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# THE EQUILIBRIUM AND THE STABILITY OF THE JEANS SPHEROIDS

S. Chandrasekhar and Norman R. Lebovitz
University of Chicago and Massachusetts Institute of Technology

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## ABSTRACT

The equilibrium and the stability of homogeneous masses distorted by the tidal effects of a secondary (of mass $M'$ at a distance $R$) are re-examined on the basis of the second-order virial equations. In agreement with known results, it is shown that, under circumstances when the figure of equilibrium is a prolate spheroid, there is a maximum value of $\mu(= GM'/R^3)$ which is compatible with equilibrium. The problem of the small oscillations of these Jeans spheroids is next considered. The characteristic frequencies of oscillation belonging to the second harmonics are determined both in case the mass is considered incompressible and in case it is considered compressible and subject to the gas laws governing adiabatic changes. In the former case, instability sets in when $\mu$ attains its maximum value; and in the latter case it sets in before that happens.

## I. INTRODUCTION

In a paper published in 1917 Jeans considered the equilibrium and the stability of tidally distorted masses in the context of certain cosmogonical speculations current at that time (see also Jeans 1919, 1929). A substantial part of Jeans's analysis was devoted to homogeneous masses under conditions in which the tidally distorted configurations have prolate spheroidal forms. While the applications of these very special considerations to real astronomical situations may be remote, the results themselves would appear to have some theoretical meaning. In this paper we shall reconsider Jeans's problem by the methods which we have recently developed based on the virial theorem and its extensions. We shall find that Jeans's theory can be completed in several respects; and, further, that some new results which emerge from the present analysis may have a wider base than the particular circumstances under which they are derived may justify.

## II. THE SECOND-ORDER VIRIAL EQUATIONS

Consider a fluid mass ("the primary") tidally distorted by a "secondary" of mass $M'$. Let the distance between the centers of mass of the two objects be $R$. Choose a co-ordinate system with the origin at the center of mass of the primary and with the $x_1$-axis pointing in the direction of the secondary.

The principal assumption of Jeans is that the variation of the tidal potential $\mathfrak{B}_T$ over the primary can be approximated by[1]

$$\mathfrak{B}_T = -\tfrac{1}{2}\mu(x_1^2 + x_2^2 + x_3^2) + \tfrac{3}{2}\mu x_1^2 \,, \tag{1}$$

where

$$\mu = \frac{GM'}{R^3} \,. \tag{2}$$

This same assumption will underlie the present investigation.

Allowing for the presence of the tidal potential $\mathfrak{B}_T$, we have the equation of motion,

$$\rho \frac{du_i}{dt} = -\frac{\partial p}{\partial x_i} + \rho \frac{\partial}{\partial x_i}(\mathfrak{B} + \mathfrak{B}_T) \,, \tag{3}$$

[1] The limitations of this approximation and the manner in which it could be refined are fully discussed by Jeans in his 1917 paper.

where $\mathfrak{V}$ is the self-gravitational potential. By multiplying equation (3) by $x_j$ and integrating over the volume $V$ occupied by the fluid, we obtain in the usual manner the tensor equation

$$\frac{d}{dt}\int_V \rho u_i x_j dx = 2\mathfrak{T}_{ij} + \mathfrak{W}_{ij} - \mu I_{ij} + 3\mu\delta_{i1}I_{1j} + \delta_{ij}\Pi, \tag{4}$$

where

$$\Pi = \int_V p dx, \tag{5}$$

and

$$\mathfrak{T}_{ij} = \tfrac{1}{2}\int_V \rho u_i u_j dx, \qquad \mathfrak{W}_{ij} = -\tfrac{1}{2}\int_V \rho \mathfrak{V}_{ij} dx, \qquad \text{and } I_{ij} = \int_V \rho x_i x_j dx \tag{6}$$

are the kinetic energy, the potential energy, and the moment of inertia tensors.

### III. THE SECOND-ORDER VIRIAL EQUATIONS GOVERNING EQUILIBRIUM

When no relative motions are present and hydrostatic equilibrium prevails, equation (4) becomes

$$\mathfrak{W}_{ij} - \mu I_{ij} + 3\mu\delta_{i1}I_{1j} = -\Pi\delta_{ij}. \tag{7}$$

The diagonal elements of this relation give

$$\mathfrak{W}_{11} + 2\mu I_{11} = \mathfrak{W}_{22} - \mu I_{22} = \mathfrak{W}_{33} - \mu I_{33} = -\Pi, \tag{8}$$

while the non-diagonal elements give

$$\mathfrak{W}_{12} = \mathfrak{W}_{13} = 0, \quad I_{12} = I_{13} = 0, \quad \text{and} \quad \mathfrak{W}_{23} = \mu I_{23}. \tag{9}$$

Equations (9) can be satisfied identically if the configuration is assumed to have symmetry about the co-ordinate planes; for in that case the tensors $\mathfrak{W}_{ij}$ and $I_{ij}$ will be diagonal in the chosen representation and

$$\mathfrak{W}_{ij} = 0 \quad \text{and} \quad I_{ij} = 0 \qquad (i \neq j). \tag{10}$$

We shall suppose that equations (9) are satisfied identically in this manner.

In considering equation (8), we shall suppose, in addition, that the configurations have ellipsoidal forms. It can be readily shown (cf. Jeans 1917) that this supposition is consistent with the equations of hydrostatic equilibrium and the condition which requires the pressure to vanish on the bounding surface.

We shall first show that, on the assumptions made, the equilibrium configurations must be prolate spheroids and not true triaxial ellipsoids.

By making use of the identities (see Chandrasekhar and Lebovitz 1962*a*, eqs. [32] and [35])

$$\mathfrak{W}_{22;33} = \mathfrak{W}_{33;22} = \mathfrak{W}_{23;23} + \mathfrak{W}_{22} = \mathfrak{W}_{32;32} + \mathfrak{W}_{33} \tag{11}$$

between the elements of the supermatrix,

$$\mathfrak{W}_{pq;ij} = \int_V \rho x_p \frac{\partial \mathfrak{V}_{ij}}{\partial x_q} dx, \tag{12}$$

we can rewrite the first two equalities in (8) in the form

$$\mathfrak{W}_{11} - \mathfrak{W}_{22;33} + 2\mu I_{11} = -\mathfrak{W}_{23;23} - \mu I_{22} = -\mathfrak{W}_{32;32} - \mu I_{33}. \tag{13}$$

If $I_{22} \neq I_{33}$, then we can deduce from the second equality in (13) that

$$-\mu = \frac{\mathfrak{W}_{23;23} - \mathfrak{W}_{32;32}}{I_{22} - I_{33}} . \tag{14}$$

But, for an ellipsoid (Chandrasekhar and Lebovitz 1962*c*, eq. [66]),

$$\frac{\mathfrak{W}_{23;23}}{I_{22}} = \frac{\mathfrak{W}_{32;32}}{I_{33}} ; \tag{15}$$

and we should conclude that

$$-\mu = \frac{\mathfrak{W}_{23;23}}{I_{22}} = \frac{\mathfrak{W}_{32;32}}{I_{33}} , \tag{16}$$

which is impossible, since $\mu$, by definition, is positive and the matrix elements $\mathfrak{W}_{ij;ij}$ are also positive. Hence the second equality in (13) (and, therefore, also in [8]) can be satisfied only *identically* under circumstances when $I_{22} = I_{33}$. In other words, on the assumption of an ellipsoidal form, the equilibrium configurations can only be prolate spheroids. The geometry of these spheroids will be determined by the first equality in (8), namely,

$$(2I_{11} + I_{22})\mu = \mathfrak{W}_{22} - \mathfrak{W}_{11} . \tag{17}$$

Expressions for the various tensors associated with homogeneous ellipsoids have been given in an earlier paper (Chandrasekhar and Lebovitz 1962*c*). For a prolate spheroid we have, in particular (cf. *loc. cit.*, eq. [57])

$$\frac{\mathfrak{W}_{22}}{\pi G \rho a_1 a_2^2} = -2A_2 I_{22} \quad \text{and} \quad \frac{\mathfrak{W}_{11}}{\pi G \rho a_1 a_2^2} = -2A_1 I_{11} . \tag{18}$$

Inserting these expressions in equation (17), we have

$$\frac{\mu}{\pi G \rho} = 2 a_1 a_2^2 \frac{A_1 a_1^2 - A_2 a_2^2}{2 a_1^2 + a_2^2} . \tag{19}$$

Using the relation

$$A_1 + 2A_2 = \frac{2}{a_1 a_2^2} \tag{20}$$

(valid for prolate spheroids), we can rewrite equation (19) in the form

$$\frac{\mu}{\pi G \rho} = A_1 a_1 a_2^2 - \frac{2 a_2^2}{2 a_1^2 + a_2^2} . \tag{21}$$

Now it is known that for a prolate spheroid

$$A_1 = \frac{1}{a_1 a_2^2} \frac{1 - e^2}{e^2} \left( \frac{1}{e} \log \frac{1+e}{1-e} - 2 \right) \tag{22}$$

and

$$A_2 = A_3 = \frac{1}{a_1 a_2^2} \frac{1}{e^2} \left( 1 - \frac{1-e^2}{2e} \log \frac{1+e}{1-e} \right) , \tag{23}$$

where $e$ is the eccentricity defined by the relation

$$a_2^2 = a_1^2 (1 - e^2) . \tag{24}$$

On substituting for $A_1$ from equation (22) in equation (21), we recover Jeans's relation:

$$\frac{\mu}{\pi G \rho}=\frac{1-e^2}{e^3}\log\frac{1+e}{1-e}-\frac{6(1-e^2)}{e^2(3-e^2)}. \tag{25}$$

This relation, as Jeans has already noted, predicts a maximum value for $\mu$ which is consistent with equilibrium and the assumed ellipsoidal form. The maximum value occurs at

$$e=0.883026\,,\qquad \text{where}\qquad \frac{\mu}{\pi G\rho}=0.125536\,. \tag{26}$$[2]

Table 1 in Section V includes a column of values of $\mu/\pi G\rho$ which exhibits the relationship (25) (see also Fig. 1).

## IV. THE SECOND-ORDER VIRIAL EQUATIONS GOVERNING SMALL OSCILLATIONS ABOUT EQUILIBRIUM

The characteristic frequencies belonging to the different second harmonic modes of oscillation of the Jeans spheroid can be determined by a consideration of the linearized form of the second-order virial equations (cf. Lebovitz 1961 for a similar treatment of the Maclaurin spheroid).

Suppose, then, that the equilibrium spheroid considered in Section III is slightly perturbed; and, further, that the ensuing motions are described by a Lagrangian displacement of the form

$$\xi(x)\,e^{i\sigma t}\,, \tag{27}$$

where $\sigma$ denotes the characteristic frequency of oscillation to be determined. To the first order in $\xi$, the virial equation (4) gives

$$-\sigma^2 V_{i;j}=\delta\mathfrak{W}_{ij}-\mu\delta I_{ij}+3\mu\delta_{i1}\delta I_{1j}+\delta_{ij}\delta\Pi\,, \tag{28}$$

where

$$V_{i;j}=\int_V \rho\,\xi_i x_j\,d\mathbf{x}\,, \tag{29}$$

and $\delta\Pi$, $\delta\mathfrak{W}_{ij}$, and $\delta I_{ij}$ are the first variations of $\Pi$, $\mathfrak{W}_{ij}$, and $I_{ij}$ due to the deformation of the spheroid caused by the Lagrangian displacement $\xi$.

It is convenient to introduce the symmetrized virial

$$V_{ij}=V_{i;j}+V_{j;i}\,; \tag{30}$$

in terms of it,

$$\delta I_{ij}=V_{ij}\,. \tag{31}$$

Writing equation (28) explicitly out for the different components, we have

$$-\sigma^2 V_{1;1}=\delta\mathfrak{W}_{11}+2\mu V_{11}+\delta\Pi\,, \tag{32}$$

$$-\sigma^2 V_{2;2}=\delta\mathfrak{W}_{22}-\mu V_{22}+\delta\Pi\,, \tag{33}$$

$$-\sigma^2 V_{3;3}=\delta\mathfrak{W}_{33}-\mu V_{33}+\delta\Pi\,, \tag{34}$$

$$-\sigma^2 V_{1;2}=\delta\mathfrak{W}_{12}+2\mu V_{12}\,;\qquad -\sigma^2 V_{2;1}=\delta\mathfrak{W}_{12}-\mu V_{12}\,, \tag{35}$$

$$-\sigma^2 V_{1;3}=\delta\mathfrak{W}_{13}+2\mu V_{13}\,;\qquad -\sigma^2 V_{3;1}=\delta\mathfrak{W}_{13}-\mu V_{13}\,, \tag{36}$$

$$-\sigma^2 V_{2;3}=\delta\mathfrak{W}_{23}-\mu V_{23}\,;\qquad -\sigma^2 V_{3;2}=\delta\mathfrak{W}_{23}-\mu V_{23}\,. \tag{37}$$

[2] Jeans gives $e = 0.88258$ and $\mu/\pi G\rho = 0.125504$; but we have not been able to confirm his values.

By suitably combining these equations, we obtain the following equivalent set of equations:

$$(\sigma^2+\mu)V_{12} = -2\delta\mathfrak{W}_{12}\,; \qquad (\sigma^2+\mu)V_{13} = -2\delta\mathfrak{W}_{13}\,, \tag{38}$$

$$\sigma^2(V_{1;2}-V_{2;1}) = -3\mu V_{12}\,; \qquad \sigma^2(V_{1;3}-V_{3;1}) = -3\mu V_{13}\,, \tag{39}$$

$$(\sigma^2-2\mu)V_{23} = -2\delta\mathfrak{W}_{23}\,; \qquad \sigma^2(V_{2;3}-V_{3;2}) = 0\,, \tag{40}$$

$$-\tfrac{1}{2}\sigma^2 V_{11} = \delta\mathfrak{W}_{11} + 2\mu V_{11} + \delta\Pi\,, \tag{41}$$

$$-\tfrac{1}{2}\sigma^2 V_{22} = \delta\mathfrak{W}_{22} - \mu V_{22} + \delta\Pi\,, \tag{42}$$

$$-\tfrac{1}{2}\sigma^2 V_{33} = \delta\mathfrak{W}_{33} - \mu V_{33} + \delta\Pi\,. \tag{43}$$

### a) *The Expression of $\delta\mathfrak{W}_{ij}$ in Terms of $V_{ij}$*

First, we may recall that (Chandrasekhar 1961, p. 584, eq. [48])

$$\delta\mathfrak{W}_{ij} = -\int_V \rho\xi_l \frac{\partial\mathfrak{B}_{ij}}{\partial x_l}\,dx\,. \tag{44}$$

And it is known that for homogeneous ellipsoids (cf. Chandrasekhar and Lebovitz 1962*c*, eq. [51])

$$\mathfrak{B}_{ij} = 2B_{ij}x_ix_j + a_i^2\delta_{ij}\left(A_i - \sum_{l=1}^{3} A_{il}x_l^2\right), \tag{45}$$

where

$$B_{ij} = A_i - a_j^2 A_{ij} = A_j - a_i^2 A_{ij} \tag{46}$$

(no summation over repeated indices in eqs. [45] and [46]) ,

and the remaining symbols have their standard meanings. (*Note that in writing eq.* [*45*] *a common factor $\pi G\rho a_1a_2a_3$ has been suppressed.*)

With $\mathfrak{B}_{ij}$ given by equation (45) we find, in accordance with equation (44), that

$$\delta\mathfrak{W}_{ij} = -2B_{ij}V_{ij} \qquad (i\neq j)\,, \tag{47}$$

and

$$\delta\mathfrak{W}_{ii} = -\,(2B_{ii} - a_i^2 A_{ii})\,V_{ii} + a_i^2 \sum_{l\neq i} A_{il}V_{ll} \tag{48}$$

(no summation over repeated indices in eqs. [47] and [48]) .

The expressions for $\delta\mathfrak{W}_{ij}$ given in equations (47) and (48) are valid for general triaxial ellipsoids. For the particular case of prolate spheroids, in which we are presently interested, there are simplifications arising from the equality of $a_2$ and $a_3$. Thus the value of any of the symbols $A_{ij}$ and $B_{ij}$ will be unaltered if the index 2 (wherever it may occur) is replaced by the index 3 (and conversely). On this account we may now write

$$\delta\mathfrak{W}_{12} = -2B_{12}V_{12}\,, \quad \delta\mathfrak{W}_{13} = -2B_{12}V_{13}\,, \quad \delta\mathfrak{W}_{23} = -2B_{22}V_{23}\,, \tag{49}$$

$$\delta\mathfrak{W}_{11} = -(2B_{11} - a_1^2A_{11})V_{11} + a_1^2A_{12}(V_{22}+V_{33})\,, \tag{50}$$

$$\delta\mathfrak{W}_{22} = -(2B_{22} - a_2^2A_{22})V_{22} + a_2^2A_{22}V_{33} + a_2^2A_{12}V_{11}\,, \tag{51}$$

and

$$\delta\mathfrak{W}_{33} = -(2B_{22} - a_2^2A_{22})V_{33} + a_2^2A_{22}V_{22} + a_2^2A_{12}V_{11}\,. \tag{52}$$

*b) The Divergence Condition*

While equations (38)–(43) are of general applicability, the expressions for $\delta \mathfrak{W}_{ij}$ given in Section IV*a* above are valid only for homogeneous configurations. If we should now suppose that the fluid is in addition incompressible, then there is a further relation among the virials which follows from the solenoidal condition on $\boldsymbol{\xi}$; it is (cf. Lebovitz 1961, eq. [83])

$$\frac{V_{11}}{a_1^2}+\frac{V_{22}}{a_2^2}+\frac{V_{33}}{a_3^2}=0\,. \tag{53}$$

## V. THE CHARACTERISTIC FREQUENCIES OF OSCILLATION BELONGING TO THE SECOND HARMONICS; THE ONSET OF INSTABILITY AT $\mu_{max}$

The characteristic frequencies of the Jeans spheroid belonging to the second harmonics can now be deduced quite readily from the equations assembled in Section IV. Thus from equations (38) and (49) it follows that

$$\sigma^2 = 4B_{12} - \mu = \sigma_1^2 \text{ (say) is a double root}\,. \tag{54}$$[3]

Now, putting $V_{12} = V_{13} = 0$ (to be consistent with eqs. [38]) in equations (39), we find that

$$\sigma^2 = 0 \text{ is a double root}\,. \tag{55}$$

Similarly, equations (40) lead to the roots

$$\sigma^2 = 4B_{22} + 2\mu = \sigma_2^2 \qquad \text{(say) and } \sigma^2 = 0\,. \tag{56}$$

Next, eliminating $\delta\Pi$ from equations (42) and (43), we have

$$-\tfrac{1}{2}\sigma^2(V_{22} - V_{33}) = \delta\mathfrak{W}_{22} - \delta\mathfrak{W}_{33} - \mu(V_{22} - V_{33})\,, \tag{57}$$

whereas, from equations (51) and (52),

$$\delta\mathfrak{W}_{22} - \delta\mathfrak{W}_{33} = -2B_{22}(V_{22} - V_{33})\,. \tag{58}$$

We are thus led once more to the root

$$\sigma^2 = 2(2B_{22} + \mu) = \sigma_2^2\,. \tag{59}$$

The roots $\sigma_1^2$ and $\sigma_2^2$ are each of multiplicity 2; and the root $\sigma^2 = 0$ is of multiplicity 3. A last root remains to be determined.

Considering equations (41) and (42) and eliminating $\delta\Pi$, we have

$$-\tfrac{1}{2}\sigma^2(V_{11} - V_{22}) = \delta\mathfrak{W}_{11} - \delta\mathfrak{W}_{22} + \mu(2V_{11} + V_{22})\,. \tag{60}$$

In substituting for $\delta\mathfrak{W}_{11} - \delta\mathfrak{W}_{22}$ in equation (60) in accordance with equations (50) and (51), we can put $V_{22} = V_{33}$ to exclude the root $\sigma_2^2$ and be consistent with equation (57). In this manner we find

$$\begin{aligned}\tfrac{1}{2}\sigma^2(V_{11} - V_{22}) &- (2B_{11} - a_1^2A_{11} + a_2^2A_{12})V_{11} \\ &+ 2(B_{22} - a_2^2A_{22} + a_1^2A_{12})V_{22} + \mu(2V_{11} + V_{22}) = 0\,.\end{aligned} \tag{61}$$

[3] The suppression of the factor $\pi G\rho a_1a_2a_3$ in eq. (45) for $\mathfrak{B}_{ij}$ has the consequence that the formulae for $\sigma^2$ given in this section require to be multiplied by $\pi G\rho a_1 a_2^2$.

Supplementing equation (61) by the divergence condition

$$V_{11} = -\frac{a_1^2}{a_2^2}(V_{22} + V_{33}) = -2\frac{a_1^2}{a_2^2}V_{22}, \tag{62}$$

we obtain the last of the characteristic roots:

$$\sigma^2 = \frac{4a_1^2}{2a_1^2 + a_2^2}\left[2B_{11} - a_1^2 A_{11} + a_2^2 A_{12} + \frac{a_2^2}{a_1^2}(B_{22} - a_2^2 A_{22} + a_1^2 A_{12}) - \mu\left(2 - \frac{a_2^2}{2a_1^2}\right)\right] = \sigma_3^2 \text{ (say)}. \tag{63}$$

TABLE 1

THE SQUARES OF THE CHARACTERISTIC FREQUENCIES BELONGING TO THE SECOND HARMONICS

($\sigma^2$ Is Listed in the Unit $\pi G\rho$)

| $e$ | $\mu/(\pi G\rho)$ | $\sigma_1^2$ | $\sigma_2^2$ | $\sigma_3^2$ | $e$ | $\mu/(\pi G\rho)$ | $\sigma_1^2$ | $\sigma_2^2$ | $\sigma_3^2$ |
|---|---|---|---|---|---|---|---|---|---|
| 0.... | 0 | 1 06667 | 1 06667 | 1 06667 | 0 75 . | 0 103451 | 0 79312 | 1 52033 | +0 42239 |
| 0 05 . | 0.000445 | 1 06692 | 1 06805 | 1 06342 | 80 . | 114839 | .73210 | 1.59802 | + 28087 |
| .10 .. | .001781 | 1 06326 | 1 07332 | 1 06016 | 82 | 118754 | 70341 | 1 63172 | + 21799 |
| .15 . | .004017 | 1.05914 | 1 08165 | 1 05151 | .84 | .122038 | 67162 | 1 66700 | + 15169 |
| .20 . | .007164 | 1 05316 | 1 09343 | 1 03917 | .86 . | 124419 | .63609 | 1 70389 | + 08239 |
| .25 . | .011240 | 1 04528 | 1 10881 | 1 02273 | 88.... | 125514 | .59593 | 1 74242 | + 01092 |
| .30 . | 016262 | 1 03535 | 1 12793 | 1 00167 | 8830265 | 125536 | 58938 | 1.74839 | 0 |
| 35 | .022254 | 1 02323 | 1 15098 | 0 97525 | 90 | 124760 | 54987 | 1.78259 | − .06117 |
| .40 . | .029235 | 1 00860 | 1 17823 | 0 94275 | 92 | 121293 | .49595 | 1.82434 | − .13103 |
| .45.. | .037222 | 0 99121 | 1 20996 | 0 90304 | 94 ... | .113683 | .43090 | 1.86754 | − 19319 |
| 50 . | .046219 | 0.97060 | 1 24656 | 0 85493 | .96 | 099288 | .34854 | 1 91189 | − 23617 |
| .55... | .056209 | 0.94627 | 1.28844 | 0 79684 | .98. | .072040 | .23394 | 1 95671 | − 23051 |
| 60 | .067135 | 0 91751 | 1 33612 | 0 72689 | 995 | 030569 | .09355 | 1 98961 | − 12693 |
| 65 . | .078864 | 0 88337 | 1 39020 | 0 64289 | 999. | 009236 | 02782 | 1 99798 | − .04361 |
| 0.70. | 0 091137 | 0 84255 | 1 45134 | 0 54229 | 0 9999.. | 0.001381 | 0 00415 | 1 99980 | −0 00709 |

After some minor simplifications we can write, alternatively,

$$\sigma_3^2 = \frac{4a_1^2}{2a_1^2 + a_2^2}\left[2A_1 + 2a_2^2 A_{12} - 3a_1^2 A_{11} + \frac{a_2^2}{a_1^2}(A_2 - 2a_2^2 A_{22}) - \mu\left(2 - \frac{a_2^2}{2a_1^2}\right)\right]. \tag{64}$$

In Table 1, the roots $\sigma_1^2$, $\sigma_2^2$, and $\sigma_3^2$ determined in accordance with equations (54), (59), and (64) are listed; and in Figure 1 their variations along the Jeans sequence is illustrated.

From Table 1 it is apparent that *while the modes belonging to $\sigma_1^2$ and $\sigma_2^2$ are stable, the mode belonging to $\sigma_3^2$ becomes unstable at the point where $\mu$ attains its maximum value.*

That a neutral mode can occur only where $\mu = \mu_{\max}$ can be seen as follows. According to equation (60), the vanishing of $\sigma^2$ requires

$$\delta(\mathfrak{W}_{11} - \mathfrak{W}_{22}) + \mu\delta(2I_{11} + I_{22}) = 0, \tag{65}$$

whereas, according to equation (17), equilibrium requires

$$\mathfrak{W}_{11} - \mathfrak{W}_{22} + \mu(2I_{11} + I_{22}) = 0. \tag{66}$$

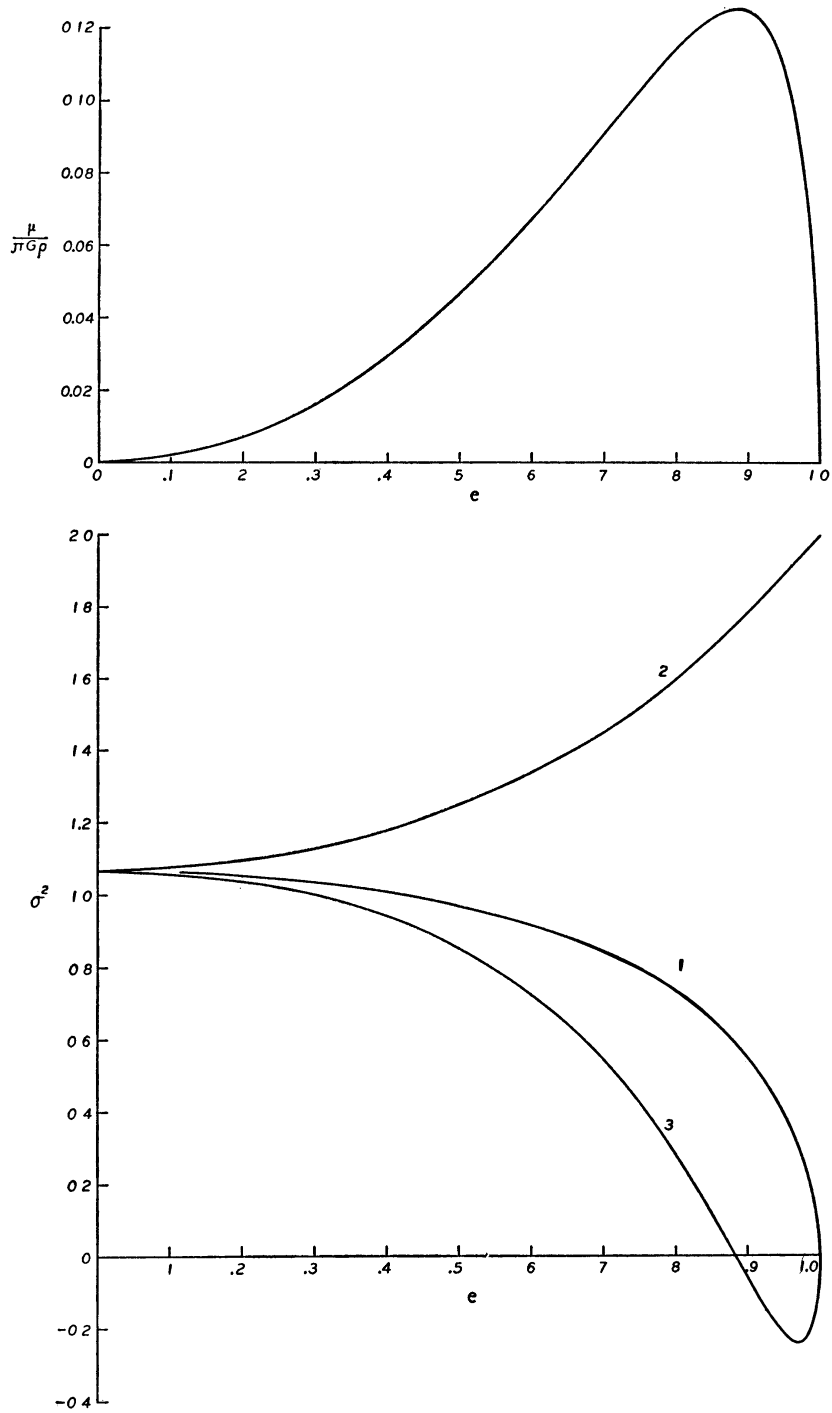


FIG 1.—*Top:* the variation of $\mu/\pi G\rho$ along the Jeans sequence. It attains its maximum at $e = 0.88303$. *Bottom:* the squares of the characteristic frequencies of oscillation belonging to the second harmonics; the labeling of the curves corresponds to the enumeration in Table 1. The mode "3" becomes unstable when $\mu$ attains its maximum.

Clearly, equations (65) and (66) can be satisfied simultaneously only where $\mu$ attains its maximum.

Finally, we may note that, according to equation (64), the explicit condition for the occurrence of the neutral mode is

$$\frac{\mu}{\pi G\rho}=a_1a_2^2\frac{2a_1^2}{4a_1^2-a_2^2}\left[2A_1+2a_2^2A_{12}-3a_1^2A_{11}+\frac{a_2^2}{a_1^2}(A_2-2a_2^2A_{22})\right], \qquad (67)$$

where the factor $\pi G\rho a_1 a_2^2$, which had been suppressed in writing the expression for $\mathfrak{B}_{ij}$, has been restored.

## VI. THE EFFECT OF COMPRESSIBILITY ON THE STABILITY OF THE JEANS SPHEROID

The analysis in the preceding section can be readily extended to determine the effect of compressibility on the stability of the Jeans spheroid. Specifically, the problem to be considered is that of the adiabatic oscillations of a tidally distorted homogeneous gaseous configuration. The assumption of *homogeneity* insures that in the equilibrium state the configurations will be indistinguishable from the incompressible Jeans spheroids. But the assumption that the configuration is *gaseous* has the consequence that the Lagrangian displacement describing a deformation can no longer be restricted to be solenoidal; instead, we must apply the laws appropriate to a gas which is subject to adiabatic changes. If the gas is assumed to have a ratio of specific heats $\gamma$, then the condition div $\xi = 0$ must be replaced by the condition

$$\frac{\delta p}{p}=\gamma\frac{\delta\rho}{\rho}=-\gamma\,\text{div}\,\xi\,. \qquad (68)$$

These relations enable us to express the first variation of $\Pi(=\int p dx)$ in terms of $\xi$; we have (cf. Chandrasekhar 1961, p. 584, eqs. [50] and [51])

$$\delta\Pi=(\gamma-1)\int_V \xi\cdot\text{grad}\,p dx\,, \qquad (69)$$

where $p$ denotes the pressure in the equilibrium state.

For the case under consideration,

$$\text{grad}\,p = \rho\,\text{grad}\,[I - A_1x_1^2 - A_2(x_2^2+x_3^2)+\mu x_1^2-\tfrac{1}{2}\mu(x_2^2+x_3^2)]\,; \qquad (70)$$

and equation (69) gives

$$\delta\Pi = -(\gamma-1)[(A_1-\mu)V_{11}+(A_2+\tfrac{1}{2}\mu)(V_{22}+V_{33})]\,. \qquad (71)$$

Returning to the problem of determining the characteristic frequencies of oscillation under the present more general circumstances, we first observe that the roots designated by $\sigma_1^2$ and $\sigma_2^2$ will be unaffected: for, in their derivation no use was made of the divergence condition (53). However, the root designated by $\sigma_3^2$ will be affected; and the place at which the analysis must be modified is where the divergence condition (62) was used to eliminate $V_{22}$ from equation (61). Instead, we must now consider, along with equation (61) (which was obtained by eliminating $\delta\Pi$ from eqs. [41] and [42]), equation (41) or (42) and give to $\delta\Pi$ its present value (71). Choosing equation (41) as the second equation and remembering that we must now put $V_{22} = V_{33}$, we have (cf. eq. [50])

$$\begin{aligned}-\tfrac{1}{2}\sigma^2V_{11} &= -(2B_{11}-a_1^2A_{11})V_{11}+2a_1^2A_{12}V_{22}+2\mu V_{11}\\ &\quad -(\gamma-1)[(A_1-\mu)V_{11}+(2A_2+\mu)V_{22}]\,.\end{aligned} \qquad (72)$$

Equations (61) and (72) will lead to the desired characteristic equation. It is, however, more convenient to consider, together with equation (72), the equation

$$
\begin{aligned}
-\tfrac{1}{2}\sigma^2 V_{22} = & -2(B_{22} - a_2^2 A_{22})V_{22} + a_2^2 A_{12} V_{11} - \mu V_{22} \\
& -(\gamma - 1)[(A_1 - \mu)V_{11} + (2A_2 + \mu)V_{22}] ,
\end{aligned}
\tag{73}
$$

obtained in a similar way from equations (42) and (71). Equations (72) and (73) now lead to the characteristic equation

$$
\begin{Vmatrix}
\tfrac{1}{2}\sigma^2 - (2B_{11} - a_1^2 A_{11}) + 2\mu - (\gamma - 1)(A_1 - \mu) & 2a_1^2 A_{12} - (\gamma - 1)(2A_2 + \mu) \\
a_2^2 A_{12} - (\gamma - 1)(A_1 - \mu) & \tfrac{1}{2}\sigma^2 - 2(B_{22} - a_2^2 A_{22}) - \mu - (\gamma - 1)(2A_2 + \mu)
\end{Vmatrix} = 0 .
\tag{74}
$$

The two roots for $\sigma^2$ ($\sigma_R^2$ and $\sigma_S^2$) provided by equation (74) represent a coupling of two modes (the $R$- and the $S$-modes) which are, in the limit $\mu = 0$, purely radial and purely non-radial (and volume-preserving). The corresponding limits to which the two roots tend are

$$
\sigma_R^2 \to \tfrac{4}{3}(3\gamma - 4)\pi G\rho \quad \text{and} \quad \sigma_S^2 \to \tfrac{16}{15}\pi G\rho \quad (\mu \to 0) .
\tag{75}
$$

Equation (74) has been solved for a number of different cases to determine the behavior of $\sigma_R^2$ and $\sigma_S^2$ along the Jeans sequence and their dependence on the value of $\gamma$. The results of the calculations are given in Table 2; and they are illustrated in Figure 2.

We observe that the principal effect of compressibility is to enhance the instability which is already present: configurations which are normally stable when $\gamma > \frac{4}{3}$ become unstable under the influence of tidal action if it is strong enough. And it is important to note that the effect of compressibility is to make the Jeans spheroid unstable even before $\mu$ attains its maximum.

From Table 2 and Figure 2 it is also apparent that $\gamma = 1.6$ plays a critical role in this problem, which is similar to the one it plays in the rotational problem (cf. Chandrasekhar and Lebovitz 1962*d*). The origin of the critical role in both cases is the same: the equality of $\sigma_R^2$ and $\sigma_S^2$ in the "zero" limit when $\gamma = 1.6$ (cf. eq. [75]). The particular manifestation of this "degeneracy" in the present problem is that, while it is the $R$-mode which becomes unstable when $\frac{4}{3} < \gamma < 1.6$, it is the $S$-mode which becomes unstable when $\gamma > 1.6$. When $\gamma = 1.6$, neither of the two "pure" modes which one obtains in the limit $\mu \to 0$ will be purely radial; the phenomenon is the same as that encountered in the rotational problem, which has been discussed in that connection (see Chandrasekhar and Lebovitz 1962*e*).

## VII. CONCLUDING REMARKS

A result which bears on some of Jeans's conclusions and analysis relative to his spheroids is their instability to small oscillations when $\mu$ ($= GM'/R^3$) attains its maximum value $\mu_{\max} = 0.12554\ \pi G\rho$. While Jeans states in a number of places that the spheroids become "unstable" when $\mu$ becomes equal to $\mu_{\max}$, it is not always clear from his discussion that he fully appreciated the true nature of the instability, since he did not investigate the problem of the small oscillations.

Let us consider, then, what would happen if $\mu$ were to increase very gradually from zero. And let us suppose, also following Jeans, that the evolution, at first, is through a sequence of equilibrium prolate spheroidal figures. This supposition is a reasonable one if the time scale in which $\mu$ changes is long compared with the normal periods of oscilla-

TABLE 2

THE CHARACTERISTIC ROOTS $\sigma_R^2$ AND $\sigma_S^2$

($\sigma^2$ Is Listed in the Unit $\pi G\rho$)

| $e$ | $\gamma=1\ 3$ | | $\gamma=\frac{4}{3}$ | | $\gamma=1.4$ | |
|---|---|---|---|---|---|---|
| | $\sigma_S^2$ | $\sigma_R^2$ | $\sigma_S^2$ | $\sigma_R^2$ | $\sigma_S^2$ | $\sigma_R^2$ |
| 0.. | 1.06667 | −0 13333 | 1.06667 | 0 | 1 06667 | +0 26667 |
| 0 20 . | 1 04135 | − 13454 | 1 04163 | −0.00149 | 1 04247 | + .26434 |
| .25 | 1 02817 | − 13635 | 1 02887 | − 00372 | 1.03100 | + .26082 |
| .30. | 1 01320 | − .13976 | 1 01470 | − 00793 | 1 01927 | + .25417 |
| .35. | 0 99718 | − .14560 | 1 00007 | − .01516 | 1 00882 | + 24277 |
| .40 | 0 98129 | − 15495 | 0 98639 | − .02672 | 1 00167 | + 22467 |
| .45. | 0 96679 | − 16913 | 0 97520 | − 04420 | 0 99994 | + 19772 |
| .50 | 0 95536 | − .18969 | 0 96843 | − .06942 | 1 00586 | + .15982 |
| .55. | 0 94884 | − .21827 | 0 96808 | − 10418 | 1 02126 | + .10931 |
| 60 | 0 94916 | − .25643 | 0 97612 | − .15006 | 1 04752 | + 04521 |
| .65. | 0 95818 | − .30532 | 0.99423 | − .20804 | 1 08547 | − 03261 |
| .70 | 0.97743 | − 36522 | 1 02364 | − .27809 | 1.13555 | − .12334 |
| .75 | 1 00793 | − .43484 | 1 06506 | − .35864 | 1 19792 | − .22483 |
| .80 | 1 04999 | − .51023 | 1 11857 | − .44547 | 1 27249 | − .33272 |
| .82 | 1 06992 | − .54019 | 1 14320 | − .48014 | 1 30562 | − .37589 |
| .84 | 1 09146 | − .56870 | 1 16953 | − .51344 | 1 34053 | − .41777 |
| .86 | 1 11439 | − 59435 | 1 19736 | − .54399 | 1 37708 | − .45704 |
| .88 | 1 13839 | − 61510 | 1.22640 | − .56977 | 1 41505 | − .49175 |
| .90 | 1 16294 | − .62785 | 1.25617 | − .58775 | 1 45406 | − 51897 |
| .92 | 1 18718 | − 62775 | 1 28590 | − .59314 | 1.49351 | − .53408 |
| .94 | 1 20962 | − .60651 | 1 31427 | − .57782 | 1 53233 | − 52921 |
| .96 | 1.22740 | − 54826 | 1 33868 | − 52621 | 1 56842 | − 48928 |
| 0 98 | 1 23371 | −0 41502 | 1.35313 | −0 40111 | 1 59703 | −0 37834 |

| $e$ | $\gamma=1\ 5$ | | $\gamma=1\ 6$ | | $\gamma=\frac{5}{3}$ | |
|---|---|---|---|---|---|---|
| | $\sigma_S^2$ | $\sigma_R^2$ | $[\sigma_{R(\text{or } S)}]^2$ | $[\sigma_{S(\text{or } R)}]^2$ | $\sigma_R^2$ | $\sigma_S^2$ |
| 0 .. | 1.06667 | +0.66667 | 1 06667 | +1.06667 | 1.33333 | +1.06667 |
| 0.20 | 1.04592 | + .66088 | 1.10594 | +1 00087 | 1 34275 | +1 03073 |
| 25 | 1 03972 | + .65209 | 1 12856 | +0.96325 | 1.35471 | +1.00377 |
| .30 | 1 03764 | + .63580 | 1.15666 | +0.91679 | 1 37380 | +0.96631 |
| .35 | 1 04240 | + .60918 | 1.19053 | +0.86105 | 1 40078 | +0.91747 |
| .40 | 1.05628 | + 57006 | 1 23059 | +0 79575 | 1.43604 | +0.85697 |
| .45 | 1 08027 | + 51739 | 1.27724 | +0 72041 | 1 47978 | +0 78454 |
| .50 | 1 11466 | + 45101 | 1 33108 | +0 63460 | 1 53229 | +0 70005 |
| .55 | 1 15936 | + 37121 | 1 39269 | +0 53788 | 1 59397 | +0 60327 |
| .60 | 1 21428 | + 27845 | 1 46283 | +0 42990 | 1 66539 | +0 49400 |
| 65 | 1 27955 | + .17331 | 1 54235 | +0 31051 | 1 74733 | +0 37220 |
| .70 | 1 35551 | + 05670 | 1 63223 | +0 17998 | 1 84072 | +0 23816 |
| .75 | 1.44266 | − .06958 | 1.73352 | +0 03956 | 1 94666 | +0 09309 |
| .80 | 1.54153 | − 20177 | 1 84732 | −0 10755 | 2.06634 | −0 05991 |
| .82 | 1 58444 | − 25471 | 1 89655 | −0 16682 | 2 11831 | −0 12191 |
| .84 | 1 62925 | − .30650 | 1 94796 | −0 22520 | 2 17271 | −0 18329 |
| .86 | 1 67592 | − 35588 | 2 00156 | −0 28152 | 2 22958 | −0 24287 |
| 88 | 1 72431 | − 40102 | 2 05729 | −0 33399 | 2 28889 | −0 29892 |
| 90 | 1 77421 | − 43912 | 2 11502 | −0 37993 | 2 35055 | −0 34880 |
| .92 | 1 82518 | − .46575 | 2 17447 | −0 41504 | 2 41437 | −0 38827 |
| .94 | 1 87645 | − 47334 | 2 23508 | −0 43196 | 2 47988 | −0.41010 |
| .96 | 1 92651 | − 44737 | 2 29571 | −0 41657 | 2 54614 | −0 40033 |
| 0.98 | 1 97187 | −0.35318 | 2.35374 | −0.33505 | 2.61095 | −0 32559 |

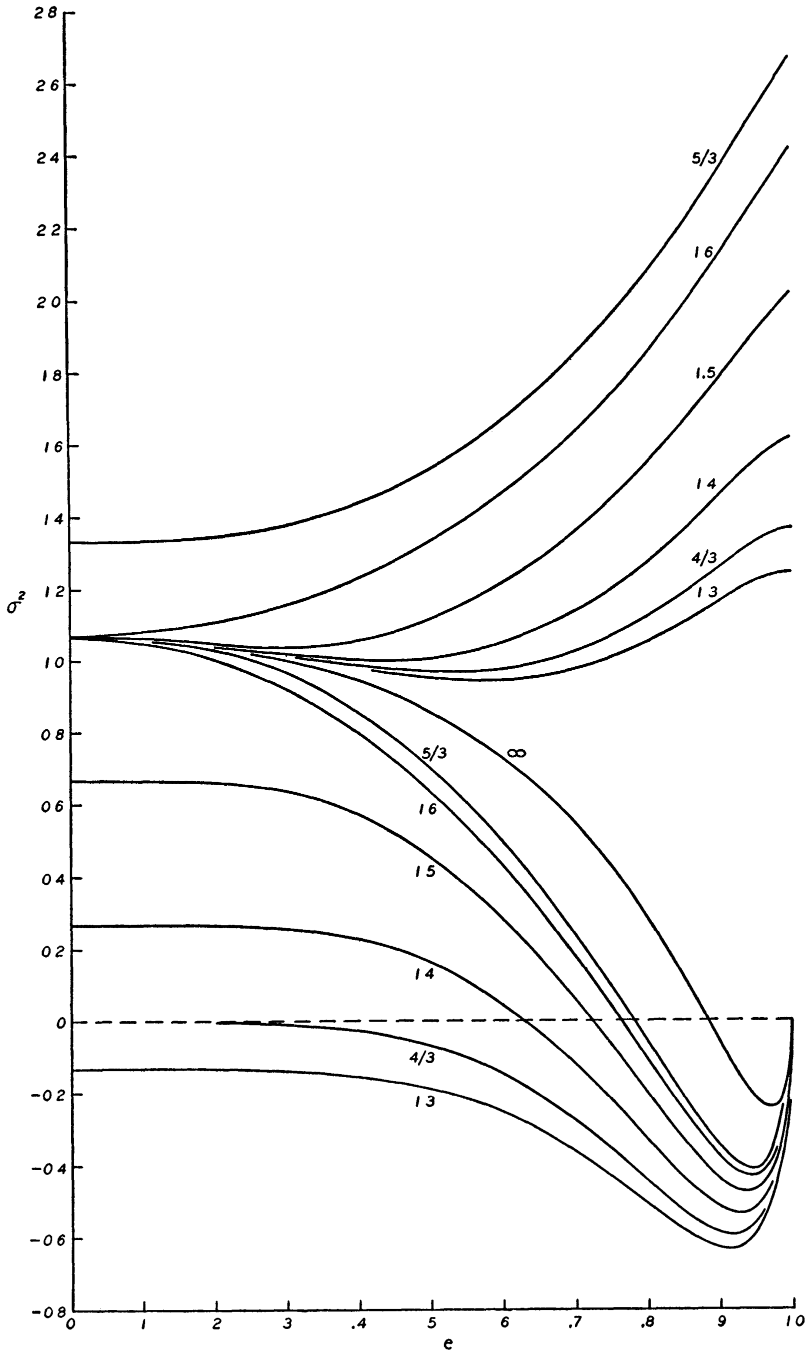


FIG. 2.—The squares of the characteristic frequencies belonging to the $R$- and the $S$-modes for the compressible Jeans spheroids. The curves are labeled by the values of $\gamma$ to which they belong The curve belonging to the incompressible case is labeled by $\infty$. For $\gamma < 1\ 6$, the $R$-mode is the one which becomes unstable, while for $\gamma > 1\ 6$ it is the $S$-mode which becomes unstable.

tion of the system. Under these conditions, the eccentricity of the spheroid will increase in step with $\mu$ until $\mu$ attains its maximum value when $e = 0.88303$. When this happens, the character of the problem will change into a truly dynamical one. Jeans derives an equation of motion for $e$ on the assumption that the configuration continues to evolve along a sequence of prolate spheroidal forms and $\mu$ varies in some prescribed manner (Jeans 1917, eq. [82]). Jeans further supposes that, in accordance with his equation of motion, $e$ can become as high as 0.94774 (where a second point of neutral stability occurs along the Jeans sequence); and he invokes in his discussion the instability of the equilibrium spheroids (with respect to a mode of oscillation belonging to the third harmonics) which sets in at $e = 0.94774$. It is difficult to see why, under these same circumstances, the instability (with respect to a mode of oscillation belonging to the second harmonics) which sets in at $e = 0.88303$ should not have intervened already. It is important to note in this connection that the configuration will become unstable *before* $\mu$ attains its maximum value if allowance is made for compressibility; and we do not, then, have to face the particular difficulty of treating a "singular" case in which both stability and available equilibrium forms cease simultaneously.

The accentuation of the instability of a gaseous configuration by tidal action has been established only for homogeneous configurations. But it would not seem that the phenomenon is peculiar to them. Clearly, the tidally induced instability of the modes which are mainly radial for $\frac{4}{3} < \gamma < 1.6$, if a general phenomenon, must have some cosmogonical meaning.

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