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On a problem of G Fejes Toth

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Dedicated to the memory of Professor K G Ramanathan

Abstract. A solution is given for the following Problem of G Fejes Toth: In 3-space find the thinnest lattice of balls such that every straight line meets one of the balls.

Keywords. Spheres (balls); lattices; thinnest arrangements.

1. Introduction

1.1 The object of this note is to give a solution of the following problem of G Fejes Toth [2]:

In 3-space find the thinnest lattice arrangement of closed balls such that every straight line meets these balls.

As pointed out by G Fejes Toth himself this is in some sense the first unsolved case of the more general problem:

In n-space find the thinnest lattice arrangement of closed balls such that every k-dimensional $(0 \leqslant k \leqslant n-1)$ *flat meets one of these balls.*

For $k=0$, this is the problem of thinnest lattice coverings by spheres, while for $k=n-1$, Makai [4] has shown that the problem can be reduced to that of the closest lattice packings of spheres. Thus the solution is known for $k=0$, $n \leqslant 5$ and for $0 \leqslant k = n-1 \leqslant 7$. (See any book dealing with packings and coverings, e.g. Lekkerkerker and Gruber [3]). The problem above can be generalised to one for other "bodies" also. In the case of convex bodies, Makai [4] has shown that a theorem analogous to the one for spheres holds if $k=n-1$. Our solution to the Fejes Toth problem stated in the beginning is contained in the following Theorems I and II and the remark after Theorem II.
(We shall throughout be working in the three-dimensional Euclidan space R^3).

Theorem I. *Let K be the sphere* $|x| \leqslant 1$. *Let* Λ *be a lattice with determinant* $d(\Lambda)$. *If every straight line meets a ball* $K + A$, $A \in \Lambda$, *then* $d(\Lambda) \leqslant 2(4/3)^3$.

Theorem II. *Let K be the sphere* $|x| \leqslant 1$ *and* Λ *be the lattice generated by* $4/3(1,1,0)$, $4/3(0,1,1)$ *and* $4/3(1,0,1)$. *Then every straight line meets a sphere* $K + A$, $A \in \Lambda$.

Remark Our proof of Theorem I (see §4.4) shows that "up to" orthogonal transformations the lattice Λ of Theorem II is the only "critical" lattice.

For convenience we replace Theorems I and II by the equivalent Theorems I′, II′:

Theorem I′. *Let K be the sphere $|x| \leqslant 3/4$ and Λ a lattice with determinant $d(\Lambda)$. If every straight line meets a ball $K + A$, $A \in \Lambda$, then $d(\Lambda) \leqslant 2$.*

Theorem II′. *Let K be the sphere $|x| \leqslant 3/4$ and Λ the lattice generated by $(1,1,0)$, $(0,1,1)$ and $(1,0,1)$. Then every straight line meets a $K + A$, $A \in \Lambda$.*

2. Proof of Theorem I′

2.1. Let K be the sphere $|x| \leqslant 3/4$ and Λ a lattice. Let $A_1 \in \Lambda$. Let Π be the plane through O perpendicular to OA_1. Let Λ^* be the (orthogonal) projection of Λ on Π. Let C be the circle $K \cap \Pi$. All lines parallel to OA_1 meet a $K + A$, $A \in \Lambda$ is equivalent to the statement: the circles $C + A^*$, $A^* \in \Lambda^*$ cover Π, i.e. the "covering radius" $\rho(\Lambda^*)$ of Λ^* is $\leqslant 3/4$.

2.2. Let A_1, A_2, A_3 be a basis of Λ. Let L be the matrix (A_1, A_2, A_3) with A_1, A_2, A_3 written as column vectors. The positive definite quadratic form $f(x) = f(x_1, x_2, x_3) = X' L' LX$, where $X' = (x_1, x_2, x_3)$ is called the quadratic form of Λ w.r.t. the basis A_1, A_2, A_3. Its determinant $d(f) = \det(L' L) = d^2(\Lambda)$. If $(B_1, B_2, B_3) = (A_1, A_2, A_3)U$ is any other basis of Λ, the $U \in GL(3, Z)$ and the corresponding quadratic form $X'U' L' LUX$ is equivalent to $f(X)$. In fact the quadratic forms corresponding to different bases of Λ consist of the class of quadratic forms equivalent to f.

Again if $f(x) = X' L' L X = X'M' MX$, then $M = TL$, where T is orthogonal and the lattice $T\Lambda$ with basis TA_1, TA_2, TA_3 is an orthogonal transform of Λ. We may note that $TK = K$, and Λ has the property of Theorem I′ if and only if $T\Lambda$ has.

2.3. Let $f(x) = \Sigma a_{ij} x_i x_j$, $a_{ij} = a_{ji}$ be the real positive definite quadratic form corresponding to a basis A_1, A_2, A_3 of Λ. Write

$$f = a_{11}\left(x_1 + \frac{a_{12}}{a_{11}} x_2 + \frac{a_{13}}{a_{11}} x_3\right)^2 + g(x_2, x_3)$$
$$= (\alpha_{11} x_1 + \alpha_{12} x_2 + \alpha_{13} x_3)^2 + (\alpha_{22} x_2 + \alpha_{23} x_3)^2 + (\alpha_{32} x_2 + \alpha_{33} x_3)^2,$$

and f is the quadratic form of a lattice $\Lambda_1 = T\Lambda$, T orthogonal, with respect to the basis $B_1 = TA_1, B_2 = TA_2, B_3 = TA_3$, and $B_1 = (\alpha_{11}, 0, 0)$, $B_2 = (\alpha_{12}, \alpha_{22}, \alpha_{32})$, $B_3 = (\alpha_{13}, \alpha_{23}, \alpha_{33})$. Every line parallel to OA_1 meets a $K + A$, $A \in \Lambda$ if and only if every line parallel to OB_1 meets a $K + B$, $B \in \Lambda_1$. Since B_1 is the point $(\alpha_{11}, 0, 0)$, the plane Π of 2.1 is $x_1 = 0$ and the projection Λ^* of Λ_1 on Π is the lattice generated by $(0, \alpha_{22}, \alpha_{32})$ and $(0, \alpha_{23}, \alpha_{33})$, while

$$g(x_2, x_3) = (\alpha_{22} x_2 + \alpha_{23} x_3)^2 + (\alpha_{32} x_2 + \alpha_{33} x_3)^2.$$

Let $\rho = \rho(\Lambda^*)$ be the covering radius of Λ^* and $R(g) = \rho^2$. ($R(g)$ depends only on g, because if g is a quadratic form of another lattice Λ_1^*, then $\Lambda_1^* = T\Lambda^*$, where T is orthogonal and the covering radius of Λ_1^* is the same as that of Λ^*.)

By § 2.1 all lines parallel to OA_1 meet a $K + A$, $A \in \Lambda$ if and only if $\rho(\Lambda)^* \leqslant 3/4$, if and only if $R(g) \leqslant 9/16$. Since every primitive lattice point can be extended to a basis of Λ, all lines parallel to lines OA, $A \in \Lambda$ meet the balls $K + P$, $P \in \Lambda$ if and only if for all forms $f' \sim f$, the corresponding "sections" $g'(x_2, x_3)$ have $R(g') \leqslant 9/16$. More precisely, the hypothesis of Theorem I′ implies the following:

Let Λ be a lattice. Let $f(x) = \Sigma a_{ij} x_i x_j, a_{ij} = a_{ji}$ be any quadratic form of Λ. Let

$$f(x) = a_{11}\left(x_1 + \frac{a_{12}}{a_{11}}x_2 + \frac{a_{13}}{a_{11}}x_3\right)^2 + g(x_2, x_3).$$

Then

$$R(g) \leqslant 9/16.$$

To prove Theorem I′ it is enough to prove

Theorem IA. *Let $f(x) = \Sigma a_{ij} x_1 x_j$, $a_{ij} = a_{ji}$ be a real positive definite quadratic form with determinant $d(f)$. Let $f' \sim f$; write*

$$f'(x) = a'_{11}\left(x_1 + \frac{a'_{12}}{a'_{11}}x_2 + \frac{a'_{13}}{a'_{11}}x_3\right)^2 + g'(x_2, x_3).$$

If $R(g') \leqslant 9/16$ for each $f' \sim f$, then $d(f) \leqslant 4$.

2.4. Let $f(x) = \Sigma a_{ij} x_i x_j$, $a_{ij} = a_{ji}$ be a positive definite quadratic form. Let

$$f(x) = a_{11}\left(x_1 + \frac{a_{12}}{a_{11}}x_2 + \frac{a_{13}}{a_{11}}x_3\right)^2 + g(x_2, x_3).$$

Then

$$\begin{aligned} a_{11} g &= (a_{11}a_{22} - a_{12}^2)x_2^2 + 2(a_{11}a_{23} - a_{12}a_{13})x_2 x_3 + (a_{11}a_{33} - a_{13}^2)x_3^2 \\ &= A_{33}x_2^2 - 2A_{23}x_2 x_3 + A_{22}x_3^2 \\ &= G', \text{ say,} \end{aligned}$$

where A_{ij} are the entries of the matrix adjoint to (a_{ij}). Since $g = a_{11}^{-1} G'$, $R(g) = a_{11}^{-1} R(G')$. If

$$G = A_{22}x_2^2 + 2A_{23}x_2 x_3 + A_{33}x_3^2,$$

then $G \sim G'$ and $R(G) = R(G')$, and

$$R(g) = a_{11}^{-1} R(G). \tag{a}$$

Let $A = (a_{ij})$, adj $A = (A_{ij})$. Then $A \operatorname{adj} A = \det(A)I$, and $\det(\operatorname{adj} A) = (\det A)^2$. Write

$$F(x) = \operatorname{adj} f(x) = \Sigma A_{ij} x_i x_j$$

Then

$$d(F) = \det(A_{ij}) = (\det A)^2 = d^2(f). \tag{b}$$

Since

$$A(\operatorname{adj} A) = (\det A)I = d(f)I, \text{ and } (\operatorname{adj} A) \operatorname{adj}(\operatorname{adj} A) = d(F)I = d^2(f)I,$$

we have

$$\frac{1}{d(f)}\mathrm{A} = \frac{1}{d^2(f)}\mathrm{adj}(\mathrm{adj}\,\mathrm{A})$$

i.e.

$$\frac{1}{d(f)}(a_{ij}) = \frac{1}{d^2(f)}\mathrm{adj}(\mathrm{A}_{ij})$$

Equating elements in the leading position, we get

$$\frac{1}{d(f)}a_{11} = \frac{1}{d^2(f)}(\mathrm{A}_{22}\mathrm{A}_{33} - \mathrm{A}_{23}^2)$$

$$= \frac{1}{d^2(f)}d(G),$$

and $a_{11}^{-1} = d(f)/d(G) = \sqrt{d(F)}/d(G)$, and, by (a),

$$R(g) = R(G)\sqrt{d(F)}/d(G).$$

Therefore,

$$R(g) \leqslant 9/16 \text{ iff } R(G) \leqslant 9/16\ d(G)/d(F)^{1/2} \tag{c}$$

and

$$d(F) = d^2(f). \tag{d}$$

It is well known that if $f \sim f'$, then adj $f \sim$ adj f' and vice versa, i.e., the class of forms equivalent to adj f is the class of adjoints of forms $\sim f$.

Let $F(x_1, x_2, x_3) = \Sigma \mathrm{A}_{ij}x_i x_j$ be a definite quadratic form and $F_1 \sim F$. Let $G(x_2, x_3) = F_1(0, x_2, x_3)$ be called a partial sum of F and let S be the set of partial sums of F. Since $F(x_1, x_2, x_3) \sim F(x_3, x_1, x_2)$ the set of partial sums of F consists of the forms $G(x_1, x_2) = F'(x_1, x_2, 0)$ for all forms $F' \sim F(x)$.

We can replace Theorem IA by (see (c) and (d) above).

Theorem IB. *Let $F(x_1, x_2, x_3) = \Sigma \mathrm{A}_{ij}x_i x_j$, $\mathrm{A}_{ij} = \mathrm{A}_{ji}$ be a positive definite quadratic form. Suppose for every partial sum G of F we have $R(G) \leqslant 9/16\ d(G)/\sqrt{d(F)}$. Then $d(F) \leqslant 16$.*

It is clear that we can replace F by any equivalent form without affecting the hypothesis or conclusion of the theorem. For convenience we alter the notation a little bit and state Theorem IB as:

Theorem IC. *Let $f(x_1, x_2, x_3) = \Sigma a_{ij}x_i x_j$, $a_{ij} = a_{ji}$ be a positive definite quadratic form. Suppose for every partial sum $g(x_1, x_2) = f'(x_1, x_2, 0)$, where $f' \sim f$, we have $R(g) \leqslant 9/16\ d(g)/\sqrt{d(f)}$, then $d(f) \leqslant 16$.*

3. Proof of Theorem IC

3.1 A basis A, B of a two-dimensional lattice Λ is said to be reduced if the angle O of the Δ OAB is largest and lies between 60° and 90°, equivalently Δ OAB is acute angled with largest angle at O. In this case the covering radius of Λ is the circumradius of Δ OAB. (see e.g. Dickson [1], pp. 160).

Now suppose A, B generate a two-dimensional lattice and Δ OAB is acute angled. Then $(A_1, B_1) = (A, B)$ or $(-A, B-A)$ or $(-B, A-B)$ is a reduced basis of Λ and its covering radius is the circumradius of $\Delta OA_1B_1 =$ the circumradius of Δ OAB. Thus if A, B generate Λ and Δ OAB is acute angled, then the covering radius $\rho(\Lambda)$ of Λ is the circumradius of Δ OAB.

Let $g(x, y) = ax^2 + 2bxy + cy^2$ be positive definite. Let $g(x, y) = (\alpha x + \beta y)^2 + (\gamma x + \delta y)^2$.

Let $A = (\alpha, \gamma)$, $B = (\beta, \delta)$. Then A, B generate a lattice Λ and $R(g) = \rho^2(\Lambda)$. The triangle **OAB** is acute angled if the square of each side $\leqslant$ sum of squares of the other two sides, i.e., if

$$a \leqslant c + (a + c - 2b),$$

$$c \leqslant a + (a + c - 2b),$$

$$a + c - 2b \leqslant a + c,$$

i.e.

$$b \leqslant c,\ b \leqslant a,\ b \geqslant 0, \text{ i.e.}$$

$$0 \leqslant b \leqslant \min(a, c).$$

Therefore, if $0 \leqslant b \leqslant \min(a, c)$, then
$R(g) =$ (circumradius of triangle OAB) $= ac(a + c - 2b)/4\,d(g)$. (If ABC is an acute angle triangle with sides a, b, c circumradius ρ and area Δ, then

$$\rho = \frac{a}{2\sin A} = \frac{b}{2\sin B} = \frac{c}{2\sin C},$$

$$\rho^3 = \frac{abc}{8\sin A\sin B\sin C} = \frac{a^3b^3c^3}{64(1/2\,bc\sin A)(1/2ca\sin B)(1/2ab\sin C)}$$

$$= \frac{a^3b^3c^3}{64\Delta^3}$$

so that

$$\rho^2 = \frac{a^2b^2c^2}{4(2\Delta)^2}.$$

3.2 Let $f(x1) = \Sigma a_{ij}x_ix_j$, $a_{ij} = a_{ji}$ be a positive definite form, all of whose partial sums $g(x_1, x_2)$ have $R(g) \leqslant 9/16\ d(g)/\sqrt{d(f)}$. We have to show $d(f) \leqslant 16$.

By replacing f, by an equivalent form reduced in the sense of Gauss and Sieber (see, e.g. Dickson [1], Th 103, pp. 171), we can suppose

$$0 < a_{11} \leqslant a_{22} \leqslant a_{33},$$

$$2|a_{12}| \leqslant a_{11},\ 2|a_{13}| \leqslant a_{11},\ 2|a_{23}| \leqslant a_{22}, \text{ and} \qquad \text{(A)}$$

$$a_{ij},\ i \neq j, \text{ all have the same sign,}$$

$$a_{11} + a_{22} + 2(a_{12} + a_{13} + a_{23}) \geqslant 0.$$

We divide the proof into two cases:

case I: all $a_{ij}, i \neq j$, are negative (or 0),
case II: all $a_{ij}, i \neq j$, are positive (or 0).

4. Proof of Theorem IC Case I

4.1 Clearly $g_1 = f(0, x_2, x_3)$, $g_2 = f(x_1, 0, x_3)$ and $g_3 = f(x_1, x_2, 0)$ are all partial sums of f. If $\Sigma A_{ij} x_i x_j$ is adjoint to f, then

$$d(g_1) = A_{11}, d(g_2) = A_{22}, d(g_3) = A_{33}.$$

Also each g is equivalent to one with the cross term of opposite sign. Therefore, applying the formula of § 3.1,

$$R(g_1) = a_{22}a_{33}(a_{22} + a_{33} + 2a_{23})/4\,A_{11},$$
$$R(g_2) = a_{33}a_{11}(a_{33} + a_{11} + 2a_{31})/4\,A_{22}, \text{ and}$$
$$R(g_3) = a_{11}a_{22}(a_{11} + a_{22} + 2a_{12})/4\,A_{33}$$

By the hypothesis $R(g_i) \leqslant 9/16\; d(g_i)/\sqrt{d(f)}$, and we have

$$a_{22}a_{23}(a_{22} + a_{33} + 2a_{23})/4\,A_{11} \leqslant 9/16\,A_{11}/\sqrt{d(f)}$$

or

$$4\,a_{22}a_{33}(a_{22} + a_{33} + 2a_{23})\sqrt{d(f)} \leqslant 9\,A_{11}^2. \quad (1)$$

Similarly,

$$4\,a_{33}a_{11}(a_{33} + a_{11} + 2a_{13})\sqrt{d(f)} \leqslant 9\,A_{22}^2, \quad (2)$$

and

$$4\,a_{11}a_{22}(a_{11} + a_{22} + 2a_{12})\sqrt{d(f)} \leqslant 9\,A_{33}^2. \quad (3)$$

4.2 Define $\beta_{12}, \beta_{23}, \beta_{13}$ by

$$a_{12} = -\beta_{12}\sqrt{a_{11}a_{22}},\; a_{13} = -\beta_{13}\sqrt{a_{11}a_{33}},$$
$$a_{23} = -\beta_{23}\sqrt{a_{22}a_{33}}, \quad (4)$$

and put

$$t_1 = (a_{11}/a_{22})^{1/2},\; t_2 = (a_{22}/a_{33})^{1/2}. \quad (5)$$

The reduction conditions (A) of § 3.2 give

$$0 \leqslant t_1,\; t_2 \leqslant 1 \quad (6)$$

$$0 \leqslant \beta_{12} \leqslant \frac{1}{2}t_1,\; 0 \leqslant \beta_{13} \leqslant \frac{1}{2}t_1 t_2,\; 0 \leqslant \beta_{23} \leqslant \frac{1}{2}t_2, \quad (7)$$

and

$$a_{11} + a_{22} + 2(a_{12} + a_{13} + a_{23}) \geqslant 0 \text{ becomes}$$
$$a_{11} + a_{22} \geqslant 2(\beta_{12}\sqrt{a_{11}a_{22}} + \beta_{13}\sqrt{a_{11}a_{33}} + \beta_{23}\sqrt{a_{22}a_{33}}),$$

so that, dividing by $\sqrt{a_{22}a_{33}}$, we get

$$t_1^2 t_2 + t_2 \geqslant 2(\beta_{12}t_1t_2 + \beta_{13}t_1 + \beta_{23}). \tag{8}$$

Now, if we write

$$g(t_1, t_2) = t_1{}^2 t_2 + t_2 - 2(\beta_{12}t_1t_2 + \beta_{13}t_1 + \beta_{23}),$$

then

$$\frac{\partial g}{\partial t_1} = 2t_1t_2 - 2\beta_{12}t_2 - 2\beta_{13} \geqslant 2t_1t_2 - t_1t_2 - t_1t_2 \quad \text{(By (7))}$$
$$\geqslant 0,$$
$$\frac{\partial g}{\partial t_2} = t_1^2 + 1 - 2\beta_{12}t_1$$
$$= 1 + t_1(t_1 - 2\beta_{12}) \geqslant 1 > 0. \quad \text{(By (7))}$$

Therefore, (8) remains true if we replace t_1, t_2 by 1, i.e.

$$\beta_{12} + \beta_{13} + \beta_{23} \leqslant 1. \tag{B}$$

Also,

$$d(f) = a_{11}a_{22}a_{33} + 2a_{12}a_{13}a_{23} - a_{33}a_{12}^2 - a_{11}a_{23}^2 - a_{12}a_{13}^2$$
$$= a_{11}a_{22}a_{33}(1 - 2\beta_{12}\beta_{13}\beta_{23} - \beta_{12}^2 - \beta_{13}^2 - \beta_{23}^2)$$
$$= a_{11}a_{22}a_{23}\Delta, \text{ say.} \tag{C}$$

4.3 Using inequality 1 of § 4.1, together with the arithmetic geometric mean inequality, we get

$$9\,A_{11}^2 \geqslant 4a_{22}a_{33}(a_{22} + a_{33} + 2a_{23})\sqrt{d(f)}$$
$$\geqslant 8a_{22}a_{33}(\sqrt{a_{22}a_{33}} + a_{23})\sqrt{d(f)}$$
$$= 8a_{22}a_{33}\sqrt{a_{11}a_{22}a_{33}\Delta}(\sqrt{a_{22}a_{33}} + a_{23})$$
$$= 8\sqrt{a_{11}\Delta}(a_{22}a_{33})^{3/2}(\sqrt{a_{22}a_{23}} + a_{23}),$$

so that

$$8\sqrt{a_{11}\Delta} \leqslant 9(a_{22}a_{33} - a_{23}^2)^2/(a_{22}a_{33})^{3/2}(\sqrt{a_{22}a_{33}} + a_{23})$$
$$= 9\left\{1 - \frac{a_{23}^2}{a_{22}a_{23}}\right\}^2 \Big/ \left\{1 + \frac{a_{23}}{\sqrt{a_{22}a_{23}}}\right\}$$
$$= 9(1 - \beta_{23}^2)^2/(1 - \beta_{23})$$
$$= 9(1 - \beta_{23})(1 + \beta_{23})^2, \text{ and}$$
$$\sqrt{a_{11}\Delta} \leqslant \frac{9}{8}(1 - \beta_{23})(1 + \beta_{23})^2. \tag{9}$$

Similarly, (2), (3) give

$$\sqrt{a_{22}\Delta} \leqslant \frac{9}{8}(1 - \beta_{31})(1 + \beta_{31})^2, \tag{10}$$

and

$$\sqrt{a_{33}\Delta} \leqslant \frac{9}{8}(1-\beta_{12})(1+\beta_{12})^2 \tag{11}$$

Multiplying (9), (10), and (11), we get

$$\begin{aligned}\sqrt{d(f)} = \sqrt{a_{11}a_{22}a_{33}\Delta} &\leqslant (9/8)^3(1-\beta_{12})(1-\beta_{23})(1-\beta_{13}) \\ &(1+\beta_{12})^2(1+\beta_{23})^2(1+\beta_{13})^2/\Delta \\ &= h(\beta_{12},\beta_{23},\beta_{13}), \text{ say} \end{aligned} \tag{D}$$

4.4 Our object now is to use (D) above to show that the condition (B) of §4.2 (i.e. $\beta_{12}+\beta_{23}+\beta_{13} \leqslant 1$) implies $\sqrt{d(f)} \leqslant 4$. (This will, of course, prove theorem IC in case I).

We note that if $\beta_{12}+\beta_{23}+\beta_{13} \leqslant 1$, one of the β's must be $\leqslant 1/3$. Increasing the β increases the numerator of h and decreases its denominator

$$\Delta = (1-2\beta_{12}\beta_{23}\beta_{13}-\beta_{12}^2-\beta_{13}^2-\beta_{23}^2),$$

because

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}x}(1-x)(1+x)^2 = -(1+x)^2+2(1-x^2) \\ &= (1+x)(1-3x) \geqslant 0 \text{ if } x \leqslant 1/3.\end{aligned}$$

Increasing the β's appropriately, we can assume

$$\beta_{12}+\beta_{23}+\beta_{13} = 1. \tag{E}$$

Putting $\beta_{23} = 1-\beta_{12}-\beta_{13}$, we have

$$\begin{aligned}\Delta &= 1-2\beta_{12}\beta_{13}\beta_{23}-\beta_{12}^2-\beta_{13}^2-\beta_{23}^2 \\ &= 1-2\beta_{12}\beta_{13}(1-\beta_{12}-\beta_{13})-\beta_{12}^2-\beta_{13}^2-(1-\beta_{12}-\beta_{13})^2 \\ &= 1-2\beta_{12}\beta_{13}+2\beta_{12}\beta_{13}(\beta_{12}+\beta_{13})-\beta_{12}^2-\beta_{13}^2 \\ &\quad -1+2(\beta_{12}+\beta_{13})-(\beta_{12}+\beta_{13})^2 \\ &= 2(\beta_{12}+\beta_{13})(1+\beta_{12}\beta_{13}-\beta_{12}-\beta_{13}) \\ &= 2(\beta_{12}+\beta_{13})(1-\beta_{12})(1-\beta_{13}),\end{aligned} \tag{12}$$

while

$$\begin{aligned}&(1-\beta_{12})(1-\beta_{13})(1-\beta_{23})(1+\beta_{12})^2(1+\beta_{13})^2(1+\beta_{23})^2 \\ &\quad = (1-\beta_{12})(1-\beta_{13})(\beta_{12}+\beta_{13})(1+\beta_{12})^2(1+\beta_{13})^2(2-\beta_{12}-\beta_{13})^2,\end{aligned}$$

so that (*D*) gives

$$\begin{aligned}\sqrt{d(f)} &\leqslant (9/8)^3(1-\beta_{12})(1-\beta_{13})(\beta_{12}+\beta_{13})(1+\beta_{12})^2(1+\beta_{13})^2 \\ &\quad (2-\beta_{12}-\beta_{13})^2/2(\beta_{12}+\beta_{13})(1-\beta_{12})(1-\beta_{13}) \\ &= (9^3/2^{10})(1+\beta_{12}^2)(1+\beta_{13}^2)(2-\beta_{12}-\beta_{13})^2.\end{aligned} \tag{F}$$

Also (7) gives $0 \leqslant \beta_{12} \leqslant 1/2$, $0 \leqslant \beta_{13} \leqslant 1/2$. We now observe

Lemma. The maximum of $f(x, y) = (1 + x)(1 + y)(2 - x - y)$, *subject to* $0 \leqslant x, y \leqslant 1$ *is attained only when* $x = y = 1/3$ *and has the value* $(4/3)^3$.

Proof. By the inequality of arithmetic geometric mean

$$f(x, y) = (1 + x)(1 + y)(2 - x - y) \leqslant \left(\frac{1 + x + 1 + y + 2 - x - y}{3}\right)^3 = (4/3)^3,$$

and the equality occurs if $1 + x = 1 + y = 2 - x - y = 4/3$, i.e. $x = y = 1/3$.

Using the Lemma in (F), we get

$$\sqrt{d(f)} \leqslant \frac{9^3}{2^{10}}(4/3)^6 = 2^2 = 4,$$

which proves Theorem I(C) in this case.

We also note that $d(f)$ can be 16 only if

$$\beta_{12} = 1/3, \; \beta_{13} = 1/3, \; \beta_{23} = 1/3,$$

$$\Delta = 2\frac{2}{3}\frac{2}{3}\frac{2}{3} = 2(2/3)^3,$$

and by (9), (10), (11)

$$\sqrt{a_{ii}\Delta} = \frac{9}{8}2/3 \; (4/3)^2$$

i.e.

$$a_{ii} = (4/3)^2\frac{3^3}{16} = 3,$$

i.e.,

$$f(x_1, x_2, x_3) = 3 \sum_{1 \leqslant i \leqslant 3} x_i^2 - 2 \sum_{1 \leqslant i < j \leqslant 3} x_i x_j$$

5. Proof of Theorem IC, Case II

5.1 In this case $f = \Sigma a_{ij} x_i x_j$, $a_{ij} = a_{ji}$; and

$$0 < a_{11} \leqslant a_{22} \leqslant a_{33},$$

$$0 \leqslant 2a_{12}, 2a_{13} \leqslant a_{11}, 0 \leqslant 2a_{23} \leqslant a_{22}.$$

Writing

$$a_{ij} = \beta_{ij}\sqrt{a_{ii}a_{jj}}, \; i \neq j$$

We have

$$0 \leqslant \beta_{ij} \leqslant \frac{1}{2}.$$

We divide this case into two subcases:

(a) at least one $\beta_{ij} \leqslant 0.459$, $i \neq j$,
(b) $0.459 < \beta_{ij} \leqslant 1/2$ for $i, j, i \neq j$.

6. Proof of Theorem IC Case II (a)

6.1 As in § 4.1, considering the partial sums $f(0, x_2, x_3)$, $f(x_1, 0, x_3)$, $f(x_1, x_2, 0)$, and noting $a_{ij} \geqslant 0$, we get

$$4a_{22}a_{33}(a_{22} + a_{33} - 2a_{23})\sqrt{d(f)} \leqslant 9A_{11}^2, \tag{1'}$$

$$4a_{33}a_{11}(a_{33} + a_{11} - 2a_{13})\sqrt{d(f)} \leqslant 9A_{22}^2, \tag{2'}$$

and

$$4a_{11}a_{22}(a_{11} + a_{22} - 2a_{12})\sqrt{d(f)} \leqslant 9A_{33}^2 \tag{3'}$$

Also

$$\begin{aligned} d(f) &= a_{11}a_{22}a_{33} + 2a_{12}a_{13}a_{23} - a_{11}a_{23}^2 - a_{22}a_{13}^2 - a_{33}a_{12}^2 \\ &= a_{11}a_{22}a_{23}(1 + 2\beta_{12}\beta_{22}\beta_{33} - \beta_{12}^2 - \beta_{23}^2 - \beta_{31}^2) \\ &= a_{11}a_{22}a_{33}\Delta', \text{ say} \end{aligned} \tag{C'}$$

from (1′) and (C′) we get, applying A–G mean inequality,

$$\begin{aligned} 9A_{11}^2 &\geqslant 8a_{22}a_{23}(\sqrt{a_{22}a_{33}} - a_{23})\sqrt{d(f)} \\ &= 8\sqrt{a_{11}\Delta'}(a_{22}a_{33})^2(1 - \beta_{23}), \end{aligned}$$

so that

$$\begin{aligned} 8\sqrt{a_{11}\Delta'} &\leqslant 9(a_{22}a_{33} - a_{23}^2)^2/(a_{22}a_{33})^2(1 - \beta_{23}) \\ &= 9(1 - \beta_{23}^2)^2/(1 - \beta_{23}) \\ &= 9(1 - \beta_{23})(1 + \beta_{23})^2 \end{aligned} \tag{4'}$$

Similarly, (2′), (3′) and (C′) give

$$8\sqrt{a_{22}\Delta'} \leqslant 9(1 - \beta_{13})(1 + \beta_{13})^2 \tag{5'}$$

$$8\sqrt{a_{33}\Delta'} \leqslant 9(1 - \beta_{12})(1 + \beta_{12})^2. \tag{6'}$$

Multiplying (4′), (5′), (6′), we get

$$\begin{aligned} 8^3\sqrt{d(f)}\Delta' \leqslant 9^3(1 - \beta_{12})(1 - \beta_{13})(1 - \beta_{23}) \\ (1 + \beta_{12})^2(1 + \beta_{13})^2(1 + \beta_{23})^2, \end{aligned}$$

and

$$\begin{aligned} \sqrt{d(f)} &\leqslant (9/8)^3(1 - \beta_{12})(1 - \beta_{13})(1 - \beta_{23}) \\ &\quad (1 + \beta_{12})^2(1 + \beta_{13})^2(1 + \beta_{23})^2/1 + 2\beta_{12}\beta_{13}\beta_{23} - \beta_{12}^2 - \beta_{13}^2 - \beta_{23}^2 \\ &= F, \text{ say}. \end{aligned} \tag{F'}$$

Make the substitution

$$x_1 = 1 + \beta_{12},\ x_2 = 1 + \beta_{13},\ x_3 = 1 + \beta_{23}.$$

Then
$1 \leqslant x_i \leqslant 3/2$, and at least one $x_i \leqslant 1.459$.
Noting

$$\begin{aligned}&2x_1x_2x_3 - (x_1 + x_2 + x_3 - 2)^2\\&= 2(1 + \beta_{12})(1 + \beta_{13})(1 + \beta_{23}) - (1 + \beta_{12} + \beta_{13} + \beta_{23})^2\\&= 1 + 2\beta_{12}\beta_{13}\beta_{23} - \beta_{12}^2 - \beta_{13}^2 - \beta_{23}^2 = \Delta',\end{aligned}$$

We get, from (F′),

$$\begin{aligned}\sqrt{d(f)} &\leqslant (9/8)^3(2 - x_1)(2 - x_2)(2 - x_3)x_1^2x_2^2x_3^2/\\&\quad 2x_1x_2x_3 - (x_1 + x_2 + x_3 - 2)^2.\\&= F(x_1, x_2, x_3), \text{ say.}\end{aligned}$$

It is, therefore, enough to prove that if $1 \leqslant x_i \leqslant 3/2$ and at least one $x_i \leqslant 1{\cdot}459$, then $F(x_1, x_2, x_3) \leqslant 4$.

Now $\partial F/\partial x_1$ has the same sign as

$$\begin{aligned}&(4x_1 - 3x_1^2)(2x_1x_2x_3 - (x_1 + x_2 + x_3 - 2)^2)\\&\quad - (2x_2x_3 - 2(x_1 + x_2 + x_3 - 2))x_1^2(2 - x_1),\end{aligned}$$

which has the same sign as

$$\begin{aligned}&(4 - 3x_1)(2x_1x_2x_3 - (x_1 + x_2 + x_3 - 2)^2)\\&\quad - 2x_1(2 - x_1)(x_2x_3 - x_1 - x_2 - x_3 + 2)\\&= 4x_1x_2x_3(1 - x_1) + (x_1 + x_2 + x_3 - 2)\\&\quad \{4x_1 - 2x_1^2 - (4 - 3x_1)(x_1 + x_2 + x_3 - 2)\}\\&= 4x_1x_2x_3(1 - x_1) + (x_1 + x_2 + x_3 - 2)\\&\quad \{x_1^2 - (4 - 3x_1)(x_1 + x_3 - 2)\}\\&= G(x_1, x_2, x_3), \text{ say.}\end{aligned}$$

Writing $x = ((x_2 + x_3)/2)$, and noting,

$$\begin{aligned}&x_2x_3 \leqslant ((x_2 + x_3)/2)^2 = x^2, \quad 1 - x_1 \leqslant 0,\\&G(x_1, x_2, x_3) \geqslant 4x_1x^2(1 - x_1) + (x_1 + 2x - 2)\\&\qquad\qquad \{x_1^2 - (4 - 3x_1)(2x - 2)\}\\&= (x_1 - 2)^2\{x_1 - 4(x - 1)^2\}\\&= (x_1 - 2)^2\{x_1 - 1 + 1 - 4(x - 1)^2\}\\&\geqslant (x_1 - 2)^2(x_1 - 1), \quad (\text{because } 0 \leqslant x - 1 \leqslant \tfrac{1}{2})\\&\geqslant 0.\end{aligned}$$

Therefore, $(\partial F/\partial x_1)\geqslant 0$. Similarly $(\partial F/\partial x_2)\geqslant 0$, $(\partial F/\partial x_3)\geqslant 0$, and the maximum of F will occur at $x_1=1{\cdot}459, x_2=1{\cdot}5, x_3=1{\cdot}5$, so that $F\leqslant F(1{\cdot}459, 1{\cdot}5, 1{\cdot}5)=3{\cdot}99\ldots<4$, and the Theorem is proved in this case.

7. Proof of Theorem IC Case II (b)

7.1 In this case $0{\cdot}459\leqslant \beta_{ij}\leqslant 0{\cdot}5$ for all i,j, $i\neq j$. We first note that the inequality (1′), (2′), (3′) of § 6.1 is valid in this case also.

Since

$$f(x_1,x_2,x_3)\sim f(x_1-x_2,x_2,x_3),$$

The form

$$g(x_2,x_3)=f(-x_2,x_2,x_3)=(a_{11}+a_{22}-2a_{12})x_2^2+2(a_{23}-a_{13})x_2x_3+a_{33}x_3^2$$

is a partial sum of f.

Since

$$g(x_2,x_3)\sim g(x_2,-x_3),$$

$$g(x_2,x_3)\sim(a_{11}+a_{22}-2a_{12})x_2^2-2|a_{23}-a_{13}|x_2x_3+a_{33}x_3^2=g'(x_2x_3), \text{ say.}$$

Then $R(g)=R(g')$.

Since

$$0\leqslant 2|a_{23}-a_{13}|\leqslant \max(2a_{23},2a_{13})\leqslant a_{22}\leqslant a_{22}+a_{11}-2a_{12},$$

and

$$|2(a_{23}-a_{13})|\leqslant a_{22}\leqslant a_{33},$$

$$R(g)=R(g')=a_{33}(a_{11}+a_{22}-2a_{12})$$

$$(a_{11}+a_{22}-2a_{12}+a_{33}-2|a_{23}-a_{13}|)/4d(g),$$

where

$$\begin{aligned}d(g)&=(a_{11}+a_{22}-2a_{12})a_{33}-(a_{23}-a_{13})^2\\&=\mathrm{A}_{11}+\mathrm{A}_{22}+2(a_{23}a_{13}-a_{12}a_{33})\\&=\mathrm{A}_{11}+\mathrm{A}_{22}+2\mathrm{A}_{12}.\end{aligned}$$

Since

$$R(g)\leqslant\frac{9}{16}d(g)/\sqrt{d(f)},$$

we have

$$a_{33}(a_{11}+a_{22}-2a_{12})(a_{11}+a_{22}+a_{33}-2a_{12}-2|a_{23}-a_{13}|)$$
$$\sqrt{d(f)}\leqslant\frac{9}{4}(\mathrm{A}_{11}+\mathrm{A}_{22}+2\mathrm{A}_{12})^2. \tag{13}$$

Permuting x_1,x_2,x_3, we get two similar inequalities.

Using

$$\beta_{ij}(a_{ii}a_{jj})^{1/2}=a_{ij},\ t_1=\sqrt{a_{11}/a_{22}},\ t_2=\sqrt{a_{22}/a_{33}},\text{ we have}$$

$$(a_{11}+a_{22}-2a_{12})=(a_{11}a_{22})^{1/2}(t_1+t_1^{-1}-2\beta_{12}),$$

$$(a_{11}+a_{22}+a_{33}-2a_1a_2-2|a_{23}-a_{13}|)$$
$$=(a_{11}a_{22})^{1/2}\left\{t_1+t_1^{-1}+\frac{1}{t_1t_2^2}-2\beta_{12}-\frac{2}{t_1t_2}|\beta_{23}-\beta_{13}t_1|\right\},$$
$$\mathrm{A}_{11}+\mathrm{A}_{22}+2\mathrm{A}_{12}=a_{22}a_{33}-a_{23}^2+a_{11}a_{33}-a_{22}^2$$
$$+2(a_{23}a_{13}-a_{12}a_{33})$$
$$=a_{22}a_{33}(1-\beta_{23}^2)+a_{11}a_{33}(1-\beta_{13}^2)$$
$$+2(a_{11}a_{22})^{1/2}(\beta_{23}\beta_{13}a_{33}-\beta_{12}a_{33})$$
$$=a_{33}(a_{11}a_{22})^{1/2}\{t_1^{-1}(1-\beta_{23}^2)+t_1(1-\beta_{13}^2)+2(\beta_{13}\beta_{23}-\beta_{12})\},$$

and (13) becomes

$$\sqrt{d(f)}\leqslant\frac{9}{4}a_{33}^2a_{11}a_{22}[t_1(1-\beta_{13}^2)+t_1^{-1}(1-\beta_{23}^2)$$
$$+2(\beta_{13}\beta_{23}-\beta_{12})]/a_{33}a_{11}a_{22}(t_1+t_1^{-1}-2\beta_{12})\left(t_1+t_1^{-1}+\frac{1}{t_1t_2^2}\right.$$
$$\left.-2\beta_{12}-\frac{2}{t_1t_2}|\beta_{23}\beta_{13}t_1|\right)$$

or

$$\sqrt{d(f)}\leqslant\frac{9}{4}a_{33}[t_1(1-\beta_{13}^2)+t_1^{-1}(1-\beta_{23}^2)$$
$$+2(\beta_{13}\beta_{23}-\beta_{12})]^2/(t_1+t_1^{-1}-2\beta_{12})$$
$$\left(t_1+t_1^{-1}+\frac{1}{t_1t_2^2}-2\beta_{12}-\frac{2}{t_1t_2}|\beta_{23}-\beta_{13}t_1|\right). \tag{14}$$

Now (3′) can be written as

$$\sqrt{d(f)}\leqslant\frac{9}{4}\mathrm{A}_{33}^2/a_{11}a_{22}(a_{11}+a_{22}-2a_{12})$$
$$=\frac{9}{4}(a_{11}a_{22}-a_{12}^2)^2/a_{11}a_{22}(a_{11}+a_{22}-2a_{12})$$
$$=\frac{9}{4}(a_{11}a_{22})^{1/2}(1-\beta_{12}^2)^2/(t_1+t_1^{-1}-2\beta_{12}),$$

using (C′), we have

$$\sqrt{a_{11}a_{22}a_{33}\Delta'}\leqslant\frac{9}{4}(a_{11}a_{22})^{1/2}(1-\beta_{12}^2)^2/(t_1+t_1^{-1}-2\beta_{12}),$$

so that

$$a_{33}\leqslant(9/4)^2(1-\beta_{12}^2)^4/(t_1+t_1^{-1}-2\beta_{12})^2\Delta'.$$

Substituting in (14), we get

$$\sqrt{d(f)} \leqslant (9/4)^3(1-\beta_{12}^2)^4[t_1(1-\beta_{13}^2) + t_1^{-1}(1-\beta_{23}^2)+2(\beta_{13}\beta_{23}-\beta_{12})]^2/ \Delta'(t_1+t_1^{-1}-2\beta_{12})^3\left(t_1+t_1^{-1}+\frac{1}{t_1t_2^2}-2\beta_{12}-\frac{2}{t_1t_2}|\beta_{23}-\beta_{13}t_1|\right). \tag{15}$$

Since

$$t_1 \leqslant 1; |\beta_{23}-\beta_{13}t_1| \leqslant 1,\ 1/t_2 \geqslant 1,$$

$$2x-2|\beta_{23}-\beta_{13}t_1| \geqslant 0 \text{ if } x=\frac{1}{t_2}\geqslant 1,$$

$$\sqrt{d(f)} \leqslant (9/4)^3(1-\beta_{12}^2)^4[t_1(1-\beta_{13}^2)+t_1^{-1}(1-\beta_{23}^2) +2(\beta_{13}\beta_{23}-\beta_{12})]^2/\Delta'\{(t_1+t_1^{-1}-2\beta_{12}-2t_1^{-1} |\beta_{23}-\beta_{13}t_1|(t_1+t_1^{-1}-2\beta_{12})^3\}. \tag{16}$$

Writing t for t_1 for convenience, we have

$$0 \leqslant t \leqslant 1,$$

and

$$\sqrt{d(f)} \leqslant (9/4)^3(1-\beta_{12}^2)^4[t+t^{-1}-2\beta_{12}-t\beta_{13}^2-t^{-1}\beta_{23}^2+2\beta_{13}\beta_{23}]^2/ \Delta'(t+t^{-1}-2\beta_{12})^3(t+2t^{-1}-2\beta_{12}-2t^{-1}|\beta_{23}-\beta_{13}t|). \tag{17}$$

Since

$$0 \leqslant \beta_{13}, \beta_{23}, \beta_{12} \leqslant \frac{1}{2},$$

$$\begin{aligned} & t(1-\beta_{13}^2)+t^{-1}(1-\beta_{23}^2)+2(\beta_{13}\beta_{23}-\beta_{12}) \\ & \geqslant 3/4(t+t^{-1})-2\beta_{12} \\ & \geqslant 3/2-1>0, \end{aligned}$$

$$2\beta_{13}\beta_{23} \leqslant t\beta_{13}^2+t^{-1}\beta_{23}^2,$$

we have, from (17),

$$\begin{aligned}\sqrt{d(f)} &\leqslant (9/4)^3(1-\beta_{12}^2)^4(t+t^{-1}-2\beta_{12})^2/ \\ &\quad \Delta'(t+t^{-1}-2\beta_{12})^3(t+2t^{-1}-2\beta_{12}-2t^{-1}|\beta_{23}-\beta_{13}t|) \\ &= (9/4)^3(1-\beta_{12}^2)^4/ \\ &\quad \Delta'(t+t^{-1}-2\beta_{12})(t+2t^{-1}-2\beta_{12}-2t^{-1}|\beta_{23}-\beta_{13}t|). \end{aligned} \tag{18}$$

Now, let

$$F(t)=t+\frac{2}{t}-2\beta_{12}-\frac{2}{t}|\beta_{23}-\beta_{13}t|.$$

If $\beta_{23} \geqslant \beta_{13} t$,

$$F'(t) = 1 - \frac{2}{t^2} + \frac{2\beta_{23}}{t^2} \leqslant 1 - \frac{2}{t^2} + \frac{1}{t^2}$$

$$= 1 - \frac{1}{t^2} \leqslant 0, \text{ because } t \leqslant 1,$$

while, if $\beta_{23} < \beta_{13} t$,

$$F'(t) = 1 - \frac{2}{t^2} - \frac{2\beta_{23}}{t^2} \leqslant 1 - \frac{2}{t^2} < 0.$$

Therefore, in all cases,

$$\begin{aligned} F(t) &\geqslant F(1) \\ &= 3 - 2\beta_{12} - 2|\beta_{23} - \beta_{13}| \\ &\geqslant 3 - 2\beta_{12} - 2 \times 0{\cdot}041 \\ &= 2{\cdot}918 - 2\beta_{12}, \end{aligned}$$

because $|\beta_{23} - \beta_{13}| \leqslant 0{\cdot}5 - 0{\cdot}459 = 0{\cdot}041$.

Also

$$t + \frac{1}{t} - 2\beta_{12} \geqslant 2 - 2\beta_{12}.$$

Therefore, (18) implies

$$\sqrt{d(f)} \leqslant (9/4)^3 (1 - \beta_{12}^2)^4 / (2{\cdot}918 - 2\beta_{12})(2 - 2\beta_{12})\Delta'. \tag{19}$$

Now

$$\Delta' = 1 + 2\beta_{12}\beta_{13}\beta_{23} - \beta_{12}^2 - \beta_{13}^2 - \beta_{23}^2,$$

$$\begin{aligned} \frac{\partial \Delta'}{\partial \beta_{13}} &= 2\beta_{12}\beta_{23} - 2\beta_{13} \\ &\leqslant 2\frac{1}{2}\frac{1}{2} - 2(0{\cdot}459) \\ &< 0. \end{aligned}$$

Similarly

$$\frac{\partial \Delta'}{\partial \beta_{23}} < 0,$$

therefore,

$$\begin{aligned} \Delta' &\geqslant 1 + 2\beta_{12}\frac{1}{2}\frac{1}{2} - \beta_{12}^2 - \frac{1}{4} - \frac{1}{4} \\ &= \frac{1}{2}(1 + \beta_{12} - 2\beta_{12}^2) \\ &= \frac{1}{2}(1 - \beta_{12})(1 + 2\beta_{12}). \end{aligned} \tag{20}$$

Writing β for β_{12}, for convenience, (19) gives

$$\begin{aligned}\sqrt{d(f)} &\leqslant (9/4)^3(1-\beta^2)^4/(2\cdot918-2\beta)(1+2\beta)(1-\beta)^2\\ &=\frac{1}{2}(9/4)^3\frac{(1-\beta)^2(1+\beta)^4}{(1\cdot459-\beta)(1+2\beta)}\\ &=\frac{1}{2}(9/4)^3\frac{1-\beta}{1\cdot459-\beta}\frac{(1-\beta)(1+\beta)^4}{1+2\beta}\\ &=\frac{1}{2}(9/4)^3 g(\beta)h(\beta), \text{ say.}\end{aligned} \tag{21}$$

Now

$$g(\beta)=\frac{1-\beta}{1\cdot459-\beta}=1-\frac{0\cdot459}{1\cdot459-\beta}$$

is a decreasing function of β. Therefore,

$$g(\beta)\leqslant g(0\cdot459). \tag{22}$$

Again

$$h(\beta)=(1-\beta)(1+\beta)^4/(1+2\beta),$$

$$\begin{aligned}\frac{h'(\beta)}{h(\beta)}&=\frac{-1}{1-\beta}+\frac{4}{1+\beta}-\frac{2}{1+2\beta}\\ &=\frac{4+4\beta-8\beta^2-1-3\beta-2\beta^2-2+2\beta^2}{(1-\beta)^2(1+2\beta)}\\ &=-\frac{(8\beta^2-\beta-1)}{(1-\beta)^2(1+2\beta)}<0,\end{aligned}$$

because

$$\begin{aligned}8\beta^2-\beta-1&\geqslant 8(0\cdot459)^2-(0\cdot459)-1\\ &>8(0\cdot459)^2-(0\cdot459)-1\\ &=1\cdot62-1\cdot459>0.\end{aligned}$$

Therefore

$$h(\beta)\leqslant h(0\cdot459), \text{ and}$$

$$\begin{aligned}\sqrt{d(f)}&\leqslant\frac{1}{2}(9/4)^3 g(0\cdot459)h(0\cdot459)\\ &=\frac{729}{128}\frac{(1-0\cdot459)^2(1+0\cdot459)^4}{1(1+0\cdot918)}=3\cdot93\ldots<4.\end{aligned}$$

Thus $d(f)<16$ in this case also and the proof of Theorem IC is complete.

8. Proof of Theorem II′

8.1 Let K be the sphere $|x|\leqslant 3/4$ and Λ the lattice generated by $(1,1,0)$, $(0,1,1)$, $(1,0,1)$. We have to show that every straight line l meets a $k+A$, $A\in\Lambda$.

We divide the proof into two parts:

(a) The lines l are parallel to "lattice lines" OA, $A \in \Lambda$,
(b) l is not parallel to any lattice line.

9. Proof of Theorem II′ Case (a)

9.1 The quadratic form

$$f(x_1, x_2, x_3) = (x_1 + x_2)^2 + (x_2 + x_3)^2 + (x_3 + x_1)^2$$

$$= 2\Sigma x_i^2 + 2 \sum_{1 \leqslant i < j \leqslant 3} x_i x_{ij}$$

is the quadratic form of Λ corresponding to the given basis. The adjoint of f is

$$F(x_1, x_2, x_3) = 3\Sigma x_i^2 - 2 \sum_{1 \leqslant i < j \leqslant 3} x_i x_j.$$

As explained in § 2.3, Theorem II′ in case (a) will follow if we can show that for every partial sum G of F, $R(G) \leqslant \frac{9}{16} d(G)/\sqrt{d(F)}$. We note that $F(x_1, x_2, x_3) = (x_1 + x_2 - x_3)^2 + (x_2 + x_3 - x_1)^2 + (x_3 + x_1 - x_2)^2$. For integers x_i, $x_1 + x_2 - x_3$, $x_2 + x_3 - x_1$, $x_3 + x_1 - x_2$ are all even or all odd. Therefore, the possible non-zero values of F for integers x_i are 3, 4, 8, 11,... in ascending order, i.e. the values can be 3, 4 or $\geqslant 8$.

Let $G'(x_1, x_2)$ be a partial sum of F and $G(x_1, x_2) = ax_1^2 + 2bx_1x_2 + cx_2^2$, $0 \leqslant 2b \leqslant a \leqslant c$, $a > 0$, be the reduced form equivalent to G'. Then

$$R(G') = R(G) = ac(a + c - 2b)/4(ac - b^2)$$

and we have to prove

$$ac(a + c - 2b) \leqslant 9/16\,(ac - b^2)^2, \tag{I}$$

because $d(F) = 16$.

We shall prove this by contradiction, i.e. we shall show that

$$ac(a + c - 2b) > 9/16(ac - b^2)^2$$

is not possible.

Since the values of G for integers x_i are a subset of the values of F for integers x_i, we have the following possibilities:

(i) $a = 3$, (ii) $a = 4$, (iii) $a \geqslant 8$.

(i) $a = 3$, so that $b = 0$ or 1, $c \geqslant 3$.

If $b = 0$, $ac(a + c - 2b) > 9/16(ac - b^2)^2$, then

$$3ac(3 + c) > 9/16(3c)^2$$

i.e.

$$11c^2 - 48c < 0$$

i.e.

$$c(11c-48)<0,$$

and

$$c=3 \text{ or } c=4, \text{ and}$$

$$G(x_1,x_2)=3x_1^2+3x_2^2 \text{ or } 3x_1^2+4x_2^2$$

takes the value 6 or 7 for integers x_i. Since 6, 7 are not possible values of F, this case is not possible. If

$$b=1,\ ac(a+c-2b)>9/16(ac-b^2)^2,$$

then

$$16c(1+c)>3(3c-1)^2$$

i.e.

$$11c^2-34c+3<0$$

i.e.

$$(c-3)(11c-1)<0,$$

which is impossible, because $c\geqslant 3$.
(ii) Let $a=4$, so that $b=0$, 1 or 2 and $c\geqslant 4$.
Then $ac(a+c-2b)>9/16(ac-b^2)^2$ implies

$$64c(4+c-2b)>9(4c-b^2)^2$$

or

$$80c^2-c(72b^2-128b+256)+9b^4<0.$$

$b=0$ gives

$$80c^2<256c$$

and $c<4$, which is impossible,
$b=1$ gives

$$80c^2-200c+9=80c(c-4)+120c+9<0,$$

which is not possible, because $c\geqslant 4$,
and $b=2$ gives

$$80c^2-288c+144=80c(c-4)+32c+144<0,$$

which is again not possible.
(iii) $a\geqslant 8$.
By the Theorem of Lagrange, since G is reduced,

$$ac\leqslant 4/3\,d(G)=4/3(ac-b^2),$$

so that

$$(ac-b^2)\geqslant 3/4ac,$$

and

$$ac(a+c-2b)>9/16(ac-b^2)^2$$

implies

$$ac(a+c-2b)>9/16\ 9/16a^2c^2$$

and

$$(a+c) > \frac{81}{256}ac,$$

so that

$$\frac{1}{a}+\frac{1}{c} \geqslant \frac{81}{256}.$$

But

$$a \geqslant 8,\ c \geqslant 8, \text{ and}$$

$$\frac{1}{a}+\frac{1}{c} \leqslant \frac{1}{8}+\frac{1}{8}=\frac{1}{4}<\frac{81}{256},$$

which shows that this case is also impossible.

We have thus completed the proof of Theorem II′ in case (a).

10. Proof of Theorem II′ Case (b)

10.1 Let l be a straight line not parallel to a lattice line. Let Π be the plane through O perpendicular to l. Let Λ_1 be the projection of Λ on Π. Then the lines parallel to l meet the spheres $K + A$, $A \in \Lambda$ if and only if the circles $C + A$, $A \in \Lambda_1$ cover Π, where C is the circle $K \cap \Pi$, i.e. C is the circle of radius 3/4. We have then to show that every point of Π is within the distance 3/4 from some point of Λ_1.

If Proj A = projection of the point A of R^3 on Π, then Proj (A − B) = Proj A − Proj B, and it follows that Λ_1 is an additive subgroup of the group Π under addition. Also, since Λ is "three-dimensional", Λ_1 is "two-dimensional". One can easily see that for Λ_1, we have the following possibilities:

(i) If O is not a limit point of Λ_1, then Λ_1 is a two-dimensional lattice, and since Proj $(mA + nB) = m$ Proj A $+ n$ Proj B, one can easily see that l is parallel to a lattice line OA of Λ, and this case does not arise,

(ii) If O is a limit point of Λ_1, and all points of Λ_1 near enough to O lie on a straight line α through O, then Λ_1 is dense on α, and consists of points lying dense on lines parallel to α at the same distance δ say, between consecutive ones, and

(iii) Λ_1 is dense everywhere in Π, in which case there is nothing to prove.

We have, therefore, to consider case (ii) only. In this case Λ is distributed in the planes orthogonal to Π through the lines parallel to α of Λ_1. These planes are at a distance δ apart (i.e. consecutive planes are at a distance δ from each other). The part of Λ in the plane through α is a two dimensional lattice Λ_2 and the parts in other planes are its translates. The determinant $d(\Lambda) = \delta.\ d(\Lambda_2)$, where $d(\Lambda_2)$ is the determinant of Λ_2.

We notice that the squares of the distances between lattice points of Λ are the values of $f = 2\Sigma x_i^2 - 2\Sigma x_i x_j$, so that these squared distances are at least 2, and Λ provides a packing for spheres of radius $(1/2)\sqrt{2}$. Therefore, Λ_2 provides a packing for circles of radius $1/\sqrt{2}$. Since the density of the closest lattice packings of circles is $\pi/2\sqrt{3}$, we get

$$\pi/2d(\Lambda_2) \leqslant \pi/2\sqrt{3}$$

and

$$d(\Lambda_2) \geqslant \sqrt{3}.$$

Since

$$d(\Lambda) = 2,\ \delta \leqslant 2/\sqrt{3} < 3/2.$$

Thus the distance δ between consecutive lines parallel to l on which Λ_1 is dense is $< 3/2$. Let $P \in \Pi$, then P is at a distance $\leqslant \delta/2 < 3/4$ from one of these lines and at a distance $< 3/4$ from some point of Λ_1, which completes the proof.

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