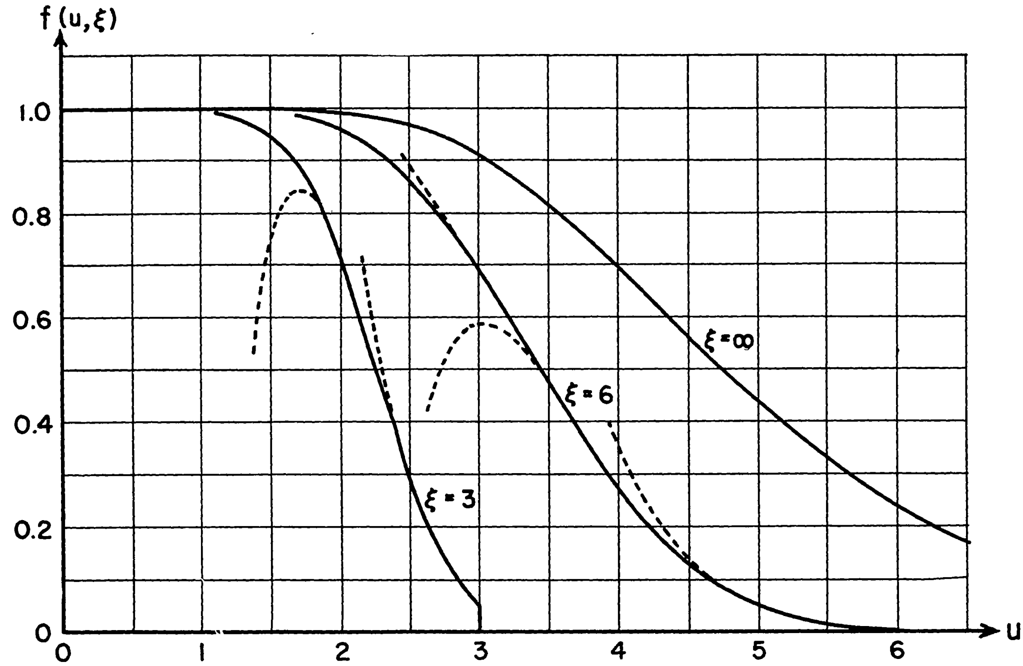

FIG. 4.—The probability distribution $f(u, \xi)$ for $q = 0.80$ and $\xi = 3$, 6, and ∞ (*full-line curves*). The solution for $f(u, \xi)$ in domains $q^n\xi \leqslant u < q^{n-1}\xi$ $(n = 1, 2, \ldots)$ has been extended to $u < q^n\xi$ (*dashed curves*) to illustrate the discontinuities of the function and/or its derivatives at $u = q^n\xi$.

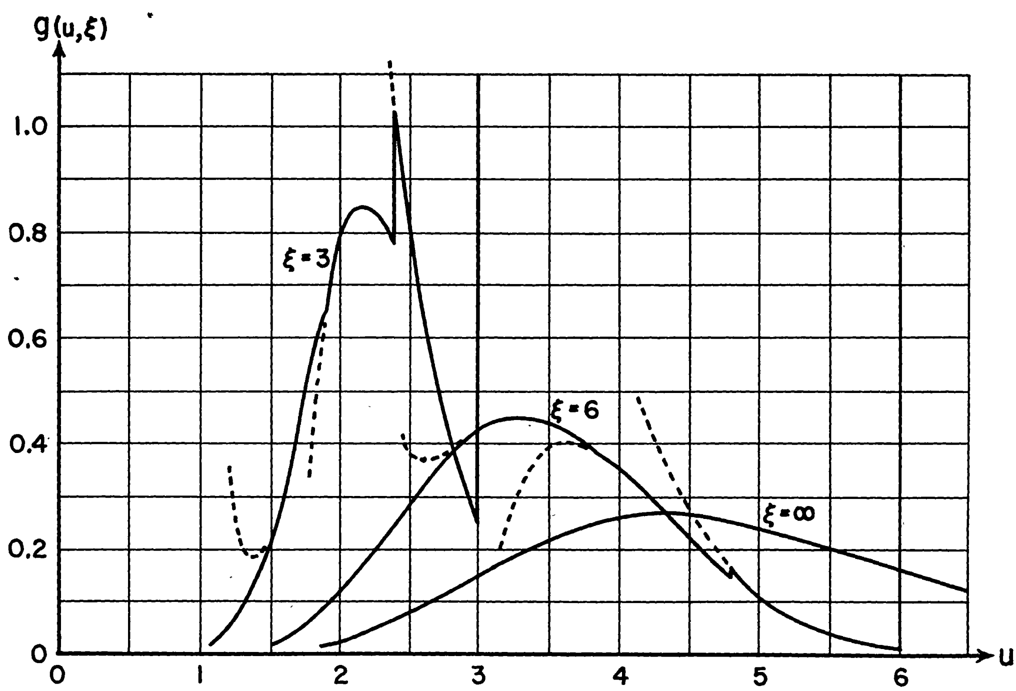

FIG. 5.—The frequency function $g(u, \xi)$ for $q = 0.80$ and $\xi = 3$, 6, and ∞ (*full-line curves*). The solution for $g(u, \xi)$ in domains $q^n\xi < u \leqslant q^{n-1}\xi$ $(n = 1, 2, \ldots)$ has been extended to $u < q^n\xi$ (*dashed curves*) to illustrate the discontinuities of the function and/or its derivatives at $u = q^n\xi$.

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According to equation (70), we may now write the complete solution for $f(u, \xi)$ in the form

$$f(u, \xi) = K e^{-\xi} \sum_{l=0}^{n-1} Q_l \sum_{k=0}^{\infty} \frac{Q_k}{q^{lk}} \exp\left[q^k\left(\xi - \frac{u}{q^l}\right)\right] \tag{77}$$

$$(q^n \xi \leqslant u < q^{n-1}\xi ; \quad n = 1, 2, \ldots).$$

It is of interest to verify that solution (77) is properly normalized. Thus, considering

$$f(0, \xi) = K e^{-\xi} \sum_{l=0}^{\infty} Q_l \sum_{k=0}^{\infty} \frac{Q_k}{q^{lk}} \exp q^k \xi , \tag{78}$$

we expand the exponential and invert the order of the summations; we find

$$f(0, \xi) = K e^{-\xi} \sum_{m=0}^{\infty} \frac{\xi^m}{m!} \sum_{k=0}^{\infty} Q_k q^{km} \sum_{l=0}^{\infty} \frac{Q_l}{q^{lk}} . \tag{79}$$

In equation (79) the summation over l leads to a nonvanishing result only for $k = 0$, and the summation over l then gives $1/K$ (cf. eqs. [53]). Thus

$$f(0, \xi) = e^{-\xi} \sum_{m=0}^{\infty} \frac{\xi^m}{m!} = 1 . \tag{80}$$

Similarly, it can be verified that solution (70) also satisfies the conditions expressed by equation (14).

Finally, the frequency function governing u can be written in the form

$$g(u, \xi) = K e^{-\xi} \sum_{l=0}^{n-1} \frac{Q_l}{q^l} \sum_{k=0}^{\infty} \frac{Q_k}{q^{(l-1)k}} \exp\left[q^k\left(\xi - \frac{u}{q^l}\right)\right] + e^{-\xi} \delta(\xi - u) \tag{81}$$

$$(q^n \xi < u \leqslant q^{n-1}\xi ; \; n = 1, \ldots).$$

Again it can be verified that this solution has the discontinuities we have enumerated in § 3.

6. *Illustrations of the derived distribution and frequency functions.*—Using formulae (77) and (81), we have computed the functions $f(u, \xi)$ and $g(u, \xi)$ for the cases

$$q = 0.75 \qquad \xi = 4 \text{ and } 2 ,$$

and (82)

$$q = 0.80 , \qquad \xi = 6 \text{ and } 3 .$$

It was found that, by retaining twelve to fourteen terms, we can preserve sufficient accuracy (three to four significant figures) in the numerical calculations. Also it was found that it was not necessary to go beyond the fourth or the fifth domain.

The computed solutions are illustrated in Figures 2-5. For comparison, the corresponding solutions for the infinite case, $\xi = \infty$ (taken from Paper II), are also shown. This comparison confirms the danger (to which we have already drawn attention in a different connection[6]) of applying the theory valid for $\xi = \infty$ even to relatively low galactic latitudes.

[6] S. Chandrasekhar and G. Münch, *Ap. J.*, **113**, 150, 1951.

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