SPACES OF RATIONAL CURVES IN COMPLETE INTERSECTIONS

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Abstract. We prove that the space of smooth rational curves of degree e in a general complete intersection of multidegree $(d_1, \dots, d_m)$ in $\mathbb{P}^n$ is irreducible of the expected dimension if $\sum_{i=1}^m d_i < \frac{2n}{3}$ and n is large enough. This generalizes the results of Harris, Roth and Starr [9], and is achieved by proving that the space of conics passing through any point of a general complete intersection has constant dimension if $\sum_{i=1}^m d_i$ is small compared to n.

1. Introduction

Throughout this paper, we work over the field of complex numbers. For a smooth projective variety $X \subset \mathbb{P}^n$ and an integer $e \geq 1$, we denote by $\text{Hilb}_{et+1}(X)$ the Hilbert scheme parametrizing subschemes of X with Hilbert polynomial $et + 1$, and we denote by $R_e(X) \subset \text{Hilb}_{et+1}(X)$ the open subscheme parametrizing smooth rational curves of degree e in X. If $X = \mathbb{P}^n$, then $R_e(X)$ is a smooth, irreducible, rational variety of dimension $(e+1)(n+1) - 4$. But already in the case of hypersurfaces in $\mathbb{P}^n$, there are many basic questions concerning the geometry of $R_e(X)$ which are still open. In this paper we address and discuss some of these questions, focusing in particular on the dimension and irreducibility of $R_e(X)$, when X is a general complete intersection in $\mathbb{P}^n$.

To study the space of smooth rational curves in X, we consider the Kontsevich moduli space of stable maps $\overline{\mathcal{M}}_{0,0}(X, e)$ which compactifies $R_e(X)$ by allowing smooth rational curves to degenerate to morphisms from nodal curves. These have certain advantages over the Hilbert Schemes for the problems studied here. We refer the reader to [1, 5, 8, 9] for detailed discussions of these moduli spaces and the comparison between them.

For every smooth hypersurface $X \subset \mathbb{P}^n$ of degree d, the dimension of every irreducible component of $\overline{\mathcal{M}}_{0,0}(X, e)$ is at least $e(n+1-d) + n - 4$, and if $d \leq n-1$, then there is at least one irreducible component whose dimension is equal to $e(n+1-d)+n-4$ (see Sections 2.1 and 6). The number $e(n+1-d)+n-4$ is referred to as *the expected dimension* of $\overline{\mathcal{M}}_{0,0}(X, e)$. If X is an arbitrary smooth hypersurface, $\overline{\mathcal{M}}_{0,0}(X)$ (or even $R_e(X)$) can be reducible and its dimension can be larger than expected (see [4], Section 1). By a result of Harris, Roth, and Starr [9], if $d < \frac{n+1}{2}$ and X is a *general* hypersurface of degree d in $\mathbb{P}^n$, then for every $e \geq 1$,

$\overline{\mathcal{M}}_{0,0}(X,e)$ is integral of the expected dimension and has only local complete intersection singularities. In this paper, we generalize this result to higher degree hypersurfaces.

Let $\overline{\mathcal{M}}_{0,1}(X,e)$ denote the moduli space of 1-pointed stable maps of degree e to X. In order to obtain the above mentioned result, Harris, Roth, and Starr show that if $d < \frac{n+1}{2}$ and X is a general hypersurface of degree d in $\mathbb{P}^n$, then the evaluation morphism

$$ev : \overline{\mathcal{M}}_{0,1}(X,e) \to X$$

is flat of relative dimension $e(n+1-d)-2$ for every $e \geq 1$ ([9, Theorem 2.1 and Corollary 5.6]). It is conjectured that the same holds for any $d \leq n-1$:

Conjecture 1.1 (Coskun-Harris-Starr [4]). *Let X be a general hypersurface of degree $d \leq n-1$ in $\mathbb{P}^n$. Then the evaluation morphism*

$$ev : \overline{\mathcal{M}}_{0,1}(X,e) \to X$$

is flat of relative dimension $e(n+1-d)-2$ for every $e \geq 1$.

The above conjecture would imply the following:

Conjecture 1.2 (Coskun-Harris-Starr [4]). *If X is a general hypersurface of degree $d \leq n-1$ in $\mathbb{P}^n$, then for every $e \geq 1$, $\overline{\mathcal{M}}_{0,0}(X,e)$ has the expected dimension $e(n+1-d)+n-4$.*

Coskun and Starr [4] show that Conjecture 1.2 holds for $d < \frac{n+4}{2}$. When $d = n-1$ and $e \geq 2$, $\overline{\mathcal{M}}_{0,0}(X,e)$ is reducible for the following reason. By Lemma 6.2, $\overline{\mathcal{M}}_{0,0}(X,e)$ has at least one irreducible component of dimension $2e+n-4$ whose general point parametrizes an embedded smooth rational curve of degree e in X. On the other hand, the space of lines in X has dimension at least $n-2$, and therefore, the space of degree e covers of lines in X has dimension $\geq (n-2)+(2e-2)$, thus $\overline{\mathcal{M}}_{0,0}(X,e)$ has at least 2 irreducible components. It is expected that if X is general, then $\overline{\mathcal{M}}_{0,0}(X,e)$ is irreducible when $d \leq n-2$, and $R_e(X)$ is irreducible when $d \leq n-1$ (see [4, Conjecture 1.3]).

In this paper, we show:

Theorem 1.3. *Suppose that X is a general hypersurface of degree d in $\mathbb{P}^n$. If*

$$n-1 < \binom{n-d}{2},$$

then the evaluation morphism $ev : \overline{\mathcal{M}}_{0,1}(X,2) \to X$ is flat of relative dimension $2n-2d$.

A smooth rational curve in X is called *free* if its normal bundle in X is globally generated. The proof of the above theorem is based on an analysis of the space of non-free conics in X passing through an arbitrary point of X. It seems quite plausible that the same approach can be applied to the case of cubics or other higher degree rational curves to prove the flatness of ev when d and n satisfy the inequality of the theorem, but we have not carried out all the details, and we restrict the discussion here to the case of conics.

Theorem 1.3 along with the results of [9] gives the following:

Theorem 1.4. *If $X \subset \mathbb{P}^n$ is a general hypersurface of degree $d < \frac{2n}{3}, n \geq 20$, then for every $e \geq 1$,*

(a) *The evaluation morphism $\overline{\mathcal{M}}_{0,1}(X, e) \to X$ is flat and of relative dimension $e(n+1-d)-2$.*
(b) *$\overline{\mathcal{M}}_{0,0}(X, e)$ is an irreducible local complete intersection stack of expected dimension $e(n+1-d)+(n-4)$.*

The above results can be generalized to the case of general complete intersections. Suppose that $X \subset \mathbb{P}^n$ is a general complete intersection of multidegree $(d_1, \ldots, d_m)$ and set $d = d_1 + \cdots + d_m$. If $n - m < \binom{n-d}{2}$, then the evaluation morphism $ev : \overline{\mathcal{M}}_{0,1}(X, 2) \to X$ is flat of relative dimension $2(n-d)$, and if d and n are in the range of Theorem 1.4, then $\overline{\mathcal{M}}_{0,0}(X, e)$ is an irreducible, local complete intersection stack of expected dimension $e(n+1-d)+n-3-m$ (Theorem 7.1).

Along the way to proving Theorem 1.3, we obtain the following result on the space of non-free lines in general hypersurfaces.

Theorem 1.5. *Let X be a general hypersurface of degree d in $\mathbb{P}^n$, and let p be an arbitrary point of X. For $1 \leq k \leq d-1$, set*

$$a_k = \min\left\{a \geq 0 \mid \binom{a+k+2}{k+1} \geq n\right\}.$$

Then the family of lines l in X passing through p with $h^1(l, N_{l/X}(-1)) \geq k$ has dimension $\leq a_k$.

In fact, we can modify the proof of the above theorem to say more in special cases. For example, it follows from the proof of the theorem that if $n \leq 5$, then the space of non-free lines through any point of X is at most zero dimensional. The proof shows that if there is a 1-parameter family of non-free lines in X through p parametrized by $C \subset \mathbb{P}^{n-1}$, then $n \geq 2 + 2\dim$ Linear Span(C) (see Proposition 6.5). Of course, a general hypersurface of degree ≥ 3 in $\mathbb{P}^n$, $n \leq 5$, does not contain any 2-plane, so the dimension of the linear span of C is at least 2. Note that for a general hypersurface X of degree $3 \leq d \leq n-1$, the non-free lines in X sweep out a divisor in X (Proposition 6.3).

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2. Background and Summary

2.1. **Preliminaries.** Fix positive integers $d_1 \leq \cdots \leq d_m$, and set $d := d_1 + \cdots + d_m$. Let X be a smooth complete intersection of multidegree $(d_1, \ldots, d_m)$ in $\mathbb{P}^n$. The Kontsevich moduli space $\overline{\mathcal{M}}_{0,r}(X, e)$ parametrizes isomorphism classes of tuples $(C, q_1, \ldots, q_r, f)$ where

(1) C is a proper, connected, at worst nodal curve of arithmetic genus 0.
(2) $q_1, \ldots, q_r$ are distinct smooth points of C.
(3) $f : C \to X$ is a map of degree e in the sense that the pull back of the hyperplane section line bundle of X has total degree e on C, which further satisfies the following stability condition: any irreducible component of C which is mapped to a point by f has at least 3 points which are either marked or nodes.

The tuples $(C, q_1, \ldots, q_r, f)$ and $(C', q'_1, \ldots, q'_r, f')$ are isomorphic if there is an isomorphism $g : C \to C'$ taking q_i to q'_i, with $f' \circ g = f$. The moduli space $\overline{\mathcal{M}}_{0,r}(X, e)$ is a proper Deligne-Mumford stack, and the corresponding coarse moduli space $\overline{M}_{0,r}(X, e)$ is a projective scheme. There is an evaluation morphism

$$ev : \overline{\mathcal{M}}_{0,r}(X, e) \to X^r$$

sending a datum $(C, q_1, \ldots, q_r, f)$ to $(f(q_1), \ldots, f(q_r))$. We refer to [1] and [8] for constructions and basic properties of these moduli spaces.

The space of first order deformations of the map f with $(C, q_1, \ldots, q_r)$ fixed can be identified with $H^0(C, f^*T_X)$. If $H^1(C, f^*T_X) = 0$ at a point $(C, q_1, \ldots, q_r, f)$, then f is unobstructed and the moduli stack is smooth at that point. In particular, $\overline{\mathcal{M}}_{0,r}(\mathbb{P}^n, e)$ is smooth of dimension $(n+1)(e+1) + r - 4$ (see Appendix A of [14] for a brief discussion of the deformation theory of $\overline{\mathcal{M}}_{0,r}(X, e)$).

Denote by $\mathcal{C} \to \overline{\mathcal{M}}_{0,0}(\mathbb{P}^n, e)$ the universal curve and by $h : \mathcal{C} \to \mathbb{P}^n$ the universal map

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{\ h\ } & \mathbb{P}^n \\ \downarrow{\scriptstyle \pi} & & \\ \overline{\mathcal{M}}_{0,0}(\mathbb{P}^n, e). & & \end{array}$$

For any $d \geq 1$, the line bundle $h^*\mathcal{O}_{\mathbb{P}^n}(d)$ is the pullback of a globally generated line bundle and so it is globally generated. The first cohomology group of a globally generated line bundle over a nodal curve of genus zero vanishes, so by the theorem of cohomology and base change [10, Theorem 12.11], $E := \pi_* h^*(\oplus_{i=1}^m \mathcal{O}_{\mathbb{P}^n}(d_i))$ is a locally free sheaf of rank $de + m$.

If $s \in \oplus_{i=1}^m H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d_i))$ is a section whose zero locus is X, then $\pi_* h^* s$ is a section of E whose zero locus as a closed substack of $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^n, e)$ is $\overline{\mathcal{M}}_{0,0}(X, e)$ ([9, Lemma 4.5]). The number

$$\dim \overline{\mathcal{M}}_{0,0}(\mathbb{P}^n, e) - (de + m) = e(n + 1 - d) + n - m - 3$$

is called the *expected dimension* of $\overline{\mathcal{M}}_{0,0}(X,e)$. It follows that the dimension of every component of $\overline{\mathcal{M}}_{0,0}(X,e)$ is at least the expected dimension, and if the equality holds, then $\overline{\mathcal{M}}_{0,0}(X,e)$ is a local complete intersection substack of $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^n,e)$.

Similarly, $\overline{\mathcal{M}}_{0,1}(\mathbb{P}^n,e)$ is a smooth stack of dimension $(e+1)(n+1)-3$, and $\overline{\mathcal{M}}_{0,1}(X,e)$ is the zero locus of a section of a locally free sheaf of rank $de+m$ on $\overline{\mathcal{M}}_{0,1}(\mathbb{P}^n,e)$. Therefore if $\dim \overline{\mathcal{M}}_{0,1}(X,e) = e(n+1-d)+n-m-2$, then it is a local complete intersection stack.

The number $e(n+1-d)+n-3-m$ can be also obtained as an Euler characteristic: if C is a smooth rational curve of degree e in X, and if $N_{C/X}$ denotes the normal bundle of C in X, then

$$\chi(N_{C/X}) = \chi(T_X|_C) - \chi(T_C) = e(n+1-d)+n-m-3.$$

2.2. **Outline of proof of Theorem 1.3.** Let X be a general hypersurface of degree d in $\mathbb{P}^n$, and consider the evaluation morphism

$$ev : \overline{\mathcal{M}}_{0,1}(X,e) \to X.$$

To show that ev is flat, it suffices to show that every fiber of ev has dimension $e(n+1-d)-2$ (see the discussion in the beginning of Section 5). Let $e=2$, and assume to the contrary that there is a point p in X and an irreducible component $\mathcal{M}$ of $ev^{-1}(p)$ whose dimension is larger than $2(n-d)$. It follows from [9, Theorem 2.1] that a general map parametrized by $\mathcal{M}$ is an isomorphism onto a smooth conic passing through p. Let $(\mathbb{P}^1, f : \mathbb{P}^1 \to X)$ be a general map parametrized by $\mathcal{M}$, and let C denote the image of f. We have

$$\chi(N_{C/X}(-p)) = \chi(T_X|_C(-p)) - \chi(T_C(-p)) = 2(n-d).$$

By [11, II, Theorem 1.7] the Zariski tangent space to $ev^{-1}(p)$ at $(\mathbb{P}^1, f)$ is isomorphic to $H^0(C, N_{C/X}(-p))$, therefore if $\dim \mathcal{M} > 2(n-d)$, then $h^0(C, N_{C/X}(-p)) > 2(n-d)$ and $H^1(C, N_{C/X}(-p)) \neq 0$.

There is a short exact sequence

$$0 \to N_{C/X}(-p) \to N_{C/\mathbb{P}^n}(-p) \to \mathcal{O}_C(d)(-p) \to 0,$$

and we show that under the above assumptions, there is a subspace W of codimension at most $n-1$ in $H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d)(-p))$ such that for any w in W and for a general stable map $(\mathbb{P}^1, f : \mathbb{P}^1 \xrightarrow{\sim} C)$ parametrized by $\mathcal{M}$, $w|_C$ is in the image of the map

$$\rho : H^0(C, N_{C/\mathbb{P}^n}(-p)) \to H^0(C, \mathcal{O}_C(d)(-p))$$

obtained from the above sequence (Proposition 3.4). We then show in Sections 4 and 5 that if $n < \min\left(\binom{n-d}{2}, (d-1)\lfloor \frac{n-d}{2} \rfloor + 1\right)$, then the existence of such W implies that for a general $(\mathbb{P}^1, f : \mathbb{P}^1 \xrightarrow{\sim} C)$ parametrized by $\mathcal{M}$, the map ρ is surjective. Applying the long exact sequence of cohomology to the above short exact sequence we get $H^1(C, N_{C/X}(-p)) = 0$, which is a contradiction.

3. Deformations of rational curves

We fix a few notations for normal sheaves first. If Y is a closed subscheme of a smooth variety Z, as usual we write $N_{Y/Z}$ for the normal sheaf of Y in Z. More generally, suppose that $f : Y \to Z$ is a morphism between quasi-projective varieties and Z is smooth. Denote by T_Y and T_Z the tangent sheaves of Y and Z, and denote by N_f the cokernel of the induced map $T_Y \to f^*T_Z$. We refer to N_f as the *normal sheaf* of f. We may sometimes write $N_{f,Z}$ instead to emphasize the range. If Y and Z are both smooth and f is generically finite then the exact sequence $T_Y \to f^*T_Z \to N_f \to 0$ is exact on the left. If $g : Z \to T$ is another morphism to a smooth variety T, then we get an exact sequence on Y of normal sheaves,

$$N_f \to N_{gf} \to f^*N_g \to 0. \tag{1}$$

Let B and X be smooth quasi-projective varieties. Suppose that $\pi : Y \to B$ is a smooth projective morphism and denote by Y_b the fiber over $b \in B$. Let $F : Y \to B \times X$ be a morphism over B such that the restriction of F to every fiber of π is generically finite. Let $f = F_{|Y_b} : Y_b \to X$ and p_B (resp. p_X) be the projections from $B \times X$ to B (resp. X). Notice that $p_B \circ F = \pi$ and $T_{B\times X}$ is naturally isomorphic to $p_B^*T_B \oplus p_X^*T_X$. Thus, we have a natural map

$$\alpha : \pi^*T_B \to N_F. \tag{2}$$

If $b \in B$, then we have $N_F|_{Y_b} = N_f$ using the following commutative diagram.

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \to & T_{Y_b} & \to & f^*T_X & \to & N_f & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow{=} & & \\
0 & \to & T_Y|_{Y_b} & \to & F^*T_{B\times X}|_{Y_b} & \to & N_F|_{Y_b} & \to & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & N_{Y_b/Y} & \xrightarrow{=} & \pi^*T_B|_{Y_b} & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

Thus we get, restricting α, a map $\pi^*T_{B,b} \to N_f$ and since one has a natural map $T_{B,b} \to H^0(Y_b, \pi^*T_{B,b})$, we get a map,

$$\alpha_b : T_{B,b} \to H^0(Y_b, N_f). \tag{3}$$

3.1. **Morphisms from $\mathbb{P}^1$ to general complete intersections.** Let $d_1 \leq \cdots \leq d_m$ be positive integers. For the rest of this section, we fix the following notation:

(1) $\mathcal{H}$ denotes the variety parametrizing complete intersections in $\mathbb{P}^n$ which are of multidegree $(d_1, \ldots, d_m)$.
(2) $\mathcal{U} \subset \mathcal{H} \times \mathbb{P}^n$ denotes the universal family over $\mathcal{H}$.
(3) $\pi_1 : \mathcal{U} \to \mathcal{H}$ and $\pi_2 : \mathcal{U} \to \mathbb{P}^n$ denote the two projection maps.

Remark 3.1. If $h \in \mathcal{H}$ and $\mathcal{U}_h \subset \mathbb{P}^n$ is the corresponding scheme, then we have the map $\alpha_h : T_{\mathcal{H},h} \to H^0(\mathcal{U}_h, N_{\mathcal{U}_h/\mathbb{P}^n})$ as defined in the previous section. (Easy to check that this is defined even though $\mathcal{U} \to \mathcal{H}$ is not smooth). This map is an isomorphism for any $h \in \mathcal{H}$.

Suppose that B is an smooth irreducible quasi-projective variety and $\psi : B \to \mathcal{H}$ a dominant morphism. Let $\mathcal{U}_B \subset B \times \mathbb{P}^n$ be the fiber product. Let $\pi : Y \to B$ be a dominant morphism whose fibers are smooth connected projective curves, and let $F : Y \to \mathcal{U}_B$ be a morphism over B which is generically finite on each fiber of π. We have an exact sequence using the sequence (1),

$$0 \to N_{F,\mathcal{U}_B} \to N_{F,B\times\mathbb{P}^n} \to F^* N_{\mathcal{U}_B/B\times\mathbb{P}^n} \to 0. \tag{4}$$

The natural map $\alpha : \pi^* T_B \to N_{F,B\times\mathbb{P}^n}$ by composing gives a map

$$\beta : \pi^* T_B \to F^* N_{\mathcal{U}_B/B\times\mathbb{P}^n}. \tag{5}$$

The dominance of ψ and the above remark show that for a general point $b \in B$ and X the corresponding complete intersection subscheme of $\mathbb{P}^n$, one has a surjection $T_{B,b} \to H^0(X, N_{X/\mathbb{P}^n})$.

Fix a point $b \in B$ and let $C \subset Y$ be the fiber over b. Let $f = F_{|C}$ and let X be the complete intersection scheme corresponding to b. Then the exact sequence (4) specializes to

$$0 \to N_{f,X} \to N_{f,\mathbb{P}^n} \to f^* N_{X/\mathbb{P}^n} = \oplus_{i=1}^m f^* \mathcal{O}_X(d_i) \to 0 \tag{6}$$

Proposition 3.2. *If b is a general point of B, then the image of the pull-back map*

$$H^0(\mathbb{P}^n, \oplus_{i=1}^m \mathcal{O}_{\mathbb{P}^n}(d_i)) \to H^0(C, \oplus_{i=1}^m f^* \mathcal{O}_X(d_i))$$

is contained in the image of the map

$$H^0(C, N_{f,\mathbb{P}^n}) \to H^0(C, \oplus_{i=1}^m f^* \mathcal{O}_X(d_i))$$

obtained from the above short exact sequence.

Proof. Since b is general, it is a smooth point of B, so we have $\alpha_b : T_{B,b} \to H^0(C, N_{f,\mathbb{P}^n})$ as before. We have a commutative diagram

$$\begin{array}{ccc} T_{B,b} & \xrightarrow{\alpha_b} & H^0(C, N_{f,\mathbb{P}^n}) \\ \downarrow{\scriptstyle d\psi} & & \downarrow \\ T_{\mathcal{H},[X]} = H^0(X, \oplus_{i=1}^m \mathcal{O}_X(d_i)) & \longrightarrow & H^0(C, \oplus_{i=1}^m f^* O_X(d_i)) \end{array}$$

and $d\psi$ is surjective since ψ is dominant and b is a general point of B, so the result follows. □

A curve $C \subset \mathbb{P}^n$ is called *d-normal* if the restriction map

$$H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d')) \to H^0(C, \mathcal{O}_C(d'))$$

is surjective for every $d' \geq d$.

Corollary 3.3. *Suppose that $X \subset \mathbb{P}^n$ is general complete intersection of multidegree $(d_1, \ldots, d_m)$, $d_1 \leq \cdots \leq d_m$. If C is any d_1-normal smooth rational curve of degree e in X, then $H^1(C, N_{C/X}) = 0$. In particular, $R_e(X)$ is smooth at $[C]$.*

Proof. If $f : \mathbb{P}^1 \to C \subset X$ is an isomorphism, then $H^1(\mathbb{P}^1, N_{f,\mathbb{P}^n}) = 0$. So applying the long exact sequence of cohomology to the sequence of normal sheaves

$$0 \to N_{f,X} \to N_{f,\mathbb{P}^n} \to \oplus_{i=1}^m f^*\mathcal{O}_X(d_i) \to 0,$$

the statement follows from Proposition 3.2. □

Now, we make a further assumption that there exists a morphism $\phi : B \to \mathcal{U}$ so that the composition with the map $\pi_1 : \mathcal{U} \to \mathcal{H}$ is ψ. This is equivalent to saying that we are given a section σ for the map $\mathcal{U}_B \to B$.

Next, for a point $([X], p)$ in the image of $\phi : B \to \mathcal{U}$, we define a subspace $W_{X,p}$ of $H^0(X, N_{X/\mathbb{P}^n})$ as follows. For $p \in \mathbb{P}^n$, let $B_p = (\pi_2 \circ \phi)^{-1}(p)$ with the reduced induced structure, and let $\mathcal{H}_p \subset \mathcal{H}$ be the closure of $\psi(B_p)$. If $([X], p)$ is in the image of ϕ , then $[X] \in \mathcal{H}_p$. We can identify $T_{\mathcal{H},[X]}$ with $H^0(X, N_{X/\mathbb{P}^n})$ and we define $W_{X,p} \subset H^0(X, N_{X/\mathbb{P}^n})$ to be $T_{\mathcal{H}_p,[X]}$ under this identification.

Assume now that b is a general point of B and $\phi(b) = ([X], p)$. Since every complete intersection which is parametrized by $\mathcal{H}_p$ contains p, we have

$$W_{X,p} \subset H^0(X, N_{X/\mathbb{P}^n}(-p)).$$

Since ψ is dominant by our assumption, the codimension of $\mathcal{H}_p$ in $\mathcal{H}$ is at most n. Thus $W_{X,p}$ is of codimension $\leq n$ in $H^0(X, N_{X/\mathbb{P}^n})$, and it is of codimension $\leq n - m$ in $H^0(X, N_{X/\mathbb{P}^n}(-p))$.

We make a further assumption on $F : Y \to \mathcal{U}_B$ that $\sigma(B) \subset F(Y)$. Let C be the fiber of π over b and $f : C \to X$ the restriction of F to C, so the image of f is a curve which passes through p. Let $D = (f^{-1}(p))_{\mathrm{red}} \subset C$. Then the image of the pullback map

$$W_{X,p} \to H^0(C, f^*N_{X/\mathbb{P}^n})$$

is contained in $H^0(C, f^*N_{X/\mathbb{P}^n}(-D))$. Consider the short exact sequence of normal sheaves (6) twisted with $\mathcal{O}_C(-D)$:

$$0 \to N_{f,X}(-D) \to N_{f,P^n}(-D) \to f^*N_{X/\mathbb{P}^n}(-D) \to 0.$$

We claim that the image of the pull-back map

$$W_{X,p} \to H^0(C, f^*N_{X/\mathbb{P}^n}(-D))$$

is contained in the image of the map

$$H^0(C, N_{f,\mathbb{P}^n}(-D)) \to H^0(C, f^*N_{X/\mathbb{P}^n}(-D))$$

obtained from the above short exact sequence. Since b is a general point, and since the morphism $B_p \to \mathcal{H}_p$ is dominant, the induced map on the Zariski tangent spaces $T_{B_p,b} \to T_{\mathcal{H}_p,[X]} = W_{X,p}$ is surjective. Set

$$Y_p = \pi^{-1}(B_p),$$

and let $\alpha_b : T_{B_p,b} \to H^0(C, N_{f,\mathbb{P}^n})$ be the natural map corresponding to the morphism $Y_p \to B_p \times \mathbb{P}^n$. Set

$$Z_p = (F^{-1}(B_p \times \{p\}))_{\mathrm{red}} \subset Y_p.$$

Then the fiber of Z_p over b is D. Let $g : D \to X$ be the restriction of f to D, and let $\beta_b : T_{B_p,b} \to H^0(D, N_{g,\mathbb{P}^n}) = H^0(D, f^*T_X|_D)$ be the natural map corresponding to the morphism $Z_p \to B_p \times \mathbb{P}^n$. Then β_b is clearly the zero map. The claim now follows from the following commutative diagram

$$\begin{array}{ccccc}
T_{B_p,b} & & \twoheadrightarrow & & W_{X,p} \subset H^0(X, N_{X/\mathbb{P}^n}) \\
\downarrow 0 & \searrow^{\alpha} & & & \downarrow \\
 & & H^0(C, N_{f,\mathbb{P}^n}) & \longrightarrow & H^0(C, f^*N_{X/\mathbb{P}^n}) \\
\downarrow & & \downarrow & & \\
f^*T_{\mathbb{P}^n}|_D & \longrightarrow & N_{f,\mathbb{P}^n}|_D. & &
\end{array}$$

We summarize the above discussion in the following proposition.

Proposition 3.4. *For a general point $([X], p)$ in the image of ϕ, there is a subspace*

$$W_{X,p} \subset H^0(X, N_{X/\mathbb{P}^n}(-p))$$

of codimension at most $n - m$ and an open subset $B_0 \subset \phi^{-1}([X], p)$ with the following property: for every $b \in B_0$, if we denote the fiber of π over b by C, the restriction of F to C by $f : C \to X$, and the inverse image of $f^{-1}(p)$ with the reduced induced structure by D, then for every $w \in W_{X,p}$, f^*w can be lifted to a section of $H^0(C, N_{f,\mathbb{P}^n}(-D))$.

4. Conics in projective space

Let $\mathrm{Hilb}_{2t+1}(\mathbb{P}^n)$ denote the Hilbert scheme of subschemes of $\mathbb{P}^n$ with Hilbert polynomial $2t+1$. In this section we prove the following:

Proposition 4.1. *Suppose that p is a point of $\mathbb{P}^n$ and $R \subset \mathrm{Hilb}_{2t+1}(\mathbb{P}^n)$ is an irreducible projective subscheme of dimension r such that every curve parametrized by R is a smooth conic through p. Let $W \subset H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d)(-p))$ be a subspace of codimension c. If*

$$c < \min\left(\binom{r+1}{2}, (d-1)\left\lfloor\frac{r+1}{2}\right\rfloor + 1\right),$$

then for a general $[C] \in R$, and for every $2 \leq k \leq 2d$, the image of the restriction map

$$W \to H^0(C, \mathcal{O}_C(d))$$

contains a section of $\mathcal{O}_C(d)$ which has a zero of order k at p.

We start with a lemma.

Lemma 4.2. *If l is a line in $\mathbb{P}^n$ through p, then there is no complete one-dimensional family of smooth conics through p all tangent to l.*

Proof. Assume to the contrary that there is such a family B. By passing to a desingularization we can assume that B is smooth. Let $Y \subset B \times \mathbb{P}^n$ be the universal family over B and $g : Y \to \mathbb{P}^n$ the projection map.

The point p gives a section σ_p of the family $Y \to B$. Fix a point $q \neq p$ on l. Then for every conic C in the family, there is a unique line $l' \neq l$ such that l' passes through q and is tangent to C. Let σ_q be the section of $Y \to B$ such that $\sigma_q([C])$ is the point of intersection of C and l'. Since if q_1 and q_2 are two distinct points of l, σ_{q_1} and σ_{q_2} are disjoint, $Y \simeq B \times \mathbb{P}^1$. Since the section σ_p is contracted by g, by the rigidity lemma [13], g factors through the projection map $B \times \mathbb{P}^1 \to \mathbb{P}^1$. Thus the image of g should be one-dimensional which is a contradiction.

Remark 4.3. The following argument shows that the set-theoretic map σ_q defined in the proof of the above lemma is a section. If $q \neq p$ on l, $B \times \{q\} \cap Y = \emptyset$. Thus the projection from q defines a morphism $g : Y \to B \times \mathbb{P}^{n-1}$. For any point $b \in B$, this map is just a map from a conic to a line and thus g is a two-to-one map to its image. Let $R \subset Y$ be the ramification locus. Then, the map $R \to B$ is a double cover. But, $B \times \{p\} \subset R$ and the residual part is a section of $Y \to B$ and just σ_q.

□

Corollary 4.4. *Suppose that $R \subset \mathrm{Hilb}_{2t+1}(\mathbb{P}^n)$ is a closed subscheme such that every curve parametrized by R is a smooth conic through p. Then,*

(1) $\dim R \leq n-1$.
(2) *If the 2-planes spanned by the curves parametrized by R all pass through a point $q \neq p$, then* $\dim R \leq 1$.

Proof. (a) If $\dim R \geq n$, since the lines through p form a $\mathbb{P}^{n-1}$, if we associate for any $r \in R$ the tangent line through p to the conic corresponding to r, then by dimension considerations, there is a positive dimensional closed subfamily of R with the same tangent line, which is impossible by the previous lemma.

(b) Set $r = \dim R$. Let l be the line through p and q, and let R' be the closed subscheme of R parametrizing conics tangent to l. By Lemma 4.2 if R' is not empty, it is zero dimensional. Since every conic parametrized by the complement of R' intersects l in a point other than p, there should be a point $q' \in l$, and a closed subvariety of dimension $r-1$ in R parametrizing conics passing through p and q'. By [9, Lemma 5.1], in any projective 1-dimensional family of conics passing through p and q, there should be reducible conics, so $r \leq 1$.

□

With the hypothesis in the Proposition 4.1, we fix the following notation. Let Γ be a hyperplane in $\mathbb{P}^n$ which does not pass through p. Choose a homogeneous system of coordinates for $\mathbb{P}^n$ so that $p = (1 : 0 : \cdots : 0)$ and $\Gamma = \{x_0 = 0\}$. For $[C] \in R$, denote by q_C the intersection of Γ with the line tangent to C at p, and denote by l_C the line of intersection of Γ with the 2-plane spanned by C. Notice that $q_C \in l_C$. Denote by Σ the cones of lines tangent at p to the conics parametrized by R and denote by Y the intersection of Σ and Γ. By Lemma 4.2, $\dim Y = r$.

For $1 \leq i \leq d$, multiplication by x_0^{d-i} identifies $H^0(\Gamma, \mathcal{O}_\Gamma(i))$ with a subspace of $H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d)(-p))$. Let $W_i \subset H^0(\Gamma, \mathcal{O}_\Gamma(i))$ be the intersection of W with $H^0(\Gamma, \mathcal{O}_\Gamma(i))$ under this identification. Since the codimension of W in $H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d)(-p))$ is c, the codimension of W_i in $H^0(\Gamma, \mathcal{O}_\Gamma(i))$ is $\leq c$.

Lemma 4.5. *If $f \in H^0(\Gamma, \mathcal{O}_\Gamma(i))$ is such that $f|_{l_C}$ has a zero of order j at q_C, then the restriction of $x_0^{d-i} f$, considered as a section of $H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d))$, to C has a zero of order $i+j$ at p.*

Proof. Let P be the two plane spanned by C. Then the divisor of x_0 in this plane is just l_C. The divisor of $f|_{l_C}$ is $jq_C + E$ where E is an effective divisor of degree $i-j$ whose support does not contain q_C. Then the divisor in P of $x_0^{d-i} f$ is $(d-i)l_C + jT + E'$ where T is the tangent line of C at p, E' is a union of $(i-j)$ lines passing through p, none of them equal to T. Thus, the order of its restriction to C at p is just $2j + (i-j) = i+j$. □

Lemma 4.6. *Suppose that Z is a k-dimensional irreducible subvariety of $\mathbb{P}^n$, and let I_Z be the ideal sheaf of Z in $\mathbb{P}^n$.*

(i) *The codimension of $H^0(\mathbb{P}^n, I_Z(t))$ in $H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(t))$ is at least $\binom{t+k}{k}$.*
(ii) *If $k \geq 1$, and if Z spans a linear subvariety of dimension s in $\mathbb{P}^n$, then the codimension of $H^0(\mathbb{P}^n, I_Z(t))$ in $H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(t))$ is at least $st+1$.*

Proof. (i) If $\pi : Z \to \mathbb{P}^k$ is a general linear projection, then π is a finite map, so the induced map $\pi^* : H^0(\mathbb{P}^k, \mathcal{O}_{\mathbb{P}^k}(t)) \to H^0(Z, \mathcal{O}_Z(t))$ is injective. We have a commutative diagram

$$\begin{array}{ccc} H^0(\mathbb{P}^k, \mathcal{O}_{\mathbb{P}^k}(t)) & \hookrightarrow & H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(t)) \\ \Big\downarrow{\scriptstyle \pi^*} & \swarrow & \\ H^0(Z, \mathcal{O}_Z(t)) & & \end{array}$$

so the codimension of the kernel of the restriction map $H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(t)) \to H^0(Z, \mathcal{O}_Z(t))$ is $\geq \dim H^0(\mathbb{P}^k, \mathcal{O}_{\mathbb{P}^k}(t)) = \binom{t+k}{k}$.

(ii) Let $C \subset Z$ be an irreducible curve whose span is equal to the span of Z, which can always be achieved by taking an irreducible curve passing through finitely many linearly independent points on Z. Since $I_Z \subset I_C$, it is clear that we need to prove the lemma only for C and thus we may assume $k = 1$.

We can assume that the codimension 2 subvariety of $\mathbb{P}^n$ defined by $\{x_0 = x_1 = 0\}$ does not intersect Z. Then the surjective map

$$\mathcal{O}_Z^{\oplus t} \xrightarrow{(x_0^{t-1}, x_0^{t-2}x_1, \ldots, x_1^{t-1})} \mathcal{O}_Z(t-1)$$

gives a short exact sequence

$$0 \to \mathcal{O}_Z^{\oplus t-1} \to \mathcal{O}_Z(1)^{\oplus t} \to \mathcal{O}_Z(t) \to 0.$$

Since Z spans a linear subvariety of dimension s, the image of the restriction map $H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1)^{\oplus t}) \to H^0(Z, \mathcal{O}_Z(1)^{\oplus t})$ has dimension at least $(s+1)t$, so the image of the map $H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1)^{\oplus t}) \to H^0(Z, \mathcal{O}_Z(t))$ has dimension at least $(s+1)t - (t-1) = st + 1$. Therefore the image of the restriction map $H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(t)) \to H^0(Z, \mathcal{O}_Z(t))$ is at least $(st+1)$-dimensional. □

Proof of Proposition 4.1. Since $\dim Y = r$, by part (i) of the above lemma, the codimension of the space of sections in $H^0(\Gamma, \mathcal{O}_\Gamma(i))$ which vanish on Y is at least $\binom{r+2}{2}$ for $2 \leq i \leq d$. Since the codimension of W_i in $H^0(\Gamma, \mathcal{O}_\Gamma(i))$ is smaller than $\binom{r+2}{2}$, for every $2 \leq i \leq d$, there is $g_i \in W_i$ which does not vanish on Y, so for a general C parametrized by R, g_i does not vanish at q_C. Hence $x_0^{d-i} g_i \in W$, and its restriction to C has a zero of order i at p by lemma 4.5.

To complete the proof of Proposition 4.1, we show that for every $1 \leq j \leq d$, and for a general conic C parametrized by R, there is $f_j \in W_d$ such that $f_j|_{l_C}$ has a zero of order j at q_C, so $f_j|_C$ has a zero of order $d + j$ at p by Lemma 4.5. This is proved in the next proposition.

□

Proposition 4.7. *Let p be a point of $\mathbb{P}^n$ and Γ a hyperplane in $\mathbb{P}^n$ which does not pass through p. Suppose that $R \subset \mathrm{Hilb}_{2t+1}(\mathbb{P}^n)$ is an irreducible closed subscheme of dimension r such that every curve parametrized by R is* *smooth and passes through p. Let W_d be a subspace of $H^0(\Gamma, \mathcal{O}_\Gamma(d))$, $d \geq 2$, of codimension c'. If*

$$c' < \min\left(\binom{r+1}{2}, (d-1)\left\lfloor \frac{r+1}{2} \right\rfloor + 1\right),$$

then for a general $[C]$ parametrized by R, and for every $0 \leq k \leq d$, there is $f_k \in W_d$ such that $f_k|_{l_C}$ has a zero of order equal to k at q_C.

Proof. Denote by $Y \subset \Gamma$ the subvariety swept out by the points q_C for $[C] \in R$ as before. By Lemma 4.2, $\dim Y = r$.

Assume first that $0 \leq k \leq d-2$. Let H be a general hyperplane in Γ, and let $Y' = Y \cap H$. Let R' be the locus in R parametrizing conics C for which $q_C \in Y'$. Then Lemma 4.2 shows that $\dim R' = \dim Y' = r - 1$. Choose a system of homogeneous coordinates for $\mathbb{P}^n$ as before so that $p = (1 : 0 : \cdots : 0)$, Γ is given by $x_0 = 0$, and H is given by $x_0 = x_1 = 0$. Consider the vector space U of all polynomials of the form $x_1^k f$ where f is a homogeneous polynomial of degree $d - k \geq 2$ in $x_2, \ldots, x_n$, and let U_0 be the subspace of U consisting of those $x_1^k f$ where f vanishes on Y'. Then the codimension of U_0 in U is $\geq \binom{(r-1)+2}{2}$ by Lemma 4.6 (i). Therefore if $\binom{r+1}{2}$ is greater than the codimension of W_d in $H^0(\Gamma, \mathcal{O}_\Gamma(d))$, then there is an element of the form $x_1^k f$ in W_d such that f does not vanish on Y'. So for a general C parametrized by R', f does not vanish on q_C. Also, x_1 does not vanish on l_C since by the next lemma l_C does not lie on H. Therefore, the restriction of $x_1^k f$ to l_C has a zero of order k at q_C for a general C parametrized by R'.

Lemma 4.8. *Suppose that R is an r-dimensional family of smooth conics through p. Then for a general codimension m linear subvariety Λ of Γ, the locus in R parametrizing conics C for which l_C lies on Λ has dimension $\leq r - 2m$.*

Proof. Denote by $G := \mathrm{Gr}(n-1-m, n-1)$ the Grassmannian of linear subvarieties of codimension m in Γ. Consider the incidence correspondence

$$\begin{array}{ccc} & I = \{([C], \Lambda), l_C \subset \Lambda\} \subset R \times G & \\ \swarrow^{\pi_1} & & \searrow^{\pi_2} \\ R & & G \end{array}$$

where π_1 and π_2 are the projection maps. The fibers of π_1 are of dimension $m(n-m-2)$ and $\dim G = m(n-m)$. So for a general $[\Lambda] \in G$, $\dim \pi_2^{-1}([\Lambda]) \leq \dim I - \dim G = \dim R + m(n-m-2) - m(n-m) = r - 2m$. □

Next we prove the statement for $k = d-1$ or d. We need the following lemma.

Lemma 4.9. *Suppose that $R \subset \mathrm{Hilb}_{2t+1}(\mathbb{P}^n)$ is a closed subscheme such that every curve parametrized by R is a smooth conic through p. Let $\dim R = r$,* *and let* $s = \lfloor \frac{r+1}{2} \rfloor$*. If* $C_1, \dots, C_s$ *are general conics parametrized by* R*, then* $l_{C_1}, \dots, l_{C_s}$ *are linearly independent, i.e. they span a linear subvariety of dimension* $2s-1$*.*

Proof. Assume that s' is the largest number for which there are conics $C_1, \dots, C_{s'}$ parametrized by R such that $l_{C_1}, \dots, l_{C_{s'}}$ are linearly independent. Let Σ be the linear span of $l_{C_1}, \dots, l_{C_{s'}}$. Then for any curve C parametrized by R, l_C should intersect Σ. By Corollary 4.4, for every point q in Σ, there is at most a 1-dimensional subscheme of R parametrizing conics C such that l_C passes through q. Therefore, $\dim R \leq \dim \Sigma + 1 = 2s'$, and so $s = \lfloor \frac{r+1}{2} \rfloor \leq s'$. Hence there are s conics parametrized by R whose corresponding l_C's are linearly independent, and so the same is true for s general conics parametrized by R. □

If $r = 1$, then $c' = 0$ by our assumption and there is nothing to prove. So assume $r \geq 2$, and put $s = \lfloor \frac{r+1}{2} \rfloor$. Let $[C_1], \dots, [C_s]$ be general points of R. By the above lemma, $l_{C_1}, \dots, l_{C_s}$ are linearly independent. Choose points $q'_{C_i} \neq q_{C_i}$ in l_{C_i}. Denote by H' the $(s-1)$-dimensional linear subvariety spanned by the points q'_{C_i}, $1 \leq i \leq s$, and let H be a general linear subvariety of Γ of codimension s containing $q_{C_1}, \dots, q_{C_s}$ (note that $n-1-s \geq s-1$ by Corollary 4.4, so such H exists). Then H and H' are disjoint. Since $C_1, \dots, C_s$ and H are general and $r - s \geq 1$, by Bertini theorem the locus R' in R parametrizing curves C such that $q_C \in H$ is irreducible.

For $[C] \in R'$, let Σ_C be the linear subvariety spanned by H and l_C. Since for each $1 \leq i \leq s$, l_{C_i} does not lie on H, for a general $[C] \in R'$, l_C does not lie on H, and so Σ_C is of codimension $s-1$ in Γ and intersects H' at a point q'_C. Since R' is irreducible, the points q'_C span an irreducible quasi-projective subvariety Z of H'. Since Z is irreducible and contains $q'_{C_1}, \dots, q'_{C_s}$, it is non-degenerate in H' and has dimension at least 1.

Set $U := H^0(H', \mathcal{O}_{H'}(d))$ which can be considered as a subspace of $H^0(\Gamma, \mathcal{O}_\Gamma(d))$. By Lemma 4.6 (ii), forms of degree d on H' which vanish on Z form a subspace of codimension at least $d(s-1)+1$ in U. Therefore if $d(s-1)+1 > c'$, there is a form $f \in W_d \cap U$ which does not vanish at the generic point of Z. If $[C]$ is such that f does not vanish at q'_C, then it does not vanish at any point of Σ_C which is not on H, so f cannot be identically zero on l_C. Hence $f \in W_d$ and $f|_{l_C}$ has a zero of order d at q_C.

Repeating the same argument with d replaced by $d-1$ and choosing a form h of degree 1 on Γ which does not vanish on H, we see that if $(d-1)(s-1)+1 > c'$, then there is a form g of degree $d-1$ in $H^0(H', \mathcal{O}_{H'}(d-1))$ such that $gh \in W_d$ and $gh|_{l_C}$ has a zero of order $d-1$ at q_C. This completes the proof. □

5. Proof of Theorem 1.3

Let X be a general hypersurface of degree d in $\mathbb{P}^n$. In this section we show that the evaluation map

$$ev : \overline{\mathcal{M}}_{0,1}(X, 2) \to X$$

is flat of constant fiber dimension $2(n-d)$ if d is in the range of Theorem 1.3. Recall that $\overline{\mathcal{M}}_{0,1}(\mathbb{P}^n, 2)$ is a smooth stack of dimension $3(n+1) - 3 = 3n$ and that $\overline{\mathcal{M}}_{0,1}(X, 2)$ is the zero locus of a section of a locally free sheaf of rank $2d+1$ over $\overline{\mathcal{M}}_{0,1}(\mathbb{P}^n, 2)$. If the fibers of ev are of expected dimension $2(n-d)$, then $\overline{\mathcal{M}}_{0,1}(X, 2)$ has dimension

$$2(n-d) + n - 1 = 3n - (2d+1),$$

so it is a local complete intersection and in particular a Cohen-Macaulay substack of $\overline{\mathcal{M}}_{0,1}(\mathbb{P}^n, 2)$. Since a map from a Cohen-Macaulay scheme to a smooth scheme is flat if and only if it has constant fiber dimension ([12, Theorem 23.1]), to prove the theorem, it is enough to show that ev has constant fiber dimension $2(n-d)$. Note that $\dim \overline{\mathcal{M}}_{0,1}(X, 2)$ is at least $3n - (2d+1)$, and ev is surjective, so every irreducible component of any fiber of ev has dimension at least $2(n-d)$.

Let p be a point in X and $\mathcal{M}$ an irreducible component of $ev^{-1}(p)$. Since $\overline{\mathcal{M}}_{0,1}(\mathbb{P}^n, 2)$ is smooth of dimension $3n$ (see Section 2), and since the fibers of the evaluation map

$$\widetilde{ev} : \overline{\mathcal{M}}_{0,1}(\mathbb{P}^n, 2) \to \mathbb{P}^n$$

are all isomorphic, $\widetilde{ev}^{-1}(p)$ is smooth of dimension $2n$. Since the space of lines through every point of $\mathbb{P}^n$ has dimension $n-1$, the space of reducible conics through every point of $\mathbb{P}^n$ has dimension $2n-1$. Thus the stable maps with reducible domains which are parametrized by $\widetilde{ev}^{-1}(p)$ form a divisor in $\widetilde{ev}^{-1}(p)$. Hence there are 3 possibilities for the locus $\mathcal{M}'$ in $\mathcal{M}$ parametrizing stable maps with reducible domains: 1) $\dim \mathcal{M}' = \dim \mathcal{M} - 1$, 2) $\dim \mathcal{M}' = \dim \mathcal{M}$, 3) $\mathcal{M}'$ is empty. Since the space of lines through every point p in X has dimension $n-d-1$ ([9, Theorem 2.1]), the space of reducible conics through p has dimension $2n-2d-1$, and hence in the first case, $\dim \mathcal{M} = 2(n-d)$ which is the expected dimension. The second case cannot happen since in this case $\dim \mathcal{M}$ would be equal to $2(n-d)-1$ which is smaller than expected.

So assume that $\mathcal{M}$ parametrizes only stable maps with irreducible domains, and assume to the contrary that $\dim \mathcal{M} \geq 2(n-d)+1$. If $1 \leq d < \frac{n+1}{2}$, then by [9, Theorem 1.2 and Corollary 5.5], the space of conics through every point of X has dimension $2(n-d)$, hence $\frac{n+1}{2} \leq d$.

Any map parametrized by $\mathcal{M}$ is either an isomorphism onto a smooth conic through p or a double cover of a line through p. For any line $l \subset X$ through p, there is a 2-parameter family of degree 2 covers of l (determined by the 2 branch points), and by [9, Theorem 2.1], the family of lines through p in X has dimension $n-d-1$. So the substack of $\mathcal{M}$ parametrizing double covers of lines has dimension at most $n-d+1$, and therefore, there is an irreducible closed subscheme R of $\mathcal{M}$ of dimension $n-d-1$ which parametrizes only smooth embedded conics through p in X.

Let $\mathcal{H}$ be the projective space of hypersurfaces of degree d in $\mathbb{P}^n$, and let B be the closed subscheme of $\mathcal{H}\times\mathbb{P}^n\times\mathrm{Hilb}(\mathbb{P}^n)$ parametrizing all the points $([X],p,[C])$ such that C is a smooth conic in X through p which belongs to a larger than expected component R of $ev^{-1}(p)$. Then since by our assumption the projection map $B\to\mathcal{H}$ is dominant and X is general, by Proposition 3.4, there is a subspace $W_{X,p}\subset H^0(X,\mathcal{O}_X(d)(-p))$ of codimension at most $n-1$ such that for a general curve $[C]$ parametrized by R and for every $w\in W_{X,p}$, $w|_C$ can be lifted to a section of $N_{C/\mathbb{P}^n}(-p)$ under the map

$$\rho: H^0(C,N_{C/\mathbb{P}^n}(-p))\to H^0(C,\mathcal{O}_C(d)(-p))$$

obtained from the short exact sequence

$$0\to N_{C/X}(-p)\to N_{C/\mathbb{P}^n}(-p)\to\mathcal{O}_C(d)(-p)\to 0. \tag{7}$$

We show that this implies ρ is surjective. Since $\frac{n+1}{2}\le d$, we have

$$\binom{n-d}{2}\le(d-1)\left\lfloor\frac{n-d}{2}\right\rfloor+1.$$

Hence by Proposition 4.1, if $n\le\binom{n-d}{2}$, then for every $2\le i\le 2d$, there exists $s_i\in H^0(C,\mathcal{O}_C(d))$ which has a zero of order i at p and can be lifted to a section of $N_{C/\mathbb{P}^n}(-p)$. So to show ρ is surjective, it is enough to show that there is also a section of $H^0(C,\mathcal{O}_C(d))$ which has a zero of order 1 at p and can be lifted to a section of $N_{C/\mathbb{P}^n}(-p)$.

Suppose that X is given by $f=0$ in $\mathbb{P}^n$, and consider the commutative diagram

$$\begin{array}{ccc}\mathcal{O}_C(1)^{n+1} & \xrightarrow{(\frac{\partial f}{\partial x_0},\dots,\frac{\partial f}{\partial x_n})} & \mathcal{O}_C(d)\\ \downarrow & & \uparrow\\ T_{\mathbb{P}^n}|_C & \longrightarrow & N_{C/\mathbb{P}^n}\end{array}$$

Since X is smooth at p, there is an i, $0\le i\le n$, such that $\frac{\partial f}{\partial x_i}$ does not vanish at p. Hence there is a section of $\mathcal{O}_C(d)$ which has a simple zero at p and is in the image of the map $H^0(C,\mathcal{O}_C(1)^{n+1})\to H^0(C,\mathcal{O}_C(d))$. Such a section can be lifted to a section of $N_{C/\mathbb{P}^n}(-p)$. Hence ρ is surjective.

Applying the long exact sequence of cohomology to sequence (7), we get $H^1(C,N_{C/X}(-p))=0$, thus

$$h^0(C,N_{C/X}(-p))=\chi(N_{C/X}(-p))=\chi(T_X|_C(-p))-\chi(T_C(-p))=2(n-d).$$

On the other hand, the Zariski tangent space to $ev^{-1}(p)$ at $[C]$ is isomorphic to $H^0(C,N_{C/X}(-p))$, thus $\dim\mathcal{M}$ should be at most $2(n-d)$, which is a contradiction.

6. Non-free lines in complete intersections

By a *non-free line* in a subvariety X of $\mathbb{P}^n$ we mean a line l which is contained in the smooth locus of X and its normal bundle in X is not globally generated. For a line l contained in the smooth locus of X, this is equivalent to saying that for any point (or some point) $p \in l$, the natural map $H^0(l, N_{l/X}) \to N_{l/X}|p$ is not surjective.

Lemma 6.1. *If X is a complete intersection of multidegree $(d_1, \ldots, d_m)$ in $\mathbb{P}^n$ such that $d_1 + \cdots + d_m \leq n-1$ then X is covered by lines and the set of non-free lines contained in the smooth locus of X do not cover a dense subset of X.*

Proof. The first part is proved in [2, Proposition 2.13]. We just repeat the proof here. Let $p \in X$ and without loss of generality we may assume that $p = (1 : 0 : \cdots : 0)$ and X is defined by $0 = f_i = a_{i1}x_0^{d_i-1} + \cdots + a_{id_i}$ for $1 \leq i \leq m$, a_{ij} homogeneous polynomials in $x_1, \ldots, x_n$. Since there are $\sum d_i$ of these a_{ij}s, they have a common non-trivial zero in $\mathbb{P}^{n-1} = \{x_0 = 0\}$ since $n-1 \geq \sum d_i$ and then the line joining p and this point in $\mathbb{P}^{n-1}$ is contained in X. Since p was arbitrary, we see that X is covered by lines.

Let $\mathcal{J}$ be the locally closed subscheme (with the reduced induced structure) of $R_1(X)$ (the set of lines in X) parametrizing non-free lines contained in the smooth locus of X. Let $\mathcal{I} \subset \mathcal{J} \times X$ denote the incidence correspondence. We wish to show that the projection $\mathcal{I} \to X$ is not dominant. Let $([l], p)$ be a general point of $\mathcal{I}$. Then we have $T_{R_1(X),[l]} \cong H^0(l, N_{l/X})$, and since l is contained in the smooth locus of X, $T_{X,p} \to N_{l/X}|_p$ is surjective.

We have a commutative diagram

$$\begin{array}{ccc} T_{\mathcal{I},([l],p)} & \longrightarrow & T_{X,p} \\ \downarrow & & \downarrow \\ T_{\mathcal{J},[l]} \longrightarrow & T_{R_1(X),[l]} = H^0(l, N_{l/X}) & \longrightarrow N_{l/X}|_p \end{array}$$

Since l is not free, the map $H^0(l, N_{l/X}) \to N_{l/X}|_p$ is not surjective, so $T_{\mathcal{I},([l],p)} \to T_{X,p}$ is not surjective and thus $\mathcal{I} \to X$ is not dominant. $\square$

Lemma 6.2. *If $X \subset \mathbb{P}^n$ is a smooth complete intersection of multidegree $(d_1, \ldots, d_m)$ such that $d_1 + \cdots + d_m \leq n-1$, then $R_e(X)$ (and hence $\overline{\mathcal{M}}_{0,0,}(X, e)$) has at least one irreducible component of the expected dimension for every $e \geq 1$.*

Proof. We first show that smooth rational curves of degree e in X sweep out a dense subset of X. From the previous lemma, X is covered by lines and any line passing through a general point is free. Hence there is a chain of e free lines in X for every $e \geq 1$. By [11, Theorem 7.6], this chain of lines can be deformed to a smooth free rational curve C of degree e in X. Since C is free, its flat deformations in X sweep out X.

If R is a component of $R_e(X)$ such that the curves parametrized by its points sweep out a dense subset of X, then the argument in the previous lemma 6.1 shows that the normal bundle of a general curve C parametrized by R should be globally generated. We have

$$\begin{aligned}\dim T_{R_e(X),[C]} &= h^0(C, N_{C/X}) \\ &= \chi(N_{C/X}) + h^1(C, N_{C/X}) \\ &= e(n+1-d) + (n-m-3).\end{aligned}$$

On the other hand the dimension of R is at least the expected dimension $e(n+1-d)+n-m-3$, so it should be of the expected dimension and smooth at $[C]$. □

Proposition 6.3. *If X is a general hypersurface of degree $3 \le d \le n-1$ in $\mathbb{P}^n$, then the non-free lines in X sweep out a divisor in X.*

Remark 6.4. If $d = 1$, clearly this set is empty. Same is true if $d = 2$. We have already seen that the codimension of this set is at least one in lemma 6.1.

Proof. Let $\mathcal{H}$ be the projective space of hypersurfaces of degree d in $\mathbb{P}^n$. Consider the subvariety $I \subset \mathbb{P}^n \times \mathcal{H}$ consisting of pairs $(p, [X])$ such that there is either a non-free line in X through p or a line in X through p which intersects the singular locus of X. Denote by π_1 and π_2 the projection maps from I to $\mathbb{P}^n$ and $\mathcal{H}$. We show the fibers of π_1 are of dimension $= \dim \mathcal{H} + n - 2$.

Since all the fibers of π_1 are isomorphic, we can assume $p = (1 : 0, \cdots : 0)$. A hypersurface X which contains p is given by an equation of the form $x_0^{d-1} f_1 + \cdots + f_d = 0$ where f_i is homogenous of degree i in $x_1, \ldots, x_n$ for $1 \le i \le d$. The space of lines through p in X, which we denote by $F_p(X)$, is isomorphic to the scheme $V(f_1, \ldots, f_d)$ in $\mathbb{P}^{n-1}$, so $\dim F_p(X) \ge n-d-1$. If $F_p(X)$ is singular at $[l]$, then we see that $([X], p) \in \pi_1^{-1}(p)$ as follows. Since

$$T_{F_p(X),[l]} \cong H^0(l, N_{l/X}(-p)),$$

if $F_p(X)$ is singular at $[l]$, then

$$h^0(l, N_{l/X}(-p)) > \dim F_p(X) \ge n - d - 1.$$

So either l is not contained in the smooth locus of X, or it is contained in the smooth locus of X and

$$\begin{aligned}h^1(l, N_{l/X}(-p)) &= h^0(l, N_{l/X}(-p)) - \chi(N_{l/X}(-p)) \\ &= h^0(l, N_{l/X}(-p)) - (n-d-1) \\ &> 0,\end{aligned}$$

so l is non-free.

If $2 \le d \le n-1$, and if f_i is a general homogeneous polynomial of degree i in $x_1, \ldots, x_n$ for $2 \le i \le d$, then the intersection $Y := V(f_2, \ldots, f_d)$ is a smooth complete intersection subvariety of $\mathbb{P}^{n-1}$ of dimension $n-d \geq 1$. By [7, Proposition 3.1], the dual variety of Y in $\mathbb{P}^{n-1\vee}$ is a hypersurface, so there is a codimension 1 subvariety of $H^0(\mathbb{P}^{n-1}, \mathcal{O}_{\mathbb{P}^{n-1}}(1))$ consisting of forms f_1 such that $Y \cap \{f_1 = 0\}$ is singular. This shows that the space of tuples $(f_1, \ldots, f_d)$ for which the scheme $V(f_1, \ldots, f_d)$ is singular is of codimension 1 in the space of all tuples $(f_1, \ldots, f_d)$. So the fibers of π_1 over p form a subvariety of codimension at most 1 in the space of all hypersurfaces which contain p, and $\dim I \geq \dim \mathcal{H} + n - 2$.

Consider now the map $\pi_2 : I \to \mathcal{H}$. Since the fibers of π_2 have dimension at most $n-1$, either π_2 is dominant or its image is of codimension 1 in $\mathcal{H}$. We show the latter cannot happen. For any hypersurface X, the space of lines which are contained in the smooth locus of X and are not free cannot sweep out a dense subset in X by Lemma 6.1, so if $\dim \pi_2^{-1}([X]) = n-1$, then the lines passing through the singular points of X should sweep out X. The locus of hypersurfaces which are singular at least along a curve is of codimension greater than 1 in $\mathcal{H}$, and so is the locus of hypersurfaces which are cones over hypersurfaces in $\mathbb{P}^{n-1}$ when $d \geq 3$. Therefore π_2 is dominant, and $\dim I = \dim \mathcal{H} + n - 2$, so a general fiber of π_2 has dimension $n-2$.

□

Proof of Theorem 1.5. Assume to the contrary that every smooth hypersurface of degree d has a family of lines of dimension $(a_k + 1)$ passing through one point such that for every line l in the family, $h^1(l, N_{l/X}(-1)) \geq k$. Let $\mathcal{H}$ be the projective space of hypersurfaces of degree d in $\mathbb{P}^n$ and $\mathcal{U}$ the universal hypersurface over $\mathcal{H}$. Let $F_{p,k}(X)$ be the subvariety of the Grassmannian of lines in X passing though p parametrizing lines l with $h^1(l, N_{l/X}(-1)) \geq k$, and let B be the closed subvariety of $\mathcal{U} \times \mathrm{Gr}(1,n)$ consisting of all the points $([X], p, [l])$ such that $\dim F_{p,k}(X)$ at $[l]$ is larger than a_k.

Denote by $\phi : B \to \mathcal{U}$ and $\pi_1 : \mathcal{U} \to \mathcal{H}$ the projection maps. By our assumption $\psi = \pi_1 \circ \phi$ is dominant. We replace B by an irreducible component of B for which ψ is still dominant. By Proposition 3.4, for a general point $(X, [p])$ in the image of ϕ, there is a nonempty open subset B_0 of $\phi^{-1}([X], p)$ and a subspace

$$W_{X,p} \subset H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d)(-p))$$

of codimension at most $n-1$ such that for every $b = ([X], p, [l]) \in B_0$ and $w \in W_{X,p}$, $w|_l$ can be lifted to a section of $N_{l/\mathbb{P}^n}(-p)$ under the map obtained from the short exact sequence,

$$0 \to N_{l/X}(-p) \to N_{l/\mathbb{P}^n}(-p) \to \mathcal{O}_l(d)(-p) \to 0.$$

Suppose that $([X], p, [l])$ is a point of B_0. Let Γ be a hyperplane in $\mathbb{P}^n$ which does not pass through p. Choose a system of coordinates for $\mathbb{P}^n$ so that $p = (1 : 0 : \cdots : 0)$ and Γ is given by $x_0 = 0$. Let F be a larger than expected dimension irreducible component of $F_{p,k}(X)$ to which $[l]$ belongs. The cone of lines parametrized by F intersects Γ along a subvariety Y of dimension $\geq a_k + 1$.

For every $i \geq 1$, multiplication by x_0^{d-i} identifies $H^0(\Gamma, \mathcal{O}_\Gamma(i))$ with a subspace of $H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d)(-p))$. Set

$$W_i := W_{X,p} \cap \ x_0^{d-i} H^0(\Gamma, \mathcal{O}_\Gamma(i)).$$

Since the codimension of $W_{X,p}$ in $H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d)(-p))$ is at most $n-1$, the codimension of W_i in $x_0^{d-i}H^0(\Gamma, \mathcal{O}_\Gamma(i))$ is at most $n-1$. By definition of a_k, for every i with $k+1 \leq i \leq d$, $n-1 < \binom{a_k+1+i}{i}$. Since $\dim Y \geq a_k+1$, by Lemma 4.6 (i), there is $f_i = x_0^{d-i} g_i$ in W_i such that g_i does not vanish on Y. So for every $k+1 \leq i \leq d$, $f_i \in W_{X,p} \subset H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d)(-p))$, it vanishes to order exactly i at p, and it is in the image of the map

$$\rho : H^0(l, N_{l/\mathbb{P}^n}(-p)) \to H^0(l, \mathcal{O}_l(d)(-p)).$$

The argument at the end of the proof of Theorem 1.3 shows that there is a section of $\mathcal{O}_l(d)$ which vanishes to order 1 at p and can be lifted to a section of $N_{l/\mathbb{P}^n}(-p)$. Thus the dimension of the image of ρ is at least $d-k+1$ and $h^1(l, N_{l/X}(-p)) \leq k-1$, which is a contradiction. $\square$

The proof of Theorem 1.5 yields stronger results if we know the dimension of the linear span of Y defined in the proof of the theorem. The next proposition gives such a result when $k = 1$.

Proposition 6.5. *Suppose that X is a general hypersurface of degree d in $\mathbb{P}^n$ and Σ is a cone of lines in X over a curve $Y \subset \mathbb{P}^{n-1}$. If the linear span of Y has dimension $> (n-2)/2$, then a general line parametrized by Y is free.*

Proof. The proof is similar to that of Theorem 1.5 except that we apply part (ii) of Lemma 4.6 to Y. Let s be the dimension of the linear span of Y, and let p be the vertex of the cone over Y. Since X is general, there is a subspace $W_{X,p} \subset H^0(X, \mathcal{O}_X(d)(-p))$ of codimension at most $n-1$ such that for every $w \in W$ and a general line l parametrized by Y, $w|_l$ can be lifted to a section of $N_{l/\mathbb{P}^n}(-p)$ under the map

$$\rho : H^0(l, N_{l/\mathbb{P}^n}(-p)) \to H^0(l, \mathcal{O}_l(d)(-p)).$$

Let W_i be defined as in the proof of Theorem 1.5. By Lemma 4.6 (ii), if $2s+1 > n-1$, then for every $2 \leq i \leq d$, there is a section $f_i = x_0^{d-i} g_i \in W_{X,p}$ such that g_i does not vanish on Y. So f_i has a zero of order i at p and is contained in the image of ρ. The image of ρ also contains a section which has a simple zero at p by the same reasoning as in the proof of Theorem 1.5, so ρ is surjective and $H^1(l, N_{l/X}(-p)) = 0$.

$\square$

Assume now that X is a general complete intersection of multidegree $(d_1, \ldots, d_m)$ in $\mathbb{P}^n$, and put $d := d_1 + \cdots + d_m$. It is likely that a similar strategy as in the proof of [9, Theorem 2.1] could be applied to show that if $\sum_i d_i \leq n-1$, the space of lines in X through every point of X has dimension equal to $n - \sum_i d_i - 1$, but here we get a weaker result as an immediate corollary to a generalization of Theorem 1.5.

Proposition 6.6. *Let $X \subset \mathbb{P}^n$ be a general complete intersection of multidegree $(d_1, \ldots, d_m)$. Set $d = d_1 + \cdots + d_m$.*

(i) *If there is a family of dimension r of non-free lines through a point p in X, then $\binom{r+2}{2} \leq n - m$.*

(ii) *If $\binom{n-d+2}{2} > n - m$, then the evaluation map $ev : \overline{\mathcal{M}}_{0,1}(X, 1) \to X$ is flat and has relative dimension $n - d - 1$.*

Proof. (i) The proof is similar to the proof of Theorem 1.3. Let s be the dimension of the largest family of non-free lines in a general complete intersection of multidegree $(d_1, \ldots, d_m)$ which all pass through the same point. We show $\binom{s+2}{2} \leq n - m$. Assume to the contrary that the inequality does not hold. Let $\mathcal{H}$ denote the variety parametrizing complete intersections of multidegree $(d_1, \ldots, d_m)$ in $\mathbb{P}^n$, and let $\mathcal{U}$ denote the universal family over $\mathcal{H}$. Let $B \subset \mathcal{U} \times \mathrm{Gr}(1, n)$ be the subvariety parametrizing triples $([X], p, [l])$ such that l belongs to a s-dimensional family of lines through p in X, and let $([X], p)$ be a general point in the image of the projection map $\phi : B \to \mathcal{U}$.

The existence of a subspace $W_{X,p} \subset H^0(X, \mathcal{O}_X(-p))$ of codimension at most $n-m$ as in the proof of Theorem 1.5 and the assumption that $\binom{s+2}{2} > n-m$ show that if $([X], p, l)$ is a general point in $\phi^{-1}([X], p)$, then for every $1 \leq j \leq m$, and every $k \geq 2$, there is a global section $f_{j,k}$ of $N_{X/\mathbb{P}^n} = \oplus_{i=1}^m \mathcal{O}_l(d_i)$ such that

(1) The i-th component of $f_{j,k}$ is zero for $i \neq j$,
(2) The j-th component of $f_{j,k}$ has a zero of order equal to k at p,
(3) $f_{j,k}$ can be lifted to a section of $N_{l/\mathbb{P}^n}(-p)$ via the map obtained from the exact sequence:

$$0 \to N_{l/X}(-p) \to N_{l/\mathbb{P}^n}(-p) \to \oplus_{i=1}^m \mathcal{O}_l(d_i)(-p) \to 0.$$

On the other hand, the surjectivity of the map of sheaves $\mathcal{O}_l(1)^{n+1} \to \oplus_{i=1}^m \mathcal{O}_l(d_i)$ implies that for each $1 \leq j \leq m$, there is a global section f_j of $\oplus_{i=1}^m \mathcal{O}_l(d_i)$ whose j-th component has a simple zero, and whose other components have zeros of order at least 2 at p, such that f_j is in the image of the map

$$H^0(l, \mathcal{O}_l(1)^{n+1}(-p)) \to H^0(l, \oplus_{i=1}^m \mathcal{O}_l(d_i)(-p)) \subset H^0(l, \oplus_{i=1}^m \mathcal{O}_l(d_i)).$$

This shows that the map $H^0(l, N_{l/X}(-p)) \to H^0(l, \oplus_{i=1}^m \mathcal{O}_l(d_i)(-p))$ is surjective, a contradiction.

(ii) As was shown in the proof of Theorem 1.3, to prove the flatness of ev, it suffices to show that the fibers of ev have constant dimension $n - d - 1$. But the fibers of ev have dimension at least $n - d - 1$, and if there is an irreducible component $\mathcal{M}$ of $ev^{-1}(p)$ whose dimension is larger than $n-d-1$, then every line parametrized by $\mathcal{M}$ should be non-free. This is not possible by part (i). □

7. Dimension and irreducibility of $\overline{\mathcal{M}}_{0,0}(X,e)$

Suppose that X is a complete intersection of multidegree $(d_1,\dots,d_m)$ in $\mathbb{P}^n$, and set $d := d_1+\cdots+d_m$. If $d \le n$, then the *threshold degree* of X is defined to be

$$E(X) := \left\lfloor \frac{n+1}{n+1-d} \right\rfloor.$$

Theorem 7.1. *Let $X \subset \mathbb{P}^n$ be a general complete intersection of multidegree $(d_1,\dots,d_m)$. If $d < 2n/3, n \ge 20$, then the evaluation map $ev : \overline{\mathcal{M}}_{0,1}(X,e) \to X$ is flat and has relative dimension $e(n+1-d)-2$ for every $e \ge 1$.*

Proof. By [9, Corollary 5.5], if the evaluation map $ev : \overline{\mathcal{M}}_{0,1}(X,e) \to X$ is flat of relative dimension $e(n+1-d)-2$ for every $1 \le e \le E(X)$, then it is flat of relative dimension $e(n+1-d)-2$ for every $e \ge 1$. If $d < \frac{2n}{3}$, then $E(X) \le 2$, so to prove the statement it is enough to prove it for $e = 1, 2$. If $e = 1$, our assumptions on n and d imply that $\binom{n-d+2}{2} > n-m$, so by Proposition 6.6, ev is flat of relative dimension $n-d-1$. If $e = 2$, and n, d satisfy the given inequalities, then $n-1 < \binom{n-d}{2}$, so Theorem 1.3 shows that ev is flat of the expected relative dimension when X is a hypersurface. The same proof can be extended to the case of general complete intersections with $n-m < \binom{n-d}{2}$. □

Theorem 7.2. *With the same assumptions as in Theorem 7.1, $\overline{\mathcal{M}}_{0,0}(X,e)$ is an irreducible complete intersection stack of dimension $e(n+1-d)+n-m-3$ for every $e \ge 1$.*

Proof. By the previous theorem,

$$\dim \overline{\mathcal{M}}_{0,0}(X,e) = \dim \overline{\mathcal{M}}_{0,1}(X,e) - 1 = e(n+1-d)+n-m-3.$$

The stack $\overline{\mathcal{M}}_{0,0}(X,e)$ is the zero locus of a section of a locally free sheaf of rank $de+m$ over the smooth stack $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^n,e)$ (see Section 2.1). Since $\dim \overline{\mathcal{M}}_{0,0}(\mathbb{P}^n,e) = (e+1)(n+1)-4$, and since $\overline{\mathcal{M}}_{0,0}(X,e)$ has the expected dimension $= \dim \overline{\mathcal{M}}_{0,0}(\mathbb{P}^n,e) - (de+m)$, it is a local complete intersection stack.

Next we prove that $\overline{\mathcal{M}}_{0,0}(X,e)$ is irreducible. By [9, Corollary 6.9], if X is a smooth complete intersection, then $\overline{\mathcal{M}}_{0,0}(X,e)$ is irreducible for every $e \ge 1$ if

- (i) The evaluation map $ev : \overline{\mathcal{M}}_{0,1}(X,e) \to X$ is flat of relative dimension $e(n+1-d)-2$ for every $e \ge 1$.
- (ii) General fibers of ev are irreducible.
- (iii) There is a free line in X.
- (iv) $\mathcal{M}_{0,0}(X,e)$ is irreducible for every $1 \le e \le E(X)$ where $\mathcal{M}_{0,0}(X,e)$ denotes the stack of stable maps of degree e with irreducible domains.

By Theorem 7.1 the first property is satisfied, and property (iii) holds for every smooth complete intersection which is covered by lines, that is every smooth complete intersection with $d \leq n-1$. By Corollary 3.3, for every line l in X, $H^1(l, N_{l/X}) = 0$, so $\overline{\mathcal{M}}_{0,0}(X,1)$ and hence $\overline{\mathcal{M}}_{0,1}(X,1)$ are smooth. Therefore, by generic smoothness, a general fiber of $ev : \overline{\mathcal{M}}_{0,1}(X,1) \to X$ is smooth. Since every fiber of this map has the expected dimension $n-d-1$, it is a complete intersection of dimension ≥ 1 in $\mathbb{P}^{n-1}$, so it is also connected and therefore irreducible.

To show property (iv) holds, we need to show $\mathcal{M}_{0,0}(X,e)$ is irreducible for $e = 1, 2$. $\mathcal{M}_{0,0}(X,1)$ is the space of lines in X which is irreducible by ([11, V.4.3]) when X is a smooth hypersurface of degree $\leq 2n-4$ in $\mathbb{P}^n$, $n \geq 4$. The same proof can be generalized to the case of complete intersections. The irreducibility of the space of lines in general complete intersections with $d \leq n-1$ is also proved in [6, Corollary 4.5].

It is proved in [3] that $\mathcal{M}_{0,0}(X,2)$ is irreducible for a general hypersurface of degree $\leq n-2$ in $\mathbb{P}^n$. Let us explain how one can generalizes the same argument to the case of general complete intersections with $d \leq n-2$. Note that since the dimension of $\mathcal{M}_{0,0}(X,2)$ is $3n-2d-m-1$, and since the space of lines passing through any point of X has dimension $n-d-1$, the space of reducible conics in X has dimension

$$\dim \mathcal{M}_{0,0}(X,1) + 1 + n - d - 1 = 3n - 2d - m - 2 < \dim \mathcal{M}_{0,0}(X,2),$$

and the space of double covers of lines in X has dimension

$$\dim \mathcal{M}_{0,0}(X,1) + 2 = 2n - d - m < 3n - 2d - m - 1,$$

every irreducible component of $\mathcal{M}_{0,0}(X,2)$ contains an open subscheme parametrizing smooth embedded conics in X. Therefore, to prove $\mathcal{M}_{0,0}(X,2)$ is irreducible, it is enough to show that $\mathrm{Hilb}_{2t+1}(X)$ is irreducible. To this end, let $I \subset \mathrm{Hilb}_{2t+1}(\mathbb{P}^n) \times \mathcal{H}$ be the incidence correspondence parametrizing pairs $([C],[X])$ such that C is a conic in X, and let $\pi_1 : I \to \mathrm{Hilb}_{2t+1}(\mathbb{P}^n)$ and $\pi_2 : I \to \mathcal{H}$ denote the two projection maps. Since $\mathrm{Hilb}_{2t+1}(\mathbb{P}^n)$ is smooth and irreducible, and since the fibers of π_1 are product of projective spaces, I is smooth and irreducible.

Let J be the closed subscheme of I parametrizing pairs $([C],[X])$ such that C is a non-reduced conic, so the support of C is a line in X. Then J is smooth and irreducible since J maps to the Grassmannian of lines in $\mathbb{P}^n$ and the fibers are smooth and irreducible. Let $\pi_2' : J \to \mathcal{H}$ be the projection map. Note that for any smooth complete intersection X and $l \subset X$, the space of non-reduced conics in X whose support is l can be identified with $\mathbb{P}(H^0(l, N_{l/X}(-1)))$. If $[X] \in \mathcal{H}$ is general, then the space of lines in X is irreducible, thus the fiber of π_2' over $[X]$ is connected. By generic smoothness, $\pi_2'^{-1}([X])$ is smooth and therefore irreducible.

By [6, Lemma 3.2], if $i : N \to M$ and $e : M \to Y$ are morphisms of irreducible schemes and i maps the generic point of N to a normal point of M, then e has irreducible general fibers provided that $e \circ i$ is dominant with irreducible general fibers. We apply this result to $N = J$, $M = I$, $Y = \mathcal{H}$, $i =$ the inclusion map, and $e = \pi_2$. Since $d \leq n-2, h^0(l, N_{l/X}(-1)) \geq 1$

for any smooth X parametrized by $\mathcal{H}$ and any line $l \subset X$, so $e \circ i = \pi_2'$ is dominant and we have shown its general fibers are irreducible. Since I is smooth, a general fiber of π_2 should be irreducible.

□

References

[1] K. Behrend and Yu. Manin, Stacks of stable maps and Gromov-Witten invariants, Duke Math. J. 85 (1996), no. 1, 1–60.

[2] O. Debarre, Higher dimensional algebraic geometry, Springer-Verlag, New York, 2001.

[3] M. Deland, Ph. D. thesis, Columbia University, 2004.

[4] I. Coskun and J. Starr, Rational curves on smooth cubic hypersurfaces, Int. Math. Res. Not. 2009, no. 24, 4626–4641.

[5] A. J. de Jong and J. Starr, Cubic fourfolds and spaces of rational curves, Illinois J. Math., 48 (2004), no. 2, 415–450.

[6] A. J. de Jong and J. Starr, Low degree complete intersections are rationally simply connected, preprint.

[7] L. Ein, Varieties with small dual varieties I, Invent. Math. (1986) 63–74.

[8] W. Fulton and R. Pandharipande, Notes on stable maps and quantum cohomology, in Algebraic geometry Santa Cruz 1995, J. Kollár, R. Lazarsfeld, D. Morrison eds., AMS: Providence, 1997.

[9] J. Harris, M. Roth, and J. Starr, Rational curves on hypersurfaces of low degree, J. Reine Angew. Math. 571 (2004), 73–106.

[10] R. Hartshorne, Algebraic geometry, volume 52 of Graduate Texts in Mathematics, Springer-Verlag, 1977.

[11] J. Kollár, Rational curves on algebraic varieties, volume 32 of Ergebnisse der Mathematik und ihrer Grenzgebiete, 3. Folge. Springer-Verlag, 1996.

[12] H. Matsumura, Commutative ring theory, volume 8, Cambridge studies in advanced mathematics, Cambridge University Press, 1986.

[13] D. Mumford, Abelian varieties, volume 5, Tata Institute of Fundamental Research Studies in Mathematics, Oxford University Press, 1970.

[14] R. Vakil, The enumerative geometry of rational and elliptic curves in projective space, J. Reine Angew. Math. 529 (2000), 101–153.

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