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Finite Strain in Elastic Problems—II

By W. M. Shepherd and B. R. Seth

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1—Introduction

In a recent paper* one of the authors has developed the theory of finite strain in elastic problems on the hypothesis that the second order terms in the components of strain may not be neglected. Owing† to the magnitude of the displacements considered, it has been necessary to use throughout the coordinates of the points of the body after strain instead of those before strain, as is always done in the theory of small strains. The method has already been applied, in the paper mentioned above, to a few particular cases, viz., (i) a cylinder subjected to uniform tension; (ii) a rectangular plate bent into the form of a right cylinder; and (iii) the torsion of a right circular cylinder. As may be expected, the tension stretch curve is not now a straight line as in the ordinary theory, but is more like that found in practice for some materials.

The object of the present paper is to apply the hypothesis to the following further problems:—

(i) a thick spherical shell subjected to uniform, but not necessarily equal, normal tractions on the inner and outer surfaces;

(ii) a thick cylindrical shell or tube under the same type of internal and external traction as in (i), but which is constrained by end tractions to remain in a state of plane strain;

(iii) a thick cylindrical tube turned inside out under the same types of traction as in (ii);

(iv) a thick cylindrical tube turned inside out under no surface tractions;

(v) the particular forms of problems (i) and (ii) arising when the shells are thin.

2—General Formulae

Let (u, v, w) be the components of the displacement of a point whose coordinates in the strained state are (x, y, z). The strains are then given

* Seth, 'Phil. Trans.,' A, vol. 234, pp. 231–264 (1935).
† Coker and Filon, "Photo-elasticity," p. 188.

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by equations of the types*

$$S_x = \frac{\partial u}{\partial x} - \tfrac{1}{2}\left\{\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial x}\right)^2\right\},$$

$$\sigma_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} - \left\{\frac{\partial u}{\partial y}\frac{\partial u}{\partial z} + \frac{\partial v}{\partial y}\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\frac{\partial w}{\partial z}\right\}.$$

The stress-strain relations are the same as those holding when the strains are small, viz.,

$$\widehat{xx} = \lambda\,\delta + 2\mu S_x,$$

$$\widehat{yz} = \mu\sigma_{yz}, \text{ etc.}$$

3—The Solution for the Thick Spherical Shell

Let us take rectangular cartesian coordinates referred to an origin at the centre of the shell and spherical polar coordinates connected with them by the relations

$$x = r\sin\theta\cos\phi, \qquad y = r\sin\theta\sin\phi, \qquad z = r\cos\theta.$$

It is clear from the symmetry of the problem that the displacements are purely radial and a function of r only. Let this radial displacement be u_r and assume that

$$u_r = (1 - P)\,r,$$

where P is a function of r only.

The displacements in the directions Ox, Oy, Oz are then given by

$$u = (1 - P)\,x, \qquad v = (1 - P)\,y, \qquad w = (1 - P)\,z,$$

and we find that the stresses are given by equations of the types

$$\widehat{xx} = \tfrac{1}{2}(3\lambda + 2\mu)(1 - P^2) - (\lambda r^2 + 2\mu x^2)\left\{\frac{P}{r}\frac{dP}{dr} + \tfrac{1}{2}\left(\frac{dP}{dr}\right)^2\right\}, \tag{3.11}$$

$$\widehat{xy} = -\,2\mu\,xy\left\{\frac{P}{r}\frac{dP}{dr} + \tfrac{1}{2}\left(\frac{dP}{dr}\right)^2\right\}. \tag{3.12}$$

Each body stress equation leads to the same equation for P, viz.,

$$4(\lambda + 2\mu)\frac{P}{r}\frac{dP}{dr} + 2(\lambda + 3\mu)\left(\frac{dP}{dr}\right)^2 + (\lambda + 2\mu)\left(r\frac{dP}{dr} + P\right)\frac{d^2P}{dr^2} = 0. \tag{3.21}$$

On putting

$$\frac{r}{P}\frac{dP}{dr} = V, \qquad \frac{\lambda + 3\mu}{\lambda + 2\mu} = F,$$

* Coker and Filon, "Photo elasticity," p. 188.

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equation (3.21) becomes

$$(1 + V) P \frac{dV}{dP} + V^2 + 2FV + 3 = 0, \tag{3.22}$$

so that

$$\log P = - \int \frac{(1 + V) dV}{V^2 + 2FV + 3}. \tag{3.23}$$

Now

$$F = \frac{\lambda + 3\mu}{\lambda + 2\mu} = \frac{3 - 4\eta}{2(1 - \eta)},$$

where η is Poisson's ratio, and since $0 < \eta < \frac{1}{2}$ it follows that

$$1 < F < 3/2.$$

Making use of this result and integrating equation (3.23) we obtain

$$\log P = \frac{F - 1}{\sqrt{(3 - F^2)}} \tan^{-1} \left\{ \frac{V + F}{\sqrt{(3 - F^2)}} \right\} - \tfrac{1}{2} \log (V^2 + 2FV + 3) + H, \tag{3.24}$$

where H is a constant.

Since

$$r \frac{dV}{dr} = PV \frac{dV}{dP}, \tag{3.25}$$

we have, using equation (3.22),

$$\log r = - \int \frac{(1 + V) dV}{V(V^2 + 2FV + 3)},$$

i.e.,

$$\log r = \tfrac{1}{6} \log (V^2 + 2FV + 3) - \tfrac{1}{6} \log (V^2) - \frac{3 - F}{3\sqrt{(3 - F^2)}} \tan^{-1} \left\{ \frac{V + F}{\sqrt{(3 - F^2)}} \right\} + K, \tag{3.31}$$

where K is a constant.

Equations (3.24) and (3.31) give the relation between P and r in terms of the parameter V. The two constants K and H must be determined from the boundary conditions.

The values of V are not wholly unrestricted. If r' is the distance of a point of the shell from the centre in the unstrained state, which is at a distance r in the strained state, then r and r' must increase together, *i.e.*, dr'/dr is positive throughout. But

$$r' = r - u_r = Pr,$$

so that

$$\frac{dr'}{dr} = r \frac{dP}{dr} + P = P(1 + V). \tag{3.32}$$

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Since $u_r < r$, it follows that $P > 0$ and consequently we obtain the condition

$$V > -1.$$

It follows from equation (3.11) that

$$\widehat{rr} = \tfrac{1}{2}(3\lambda + 2\mu)(1 - P^2) - \tfrac{1}{2}(\lambda + 2\mu)P^2(2V + V^2). \qquad (3.41)$$

This may be written, introducing a new symbol R,

$$R \equiv 1 - \frac{2\widehat{rr}}{3\lambda + 2\mu} = \frac{P^2}{1+\eta}\{2\eta + (1-\eta)(1+V)^2\}. \qquad (3.42)$$

To go further in the solution it is necessary to assign a value to Poisson's ratio. The problem has been solved for two values of η: (i) $\eta = 0{\cdot}25$, and (ii) $\eta = 0{\cdot}49$ (india-rubber). Any other value in the range $0 < \eta < \frac{1}{2}$ might have been used without altering the method of procedure.

(*a*) $\eta = 0{\cdot}25, \quad (\lambda = \mu).$

With this value of Poisson's ratio, equations (3.24), (3.31), and (3.42) now take the forms

$$\log(AP) = -\tfrac{1}{2}\log\{(3V+4)^2 + 11\} + \frac{1}{\sqrt{11}}\tan^{-1}\left\{\frac{3V+4}{\sqrt{11}}\right\}, \qquad (3.51)$$

$$\log(Br) = -\tfrac{1}{6}\log(V^2) + \tfrac{1}{6}\log\{(3V+4)^2 + 11\} - \frac{5}{3\sqrt{11}}\tan^{-1}\left\{\frac{3V+4}{\sqrt{11}}\right\}, \qquad (3.52)$$

$$\log(A^2R) = 2\log(AP) - \log 5 + \log\{3(1+V)^2 + 2\}. \qquad (3.53)$$

In these equations and throughout the paper the logarithms are to the base *e*. A and B are new constants replacing H and K. The relation

TABLE I

V	log (AP)	log (B*r*)	log (A²R)	V	log (AP)	log (B*r*)	log (A²R)
∞	−∞	−0·423	−1·761	1/4	−1·467	0·565	−2·643
10	−3·087	−0·331	−1·883	1/6	−1·439	0·701	−2·682
7	−2·794	−0·296	−1·929	1/12	−1·411	0·934	−2·723
5	−2·538	−0·253	−1·985	0	−1·383	∞	−2·766
4	−2·382	−0·218	−2·029	−1/12	−1·355	0·940	−2·812
3	−2·198	−0·164	−2·094	−1/6	−1·328	0·713	−2·859
2	−1·978	−0·075	−2·198	−1/3	−1·276	0·496	−2·958
3/2	−1·849	−0·001	−2·276	−1/2	−1·229	0·378	−3·056
1	−1·707	0·115	−2·384	−2/3	−1·190	0·314	−3·143
3/4	−1·630	0·204	−2·455	−5/6	−1·164	0·278	−3·203
1/2	−1·550	0·334	−2·540	−1	−1·154	0·267	−3·225

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between A^2R and Br is now known in parametric form. The values of $\log(AP)$, $\log(Br)$, and $\log(A^2R)$ corresponding to certain values of V have been calculated and are given in Table I.

(*b*) $\eta = 0{\cdot}49, \quad (\lambda = 49\,\mu).$

Equations (3.24), (3.31), (3.42) now take the forms

$$\log(AP) = 0{\cdot}0140043 \tan^{-1}(0{\cdot}714214\,V + 0{\cdot}728218) - \tfrac{1}{2}\log(V^2 + 2{\cdot}039216\,V + 3), \quad (3.61)$$

$$\log(Br) = -\,0{\cdot}471475 \tan^{-1}(0{\cdot}714214\,V + 0{\cdot}728218) + \tfrac{1}{6}\log(V^2 + 2{\cdot}039216\,V + 3) - \tfrac{1}{6}\log(V^2), \quad (3.62)$$

$$\log(A^2R) = 2\log(AP) + \log\{(1+V)^2 + 1{\cdot}92157\} - 1{\cdot}0721211, \quad (3.63)$$

and the corresponding values of $\log(AP)$, $\log(Br)$, and $\log(A^2R)$ are given in Table II.

TABLE II

V	−log (AP)	log (B*r*)	−log (A²R)	V	−log (AP)	log (B*r*)	−log (A²R)
∞	∞	−0·7406	1·02813	1/4	0·6263	0·3270	1·07650
10	2·3875	−0·6460	1·03548	1/6	0·5973	0·4682	1·07799
7	2·0773	−0·6088	1·03831	1/12	0·5686	0·7064	1·07949
5	1·8026	−0·5622	1·04174	0	0·5405	∞	1·08098
4	1·6326	−0·5242	1·04445	−1/12	0·5131	0·7242	1·08247
3	1·4311	−0·4660	1·04840	−1/6	0·4867	0·5041	1·08391
2	1·1866	−0·3661	1·05455	−1/3	0·4378	0·2994	1·08661
3/2	1·0438	−0·2837	1·05899	−1/2	0·3961	0·1972	1·08890
1	0·8856	−0·1551	1·06478	−2/3	0·3639	0·1412	1·09060
3/4	0·8012	−0·0578	1·06828	−5/6	0·3435	0·1135	1·09161
1/2	0·7142	0·0834	1·07221	−1	0·3365	0·1056	1·09193

The relation between $\log(Br)$ and $\log(A^2R)$ is shown graphically in fig. 1, and that between $\log(Br)$ and $\log(AP)$ in fig. 2. These graphs are to be used for the solution of all forms of the problem arising from different boundary conditions. To obtain accurate results the reader must draw his own graphs on a large scale, using the data of the tables.

Now suppose that the boundary conditions are

$$\widehat{rr} = \alpha, \quad \text{when} \quad r = a,$$

$$\widehat{rr} = \beta, \quad \text{when} \quad r = b,$$

and that the corresponding values of R are R_α and R_β. It follows that an increment of $\log(b/a)$ in $\log(Br)$ corresponds to an increment

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$\log(R_\beta/R_\alpha)$ in $\log(A^2R)$. In a particular case these increments are represented by the line LM (fig. 1). We must now find the chord of the curve, equal and parallel to LM. This chord is HK. If H is the point where

$$\log(A^2R) = q \qquad \text{and} \qquad \log(Br) = p,$$

then A and B are given by the equations

$$2\log A = q - \log R_\alpha,$$
$$\log B = p - \log a.$$

A and B having been determined the values of R (and hence $\widehat{rr}$) at all points for which $a < r < b$ are given by the points of the curve between H and K. The stress $\widehat{\theta\theta}$ ($= \widehat{\phi\phi}$) is determined as follows. We have

$$\widehat{rr} = \tfrac{1}{2}(3\lambda + 2\mu)(1 - P^2) - \tfrac{1}{2}(\lambda + 2\mu)P^2V(V+2) \qquad (3.71)$$

$$\widehat{\theta\theta} = \widehat{\phi\phi} = \tfrac{1}{2}(3\lambda + 2\mu)(1 - P^2) - \tfrac{1}{2}\lambda P^2V(V+2). \qquad (3.72)$$

On eliminating V between these two equations we obtain the relation

$$(\lambda + 2\mu)\widehat{\theta\theta} = \mu(3\lambda + 2\mu)(1 - P^2) + \lambda\widehat{rr}. \qquad (3.73)$$

The quantity P must be found in terms of r from the graph in fig. 2. The stresses in an india-rubber shell corresponding to two sets of boundary conditions have been calculated in order to illustrate the method and to discover the characteristics of the solution.

Example I—The boundary conditions chosen are

$$a = 3, \qquad R_\alpha = 1 \cdot 02,$$
$$b = 5, \qquad R_\beta = 1 \cdot 0.$$

This means that a shell of internal radius 3 and external radius 5 (both measured after strain) is subjected to an internal pressure of amount $0 \cdot 0304\,\lambda$ and is free from traction on the outer surface. The corresponding line and chord are respectively LM and HK in fig. 1. We find that

$$\log B = -1 \cdot 490, \qquad 2\log A = -1 \cdot 0729.$$

The stresses corresponding to several values of V have been calculated by the method described above and are shown in Table III. The last figure in these values may have a small error.

It is rather unexpected to find that the stress $\widehat{\theta\theta}$ is greatest at about half-way through the material, but the total variation is not large.

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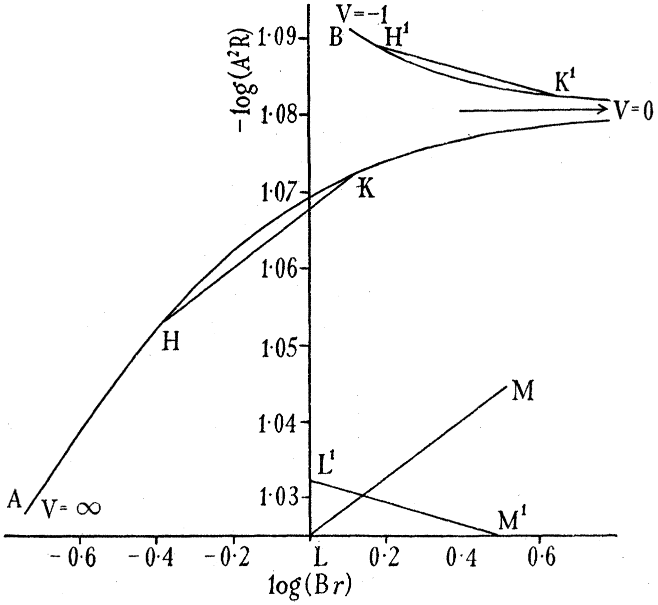

FIG. 1

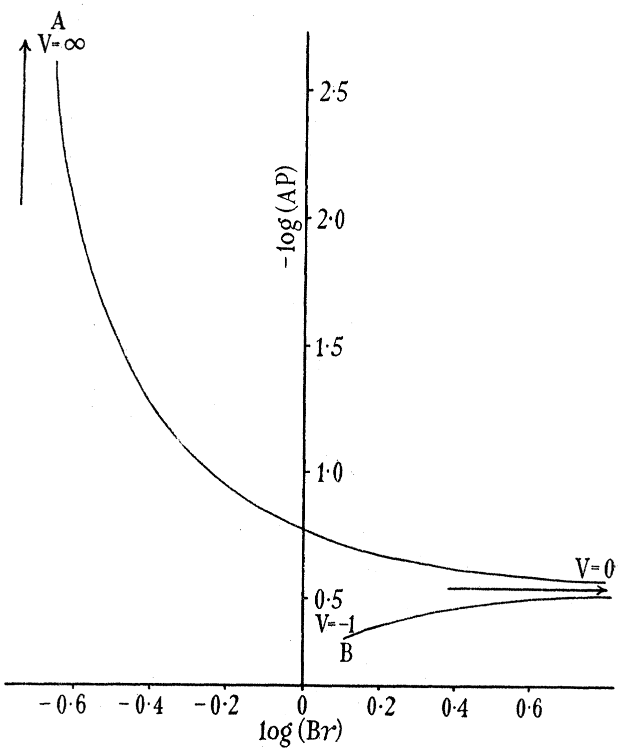

FIG. 2.

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TABLE III

r	3·00	3·25	3·50	3·75	4·00	4·25	4·50	4·75	5·00
$\widehat{\theta\theta}/\lambda$	0·0159	0·0165	0·0174	0·0179	0·0179	0·0178	0·0173	0·0165	0·0161
$-\widehat{rr}/\lambda$	0·0304	0·0236	0·0181	0·0132	0·0094	0·0062	0·0040	0·0018	0·0000

We find that the value of P at the inner surface is 0·493 and at the outer surface is 0·854, so that before strain the internal and external radii were 1·48 and 4·27. The increase in diameter of the shell is, as would be expected, accompanied by a decrease in thickness.

The magnitudes of the displacements in this case are such that no comparison with the results of the small strain theory is possible.

Example II—The boundary conditions chosen are

$$a = 3, \quad R_\alpha = 1\cdot 0,$$
$$b = 5, \quad R_\beta = 1\cdot 0075.$$

The shell is subjected to an external pressure of amount 0·0114 λ and is free from traction on the inside. The corresponding line and chord are L^1M^1 and H^1K^1 (in fig. 1) and we find that

$$\log B = -0\cdot 938, \qquad 2\log A = -1\cdot 0901.$$

Again the stresses have been calculated and are given in Table IV.

TABLE IV

r	3·00	3·25	3·50	3·75	4·00	4·25	4·50	4·75	5·00
$-\widehat{\theta\theta}/\lambda$	0·0250	0·0213	0·0193	0·0178	0·0171	0·0167	0·0162	0·0158	0·0155
$-\widehat{rr}/\lambda$	0·0000	0·0040	0·0059	0·0076	0·0091	0·0100	0·0106	0·0111	0·0114

The values of P on the inner and outer boundaries are found to be 1·191 and 1·038, so that the internal and external radii in the unstrained state are 3·57 and 5·19.

The graphs for the case of $\eta = 0\cdot 25$ are very similar to those in figs. 1 and 2 and the results found will be of the same type.

In one particular case the solution may be obtained without the use of the graphs. If the values of $\widehat{rr}$ at the inner and outer boundaries are the same, or what amounts to the same thing, there is no central hole, it is clear from the form of the graph (fig. 1) that A^2R has the value corresponding to $V = 0$. (This will be true for all values of η.) Equation (3.42) then shows that $P^2 = R$. Both R and P are constant throughout the material and we have

$$\left(\frac{r'}{r}\right)^2 = P^2 = R = 1 - \frac{2\widehat{rr}}{3\lambda + 2\mu}.$$

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From the ordinary theory

$$\left(\frac{r'}{r}\right)^2 = \left(1 - \frac{\widehat{rr}}{3\lambda + 2\mu}\right)^2,$$

so that the two are equivalent if $\widehat{rr}/(3\lambda + 2\mu)$ is sufficiently small to make its square negligible.

4—Deductions from the Solution for the Spherical Shell

It appears from the graph in fig. 1 that there are restrictions on the boundary conditions. We must consider only those sets of boundary conditions which correspond to chords of the curve. Owing to the existence of the end points A and B, corresponding to $V = \infty$ and $V = -1$, the gradient of the curve lies between certain limits, and it follows that the gradient of the chord also lies between these limits. (This is a necessary but not sufficient condition.) From equations (3.22), (3.25), (3.42) we have

$$\frac{dR}{dV} = \frac{2P^2(V+1)(V+2)}{V^2 + 2FV + 3} \cdot \frac{1-2\eta}{1+\eta},$$

and

$$\frac{1}{r}\frac{dr}{dV} = -\frac{1+V}{V(V^2 + 2FV + 3)}.$$

On combining these two results, it follows that

$$\frac{d(\log R)}{d(\log r)} = \frac{r\,dR}{R\,dr} = -\frac{2P^2V(V+2)}{R} \cdot \frac{1-2\eta}{1+\eta},$$

and, using equation (3.42),

$$\frac{d(\log R)}{d(\log r)} = -\frac{2V(V+2)(1-2\eta)}{2\eta + (1-\eta)(1+V)^2}. \tag{4.1}$$

From equations (3.71) and (3.72) we obtain

$$\widehat{\theta\theta} - \widehat{rr} = \mu P^2 V(V+2),$$

so that the gradient is given by

$$\frac{d(\log R)}{d(\log r)} = \frac{4(\widehat{rr} - \widehat{\theta\theta})}{3\lambda + 2\mu - 2\widehat{rr}}. \tag{4.2}$$

At A where $V = \infty$

$$\frac{d(\log R)}{d(\log r)} = -\frac{2(1-2\eta)}{1-\eta},$$

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and at B where $V = -1$

$$\frac{d(\log R)}{d(\log r)} = \frac{1-2\eta}{\eta}.$$

Now we have shown (3.32) that

$$\frac{dr'}{dr} = P(1+V),$$

so that at A

$$\frac{dr}{dr'} = 0,$$

and at B

$$\frac{dr'}{dr} = 0.$$

It follows that the point A corresponds to infinite contraction and the point B to infinite extension. These may be interpreted as yield points for the material.

The limitation on the values of the gradient implies that

$$-\frac{1-2\eta}{1-\eta} < \frac{2(\widehat{rr}-\widehat{\theta\theta})}{3\lambda+2\mu-2\widehat{rr}} < \frac{1-2\eta}{2\eta}, \tag{4.3}$$

or, writing this in a more symmetrical form,

$$\frac{\eta}{1-\eta} < \frac{3\lambda+2\mu-2\widehat{\theta\theta}}{3\lambda+2\mu-2\widehat{rr}} < \frac{1}{2\eta}. \tag{4.4}$$

In the form (4.3) the condition has an interesting interpretation. Two of the rival hypotheses as to the condition for the breakdown of the material appear to be combined in it. The condition may be violated owing to the greatness of one of the principal stresses or by the greatness of the principal stress difference. This matter will be referred to again in § 6 of the paper.

It is clear from the position of the end-points of the graph in fig. 1 that when breakdown occurs, either through infinite extension or infinite contraction, it must occur first on the inner surface of the shell.

There is one other case in which breakdown occurs as a result of finite surface tractions. If the internal and external surface tractions are equal we have seen that $R = P^2$, and so, if $R = 0$, *i.e.*, $\widehat{rr} = \frac{1}{2}(3\lambda + 2\mu)$, it follows that $P = 0$ and the material is infinitely stretched.

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5—The Solution for the Cylindrical Shell

The solution of the corresponding problem for a circular cylindrical shell or tube held at its ends, so that it is in a state of plane strain, is very similar to that for a spherical shell already given.

Take rectangular cartesian axes Ox, Oy, Oz such that Oz is the axis of the cylindrical surfaces and let ρ, θ, z be cylindrical coordinates related to the cartesian coordinates by the equations

$$x = \rho \cos \theta, \qquad y = \rho \sin \theta.$$

In this case assume that

$$u = (1 - Q)\, x, \qquad v = (1 - Q)\, y, \qquad w = 0,$$

where Q is a function of ρ only.

If ρ' is the distance from the axis before strain, of a particle which after strain is at a distance ρ, then $\rho' = \rho Q$. The stresses are given by the equations

$$\widehat{xx} = (\lambda + \mu)(1 - Q^2) - (\lambda\rho^2 + 2\mu x^2)\left\{\frac{Q}{\rho}\frac{dQ}{d\rho} + \tfrac{1}{2}\left(\frac{dQ}{d\rho}\right)^2\right\}, \tag{5.11}$$

$$\widehat{yy} = (\lambda + \mu)(1 - Q^2) - (\lambda\rho^2 + 2\mu y^2)\left\{\frac{Q}{\rho}\frac{dQ}{d\rho} + \tfrac{1}{2}\left(\frac{dQ}{d\rho}\right)^2\right\}, \tag{5.12}$$

$$\widehat{zz} = \lambda(1 - Q^2) - \lambda\rho^2\left\{\frac{Q}{\rho}\frac{dQ}{d\rho} + \tfrac{1}{2}\left(\frac{dQ}{d\rho}\right)^2\right\}, \tag{5.13}$$

$$\left.\begin{aligned} &\widehat{yz} = \widehat{zx} = 0, \\ &\widehat{xy} = -2\mu xy\left\{\frac{Q}{\rho}\frac{dQ}{d\rho} + \tfrac{1}{2}\left(\frac{dQ}{d\rho}\right)^2\right\} \end{aligned}\right\}. \tag{5.14}$$

One of the body stress equations is identically satisfied and the other two each give

$$\frac{3Q}{\rho}\frac{dQ}{d\rho} + \frac{2\lambda + 5\mu}{\lambda + 2\mu}\left(\frac{dQ}{d\rho}\right)^2 + \left(Q + \rho\frac{dQ}{d\rho}\right)\frac{d^2Q}{d\rho^2} = 0. \tag{5.21}$$

On putting

$$\frac{\rho}{Q}\frac{dQ}{d\rho} = U, \qquad \frac{2\lambda + 5\mu}{2(\lambda + 2\mu)} = G,$$

equation (5.21) becomes

$$(1 + U)\, Q\frac{dU}{dQ} + U^2 + 2GU + 2 = 0. \tag{5.22}$$

Since

$$\frac{2\lambda + 5\mu}{2(\lambda + 2\mu)} = \frac{5 - 6\eta}{4(1 - \eta)}$$

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and

$$0 < \eta < \tfrac{1}{2},$$

it follows that

$$1 < G < 5/4.$$

Using this condition and integrating equation (5.22), we obtain

$$\log Q = \frac{G-1}{\sqrt{(2-G^2)}} \tan^{-1}\left\{\frac{G+U}{\sqrt{(2-G^2)}}\right\} - \tfrac{1}{2}\log\{(U+G)^2 + 2 - G^2\} + L, \quad (5.23)$$

where L is a constant.

Again, since

$$\frac{\rho}{Q}\frac{dQ}{d\rho} = U,$$

we have from equation (5.22)

$$\frac{1}{\rho}\frac{d\rho}{dU} = -\frac{1+U}{U(U^2+2GU+2)}, \quad (5.31)$$

and on integration this gives

$$\log \rho = \tfrac{1}{4}\log(U^2 + 2GU + 2) - \tfrac{1}{4}\log(U^2) - \frac{2-G}{2\sqrt{(2-G^2)}} \tan^{-1}\left\{\frac{U+G}{\sqrt{(2-G^2)}}\right\} + M, \quad (5.32)$$

where M is a constant.

The relation between Q and ρ is given by the equations (5.23) and (5.32) in terms of the parameter U.

It is easily shown that

$$\widehat{\rho\rho} = \lambda + \mu - \tfrac{1}{2}Q^2\{\lambda + (\lambda + 2\mu)(U+1)^2\},$$

so that, if we put

$$R = 1 - \frac{\widehat{\rho\rho}}{\lambda + \mu},$$

we obtain

$$R = Q^2\{\eta + (1-\eta)(U+1)^2\}. \quad (5.41)$$

On giving η the value 0·25 (*i.e.*, putting $\lambda = \mu$), equations (5.23), (5.32), (5.41) take the forms

$$\log(AQ) = \frac{1}{\sqrt{23}} \tan^{-1}\left\{\frac{7+6U}{\sqrt{23}}\right\} - \tfrac{1}{2}\log\{(7+6U)^2 + 23\}, \quad (5.51)$$

$$\log(B\rho) = \tfrac{1}{4}\log\{(7+6U)^2 + 23\} - \tfrac{1}{4}\log(U^2) - \frac{5}{2\sqrt{23}} \tan^{-1}\left\{\frac{7+6U}{\sqrt{23}}\right\}, \quad (5.52)$$

$$\log(A^2R) = 2\log(AQ) - \log 4 + \log\{3(V+1)^2 + 1\}, \quad (5.53)$$

where A and B are new constants replacing L and M.

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The values of log (AQ), log (Bρ), log (A^2R) corresponding to chosen values of U are given in Table V.

TABLE V

U	log (AQ)	log (Bρ)	log (A^2R)	U	log (AQ)	log (Bρ)	log (A^2R)
∞	∞	0·078	−3·216	−1/2	−1·687	0·900	−4·200
10	−3·895	0·171	−3·278	−2/3	−1·616	0·778	−4·331
6	−3·463	0·227	−3·315	−5/6	−1·566	0·709	−4·437
4	−3·150	0·291	−3·356	−1	−1·546	0·687	−4·479
3	−2·949	0·349	−3·392	−3/2	−1·730	0·827	−4·287
2	−2·699	0·451	−3·453	−7/4	−1·913	0·940	−4·223
1	−2·375	0·680	−3·571	−2	−2·104	1·042	−4·207
1/2	−2·172	0·964	−3·682	−5/2	−2·448	1·196	−4·234
1/6	−2·018	1·475	−3·796	−3	−2·727	1·298	−4·275
1/12	−1·977	1·813	−3·832	−4	−3·142	1·418	−4·337
0	−1·936	—	−3·872	−6	−3·674	1·528	−4·404
−1/12	−1·894	1·799	−3·916	−10	−4·283	1·608	−4·455
−1/6	−1·852	1·448	−3·964	−∞	−∞	1·714	−4·526
−1/3	−1·767	1·097	−4·074				

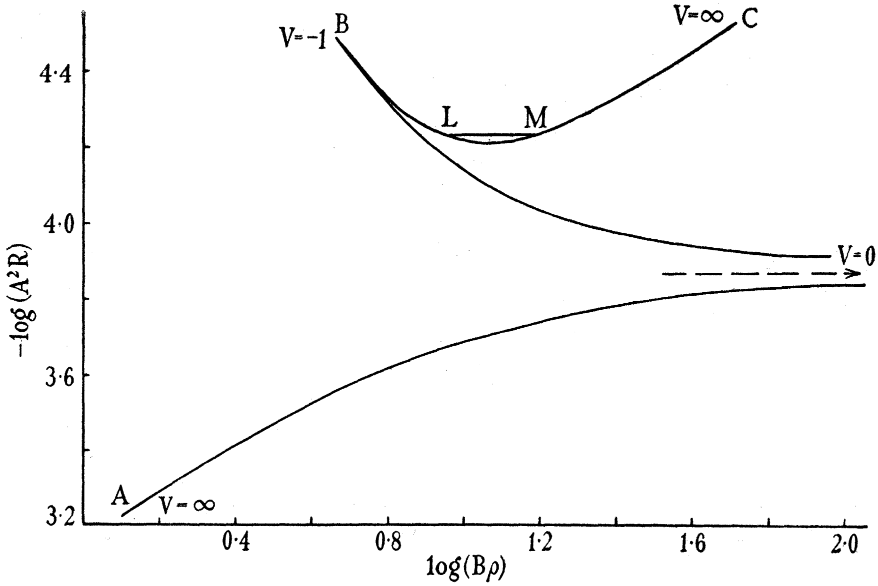

FIG. 3.

The graph of the relation between log (A^2R) and log (Bρ) is shown in fig. 3. To solve a particular problem, the reader must draw a large scale graph using the data of Table V. In this case there is no restriction on the values of U as there was on the values of V in § 3 of the paper.

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The values of U less than -1 have an interesting interpretation. We have seen that

$$\rho' = Q\rho,$$

so that

$$\frac{d\rho'}{d\rho} = Q + \rho\frac{dQ}{d\rho} = Q(1+U).$$

If $U < -1$ it follows that $d\rho'/d\rho < 0$. This can only occur if the cylinder is turned inside out. In this case the displacements will be given by

$$u = (1-Q)\,x, \qquad v = (1-Q)\,y, \qquad w = 2z.$$

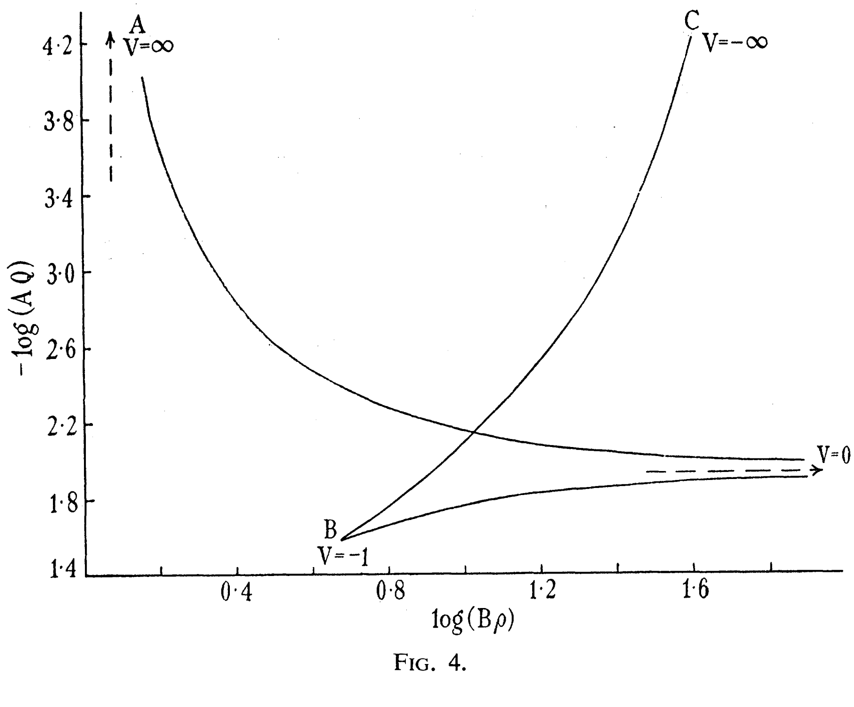

FIG. 4.

The addition of the term $w = 2z$ does not affect the stresses, so that the equations are the same as before. The part of the curve applicable in this case is the arc BC. The form of the curve in fig. 3, apart from the arc BC, is very similar to that in fig. 1, and the method of evaluating the constants A and B is precisely the same as in § 3, both in the ordinary case and when the tube is turned inside out.

The values of R, and thence $\widehat{\rho\rho}$, must be found from the graph in fig. 3, and the values of Q from the graph in fig. 4. The remaining stresses may now be calculated in terms of $\widehat{\rho\rho}$ and Q.

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We have

$$\widehat{\rho\rho} = (\lambda + \mu)(1 - Q^2) - \tfrac{1}{2}(\lambda + 2\mu) Q^2 U(U + 2), \tag{5.61}$$

$$\widehat{\theta\theta} = (\lambda + \mu)(1 - Q^2) - \tfrac{1}{2}\lambda Q^2 U(U + 2), \tag{5.62}$$

$$\widehat{zz} = \lambda(1 - Q^2) - \tfrac{1}{2}\lambda Q^2 U(U + 2), \tag{5.63}$$

and consequently

$$\widehat{\theta\theta} = 2\mu \cdot \frac{\lambda + \mu}{\lambda + 2\mu} \cdot (1 - Q^2) + \frac{\lambda}{\lambda + 2\mu}\widehat{\rho\rho} \tag{5.64}$$

$$\widehat{zz} = \widehat{\theta\theta} - \mu(1 - Q^2) = \eta(\widehat{\theta\theta} + \widehat{\rho\rho}). \tag{5.65}$$

It is of course impossible in practice to turn inside out a tube made from any of the actual materials for which $\eta = 0{\cdot}25$, so the values of $\log(AQ)$, $\log(B\rho)$, and $\log(A^2R)$ have been calculated for $\eta = 0{\cdot}49$ (india-rubber) for values of $U < -1$. This matter is more fully dealt with in § 7.

6—Deductions from the Solution for the Cylindrical Shell

As in § 4, it appears that there are restrictions on the boundary conditions. We find, using equations (5.22), (5.31), (5.41), that the gradient of the curve in fig. 3 is given by

$$\frac{d(\log R)}{d(\log r)} = \frac{\rho\, dR}{R\, d\rho} = -\frac{Q^2U(U + 2)(1 - 2\eta)}{R}. \tag{6.1}$$

Equations (5.61) and (5.62) show that

$$\widehat{\rho\rho} - \widehat{\theta\theta} = -\mu Q^2 U(U + 2),$$

so that

$$\frac{d(\log R)}{d(\log \rho)} = \frac{\widehat{\rho\rho} - \widehat{\theta\theta}}{\lambda + \mu - \widehat{\rho\rho}}. \tag{6.2}$$

On substituting for R from equation (5.41) in equation (6.1) we obtain

$$\frac{d(\log R)}{d(\log \rho)} = -\frac{U(U + 2)(1 - 2\eta)}{\eta + (1 - \eta)(U + 1)^2}. \tag{6.3}$$

At B, $U = -1$, so that

$$\frac{d(\log R)}{d(\log \rho)} = \frac{1 - 2\eta}{\eta}.$$

At A and C, $U = \pm\infty$ and

$$\frac{d(\log R)}{d(\log \rho)} = -\frac{1 - 2\eta}{1 - \eta}.$$

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It follows that

$$-\frac{1-2\eta}{1-\eta} < \frac{\widehat{\rho\rho}-\widehat{\theta\theta}}{\lambda+\mu-\widehat{\rho\rho}} < \frac{1-2\eta}{\eta}. \tag{6.4}$$

This condition is similar in type to that in the inequality (4.3) and again may be written in the more symmetrical form

$$\frac{\eta}{1-\eta} < \frac{\lambda+\mu-\widehat{\theta\theta}}{\lambda+\mu-\widehat{\rho\rho}} < \frac{1-\eta}{\eta}. \tag{6.5}$$

It is to be noted that in this case $\widehat{\rho\rho}-\widehat{\theta\theta}$ is not necessarily the greatest principal stress difference. Under certain conditions $\widehat{\rho\rho}-\widehat{zz}$ or $\widehat{\theta\theta}-\widehat{zz}$ may be numerically greater.

In the case of the cylindrical shell, as in that of the spherical shell, failure, if it occurs, must occur at the inner surface.

When the internal and external surface tractions are equal, we find that $U = 0$ and

$$\left(\frac{\rho'}{\rho}\right)^2 = Q^2 = R = 1 - \frac{\widehat{\rho\rho}}{\lambda+\mu}.$$

Both Q and R are constant throughout the material.

From the ordinary theory we obtain

$$Q^2 = \left\{1 - \frac{\widehat{\rho\rho}}{2(\lambda+\mu)}\right\}^2 = 1 - \frac{\widehat{\rho\rho}}{\lambda+\mu} + \left\{\frac{\widehat{\rho\rho}}{2(\lambda+\mu)}\right\}^2.$$

These values agree if $\{\widehat{\rho\rho}/2(\lambda+\mu)\}^2$ may be neglected in comparison with $\widehat{\rho\rho}/2(\lambda+\mu)$. This condition is very similar to that found at the end of § 3.

7—The Cylindrical Shell Turned Inside Out, Under No Tractions

When the tube is first turned inside out and then subjected to uniform internal and external surface tractions the problem may be solved in the way described in § 5. The simplest case of all is that in which the cylindrical surfaces are free from traction, when a zero increment in $\log(A^2R)$ corresponds to a given finite increment in $\log(B\rho)$. The corresponding chord of the curve (fig. 3) is of the type LM.

We can, however, go further than this and find a solution in which there is no resultant traction over the ends. If the length of the tube is great compared with its diameter and the ends are free, this solution will

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accurately represent the conditions at points of the tube not near the ends.

We may in this case assume displacements given by

$$u = x(1 - Q),$$
$$v = y(1 - Q),$$
$$w = \alpha z,$$

where Q is a function of ρ as before and α is a constant. The non-vanishing stress components now become

$$\widehat{\rho\rho} = \tfrac{1}{2}\lambda\,[3 - (\alpha - 1)^2 - Q^2\{1 + (U + 1)^2\}] + \mu\,[1 - Q^2(U + 1)^2], \quad (7.11)$$

$$\widehat{\theta\theta} = \tfrac{1}{2}\lambda\,[3 - (\alpha - 1)^2 - Q^2\{1 + (U + 1)^2\}] + \mu\,(1 - Q^2), \quad (7.12)$$

$$\widehat{zz} = \tfrac{1}{2}\lambda\,[3 - (\alpha - 1)^2 - Q^2\{1 + (U + 1)^2\}] + \mu\,[1 - (\alpha - 1)^2]. \quad (7.13)$$

One of the body stress equations is satisfied identically and the other two show that the relation between Q and ρ is that obtained in equation (5.21). Proceeding as in § 5, we obtain equations (5.23), (5.32), and (5.41) where in this case

$$R = 1 - \frac{\widehat{\rho\rho}}{\lambda + \mu} + \eta\{1 - (1 - \alpha)^2\}. \quad (7.2)$$

On giving η the value 0·49 we obtain the following equations for log (AQ), log (Bρ), and log (A^{2}R),

$$\log(AQ) = 0{\cdot}009902 \tan^{-1}(1{\cdot}010000\,U + 1{\cdot}019900) - \tfrac{1}{2}\log(U^2 + 2{\cdot}019608\,U + 2) \quad (7.31)$$

$$\log(B\rho) = \tfrac{1}{4}\log(U^2 + 2{\cdot}019608\,U + 2) - \tfrac{1}{4}\log(U^2) - 0{\cdot}500049\tan^{-1}(1{\cdot}010000\,U + 1{\cdot}019900) \quad (7.32)$$

$$\log(A^2R) = 2\log(AQ) + \log[(U + 1)^2 - 0{\cdot}960784] - 0{\cdot}673345, \quad (7.33)$$

where A and B are new constants replacing L and M. Only those values of U less than -1 are to be used and the values of log (AQ), log (Bρ), and log (A^{2}R) corresponding to chosen values of U are given in Table VI. The relations shown in Table VI are not illustrated graphically, as a more accurate method of calculation is necessary. The graphs are, however, similar in form to the arcs BC of figs. 3 and 4.

The boundary conditions are:

(i) $\widehat{\rho\rho} = 0$ on the surfaces $\rho = a$, $\rho = b$ $(b > a)$,

(ii) the ends are free from resultant traction, *i.e.*,

$$\int_a^b \rho\widehat{zz}\,d\rho = 0.$$

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TABLE VI

U	log (AQ)	log (Bρ)	log (A^2R)
−1·0	0·00991	0·00000	−0·69353
−1·5	−0·10422	0·07698	−0·69052
−1·6	−0·14740	0·10484	−0·68991
−1·75	−0·21840	0·14723	−0·68927
−2·0	−0·34445	0·21452	−0·68890
−2·2	−0·44576	0·26281	−0·68908
−2·5	−0·59148	0·32495	−0·68978
−2·545	−0·61243	0·33326	−0·68991
−3·0	−0·80979	0·40523	−0·69135
−4·0	−1·15974	0·50611	−0·69418
−5·0	−1·42686	0·56599	−0·69616
−6·0	−1·64040	0·60504	−0·69755
−10·0	−2·21663	0·68041	−0·70037
−∞	−∞	0·78547	−0·70445

The boundary condition (i) implies that an increment of amount log (b/a) in log (Bρ) corresponds to a zero increment in log (A^2R). In order to evaluate A, we must first find the value of $(1-\alpha)^2$, and this is to be done by the use of boundary condition (ii). On using equation (7.13) and the relation

$$\frac{\rho}{Q}\frac{dQ}{d\rho} = U,$$

condition (ii) gives

$$\frac{1}{2\eta}(b^2-a^2)\left[1+\eta-(1-\eta)(1-\alpha)^2\right]$$
$$-\left[b^2Q_b{}^2-a^2Q_a{}^2\right] = \tfrac{1}{2}\int_{a^2}^{b^2} U^2Q^2\,d(\rho^2), \quad (7.41)$$

where Q_a and Q_b are the values of Q when $\rho = a$ and $\rho = b$ respectively.

On the surfaces $\rho = a$ and $\rho = b$ we have, since $\widehat{\rho\rho} = 0$, $R = R_0$ where

$$R_0 = 1+\eta-\eta(1-\alpha)^2.$$

It follows that

$$A^2 = A^2R_0/\{1+\eta-\eta(1-\alpha)^2\}. \quad (7.42)$$

From equations (7.41) and (7.42) we now obtain

$$\frac{A^2R_0(b^2-a^2)\{1+\eta-(1-\eta)(1-\alpha)^2\}}{2\eta\{1+\eta-\eta(1-\alpha)^2\}}$$
$$= b^2A^2Q_b{}^2 - a^2A^2Q_a{}^2 + \tfrac{1}{2}\int_{a^2}^{b^2} A^2Q^2U^2\,d(\rho^2). \quad (7.43)$$

Owing to the fact that this equation involves small differences between

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relatively large numbers, it is necessary to obtain greater accuracy than is possible by the graphical methods used so far. By a method of trial and error, it is found that the values $U = -1 \cdot 6$ and $U = -2 \cdot 545$ each give $\log(A^2 R) = -0 \cdot 68991$. These values of U are taken as the boundary values and if we suppose that $a = 1$ then we find that $b = 1 \cdot 2566$. The boundary values of AQ are also found from the table. In order to evaluate α from equation (7.43), we must now find the value of the integral in that equation. The values of $A^2Q^2U^2$ corresponding to the values of U in the table were calculated and plotted in a graph against ρ^2. The values of $A^2Q^2U^2$ were then read from the graph for 23 equidistant values of ρ^2. These values were then smoothed by taking differences and the integral obtained by Simpson's Rule. This process was repeated to check the value. This method gives greater accuracy than might be expected owing to the small range of variation of $A^2Q^2U^2$. In this way we obtain the result

$$\frac{1 + \eta - (1 - \eta)(1 - \alpha)^2}{1 + \eta - \eta(1 - \alpha)^2} = 0 \cdot 98009.$$

From this it follows that

$$(1 - \alpha)^2 = 0 \cdot 9970,$$
$$\alpha = 1 \cdot 9985.$$

Equation (7.42) now gives the result

$$A^2 = A^2R_0/\{1 + \eta - \eta(1 - \alpha)^2\} = 0 \cdot 5009.$$

With these values of A and $(1 - \alpha)^2$ we can use Table VI and equations (7.11), (7.12), and (7.13) to obtain the results given in Table VII.

TABLE VII

U	ρ	$\widehat{zz}/\lambda$	$\widehat{\rho\rho}/\lambda$	$\widehat{\theta\theta}/\lambda$
−1·6	1·0	−0·0094	−0·0000	−0·0194
−1·75	1·0438	−0·0062	−0·0007	−0·0122
−2·0	1·1159	−0·0009	−0·0011	−0·0011
−2·2	1·1712	0·0028	−0·0010	0·0064
−2·545	1·2566	0·0082	−0·0000	0·0165

The values of Q on the inner and outer boundaries are 1·2194 and 0·7659, so that we obtain Table VIII.

TABLE VIII

	Internal radius	External radius	Thickness	Length
Right side out	0·962	1·219	0·257	1·0000
Inside out	1·000	1·257	0·257	1·0015

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The percentage increase in external radius is about 3%, and this is approximately verified by measurement. The measurement must, of course, be made at some distance from the ends which naturally turn outwards when the tube is inside out. This turning outwards of the ends is accompanied by a narrowing of the tube near the ends. This is, of course, well known but is merely an end effect.

8—Applications to Thin Shells

When applied to thin shells, the foregoing theory is considerably simplified. Consider a spherical shell of radius a and small thickness d subjected to pressures p and $p + p_1$ on the inside and outside respectively.

In equation (4.1) we have the result

$$\frac{r\,dR}{R\,dr} = -\frac{\{(V+1)^2 - 1\}\,2(1-2\eta)}{2\eta + (1-\eta)(1+V)^2}. \tag{8.1}$$

But

$$R = 1 + \frac{2p}{3\lambda + 2\mu}, \qquad dR = \frac{2p_1}{3\lambda + 2\mu},$$

$$r = a, \qquad dr = d,$$

so that

$$\frac{r\,dR}{R\,dr} = \frac{2p_1}{3\lambda + 2\mu + 2p} \cdot \frac{a}{d} = m, \quad \text{say}. \tag{8.2}$$

Since the gradient of the graph in fig. 1 is bounded it follows that $p_1/(3\lambda + 2\mu)$ is small of the order d/a.

From equations (8.1) and (8.2) we obtain

$$(1+V)^2 = \frac{2(1-2\eta) - 2m\eta}{2(1-2\eta) + m(1-\eta)}. \tag{8.3}$$

On eliminating $(1 + V)$ between equations (8.3) and (3.42) and putting $R = 1 + 2p/(3\lambda + 2\mu)$, we obtain the value of P, given by

$$\left.\begin{aligned} P^2 &= \left\{1 + \frac{2p}{3\lambda + 2\mu}\right\}\left\{1 + \frac{m(1-\eta)}{2(1-2\eta)}\right\}, \\ \text{i.e.,}\quad P^2 &= 1 + \frac{2p}{3\lambda + 2\mu} + \frac{ap_1}{d(3\lambda + 2\mu)} \cdot \frac{1-\eta}{1-2\eta}. \end{aligned}\right\} \tag{8.4}$$

Since $P = r'/r$ this equation gives the change in the radius of the shell due to the strain.

The stress $\widehat{\theta\theta}$ $(= \widehat{\phi\phi})$ may be obtained by substituting from equation

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(8.4) into equation (3 73) or by consideration of the equilibrium of a portion of the shell and the result is

$$\widehat{\theta\theta} = -p - \tfrac{1}{2}p_1 a/d.$$

The limitations on the gradient of the graph in fig. 1 take the form

$$-\frac{2(1-2\eta)}{1-\eta} < \frac{2p_1}{3\lambda+2\mu+2p}\frac{a}{d} < \frac{1-2\eta}{\eta}.$$

Consider the two cases (i) zero pressure inside and pressure Π outside, and (ii) zero pressure outside and pressure Π inside. In case (i) the first part of the inequality gives no information but the second part leads to

$$\Pi < \frac{1-2\eta}{2\eta}(3\lambda+2\mu)\frac{d}{a}. \tag{8.51}$$

In case (ii) the second part of the inequality gives no information, but the first part leads to

$$\frac{\Pi}{3\lambda+2\mu+2\Pi} < \frac{d}{a}\frac{1-2\eta}{1-\eta},$$

or, since Π is small compared with $3\lambda + 2\mu$,

$$\Pi < \frac{1-2\eta}{1-\eta}(3\lambda+2\mu)\frac{d}{a}. \tag{8.52}$$

The solution for the cylindrical shell, in a state of plane strain, is very similar to the solution for the spherical shell. Again we suppose that the radius is a, the thickness d, the internal pressure p, and the external pressure $p + p_1$. We obtain the results

$$\left(\frac{\rho'}{\rho}\right)^2 = Q^2 = 1 + \frac{p}{\lambda+\mu} + \frac{a}{d}\frac{p_1}{\lambda+\mu}\cdot\frac{1-\eta}{1-2\eta}, \tag{8.61}$$

$$\widehat{\theta\theta} = -p - p_1 a/d, \tag{8.62}$$

$$\widehat{zz} = \eta(\widehat{\theta\theta} + \widehat{\rho\rho}) = -\eta(2p + p_1 a/d). \tag{8.63}$$

The limitations on the gradient (fig. 3) take the form

$$-\frac{1-2\eta}{1-\eta} < \frac{p_1}{\lambda+\mu+p}\frac{a}{d} < \frac{1-2\eta}{\eta}.$$

Again, consider the two cases (i) zero pressure inside and pressure Π

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outside, and (ii) zero pressure outside and pressure Π inside. Case (i) leads to the condition

$$\Pi < \frac{1-2\eta}{\eta}(\lambda+\mu)\frac{d}{a}, \tag{8.71}$$

and case (ii) leads to

$$\Pi < \frac{1-2\eta}{1-\eta}(\lambda+\mu)\frac{d}{a}. \tag{8.72}$$

In conclusion, our thanks are due to Professor Filon, through whose interest in the subject of finite strain we were led to undertake this work.

On the Quantization of a Theory Arising from a Variational Principle for Multiple Integrals with Application to Born's Electrodynamics

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(*Communicated by P. A. M. Dirac, F.R.S.—Received* 18 *March*, 1936)

1—Introduction

The transition from classical mechanics to quantum mechanics has been formulated in various ways. But all these ways have the common feature that they use as starting point mechanics not in its Newtonian form, but in the form which it was given by Lagrange and Hamilton. The essence of this form consists in the fact that the differential equations of motion for a dynamical system with ν degrees of freedom can be and are represented as Euler equations of a variational principle for ν functions, the generalized coordinates, of one variable, the time.

In its purely mathematical aspect the quantization method may, therefore, be regarded as a mode of taking certain notions and certain relations arising from a variational principle for several dependent and one independent variable and of giving them a new meaning.

The partial differential equations of an electromagnetic field, as formulated by Maxwell or as generalized by Mie† and recently by Born,‡ can

† 'Ann. Physik,' vol. 37, p. 511; vol. 39, p. 1; vol. 40, p. 1 (1912–13).

‡ Born, 'Proc. Roy. Soc.,' A, vol. 143, p. 426 (1934), quoted as I. Born and Infeld, 'Proc. Roy. Soc.,' A, vol. 144, p. 425 (1934), quoted as II.